Quantum Markovian Subsystems: Invariance, Attractivity, and Control

Francesco Ticozzi and Lorenza Viola

Abstract—We characterize the dynamical behavior of continuous-time, Markovian quantum systems with respect to a subsystem of interest. Markovian dynamics describes a wide class of open quantum systems of relevance to quantum information processing, subsystem encodings offering a general pathway to faithfully represent quantum information. We provide explicit linear-algebraic characterizations of the notion of invariant and noiseless subsystem for Markovian master equations, under different robustness assumptions for model-parameter and initial-state variations. The stronger concept of an attractive quantum subsystem is introduced, and sufficient existence conditions are identified based on Lyapunov's stability techniques. As a main control application, we address the potential of output-feedback Markovian control strategies for quantum pure state-stabilization and noiseless-subspace generation. In particular, explicit results for the synthesis of stabilizing semigroups and noiseless subspaces in finite-dimensional Markovian systems are obtained.

Index Terms—Noiseless subsystems (NSs), quantum information processing (QIP).

I. INTRODUCTION

QUANTUM subsystems are the basic building block for describing composite systems in quantum mechanics [1]. From both a conceptual and practical standpoint, renewed interest toward characterizing quantum subsystems in a variety of control-theoretic settings is motivated by Quantum Information Processing (QIP) applications [2]. In order for abstractly defined QIP protocols to be useful, information needs to be represented by states of a physical system, in ways which minimize the impact of errors and decoherence due to the interaction of the system with its surrounding environment. *Subsystem encodings* provide the most general mathematical structure for realizing quantum information in terms of physical degrees of freedom, and a main tool for achieving a unified understanding of strategies for quantum error control in *open* quantum systems [3], [4]. In particular, the idea that noise-protected subsystems may be identified in the overall state space of a noisy physical system under appropriate symmetry conditions underlies the method of passive quantum stabilization

via *decoherence-free subspaces* (DFSs) [5], [6] and *noiseless subsystems* (NSs) [3], [7], [8]. In situations where no such symmetry exists, open-loop dynamical decoupling techniques can still ensure active protection through dynamical NS synthesis [9] in the *non-Markovian* regime.

While substantial progress has been made toward defining and exploiting subsystems within general quantum error correction theories (see also Section II-A), the subsystem notion and its implications have not yet reached out to the quantum control community at large. It is one of the goals of this paper to take a step in this direction, by offering a *linear-algebraic* characterization of the subsystem idea, which allows easier contact with standard system-theoretic concepts such as invariance and stability. We focus on *continuous-time Markovian* quantum dynamics [10], which both accurately describes a wide class of open quantum systems and presents distinctive quantum stabilization challenges compared to its non-Markovian counterpart. In the process, we elucidate the key role played by different *model robustness* properties in ensuring that desired dynamical features may emerge, and in influencing the interplay between Hamiltonian and dissipative components. Beside leading to a streamlined derivation of Markovian DFS- and NS-conditions which have only partially appeared in the literature [8], [11], our analysis motivates the concept of *attractive subsystem* as a strategy for "dissipation-assisted" asymptotic initialization in an intended subsystem. As a main application of our work, we begin exploring the potential of Markovian, output-feedback techniques for the robust synthesis of pure states and NSs in finite-dimensional systems—complementing the above-mentioned open-loop dynamical schemes [9] as well as closed-loop feedback approaches using continuous-time state estimation [12].

A. Notations

Consider a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. $\mathfrak{B}(\mathcal{H})$ represents the set of linear bounded operators on $\mathcal{H}$, $\mathfrak{H}(\mathcal{H})$ denoting the real subspace of Hermitian operators, with $\mathbb{I}$, $\mathbb{O}$ being the identity and the zero operator, respectively. We indicate with $A^\dagger$ the adjoint of $A \in \mathfrak{B}(\mathcal{H})$, with c^* the conjugate of $c \in \mathbb{C}$. The *commutator* and *anti-commutator* of $X, Y \in \mathfrak{B}(\mathcal{H})$ are defined as $[X, Y] := XY - YX$, and $\{X, Y\} := XY + YX$, respectively. In the special case where $\mathcal{H}$ is two-dimensional, a convenient operator basis for the traceless sector of $\mathfrak{B}(\mathcal{H})$ is given by the Pauli operators, $\sigma_\alpha, \alpha = x, y, z$, satisfying $[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$, $\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}\mathbb{I}$, ε, δ denoting the completely antisymmetric tensor and the Kronecker delta, respectively. We choose the standard representation where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Throughout the paper we shall use the Dirac notation [1]. Given the inner product $\langle , \rangle$ in $\mathcal{H}$, a natural isomorphism exists between vectors in $\mathcal{H}$ (denoted $|\psi\rangle$, and called a *ket*) and linear functionals in the dual space $\mathcal{H}^*$ (denoted $\langle\psi|$, and called a *bra*), so that $\langle\psi \,|\, \varphi\rangle = \langle\psi, \varphi\rangle$. If $A \in \mathfrak{B}(\mathcal{H})$, letting $A|\psi\rangle := A(\psi)$ and $\langle\psi|A := \langle A^\dagger(\psi), \cdot\rangle$ yields $\langle\psi|A|\varphi\rangle = \langle\phi, A(\varphi)\rangle = \langle A^\dagger(\phi), \varphi\rangle$. The "outer" product $|\psi\rangle\langle\varphi|$ stands for $\langle\varphi, \cdot\rangle\psi \in \mathfrak{B}(\mathcal{H})$. Moreover, if $|\langle\psi \,|\, \psi\rangle| = 1$, $|\psi\rangle\langle\psi| = \langle\psi, \cdot\rangle\psi$ is the orthogonal projector onto the one-dimensional subspace spanned by $|\psi\rangle$.

B. Basic Notions of Statistical Quantum Mechanics

In the standard formulation of quantum mechanics [1], a quantum system $\mathcal{Q}$ is associated with a separable Hilbert space $\mathcal{H}$, whose dimension is determined by the physics of the problem. *Physical observables* are modeled as self-adjoint operators on $\mathcal{H}$, the set of possible outcomes they can assume in a von Neumann measurement process being their spectrum. In what follows, we consider only observables with *finite* spectrum, that can be represented as Hermitian matrices acting on $\mathcal{H} \simeq \mathbb{C}^d, d < \infty$. Our (possibly uncertain) knowledge of the state of $\mathcal{Q}$ is condensed in a *density operator* ρ, with $\rho \geq 0$ and $\text{trace}(\rho) = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}(\mathcal{H})$, with one-dimensional projectors corresponding to extreme points (*pure states*, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$).

If $\mathcal{Q}$ comprises two quantum *subsystems* $\mathcal{Q}_1, \mathcal{Q}_2$, the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ [1], observables and density operators being associated with Hermitian, positive-semidefinite, normalized operators on $\mathcal{H}_{12}$, respectively. In particular, a joint pure state $\rho_{12} = |\psi\rangle_{12}\langle\psi|$ which cannot be factorized into the product of two pure states on the individual factors is called *entangled*. Let X be an element of $\mathfrak{B}(\mathcal{H}_{12})$. The *partial trace* over $\mathcal{H}_2$ is the unique linear operator $\text{trace}_2(\cdot) : \mathfrak{B}(\mathcal{H}_{12}) \to \mathfrak{B}(\mathcal{H}_1)$, ensuring that for every $X_1 \in \mathfrak{B}(\mathcal{H}_1), X_2 \in \mathfrak{B}(\mathcal{H}_2)$, $\text{trace}_2(X_1 \otimes X_2) = X_1\text{trace}(X_2)$. If ρ_{12} represents the joint density operator of $\mathcal{Q}$, the state of, say, subsystem $\mathcal{Q}_1$ alone is described by the *reduced density operator* $\rho_1 = \text{trace}_2(\rho_{12}) \in \mathfrak{H}(\mathcal{H}_1)$, so that $\text{trace}(\rho_1 X_1) = \text{trace}(\rho_{12}X_1 \otimes \mathbb{I}_2)$, for all observables $A_1 \in \mathfrak{H}(\mathcal{H}_1)$.

We consider quantum dynamics in the Schrödinger picture, with pure states and density operators evolving forward in time, and time-invariant observables. The evolution of an *isolated* (closed) quantum system is driven by the Hamiltonian, H, according to the Liouville-von Neumann equation:

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)],$$

in units where $\hbar = 1$. Thus, $\rho(t) = U_t\rho(0)U_t^\dagger$, where the unitary evolution operator (or *propagator*) $U_t = e^{-iHt}$.

In general, in the presence of internal couplings, quantum measurements, or interaction with a surrounding environment, the evolution of a subsystem of interest is no longer unitary and reversible, and the general formalism of *open quantum systems* is required [13], [10], [14]. The physically admissible discrete-time evolutions for a quantum system may be characterized axiomatically and are called *quantum operations*, or completely positive (CP) maps [15]. Let $\mathcal{I}$ denote the physical

quantum system of interest, with associated Hilbert space $\mathcal{H}_I$, $\dim(\mathcal{H}_I) = d$. The class of trace-preserving (TP) quantum operations is relevant to our purposes:

Definition 1 (TPCP Map): A TPCP-map $\mathcal{T}(\cdot)$ on $\mathcal{I}$ is a map on $\mathfrak{D}(\mathcal{H}_I)$, that satisfies

i) $\mathcal{T}(\cdot)$ is convex linear: given states $\rho_i \in \mathfrak{D}(\mathcal{H}_I)$

$$\mathcal{T}\left(\sum_i p_i\rho_i\right) = \sum_i p_i\mathcal{T}(\rho_i), \quad \sum_i p_i = 1, p_i \geq 0, \quad \forall i;$$

ii) $\mathcal{T}(\cdot)$ is trace-preserving:

$$\text{trace}(\mathcal{T}(\rho_I)) = \text{trace}(\rho_I) = 1;$$

iii) $\mathcal{T}(\cdot)$ is completely positive:

$$\mathcal{T}(\rho_I) \geq 0, \quad (\mathbb{I}_m \otimes \mathcal{T})(\rho_{IE}) \geq 0$$

for every m-dimensional ancillary space $\mathcal{H}_E$ joint to $\mathcal{H}_I$, and for every $\rho_{IE} \in \mathfrak{D}(\mathcal{H}_I \otimes \mathcal{H}_E)$[1] .

A more concrete characterization of the dynamical maps of interest is provided by the following:

Theorem 1 (Hellwig-Kraus Representation Theorem): $\mathcal{T}[\cdot]$ is a TPCP map on $\mathcal{I}$ iff for every $\rho_I \in \mathfrak{D}(\mathcal{H}_I)$:

$$\mathcal{T}(\rho_I) = \sum_k E_k\rho_I E_k^\dagger$$

where $\{E_k\}$ is a family of operators in $\mathfrak{B}(\mathcal{H}_I)$ such that $\sum_k E_k^\dagger E_k = \mathbb{I}$.

As a consequence of the above Theorem, every TPCP map $\mathcal{T}(\cdot)$ on $\mathfrak{D}(\mathcal{H}_I)$ may be extended to a well-defined linear, positive, and TP map on the whole $\mathfrak{B}(\mathcal{H}_I)$.

C. Quantum Dynamical Semigroups

A relevant class of open quantum systems obeys Markovian dynamics [10], [16], [17]. Assume that we have no access to the quantum environment surrounding $\mathcal{I}$, and that the dynamics in $\mathfrak{D}(\mathcal{H}_I)$ is continuous in time, the state change at each $t > 0$ being described by a TPCP map $\mathcal{T}_t(\cdot)$. A differential equation for the density operator of $\mathcal{I}$ may be derived provided that a forward composition law holds:

Definition 2 (QDS): A *quantum dynamical semigroup* is a one-parameter family of TPCP maps $\{\mathcal{T}_t(\cdot), t \geq 0\}$ that satisfies:

i) $\mathcal{T}_0 = \mathbb{I}$;
ii) $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}, \forall t, s > 0$;
iii) $\text{trace}(\mathcal{T}_t(\rho)X)$ is a continuous function of t, $\forall \rho \in \mathfrak{D}(\mathcal{H}_I), \forall X \in \mathcal{B}(\mathcal{H}_I)$.

Due to the trace and positivity preserving assumptions, a QDS is a semigroup of contractions[2] . It has been proved [16], [18] that the Hille-Yoshida generator for a QDS [19] exists and can be cast in the following canonical form:

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \sum_{k,l=1}^{d^2-1} \mathcal{L}_{kl}(\rho(t)) \tag{1}$$

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[1]The CP assumption is necessary to preserve positivity of arbitrary purifications of states on $\mathcal{H}_I$, including entangled states, see e.g., [2] for a discussion of a well-known counter-example, based on the transpose operation.

[2]A map f from a metric space $\mathfrak{M}$ with distance $d(\cdot, \cdot)$ to itself is a *contraction* if $d(f(X), f(Y)) \leq kd(X, Y)$ for all X, Y in $\mathfrak{M}, 0 < k \leq 1$.

$$= -i[H, \rho(t)] + \frac{1}{2} \sum_{k,l=1}^{d^2-1} a_{kl} \times (2F_k \rho(t) F_l^\dagger - \{F_k^\dagger F_l, \rho(t)\})$$

where $\{F_k\}_{k=0}^{d^2-1}$ is a basis of $\mathfrak{B}(\mathcal{H}_I)$, the space of linear operators on $\mathcal{H}_I$, with $F_0 = \mathbb{I}$. The positive definite $(d^2 - 1)$-dimensional matrix $A = (a_{kl})$, which physically specifies the relevant relaxation parameters, is also called the Gorini-Kossakowski-Sudarshan (GKS) matrix. Thanks to the Hermitian character of A, (1) can be rewritten in a symmetrized form, moving to an operator basis that diagonalizes A

$$\mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \sum_{k=1}^{d^2-1} \gamma_k \mathcal{D}(L_k, \rho(t)) \qquad (2)$$

$$= -i[H, \rho(t)] + \sum_{k=1}^{d^2-1} \gamma_k \times \left(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\} \right)$$

with $\{\gamma_k\}$ denoting the spectrum of A. The *effective Hamiltonian* H and the *Lindblad operators* L_k specify the net effect of the Markovian environment on the dynamics. In general, H is equal to the isolated system Hamiltonian, H_0, plus a correction, H_L, induced by the coupling to the environment (so-called Lamb shift). The non-Hamiltonian terms $\mathcal{D}(L_k, \rho(t))$ in (2) account for non-unitary dynamics induced by L_k.

D. Phenomenological Markovian Models and Robustness

In principle, the exact form of the generator of a QDS may be rigorously derived from the underlying Hamiltonian model for the joint system-environment dynamics under appropriate limiting conditions (the so-called "singular coupling limit" or the "weak coupling limit," respectively [10], [14]). In most physical situations, however, carrying out such a procedure is unfeasible, typically due to lack of complete knowledge of the full microscopic Hamiltonian. A Markovian generator of the form (1) is then assumed on a phenomenological basis[3]. In practice, it is often the case that direct knowledge of the noise effect is available, allowing one to specify the Markovian generator by either giving a GKS matrix in (1) or a set of noise strengths γ_k and Lindblad operators L_k (not necessarily orthogonal or complete) in (2). Each of the noise operators L_k may be thought of as corresponding to a distinct *noise channel* $\mathcal{D}(L_k, \rho(t))$, by which information irreversibly leaks from the system into the environment.

Example 1: Consider a two-level atom, with ground and excited states $|\psi_g\rangle, |\psi_e\rangle$, respectively. Assume that there is an average rate of decay from the excited to the ground state $\gamma > 0$, that is, the survival probability of the excited state $|\psi_e\rangle$ decays as $e^{-\gamma t}$. The resulting dynamics is well described by a semigroup master equation of the form

$$\frac{d}{dt}\rho(t) = -i\omega[\sigma_z, \rho(t)] + \frac{\gamma}{2}(2\sigma_-\rho(t)\sigma_+ - \{\sigma_+\sigma_-, \rho(t)\})$$

[3]The Markovian generator can be inferred from experimentally available data via quantum process tomography, see e.g., [2]. By measuring the effect of the environment on known states for fixed times, one may reconstruct the corresponding set of CP maps and the underlying infinitesimal generator. Special care must be paid to the fact that this procedure may lead to non-CP maps in the presence of measurement or numerical errors [20]. In some situations, approximate models may be obtained upon formal quantization of a classical master equation, e.g., the so-called Pauli master equation [10].

where $\omega > 0$ determines the energy splitting between the ground and excited state, and $\sigma_- = |\psi_g\rangle\langle\psi_e|$, $\sigma_+ = \sigma_-^\dagger$ are pseudo-spin lowering and raising operators, respectively. In fact, computing the probability $\mathbb{P}_e(t) = \text{trace}(\rho(0)\rho(t))$, with initial state $\rho(0) = |\psi_e\rangle\langle\psi_e|$, yields $\mathbb{P}_e(t) = e^{-\gamma t}$.

In the above example, a single noise channel, $\mathcal{D}(\sigma_-, \rho(t))$, is relevant. In physical situations, it often happens that known properties of the error process naturally restrict the relevant Lindblad operators or the admissible GKS descriptions to a *reduced form* which incorporates the existing constraints. A paradigmatic QIP-motivated example is the following:

Example 2: Consider a *quantum register* $\mathcal{Q}$, that is, a quantum system composed by q two-dimensional systems (*qubits*), with associated state $\mathcal{H}_Q = \mathbb{C}^2_{(1)} \otimes \cdots \otimes \mathbb{C}^2_{(q)}, d = 2^q$. For an arbitrary Markovian error process, combined errors on any subset of qubits may occur, corresponding to an error basis $\{F_k\}$ which spans the full traceless sector of $\mathfrak{B}(\mathcal{H}_Q)$ or, equivalently, the $(d^2 - 1)$-dimensional Lie algebra $\mathfrak{su}(d)$. Under the assumption that *linear decoherence* takes place, errors can independently affect at most one qubit at the time, reducing the relevant error set to operators of the form [6], [3]

$$F_{k,l} = \mathbb{I}^{(1)} \otimes \cdots \mathbb{I}^{(k-1)} \otimes \sigma_l^{(k)} \otimes \mathbb{I}^{(k+1)} \otimes \cdots \otimes \mathbb{I}^{(q)}$$

where $k = 1, \ldots, q$, and $l = x, y, x$. Completing these $3q$ orthonormal generators to the above basis for the traceless operators in $\mathfrak{B}(\mathcal{H}_Q)$, (1) formally holds. Clearly, the resulting matrix A differs from zero only in a $(3q \times 3q)$-dimensional block. If the noise process is additionally restricted to obey permutational symmetry (so-called *collective decoherence*), the relevant error set is further reduced to completely symmetric generators of the form

$$F_l = \sum_{k=1}^{q} \mathbb{I}^{(1)} \otimes \cdots \otimes \sigma_l^{(k)} \otimes \cdots \otimes \mathbb{I}^{(q)}, \quad l = x, y, z$$

in which case $\text{span}\{F_k\} \simeq \mathfrak{su}(2)$, and A may effectively be taken as a 3×3 positive-definite matrix.

The above examples show how, in practice, a compact version of (1) typically suffices, in terms of (orthonormal) error generators $\{F_k\}$ spanning a m-dimensional *error subalgebra*, $m \leq d^2 - 1$. The corresponding Markovian generator in (1) is then completely specified by a *reduced GKS matrix* of dimension $m \times m$.

In the following sections, we are interested at characterizing dynamical properties of finite-dimensional QDSs in terms of their generator. As in most situations only limited or approximate knowledge about the model is available, we are naturally led to consider two kinds of *structured robustness* [21], [22]:

Definition 3 (Model Robustness): Assume that a system $\mathcal{I}$ undergoes QDS dynamics, under a nominal generator of the form (1) or (2), with uncertain knowledge of the parameters $A = (a_{jk})_{j,k=1}^m$ or $\Gamma = (\gamma_1, \ldots, \gamma_m)$. Let $\mathcal{A}$ and $\mathcal{V}$ denote the uncertainty sets, that is, the sets of parameters identifying the admissible models in the form (1) and (2), respectively. A property $\mathfrak{P}$ is said to be

  i) *A-robust* if it holds for every $A = (a_{ij}) \in \mathcal{A}$;
  ii) *γ-robust* if it holds for every $\Gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{V}$.

The study of A- or γ-robustness is important to establish whether a desired feature of the model (e.g., invariance of a subsystem, existence of attractive states) may be ensured by avoiding *fine-tuning* on the noise parameters [6].

Remarks: A-robustness implies γ-robustness. However the converse is not true. In fact, γ-robustness corresponds to robustness only with respect to variation in the spectrum of A. While studying A-robustness, we shall always imply a reduced description as explained above, with $\mathcal{A} = \{A = (a_{k,l})_{k,l=1}^m\}$ denoting the set of relevant reduced GKS matrices. Clearly, only properties that are completely independent of the particular noise model can be A-robust if $\mathcal{A}$ is the whole $\mathcal{B}(\mathcal{H}_I)$.

II. THEORY

A. Quantum Subsystems and Their Role in QIP

In order for the physical system $\mathcal{I}$ to implement a QIP task, it is necessary that at every point in time well-defined "logical" degrees of freedom exist, which carry the desired quantum information and support a basic set of control capabilities. Within the standard quantum network model [2], such set must include the ability to:

- **Unitary control**: Implement a set of control actions that ensure universal control.
- **Initialization**: Realize a quantum operation that prepares the system in an intended pure state.
- **Read-out**: Perform measurements of appropriate system's observables[4].

According to the *subsystem principle* [7], [4], the most general structure which can faithfully embody quantum information is a subsystem of $\mathcal{I}$. Intuitively, a subsystem may be thought of as a "portion" of the full physical system, whose states, in the simplest setting, obey *perfectly* the criteria above. Logical subsystems may or may not directly coincide with physically natural degrees of freedom. If $\mathcal{I}$ is noisy, in particular, it suffices that the action of noise be either negligible or correctable on subsystem where the information resides. Thus, protecting information need not require the full state of the physical system to be immune to noise, although this typically involves encodings which are entangled with respect to the natural subsystem degrees of freedom. A paradigmatic example is the protected qubit encoded in three spin-$1/2$ particles subject to collective decoherence [3], [7].

Formally, the following definition is suitable to QIP settings:

Definition 4 (Quantum Subsystem): A *quantum subsystem* $\mathcal{S}$ of a system $\mathcal{I}$ defined on $\mathcal{H}_I$ is a quantum system whose state space is a tensor factor $\mathcal{H}_S$ of a subspace $\mathcal{H}_{SF}$ of $\mathcal{H}_I$

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R \tag{3}$$

for some factor $\mathcal{H}_F$ and remainder space $\mathcal{H}_R$. The set of linear operators on $\mathcal{S}$, $\mathcal{B}(\mathcal{H}_S)$, is isomorphic to the (associative) algebra on $\mathcal{H}_I$ of the form $X_I = X_S \otimes \mathbb{I}_F \oplus \mathbb{O}_R$.

Let $n = \dim(\mathcal{H}_S)$, $f = \dim(\mathcal{H}_F)$, $r = \dim(\mathcal{H}_R)$, and let $\{|\phi_j^S\rangle\}_{j=1}^n, \{|\phi_k^F\rangle\}_{k=1}^f, \{|\phi_l^R\rangle\}_{l=1}^r$ denote orthonormal bases

[4]In the simplest setting, the ability to effect strong, von Neumann projective measurements is assumed, which together with unitary control implies the ability to initialize the state.

for $\mathcal{H}_S, \mathcal{H}_F, \mathcal{H}_R$, respectively. Decomposition (3) is then naturally associated with the following basis for $\mathcal{H}_I$:

$$\{|\varphi_m\rangle\} = \left\{\left|\phi_j^S\right\rangle \otimes \left|\phi_k^F\right\rangle\right\}_{j,k=1}^{n,f} \cup \left\{\left|\phi_l^R\right\rangle\right\}_{l=1}^r.$$

This basis induces a block structure for matrices acting on $\mathcal{H}_I$:

$$X = \left(\begin{array}{c|c} X_{SF} & X_P \\ \hline X_Q & X_R \end{array} \right) \tag{4}$$

where, in general, $X_{SF} \neq X_S \otimes X_F$. Let Π_{SF} be the projection operator onto $\mathcal{H}_S \otimes \mathcal{H}_F$, that is, $\Pi_{SF} = (\mathbb{I}_{SF} \mid 0)$.

For a noisy system $\mathcal{I}$, the goal of *passive* quantum error control is to identify subsystems of $\mathcal{I}$ where the dominant error events have minimum (ideally no) effect. Loosely speaking, each error operator belonging to the fixed error set for which protection is sought must have an "identity action" once appropriately restricted to the intended subsystem. Historically, the first kind of subsystems considered to this purpose have been *noiseless subspaces* of the system's Hilbert space, often called DFSs in the relevant literature [5], [6]

$$\mathcal{H}_I = \mathcal{H}_{\text{DFS}} \oplus \mathcal{H}_R$$

which corresponds to a special instance of decomposition (3) with 1-D "syndrome" co-subsystem $\mathcal{F}$, $\mathcal{H}_F \simeq \mathbb{C}$. The possibility for genuine *noiseless subsystem*-encodings to exist and be useful was recognized in [3], in which case we specialize the notation of (3) to

$$\mathcal{H}_I = (\mathcal{H}_{\text{NS}} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R \tag{5}$$

by explicitly identifying the noiseless factor with $\mathcal{H}_{\text{NS}}$[5].

DFS and NS theory has received extensive attention to date. A relatively straightforward characterization is possible for error sets which are effectively †-closed, in which case elegant results from the representation theory of C∗-algebras apply[6]. The operator-algebraic approach is suitable for investigating NSs within both a Hamiltonian formulation of open-system dynamics and a large class of TPCP maps, see e.g., [3], [23]. However, explicit characterizations for arbitrary quantum operations and Markovian dynamics are more delicate and, to some extent, less consolidated. While a number of definitions and results are provided in [8], [11], [24], the increasingly prominent role that quantum subsystems play within quantum error correction theory [25], along with continuous experimental advances in implementing DFSs [26]–[28] and NSs [29], heighten the need for a fully consistent system-theoretic approach. It is our goal in the remaining of this Section to provide such a framework for the case of Markovian dynamics,

[5]In principle, multiple NSs may exist for a given dynamical system. While we do not explicitly address such a scenario, generalization is possible along the lines presented here.

[6]A C∗-algebra is a complex normed algebra $\mathcal{A}$ with a conjugate linear involution (∗ or †, an *adjoint* operation), which is complete, satisfies $\|AB\| \leq \|A\|\|B\|$, and $\|A^\dagger A\| = \|A\|^2$, for all $A, B \in \mathcal{A}$. Any norm-closed subspace of bounded operators on $\mathcal{H}$ is a C∗-algebra if it closed under the usual adjoint operation. Up to unitary equivalence, every finite-dimensional operator ∗-algebra is isomorphic to a unique direct sum of ampliated full matrix algebras. Such a decomposition directly reveals the supported NSs, whenever $\mathcal{A}$ represents an *error algebra* for the noisy system $\mathcal{I}$ [3].

by paying special attention to the key role played by model robustness notions as stated in Definition 3.

B. Invariant Subsystems

Definition 5 (State Initialization): The system $\mathcal{I}$ with state $\rho \in \mathfrak{D}(\mathcal{H}_I)$ is *initialized in* $\mathcal{H}_S$ *with state* $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ if the blocks of ρ satisfy:

  i) $\rho_{SF} = \rho_S \otimes \rho_F$ for some $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$;

  ii) $\rho_P = 0, \rho_R = 0$.

Condition (ii) in the above Definition guarantees that $\bar{\rho}_S = \text{trace}_F(\Pi_{SF}\rho\Pi_{SF}^\dagger)$ is a valid state of $\mathcal{S}$, while condition (i) ensures that measurements or dynamics affecting the factor $\mathcal{H}_F$ have no effect on the state in $\mathcal{H}_S$. We shall denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of states initialized in this way. The larger set of states obeying condition (ii) alone will correspondingly be denoted by $\mathfrak{I}_{SF}(\mathcal{H}_I)$.

Definition 6 (Invariance): Let $\mathcal{I}$ evolve under TPCP maps. $\mathcal{S}$ is an *invariant subsystem* if the evolution of $\rho \in \mathfrak{I}_S(\mathcal{H}_I)$ obeys:

$$\rho(t) = \left(\begin{array}{c|c} \mathcal{T}_t^S(\rho_S) \otimes \mathcal{T}_t^F(\rho_F) & 0 \\ \hline 0 & 0 \end{array} \right), \quad t \geq 0 \qquad (6)$$

$\forall \rho_S \in \mathfrak{D}(\mathcal{H}_S)$, $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$, and with $\mathcal{T}_t^S(\cdot)$ and $\mathcal{T}_t^F(\cdot)$, $t \geq 0$, being TPCP maps on $\mathcal{H}_S$ and $\mathcal{H}_F$, respectively.

Thus, a subsystem is invariant if time evolution preserves the initialization of the state, that is, the dynamics is confined within $\mathfrak{I}_S(\mathcal{H}_I)$. For Markovian evolution of $\mathcal{I}$, Definition 2 requires both $\{\mathcal{T}_t^S\}$ and $\{\mathcal{T}_t^F\}$ to be QDSs on their respective domain. We begin with the following elementary Lemma:

Lemma 1: Let a linear operator $L : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3$ be different from the zero operator. Then there exist factorized pure states in $\mathcal{H}_1 \otimes \mathcal{H}_2 \ominus \ker(L)$.

    Proof: Assume that $L|\psi\rangle = 0$ for all factorized $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Since such $|\psi\rangle$'s generate the whole $\mathcal{H}_1 \otimes \mathcal{H}_2$, then by linearity it must be $L = 0$ and we conclude by contradiction. ∎

Theorem 2 (Markovian Invariance): $\mathcal{H}_S$ supports an invariant subsystem under Markovian evolution on $\mathcal{H}_I$ iff for every initial state $\rho \in \mathfrak{I}_S(\mathcal{H}_I)$, with $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$, $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$, the following conditions hold:

$$\frac{d}{dt}\rho(t) = \left(\begin{array}{c|c} \mathcal{L}_{SF}(\rho_{SF}(t)) & 0 \\ \hline 0 & 0 \end{array} \right), \quad \forall t \geq 0, \qquad (7)$$

$$\text{trace}_F[\mathcal{L}_{SF}(\rho_{SF}(t))] = \mathcal{L}_S(\rho_S(t)), \quad \forall t \geq 0 \qquad (8)$$

where $\mathcal{L}_{SF}$ and $\mathcal{L}_S$ are QDS generators on $\mathcal{H}_S \otimes \mathcal{H}_F$ and $\mathcal{H}_S$, respectively.

    Proof: Since Definition 6 is obeyed, computing the infinitesimal generator of (6) (at $t = 0$) yields

$$\left.\frac{d}{dt}\rho(t)\right|_{t=0} = \left(\begin{array}{c|c} (\mathcal{L}_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes \mathcal{L}_F)(\rho_S \otimes \rho_F) & 0 \\ \hline 0 & 0 \end{array} \right).$$
$$(9)$$

Then the time-invariant generator must have the form (7). Take the partial trace over $\mathcal{H}_F$, and observe that $\text{trace}(\rho_F) = 1$, $\text{trace}(\mathcal{L}_F(\rho_F)) = 0$. Then (8) holds.

To prove the opposite implication, assume that (7) and (8) hold, and $\rho \in \mathfrak{I}_S(\mathcal{H}_I)$. Since ρ evolves under a QDS generator

that can be written in the form (2), with Hamiltonian H and noise operators L_k partitioned as in (4), computing the generator at a generic time t by blocks yields

$$\frac{d}{dt}\rho = \left(\begin{array}{c|c} \mathcal{L}_{SF}(\rho) & \mathcal{L}_P(\rho) \\ \hline \mathcal{L}_Q(\rho) & \mathcal{L}_R(\rho) \end{array} \right)$$

where

$$\mathcal{L}_{SF}(\rho) = -i[H_{SF}, \rho_{SF}] + \frac{1}{2}\sum_k (2L_{SF,k}\rho_{SF}L_{SF,k}^\dagger$$
$$- \left\{ L_{SF,k}^\dagger L_{SF,k} + L_{Q,k}^\dagger L_{Q,k}, \rho_{SF} \right\})$$

$$\mathcal{L}_Q(\rho) = -iH_P^\dagger \rho_{SF} + \frac{1}{2}\sum_k \left(2L_{SF,k}\rho_{SF}L_{Q,k}^\dagger \right.$$
$$\left. - \rho_{SF}(L_{SF,k}^\dagger L_{P,k} + L_{Q,k}^\dagger L_{R,k}) \right)^\dagger$$

$$\mathcal{L}_P(\rho) = i\rho_{SF}H_P + \frac{1}{2}\sum_k \left(2L_{SF,k}\rho_{SF}L_{Q,k}^\dagger \right.$$
$$\left. - \rho_{SF}(L_{SF,k}^\dagger L_{P,k} + L_{Q,k}^\dagger L_{R,k}) \right)$$

$$\mathcal{L}_R(\rho) = \frac{1}{2}\sum_k 2L_{Q,k}\rho_{SF}L_{Q,k}^\dagger.$$

In order to satisfy (7), it must be $(1/2)\sum_k 2L_{Q,k}\rho_{SF}L_{Q,k}^\dagger = 0$, for every $\rho_{SF} = \rho_S \otimes \rho_F$. Consider for example pure product states $\rho_{SF} = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$. Then, by observing that $L_{Q,k}\rho_{SF}L_{Q,k}^\dagger$ is positive for every k and by using Lemma 1, it must be $L_{Q,k} = 0$ for every k. Next, to ensure that $\rho_P(t) = 0$ for every t, the remaining contribution to $\mathcal{L}_P(\rho)$ must vanish, that is, by using Lemma 1 again, it must be

$$iH_P - \frac{1}{2}\sum_k L_{SF,k}^\dagger L_{P,k} = 0. \qquad (10)$$

This leaves a SF-block of the form

$$-i[H_{SF}, \rho_{SF}] + \frac{1}{2}\sum_k (2L_{SF,k}\rho_{SF}L_{SF,k}^\dagger$$
$$- \{L_{SF,k}^\dagger L_{SF,k}, \rho_{SF}\})$$

which indeed satisfies (7). To see this, notice that we may always write $L_{SF,k} = \sum_i M_{k,i} \otimes N_{k,i}$, with $M_{k,i}$ ($N_{k,i}$) being operators on $\mathcal{H}_S$ ($\mathcal{H}_F$), respectively (this follows e.g., from the operator Schmidt-decomposition [30]). Thus, we obtain

$$2L_{SF,k}\rho_{SF}L_{SF,k}^\dagger - \left\{ L_{SF,k}^\dagger L_{SF,k}, \rho_{SF} \right\}$$
$$= \sum_{i,j} (2M_{k,i}\rho_S M_{k,j}^\dagger \otimes N_{k,i}\rho_F N_{k,j}^\dagger$$
$$- (M_{k,j}^\dagger M_{k,i}\rho_S \otimes N_{k,j}^\dagger N_{k,i}\rho_F$$
$$+ \rho_S M_{k,j}^\dagger M_{k,i} \otimes \rho_F N_{k,j}^\dagger N_{k,i})). \qquad (11)$$

By tracing over $\mathcal{H}_F$, and using the cyclic property of the (full) trace, we obtain

$$\text{trace}_F[2L_{SF,k}\rho_{SF}L_{SF,k}^\dagger - \{L_{SF,k}^\dagger L_{SF,k}, \rho_{SF}\}] =$$
$$\sum_{i,j} b_{ij}^k (2M_{k,i}\rho_S M_{k,j}^\dagger - (M_{k,j}^\dagger M_{k,i}\rho_S + \rho_S M_{k,j}^\dagger M_{k,i}))$$

where $b_{ij}^k = \text{trace}(\rho_F N_{k,j}^\dagger N_{k,i})$. Since we require (8) to be *independent* of ρ_F, $L_{SF,k}$ must take one of the following two forms:

$$L_{SF,k} = \sum_j M_{k,j} \otimes \mathbb{I}_F = L_{S,k} \otimes \mathbb{I}_F$$

$$L_{SF,k} = \sum_j \mathbb{I}_S \otimes N_{k,j} = \mathbb{I}_S \otimes L_{F,k}.$$

A similar reasoning shows that H is constrained to

$$H = H_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes H_F.$$

Thus, the generator has the form declared in (9). ∎

As a byproduct, Theorem 2's proof gives explicit necessary and sufficient conditions for the blocks of H and L_k to ensure invariance. We collect them in the following:

Corollary 1 (Markovian Invariance): Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let $H, \{L_k\}$ be the Hamiltonian and the error generators of a Markovian QDS as in (2). Then $\mathcal{H}_S$ supports an invariant subsystem iff $\forall k$

$$L_k = \left(\begin{array}{c|c} L_{S,k} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right)$$

$$iH_P - \frac{1}{2} \sum_k (L_{S,k}^\dagger \otimes L_{F,k}^\dagger) L_{P,k} = 0$$

$$H_{SF} = H_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes H_F \tag{12}$$

where for each k either $L_{S,k} = \mathbb{I}_S$ or $L_{F,k} = \mathbb{I}_F$ (or both).

If we require parametric model robustness, as specified in Definition 3, additional constraints emerge from imposing that (7) and (8) hold irrespective of parameter uncertainties. The results may be summarized as follows.

Theorem 3 (Robust Markovian Invariance): Assume $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. (i) Let $\{F_k\}$ be the error generators in (1). Then $\mathcal{H}_S$ supports an *A-robust invariant subsystem* iff $\forall j, k$,

$$F_k = \left(\begin{array}{c|c} F_{S,k} \otimes F_{F,k} & F_{P,k} \\ \hline 0 & F_{R,k} \end{array} \right)$$

$$F_{P,k}^\dagger (F_{S,j} \otimes F_{F,j}) = 0 \tag{13}$$

$$H_{SF} = H_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes H_F, \quad H_P = 0 \tag{14}$$

where either $F_{S,k} = \mathbb{I}_S$ for every k, or $F_{F,k} = \mathbb{I}_F$ for every k, or both. (ii) If $\{L_k\}$ are the noise operators in (2), then $\mathcal{H}_{NS}$ supports a *γ-robust invariant subsystem* iff $\forall k$,

$$L_k = \left(\begin{array}{c|c} L_{S,k} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right)$$

$$L_{P,k}^\dagger (L_{S,k} \otimes L_{F,k}) = 0 \tag{15}$$

$$H_{SF} = H_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes H_F, \quad H_P = 0 \tag{16}$$

and for each k, either $L_{S,k} = \mathbb{I}_S$ or $L_{F,k} = \mathbb{I}_F$ (or both).

Proof: Consider case (i). Given Theorem 2, conditions (7)–(8) must hold irrespective of $A = (a_{jk})$ in (1). The lower-diagonal block is now $\sum_{jk} a_{jk} F_{Q,j} \rho_{SF} F_{Q,k}^\dagger$. Considering only the diagonal terms $j = k$, we are again led to require $F_{Q,k} = 0$ for every k. Thus, condition (10) must be replaced by $iH_P - \sum_{jk} a_{jk} F_{SF,k}^\dagger F_{P,j} = 0$, which is true for every

$A = (a_{jk})$ iff H_P and $F_{P,k}^\dagger (F_{S,j} \otimes F_{F,j})$ vanish independently, as stated in (13), (14).

To complete the proof, as before we write $F_{SF,k} = \sum_{k,l} M_{k,l} \otimes N_{k,l}$, with $M_{k,l}$ ($N_{k,l}$) operators on $\mathcal{H}_{NS}$ ($\mathcal{H}_F$), respectively. Then we have:

$$2 F_{SF,j} \rho_{SF} F_{SF,k}^\dagger - \{F_{SF,k}^\dagger F_{SF,j}, \rho_{SF}\}$$
$$= \sum_{l,m} (2 M_{j,l} \rho_{NS} M_{k,m}^\dagger \otimes N_{j,l} \rho_F N_{k,m}^\dagger$$
$$- (M_{k,m}^\dagger M_{j,l} \rho_{NS} \otimes N_{k,m}^\dagger N_{j,l} \rho_F$$
$$+ \rho_{NS} M_{k,m}^\dagger M_{j,l} \otimes \rho_F N_{k,m}^\dagger N_{j,l}))$$

By tracing over $\mathcal{H}_F$ and using cyclicity yields

$$\sum_{l,m} b_{lm}^{kj} (2 M_{j,l} \rho_{NS} M_{k,m}^\dagger - (M_{k,m}^\dagger M_{j,l} \rho_{NS} + \rho_{NS} M_{k,m}^\dagger M_{j,l}))$$

where $b_{lm}^{kj} = \text{trace}(\rho_F N_{k,m}^\dagger N_{j,l})$. Since we wish (8) to be independent of ρ_F, it follows that for every k, $F_{SF,k}$ takes one of two possible forms

$$F_{SF,k} = \sum_m \mathbb{I}_{NS} \otimes N_{m,k} = \mathbb{I}_{NS} \otimes F_{F,k}$$

$$F_{SF,k} = \sum_l M_{k,l} \otimes \mathbb{I}_F = F_{NS,k} \otimes \mathbb{I}_F$$

which establishes the conditions for *A-robustness*.

For *γ-robustness*, it suffices to specialize the above proof to diagonal A, that is, to consider only $j = k$. This let us identify the L_k's with the F_k's, and the result follows. ∎

C. Noiseless Subsystems

As remarked after Definition 5, given $\rho \in \mathfrak{D}(\mathcal{H}_I)$, the reduced projected state

$$\bar{\rho}_S = \text{trace}_F(\Pi_{SF} \rho \Pi_{SF}^\dagger) \tag{17}$$

need not be a valid reduced state of $\mathcal{S}$ if $\rho_R \neq 0$, since its trace might be less than one. Still, to the purposes of defining noiseless behavior, there is no reason for requiring that the evolution has to be confined to $\mathfrak{I}_S(\mathcal{H}_I)$, as long as the information encoded in the intended subsystem is preserved, i.e., it undergoes unitary evolution. This motivates a *weaker* definition of initialization:

Definition 7 (Reduced State Initialization): The system $\mathcal{I}$ with state ρ is *initialized in* $\mathcal{H}_{SF}$ *with reduced state* $\bar{\rho}_S \in \mathfrak{D}(\mathcal{H}_S)$ if the blocks of ρ satisfy:
  (i') $\text{trace}_F(\rho_{SF}) = \bar{\rho}_S$;
  (ii) $\rho_P = 0, \rho_R = 0$.

Remark: The above definition is equivalent to state initialization as in given in Definition 5 if we restrict to *pure* states in $\mathcal{H}_S$, but it allows for entangled states otherwise.

Definition 8 (Noiselessness): Let $\mathcal{I}$ evolve under TPCP maps. A subsystem $\mathcal{S}$ is a NS for the evolution if for every initial state $\rho(0)$ of $\mathcal{I}$ initialized in $\mathcal{S}$ with reduced state $\bar{\rho}_S(0)$

$$\bar{\rho}_{NS}(t) = U(t) \bar{\rho}_S(0) U(t)^\dagger, \quad t \geq 0 \tag{18}$$

where $U(t)$ is a unitary operator on $\mathcal{H}_S$, *independent of the initial state* on $\mathcal{H}_I$. If $\mathcal{H}_F \simeq \mathbb{C}$, the NS $\mathcal{S}$ reduces to a DFS.

Following [11], we shall say that *perfect NS initialization* occurs when (i′) and (ii) are obeyed, and call subsystem $\mathcal{S}$ *imperfectly initialized* whenever (i′) or (ii) is violated. Physically, Definition 8 requires the state component $\bar{\rho}_{\mathrm{NS}}$ carried by $\mathcal{H}_S$ to *evolve unitarily*, independently of the rest. The following proposition establishes how, in fact, a perfecly initialized NS is a special case of an invariant subsystem:

Proposition 1: Let $\mathcal{H}_I = (\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. Then $\mathcal{H}_{\mathrm{NS}}$ supports a NS for some given TPCP dynamics iff for all initial condition $\rho \in \mathfrak{I}_{\mathrm{NS}}(\mathcal{H}_I)$, with $\rho_{\mathrm{NS}} \in \mathfrak{D}(\mathcal{H}_{\mathrm{NS}}), \rho_F \in \mathfrak{D}(\mathcal{H}_F)$, the evolved state of $\mathcal{I}$ obeys

$$\rho(t) = \left(\begin{array}{c|c} U(t)\rho_{\mathrm{NS}}U^\dagger(t) \otimes \mathcal{T}_t^F(\rho_F) & 0 \\ \hline 0 & 0 \end{array} \right), \quad t \geq 0 \quad (19)$$

where $U(t)$ is a unitary operator on $\mathcal{H}_{\mathrm{NS}}$ and $\{\mathcal{T}_t^F\}$ are TPCP maps on $\mathcal{H}_F$ alone.

Proof: Assume $\mathcal{H}_{\mathrm{NS}}$ to support an NS and to be initialized with reduced state $\bar{\rho}_{\mathrm{NS}}$, where it could be $\rho_{NSF}(0) \neq \bar{\rho}_{\mathrm{NS}}(0) \otimes \rho_F(0)$. Let $\rho_{NSF}(0) = \sum_k S_k \otimes F_k$, with S_k and F_k operators on $\mathcal{H}_S$ and $\mathcal{H}_F$, respectively. Thus, $\bar{\rho}_{\mathrm{NS}}(0) = \sum_k \mathrm{trace}(F_k)S_k$. Notice that, by linearity

$$\mathrm{trace}_F(\Pi_{SF}\rho(t)\Pi_{SF}^\dagger)\mathrm{trace}_F \left[\sum_k U(t)S_kU^\dagger(t) \otimes \mathcal{T}_t^F(F_k) \right]$$
$$= \sum_k U(t)S_kU^\dagger(t)\mathrm{trace}\left(\mathcal{T}_t^F(F_k)\right) = U(t)\bar{\rho}_{\mathrm{NS}}(0)U^\dagger(t).$$

Thus, the condition is sufficient.

To prove the other implication, notice that if the evolution of $\bar{\rho}_{\mathrm{NS}}$ is unitary, it preserves the trace. By the properties of partial trace and by (17), it then follows that $\mathrm{trace}(\bar{\rho}_{\mathrm{NS}}(t)) = \mathrm{trace}(\rho_{NSF}(t))$. Therefore, $\rho_R = 0$ must vanish at all times in order to ensure TP-evolution in the SF-block and, similarly, $\rho_P = 0$ in order to guarantee positivity of the whole state[7]. Then for every $t \geq 0$ the evolution must take the form

$$\rho(t) = \left(\begin{array}{cc} \mathcal{T}_t^{NSF}(\rho_{NSF}) & 0 \\ 0 & 0 \end{array} \right)$$

where $\mathcal{T}_t^{NSF}$ is a TPCP map on $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_F$. Now use the Kraus representation theorem and the operator-Schmidt decomposition, by employing a basis for $\mathfrak{B}(\mathcal{H}_S)$, say $\{M_j\}$, such that, $\forall \rho_{\mathrm{NS}} \in \mathfrak{D}(\mathcal{H}_{\mathrm{NS}}), \rho_F \in \mathfrak{D}(\mathcal{H}_F)$,

$$\mathcal{T}_t^{NSF}(\rho_{\mathrm{NS}} \otimes \rho_F) = \sum_{klm} M_l\rho_{\mathrm{NS}}M_m^\dagger \otimes N_{k,l}\rho_F N_{k,m}^\dagger.$$

Thus

$$\mathrm{trace}_F[\Pi_{NSF}\rho(t)\Pi_{NSF}^\dagger] = \mathcal{T}_t^S(\rho_{\mathrm{NS}}) = \sum_{lm} \alpha_{lm}M_l\rho_{\mathrm{NS}}M_m^\dagger$$

where $\alpha_{lm} = \mathrm{trace}(\sum_k N_{k,m}^\dagger N_{k,l}\rho_F)$ is a positive matrix. By exploiting the fact that $\{M_j\}$ is a basis, and decomposing $\sum_k N_{k,m}^\dagger N_{k,l}$ in Hermitian and skew-Hermitian parts, one can see that a necessary and sufficient condition in order for $\mathcal{T}_t^S$ to be independent of ρ_F is that $\sum_k N_{k,m}^\dagger N_{k,l} = \alpha_{ml}\mathbb{1}_F$ for every j, k. By imposing that $\mathrm{trace}_F[\Pi_{SF}\rho(t)\Pi_{SF}^\dagger] = U(t)\bar{\rho}_S(0)U^\dagger(t)$, (α_{jk}) must have rank one, thus $\alpha_{jk} = \alpha_j\alpha_k$ for some $\{\alpha_j\}$.

[7]This follows from the fact that if $\rho = (\rho_{ij}) \geq 0$, then $|\rho_{ij}| \leq \sqrt{\rho_{ii}\rho_{jj}}$.

If we additionally choose the operator basis $\{M_j\}$ so that $M_1 = U(t)$, then α_{11} is the only non-zero entry, in particular, $\sum_k N_{k,l}^\dagger N_{k,l} = 0$ for every $l \neq 1$. This implies $N_{k,l} = 0$ for every $l \neq 1$, thus yielding to the desired conclusion

$$\mathcal{T}_t^{SF}(\rho_S \otimes \rho_F) = U(t)\rho_S U(t)^\dagger \otimes \sum_k N_k\rho_F N_k^\dagger.$$

∎

On one hand, as a consequence of the above Proposition, if reduced state initialization is assumed (as in Definition 7), the factor $\mathcal{H}_{\mathrm{NS}}$ *supports an NS only if it is invariant*. On the other hand, under the stronger condition of initialization of Definition 5, if $\mathcal{H}_{\mathrm{NS}}$ is *invariant and unitarily evolving*, then it supports a NS. Accordingly, most of the results concerning NSs may be derived as a specialization of conditions for invariance. Remarkably, this also implies that in the particular case of a NS, the invariance property is robust with respect to the initialization in the NSF-block, that is, condition (i) may be effectively relaxed to (i′). This is *not* true for general invariant subsystems. Explicit characterizations of the Markovian noiseless property may then be established as summarized in the rest of this Section.

Corollary 2 (Markovian NS): Let $\mathcal{H}_I = (\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. Then $\mathcal{H}_{\mathrm{NS}}$ supports a NS under Markovian evolution on $\mathcal{H}_I$ iff for every initial state $\rho \in \mathfrak{I}_{\mathrm{NS}}(\mathcal{H}_I)$, with $\rho_{\mathrm{NS}} \in \mathfrak{D}(\mathcal{H}_{\mathrm{NS}}), \rho_F \in \mathfrak{D}(\mathcal{H}_F)$, and $\forall t \geq 0$

$$\frac{d}{dt}\rho(t) = \left(\begin{array}{c|c} \mathcal{L}_{NSF}(\rho_{NSF}(t)) & 0 \\ \hline 0 & 0 \end{array} \right) \quad (20)$$

$$\mathrm{trace}_F[\mathcal{L}_{NSF}(\rho_{NSF}(t))] = -i[H_{\mathrm{NS}}, \rho_{\mathrm{NS}}(t)] \quad (21)$$

where $\mathcal{L}_{NSF}$ and $\mathcal{L}_{\mathrm{NS}}$ are QDS generators on $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_F$ and $\mathcal{H}_{\mathrm{NS}}$, respectively.

Proof: Given Proposition 1 and Theorem 2, we need to ensure that the evolution in $\mathcal{H}_{\mathrm{NS}}$ is unitary. That is, the NSF-block must be driven by an generator of the form $-i[H_{\mathrm{NS}}, \rho_{\mathrm{NS}}] \otimes \rho_F + \rho_{\mathrm{NS}} \otimes \mathcal{L}_F(\rho_F)$, which replaces (9) and ensures unitary evolution on the NS-factor, while allowing for general non-unitary Markovian dynamics on $\mathcal{H}_F$. The proof of Theorem 2 applies, with the noise operators constrained to have the form

$$L_{NSF,k} = \sum_j \mathbb{1}_S \otimes N_{k,j} = \mathbb{1}_S \otimes L_{F,k}.$$

∎

Accordingly, the necessary and sufficient conditions on the matrix blocks of H and L_k for NS-behavior are modified to:

Corollary 3: Assume $\mathcal{H}_I = (\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let $H, \{L_k\}$ be the Hamiltonian and the error generators of a Markovian QDS as in (2). Then $\mathcal{H}_{\mathrm{NS}}$ supports a NS iff $\forall k$

$$L_k = \left(\begin{array}{c|c} \mathbb{1}_{\mathrm{NS}} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right)$$

$$iH_P - \frac{1}{2}\sum_k (\mathbb{1}_{\mathrm{NS}} \otimes L_{F,k}^\dagger)L_{P,k} = 0$$

$$H_{NSF} = H_{\mathrm{NS}} \otimes \mathbb{1}_F + \mathbb{1}_{\mathrm{NS}} \otimes H_F. \quad (22)$$

While derivations differ, Corollary 3 provides the same NS-characterization of Theorem 5 in [11]. Beside more directly tying to the CP context, our approach shows how the NS notion

may emerge as a specialization of the conditions for invariance. Following the same lines as in Section II-B, we next proceed to a general result for A- and γ-robust NSs, which completes the partial conditions proposed in [8]:

Corollary 4 (Robust Markovian NS): Assume that $\mathcal{H}_I = (\mathcal{H}_{NS} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. (i) Let $\{F_k\}$ be the error generators in (1). Then $\mathcal{H}_{NS}$ supports an A-robust NS iff $\forall j, k$

$$F_k = \left(\begin{array}{c|c} \mathbb{I}_{NS} \otimes F_{F,k} & F_{P,k} \\ \hline 0 & F_{R,k} \end{array} \right)$$

$$F_{P,k}^\dagger (\mathbb{I}_{NS} \otimes F_{F,j}) = 0 \tag{23}$$

$$H_{NSF} = H_{NS} \otimes \mathbb{I}_F + \mathbb{I}_{NS} \otimes H_F, \quad H_P = 0. \tag{24}$$

(ii) If $\{L_k\}$ are the error generators in (2), then $\mathcal{H}_{NS}$ supports a γ-robust NS iff $\forall k$

$$L_k = \left(\begin{array}{c|c} \mathbb{I}_{NS} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right), \quad \forall k \tag{25}$$

$$L_{P,k}^\dagger (\mathbb{I}_{NS} \otimes L_{F,k}) = 0 \tag{26}$$

$$H_{NSF} = H_{NS} \otimes \mathbb{I}_F + \mathbb{I}_{NS} \otimes H_F, \quad H_P = 0. \tag{27}$$

Proof: Given Theorem 3, it suffices to ensure that evolution in $\mathcal{H}_{NS}$ be unitary (as in Corollary 2), for every A. This is true iff $H_P = 0$ and $F_{P,k}^\dagger (\mathbb{I}_{NS} \otimes F_{F,j}) = 0$ independently. From (17), $F_{SF,k} = \mathbb{I}_S \otimes F_{F,k}$ must hold for every k. The specialization to a γ-robust NS follows from similar observations. ∎

Clearly, an A-robust NS may exist only if the $\{F_k\}$ do not generate the whole $\mathfrak{B}(\mathcal{H}_I)$, that is, we are restricting to a set of possible noise generators as remarked in Section I-D. For applications, it may be useful to further specialize the result to the case of a γ-robust DFS, for which $\mathcal{H}_F$ is trivial:

Corollary 5 (γ-Robust DFS): Assume $\mathcal{H}_I = \mathcal{H}_{DFS} \oplus \mathcal{H}_R$. Let $\{L_k\}$ be the error generators in (2). Then $\mathcal{H}_{DFS}$ is a γ-*robust* DFS iff $\forall k$

$$H_P = 0, \quad L_k = \left(\begin{array}{c|c} c_k \mathbb{I}_{DFS} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right) \tag{28}$$

with $L_{P,k} = 0$ if $c_k \neq 0$.

Proof: In Corollary 4 above, set $\mathcal{H}_F = \mathbb{C}$. ∎

An alternative formulation of Corollary 5 also holds:

Proposition 2 (Alternative γ-Robust DFS Condition): $\mathcal{H}_{DFS} = \text{span}\{|\phi_j^{DFS}\rangle\}$ is a γ-robust Markovian DFS subspace of $\mathcal{H}_I$ iff $\forall j, k$ the following conditions hold:

$$H_P = 0, \quad L_k |\phi_j^{DFS}\rangle = c_k |\phi_j^{DFS}\rangle$$
$$L_k^\dagger L_k |\phi_j^{DFS}\rangle = |c_k|^2 |\phi_j^{DFS}\rangle.$$

Proof: Assume that $\forall k$, $L_k |\phi_j^{DFS}\rangle = c_k |\phi_j^{DFS}\rangle$. Then it must be

$$L_k = \left(\begin{array}{c|c} c_k \mathbb{I}_{DFS} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array} \right).$$

Since

$$L_k^\dagger L_k = \left(\begin{array}{c|c} c_k c_k^* \mathbb{I}_{DFS} & c_k^* L_{P,k} \\ \hline c_k L_{P,k}^\dagger & L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k} \end{array} \right)$$

then $L_k^\dagger L_k |\phi_j^{DFS}\rangle = |c_k|^2 |\phi_j^{DFS}\rangle$ is true iff the conditions of the proof of Corollary 5 are obeyed. ∎

Remark: According to the above Corollary, γ-robust DFS-states are joint (right) eigenvectors of *both* each Lindblad operator L_k and each "jump" operator $L_k^\dagger L_k$. Such characterizations of A- and γ-robust DFSs link our analysis to the definition presented in [31]. In fact, the definition of DFS property invoked there imposes more constraints than our Definition 7: it requires the Hamiltonian to preserve the DFS *independently* from the dissipative component of the generator. This may be regarded as a yet different kind of robustness, weaker than both γ- and A-robustness investigated here.

D. Imperfect Initialization

Thus far, we have addressed model robustness of the invariance and the noiselessness properties. As the relevant subsystem dynamics also depends on the initial state, it is natural to ask how critical initialization is to the purposes of ensuring a desired behavior. This motivates the introduction of a different robustness notion:

Definition 9: Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let $\mathcal{I}$ undergo QDS dynamics with a generator of the form (1) or (2). Let a given property $\mathfrak{P}_S$ hold for the dynamical model initialized in $\mathcal{H}_S$, according to Definition 5. If $\mathfrak{P}_S$ holds for $\bar{\rho}_S = \text{trace}_F(\Pi_{SF} \rho \Pi_{SF})$ for every $\rho \in \mathfrak{D}(\mathcal{H}_I)$, then $\mathfrak{P}_S$ is said to be ρ-*robust*.

This approach leads to the same conditions for imperfect-initialization Markovian NSs (*initialization-free NS*) obtained in [11]. By Definition 9, considering $\mathcal{H}_I = (\mathcal{H}_{NS} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, a subsystem supported on $\mathcal{H}_{NS}$ is a ρ-robust NS if for every $t \geq 0$, $\forall \rho(0)$ with reduced state $\bar{\rho}_{NS}(0) \in \mathfrak{D}(\mathcal{H}_{NS})$

$$\bar{\rho}_{NS}(t) = \text{trace}_F(\Pi_{NSF} \rho(t) \Pi_{NSF}^\dagger) = U(t) \bar{\rho}_{NS}(0) U^\dagger(t)$$

where $U(t)$ is unitary on $\mathcal{H}_{NS}$. This means that the dynamics of the NSF-block cannot be influenced by ρ_P, ρ_R. Notice that Proposition 1 has already clarified how imperfect initialization in the NSF block does not affect the unitary character of the evolution of the reduced state of the NS. Computing the generator $\mathcal{L}(\rho)$ by blocks, by inspection one sees that for all k it must be $L_{P,k} = 0, L_{Q,k} = 0$, hence $H_P = 0$. By the proof of Theorem 4, it also follows that $L_{NSF,k} = \mathbb{I} \otimes M_k$. Thus, the main difference with respect to the perfect NS-initialization case is the constraint $L_{P,k} = 0$, which decouples the evolution of the NSF-block from the rest. Notice that this also automatically ensures γ-robustness in our framework.

E. Attractive Subsystems

The analysis developed so far indicates how initialization requirements may be relaxed by requiring ρ-robustness. However, this implies in general tighter conditions on the noise operators, which may be demanding to ensure and leave less room for Hamiltonian compensation of the noise action (see Section III-B). In order to both address situations where such extra constraints are not met, as well as a question which is interesting on its own, we explore conditions for a NS to be not only invariant, but also attractive:

Definition 10 (Attractive Subsystem): Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. Then $\mathcal{H}_S$ supports an attractive subsystem with respect to a family $\{\mathcal{T}_t\}_{t \geq 0}$ of TPCP maps if $\forall \rho \in \mathfrak{D}(\mathcal{H}_I)$ the following condition is asymptotically obeyed:

$$\lim_{t \to \infty} \left(\mathcal{T}_t(\rho) - \left(\begin{array}{c|c} \bar{\rho}_S(t) \otimes \bar{\rho}_F(t) & 0 \\ \hline 0 & 0 \end{array} \right) \right) = 0 \qquad (29)$$

where $\bar{\rho}_S(t) = \text{trace}_F[\Pi_{SF} \mathcal{T}_t(\rho) \Pi_{SF}^\dagger]$, $\bar{\rho}_F(t) = \text{trace}_S[\Pi_{SF} \mathcal{T}_t(\rho) \Pi_{SF}^\dagger]$.

An attractive subsystem may be thought of as a subsystem that "self-initializes" in the long-time limit, by somehow reabsorbing initialization errors. Although such a desirable behavior only emerges asymptotically, for QDSs one can see that convergence is exponential, as long as the eigenvalues of $\mathcal{L}$ have strictly negative real part.

We begin with a negative result which, in particular, shows how the initialization-free and attractive characterizations are mutually exclusive.

Proposition 3: Assume $\mathcal{H}_I = (\mathcal{H}_{NS} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, $\mathcal{H}_R \neq 0$, and let $H, \{L_k\}$ be the Hamiltonian and the error generators as in (2), respectively. Let $\mathcal{H}_{NS}$ support a NS. If $L_{P,k} = L_{Q,k}^\dagger = 0$ for every k, then $\mathcal{H}_{NS}$ is not attractive.

Proof: Consider a block-diagonal state of the form

$$\rho_B = \left(\begin{array}{c|c} \rho_{SF} & 0 \\ \hline 0 & \rho_R \end{array} \right).$$

It is straightforward to see that the generator has the form

$$\frac{d}{dt} \rho_B = \left(\begin{array}{c|c} \mathcal{L}_{SF}(\rho_{SF}) & 0 \\ \hline 0 & \mathcal{L}_R(\rho_R) \end{array} \right)$$

which preserves the trace of ρ_R. Thus, if $\rho_R \neq 0$, ρ_B does not satisfy (29). ∎

Remark: The conditions of the above Proposition are obeyed, in particular, for NSs in the presence of purely *Hermitian* noise operators, that is, $L_k = L_k^\dagger$, $\forall k$. As a consequence, attractivity is never possible for this kind of *unital* Markovian noise, as defined by the requirement of preserving the fully mixed state. Still, even if the condition $L_{P,k} = L_{Q,k}^\dagger = 0$ condition holds, attractive subsystems may exist in the pure-factor case, where $\mathcal{H}_R = 0$. Sufficient conditions are provided by the following:

Proposition 4: Assume $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$ ($\mathcal{H}_R = 0$), and let $\mathcal{H}_S$ be invariant under a QDS of the form

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes \mathcal{L}_F.$$

If $\mathcal{L}_F(\cdot)$ has a unique attractive state $\hat{\rho}_F$, then $\mathcal{H}_S$ is attractive.

Proof: Let ρ be a generic state on $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$. ρ may always be expressed (recall the proof of Theorem 2) in the form $\rho = \sum_i P_i \otimes Q_i$. Without loss of generality, we may take the Q_i to be Hermitian. If this is not the case, decompose $Q_i = Q_i^H + i Q_i^A$ into Hermitian and skew-Hermitian parts, so that $P_i \otimes Q_i = P_i \otimes Q_i^H + (iP_i) \otimes Q_i^A$. Each of the $Q_i^{H,A}$ may be further decomposed in a positive and a negative part, which one may normalize to unit trace. To do so, consider the spectral representation of each Q_i, separate the positive and negative eigenvalues, and partition the matrix in a sum of two $Q_i = Q_i^+ + Q_i^-$. Normalize Q_i^+, Q_i^- to trace $1, -1$, respectively, and

reabsorb the normalization coefficients and the minus sign, in P_i. Thus, we can write $\rho = \sum_i \tilde{P}_i \otimes \rho_{F,i}$, and $\bar{\rho}_{NS} = \sum_i \tilde{P}_i$. By applying the above generator to such a state and using the linearity of the evolution

$$\lim_{t \to \infty} \rho_t = \sum_i \lim_{t \to \infty} \left(\mathcal{T}_S^t(\tilde{P}_i) \otimes \mathcal{T}_F^t(\rho_{F,i}) \right)$$
$$= \left(\sum_i \lim_{t \to \infty} \mathcal{T}_S^t(\tilde{P}_i) \right) \otimes \hat{\rho}_F = \lim_{t \to \infty} \mathcal{T}_S^t(\bar{\rho}_{NS}) \otimes \hat{\rho}_F$$

thus the desired conclusion follows. ∎

In the mathematical-physics literature, QDSs with a unique attractive stationary state are called *relaxing*, and have been mostly studied in the '70 in the context of rigorous approaches to quantum thermodynamics. Useful linear-algebraic conditions for determining whether a generator $\mathcal{L}_F(\cdot)$ is relaxing are presented in [32], [33]. The uniqueness of the stationary state turns out to be *necessary* when considering NSs:

Proposition 5: Assume $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$, ($\mathcal{H}_R = 0$), and let $\mathcal{H}_S$ support a NS under a QDS of the form

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes \mathcal{L}_F.$$

If $\mathcal{L}_F(\cdot)$ admits at least two invariant states, then $\mathcal{H}_S$ is not attractive.

Proof: It suffices to construct a state of the form:

$$\rho = p \rho_S^{(1)} \otimes \rho_F^{(1)} + (1-p) \rho_S^{(2)} \otimes \rho_F^{(2)}$$

where $\rho_S^{(1)}, \rho_S^{(2)}$ are orthogonal pure states on $\mathcal{H}_S$, and $\rho_F^{(1)}, \rho_F^{(2)}$ are the two invariant states for $\mathcal{L}_F$, and $0 < p < 1$. Again, by using the linearity of the evolution

$$\rho(t) = \left[p \mathcal{T}_S \left(\rho_S^{(1)} \right) \otimes \mathcal{T}_F \left(\rho_F^{(1)} \right) + (1-p) \mathcal{T}_S \left(\rho_S^{(2)} \right) \otimes \mathcal{T}_F \left(\rho_F^{(2)} \right) \right]$$
$$= p U_S(t) \rho_S^{(1)} U_S^\dagger(t) \otimes \rho_F^{(1)} + (1-p) U_S(t) \rho_S^{(2)} U_S^\dagger(t) \otimes \rho_F^{(2)}$$

so it follows that the state does not factorize for any $t \geq 0$. ∎

Note that proper initialization plays a more critical role in the DFS- than in the NS-context, as a consequence of Proposition 1. If $\mathcal{H}_F \neq \mathbb{C}$ and the initial state is not factorized on the NSF-block, the reduced state on $\mathcal{H}_{NS}$ still evolves unitarily, provided that the generator satisfies the conditions given in Corollary 3. Practically, this means that there is no actual need to require a (factorized) subsystem-initialized state, as long as a bounded error on the NS-component can be tolerated. For an imperfectly-initialized DFS, unitary evolution on the intended block can only occur if $L_{P,k} \equiv 0$, making attractivity a compelling option if the latter does not hold. Accordingly, our main emphasis is on attractivity in the DFS-case, which may be guaranteed by invoking a specialization of the Krasowskii-LaSalle invariance principle (see e.g., [34]).

Theorem 4 (Attractive Subspace): Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ ($\mathcal{H}_F = \mathbb{C}$), and let $\mathcal{H}_S$ support an invariant subspace under $\mathcal{L}$. Assume that there exists a continuously differentiable function $V(\rho) \geq 0$ on $\mathfrak{D}(\mathcal{H}_I)$, such that $\dot{V}(\rho) \leq 0$ on imperfectly initialized states in $\mathfrak{D}(\mathcal{H}_I) \setminus \mathfrak{I}_S(\mathcal{H}_I)$. Let

$$\mathcal{W} = \{\rho \in \mathfrak{D}(\mathcal{H}_I) | \dot{V}(\rho) = 0\}$$
$$\mathcal{Z} = \{\rho \in \mathfrak{D}(\mathcal{H}_I) | \text{trace}[\Pi_R \mathcal{L}(\rho)] = 0\}$$

where Π_R is the orthogonal projector on $\mathcal{H}_R$. If $\mathcal{W} \cap \mathcal{Z} \subseteq \mathfrak{I}_S(\mathcal{H}_I)$, then $\mathcal{H}_S$ is attractive.

Proof: Consider $V_1(\rho) = \text{trace}(\Pi_R \rho) + \text{trace}(\Pi_R \rho)V(\rho)$. It is zero iff $\rho_R = 0$, i.e., for perfectly initialized states. Computing $\mathcal{L}(\rho)$, we get for the R block:

$$\mathcal{L}(\rho)_R = -i[H_R, \rho_R] + i\rho_P^\dagger H_P - iH_P^\dagger \rho_P$$
$$- \frac{1}{2}\sum_k (2L_{R,k}\rho_R L_{R,k}^\dagger - \{L_{R,k}^\dagger L_{R,k}, \rho_R\}$$
$$+ L_{Q,k}\rho_{SF}L_{Q,k}^\dagger - 2\{L_{P,k}^\dagger L_{P,k}, \rho_R\}$$
$$+ 2L_{R,k}\rho_P^\dagger L_{Q,k}^\dagger + 2L_{Q,k}\rho_P L_{R,k}^\dagger$$
$$- \rho_P^\dagger(L_{SF,k}^\dagger L_{P,k} - L_{Q,k}^\dagger L_{R,k})$$
$$- (L_{P,k}^\dagger L_{SF,k} + L_{R,k}^\dagger L_{Q,k})\rho_P).$$

Therefore

$$\text{trace}[\Pi_R \mathcal{L}(\rho)] = -\frac{1}{2}\text{trace}\left(\left\{\sum_k L_{P,k}^\dagger L_{P,k}, \rho_R\right\}\right)$$
$$= -\text{trace}\left(\sum_k L_{P,k}^\dagger L_{P,k}\rho_R\right) \tag{30}$$

is always negative or zero. Hence

$$\dot{V}_1(\rho) = \text{trace}(\Pi_R \mathcal{L}(\rho))(1 + V(\rho)) + \text{trace}(\Pi_R \rho)\dot{V}(\rho) \le 0,$$

for every $\mathfrak{D}(\mathcal{H}_I)$, and it is zero only in $\mathcal{W} \cap \mathcal{Z} \cup \mathfrak{I}_S(\mathcal{H}_I)$. If $\mathcal{W} \cap \mathcal{Z} \subseteq \mathfrak{I}_S(\mathcal{H}_I)$, by applying Krasowskii-LaSalle invariance theorem, we conclude. ∎

The following result immediately follows:

Corollary 6: Assume that $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ ($\mathcal{H}_F = \mathbb{C}$), and let $\mathcal{H}_S$ support an invariant subspace under $\mathcal{L}$. Assume that

$$\sum_k L_{P,k}^\dagger L_{P,k} > 0 \tag{31}$$

where $>$ means strictly positive. Then $\mathcal{H}_S$ is attractive.

Proof: It suffices to note that (31) guarantees that (30) in the proof of the Theorem above is zero iff $\rho_R = 0$. The conclusion follows by taking a $V(\rho)$ constant and positive on $\mathfrak{D}(\mathcal{H}_I)$. ∎

Remark: From considerations on the rank of the l.h.s. of (31) and the $n \times r$ dimension of $L_{P,k}$, the condition of Corollary 6 may be obeyed only if $n \ge r$, i.e., $\dim(\mathcal{H}_S) \ge \dim(\mathcal{H}_R)$. An application of this result will be given in Section III-B.

Proposition 4 and Theorem 4 (or Corollary 6), may be combined in order to obtain *sufficient conditions* for attractivity in the general NS case.

Proposition 6: Let $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let $\mathcal{H}_S$ be an invariant subsystem under $\mathcal{L}$, with

$$\bar{\mathcal{L}}_F(\,\cdot\,) = \text{trace}_S(\Pi_{SF}\mathcal{L}(\,\cdot\,)\Pi_{SF}).$$

Assume that there exist a continuously differentiable functional $V(\rho) \ge 0$ on $\mathfrak{D}(\mathcal{H}_I)$, such that $\dot{V}(\rho) \le 0$ on imperfectly initialized states in $\mathfrak{D}(\mathcal{H}_I) \setminus \mathfrak{I}_{SF}(\mathcal{H}_I)$. Let $\mathcal{W}, \mathcal{Z}$ be defined as in Theorem 4. If $\bar{\mathcal{L}}_F(\,\cdot\,)$ is relaxing and $\mathcal{W} \cap \mathcal{Z} \subseteq \mathfrak{I}_{SF}(\mathcal{H}_I)$, then $\mathcal{H}_S$ is attractive.

Proof: From Corollary 1, $\bar{\mathcal{L}}_F(\,\cdot\,)$ is a Markovian generator on $\mathcal{H}_F$, thus it makes sense to require that it is relaxing. Let

$\hat{\rho}_F$ be its unique attractive state. Observe that from Theorem 4, the state will asymptotically have support only on $\mathcal{H}_S \otimes \mathcal{H}_F$. Let $\bar{\rho}_{SF}(t) = \Pi_{SF}\rho(t)\Pi_{SF}^\dagger$: By defining $\tau = t/2, s = t/2$, and by invoking the Markovian property, along with the above observation, invariance, and Proposition 4 we may write

$$\lim_{t \to \infty} \bar{\rho}_{SF}(t) = \lim_{\tau, s \to \infty} \Pi_{SF}\mathcal{T}^\tau(\rho(s))\Pi_{SF}^\dagger$$
$$= \lim_{\tau \to \infty} \mathcal{T}_S^\tau \otimes \mathcal{T}_F^\tau(\lim_{s \to \infty}\bar{\rho}_{SF}(s))$$
$$= \lim_{\tau \to \infty} \mathcal{T}_S^\tau(\lim_{s \to \infty}\bar{\rho}_{NS}(s)) \otimes \hat{\rho}_F.$$

The last equality comes from the fact that $\lim_{s \to \infty} \bar{\rho}_{SF}(s)$ is certainly bounded, and can be always written in the form $\lim_{s \to \infty} \sum_k P_{S,k}(s) \otimes \rho_{F,k}$ by choosing a basis of $\mathfrak{B}(\mathcal{H}_F)$ of density operators $\{\rho_{F,k}\}$ and by following ideas similar to those in the proof of Proposition 4. ∎

III. CONTROL APPLICATIONS

A. Quantum Trajectories and Markovian Output Feedback

Building on pioneering work by Belavkin [35], it has been long acknowledged for a diverse class of controlled quantum system that intercepting and feeding back the information leaking out of the system allow to better accomplish a number of desired control tasks (see [12] and [36]–[40] for representative contributions). This requires the ability to both effectively monitor the environment and control the target evolution through time-dependent Hamiltonian perturbations which depend upon the measurement record. We begin by recalling some well-established continuous-measurement models for the conditioned dynamics, as originally developed in the quantum-optics setting by Wiseman and Milburn [36], [37].

The basic setting is a measurement scheme which mimicks optical homo-dyne detection for field-quadrature measurements, whereby the target system (e.g., an atomic cloud trapped in an optical cavity) is indirectly monitored via measurements of the outgoing laser field quadrature [36], [41]. Let the measurement record be denoted by Y_t (e.g., a photo-current in the above setting), and let $(\Omega, \mathcal{E}, P)$ a (classical) probability space with an associated $\{W_t, t \in \mathbb{R}^+\}$ standard $\mathbb{R}$-valued Wiener process. The homo-dyne detection measurement record may then be written as the output of a stochastic dynamical system of the form

$$dY_t = \eta \,\text{tr}(M\rho_t + \rho_t M^\dagger)dt + \sqrt{\eta}dW_t \tag{32}$$

where $\rho_t \equiv \rho(t)$ is the system state at time t, M is the measurement operator determining the system-probe interaction, and $0 \le \eta \le 1$ quantifies the efficiency of the measurement.

The real-time knowledge of the photo-current provides additional information on the dynamics, leading to a *stochastic master equation* (SME) for the conditional evolution:

$$d\rho_t = (\mathcal{F}(H, \rho_t) + \mathcal{D}(M, \rho_t))dt + \mathcal{G}(M, \rho_t)dW_t$$
$$= \left(-i[H, \rho_t] + \eta(M\rho_t M^\dagger - \frac{1}{2}\{M^\dagger M, \rho_t\})\right)dt$$
$$+ \sqrt{\eta}\left[M\rho_t + \rho_t M^\dagger - \text{tr}(M\rho_t + \rho_t M^\dagger)\rho_t\right]dW_t.$$
$$\tag{33}$$

Here, $\mathcal{F}$ is the Hamiltonian generator, whereas $\mathcal{D}(M, \rho_t)$ and $\mathcal{G}(M, \rho_t)$ are the Lindblad noise channel and the "diffusion" contribution due to the weak measurement of M. Given an initial condition ρ_0, the solution ρ_t exists, is confined to $\mathfrak{D}(\mathcal{H}_I)$, and is adapted to the filtration induced by $\{W_t, t \in \mathbb{R}^+\}$ (see e.g., [42], [43]). As the SME (33) is an Itô stochastic differential equation, to obtain the average evolution generator it suffices to drop the martingale part $\mathcal{G}(M, \rho_t)dW_t$. Thus, the "unconditional" evolution obeys a deterministic QDS generator of the form (2). Notice that the diffusion term plays the role of the innovation part of a nonlinear Kalman-Bucy filter, and that the conditional state follows continuous trajectories, whereby the name of *quantum trajectories* approach in the quantum-optics literature [44].

In what follows, we assume perfect detection, that is, $\eta = 1$, unless otherwise specified (see III.E and IV). In [37], it has been argued that the photo-current can be instantaneously fed back to further modify the dynamics, still maintaining the Markovian character of the evolution. This motivates considering a *Hamiltonian feedback superoperator* of the form

$$\frac{d}{dt}\rho_t^f = \frac{d}{dt}Y_t\mathcal{F}(F, \rho_t), \quad F = F^\dagger \qquad (34)$$

which is, however, ill-defined given the stochastic nature of Y_t. In order to obtain a feedback Markovian evolution, (34) has been interpreted as an "implicit" Stratonovich stochastic differential equation [37]. Its Itô equivalent form is

$$d\rho_t^f = \mathcal{F}(F, \rho_t)dY_t + \frac{1}{2}\mathcal{F}^2(F, \rho_t)dt. \qquad (35)$$

Thus, one can consider the infinitesimal evolution resulting from the measurement followed by the feedback action $\rho_t + d\rho_t = \mathcal{T}^\mathcal{F}{}_{dt} \circ \mathcal{T}^\mathcal{M}{}_{dt}(\rho_t)$, where

$$\mathcal{T}^\mathcal{F}{}_{dt}(\rho_t) = \rho_t + \mathcal{F}(F, \rho_t)dY_t + \frac{1}{2}\mathcal{F}^2(F, \rho_t)dt$$

$$\mathcal{T}^\mathcal{M}{}_{dt}(\rho_t) = \rho_t + (\mathcal{F}(H, \rho_t) + \mathcal{D}(M, \rho_t))dt + \mathcal{G}(M, \rho_t)dW_t.$$

Substituting the definitions and using Itô's rule, it yields

$$d\rho_t = (\mathcal{F}(H, \rho_t)$$
$$+ \mathcal{D}(M, \rho_t) + \mathcal{F}(F, M\rho_t + \rho_t M^\dagger) + \frac{1}{2}\mathcal{F}^2(F, \rho_t)\Big)dt$$
$$+ (\mathcal{G}(M, \rho_t) + \mathcal{F}(F, \rho_t))dW_t. \qquad (36)$$

Dropping again the martingale part and rearranging the remaining terms leads to the Wiseman-Milburn *Markovian Feedback Master equation* (FME) [36], [37]

$$\frac{d}{dt}\rho_t = \mathcal{F}\left(H + \frac{1}{2}(FM + M^\dagger F), \rho_t\right) + \mathcal{D}(M - iF, \rho_t). \qquad (37)$$

In the following sections, we will tackle state-stabilization and NS-synthesis problems for controlled Markovian dynamics described by FMEs.

B. Control Assumptions

The feedback state-stabilization problem for Markovian dynamics has been extensively studied for the single-qubit case [45], [46]. In particular, conditions for achieving a pure steady state have been identified in [47]. In the existing literature, however, the standard approach to design a Markovian feedback strategy is to specify {both} the measurement and feedback operators M, F, and to treat the measurement strength and the feedback gain as the relevant control parameters accordingly. Here we will pretend to have more freedom, considering, for a fixed measurement operator M, *both* F and H as tunable control Hamiltonians.

Definition 11 (CHC): A controlled FME of the form (37) supports *complete Hamiltonian control* (CHC) if (i) arbitrary feedback Hamiltonians $F \in \mathfrak{H}(\mathcal{H}_I)$ may be enacted; (ii) arbitrary *constant* control perturbations $H_c \in \mathfrak{H}(\mathcal{H}_I)$ may be added to the free Hamiltonian H.

As we shall see, this leads to both new insights and constructive control protocols for systems where the noise operator is a generalized angular momentum-type observable, for generic finite-dimensional systems. While assuming that the implementation of arbitrary coherent Hamiltonians poses no problem is in line with standard universality constructions for open quantum systems [48], [49], from a physical standpoint the CHC assumption is certainly demanding and should, as such, be carefully scrutinized on a case by case basis. In particular, constraints on the allowed Hamiltonian contributions relative to the Lindblad dissipator may emerge, notably in so-called weak-coupling limit derivations of Markovian models [10]. A first, interesting consequence of assuming CHC emerges directly from the following observation:

Lemma 2: The Markovian generator

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \sum_k \mathcal{D}(L_k, \rho_t) \qquad (38)$$

is equivalent to

$$\frac{d}{dt}\rho_t = -i[H + H_c, \rho_t] + \sum_k \mathcal{D}(\tilde{L}_k, \rho_t) \qquad (39)$$

where for all k, and $c_k \in \mathbb{C}$

$$\tilde{L}_k = L_k + c_k\mathbb{1}, \quad H_c = -i\sum_k (c_k^* L_k - c_k L_k^\dagger). \qquad (40)$$

Proof: Consider $k = 1$:

$$\mathcal{D}(\tilde{L}, \rho) = \tilde{L}\rho\tilde{L}^\dagger - \frac{1}{2}\{\tilde{L}^\dagger\tilde{L}, \rho\}$$
$$= L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\} + [c^* L - cL^\dagger, \rho]$$
$$= -i[(ic^* L - icL^\dagger), \rho] + \mathcal{D}(L, \rho). \qquad (41)$$

Notice that $i(c^* L - cL^\dagger)$ is Hermitian. For $k > 1$, it suffices to add up the correction parts in (41) for different k's, and use the linearity of the commutator. ∎

Note that for Hermitian L and real c, $H_c = 0$. In general, by exploiting CHC, we may vary the trace of the Lindblad operators through transformations of the form (40)[8], and, if needed or useful, appropriately counteract the Hamiltonian correction H_c with a *constant* control Hamiltonian. This may allow to stabilize subsystems that are not invariant for the uncontrolled equation, *without directly modifying the non-unitary part*. In addition to this, restricting to such open-loop, constant control Hamiltonians avoids additional difficulties which are related to reconcile the Markovian limit with generic time-varying perturbations [10], [50].

Example 3: Consider a generator of the form:

$$\frac{d}{dt}\rho(t) = -i[\sigma_z, \rho(t)] + \left(L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\} \right)$$

where $L = \sigma_z + \sigma_+$. Suppose that the task is to make $\rho_d = \text{diag}(1,0)$ invariant. Since $H_P = 0, L_S = 1, L_p = 1$, invariance is not ensured by the uncontrolled dynamics. Using the above result, it suffices to apply a constant Hamiltonian $H_c = -i(L - L^\dagger) = \sigma_y$. The desired state turns out to be also attractive, see Proposition 7 below.

C. Pure-State Preparation With Markovian Feedback: Two-Level Systems

As mentioned, the problem of stabilizing an arbitrary pure state for a two-level atom is discussed in detail in [45] for $H = \alpha\sigma_y$, $M = \sqrt{\gamma}\sigma_-$, and $F = \lambda\sigma_y$, with the Hamiltonian, measurement, and feedback strength parameters (α, γ, λ, respectively), treated as the control design parameters. In terms of a standard Bloch sphere parametrization of the state set, $\rho = 1/2\mathbb{1}_2 + 1/2(x\sigma_x + y\sigma_y + z\sigma_z)$, with $0 \leq |(x,y,z)| \leq 1$, it is proved there that any pure state in the xz plane can be made invariant and attractive, with the only exception of the states on the equator of the sphere. The possibility of relaxing the perfect detection assumption is also addressed.

Our perspective differs not only because we mainly focus on continuous measurement of *Hermitian spin observables*, but more importantly because we start from identifying what constraints must be imposed to a Lindblad equation for a two-dimensional system as in (2) for ensuring that one of the system's pure states is an attractive equilibrium. Without loss of generality, let such a state be written as $\rho_d = \text{diag}(1,0)$, and write, accordingly

$$L_k = \begin{pmatrix} l_{k,S} & l_{k,P} \\ l_{k,Q} & l_{k,R} \end{pmatrix}, \quad H = \begin{pmatrix} h_S & h_P \\ h_P^* & h_R \end{pmatrix}.$$

Proposition 7: The pure state $\rho_d = \text{diag}(1,0)$ is a globally attractive, invariant state for a two-dimensional quantum system evolving according to (2) iff:

$$ih_P - \frac{1}{2}\sum_k l_{k,S}^* l_{k,P} = 0 \qquad (42)$$

$$l_{k,Q} = 0, \quad \forall k \qquad (43)$$

and there exists a $\bar{k}$ such that $l_{\bar{k},P} \neq 0$.

[8]Interestingly, this corresponds, in physical terms, to a variation of the local oscillator in the optical homo-dyne detection setting [37].

Proof: Equations (42)–(43) imply the invariance conditions of Corollary 1, hence ρ_d is stable. For the choice $L_k^D = \text{diag}(l_{S,k}, l_{R,k})$, every diagonal state would clearly be stationary (directly from the form of $\mathcal{L}(\cdot)$, or by Proposition 3). Hence it must be $l_{P,k} \neq 0$ for some k. To prove that ρ_d is the only attractive point for (2), it suffices to note that $l_{\bar{k},P} \neq 0$ is the two-dimensional version of the sufficient condition for attraction given in Corollary 6. ∎

Remark: Observe that, even if ρ_d is stable for the unconditional, *averaged* dynamics over the trajectories of (33), because it is pure it cannot be obtained as a convex combination of other states. Thus, ρ_d must be the *asymptotic limit of each trajectory with probability one*. In fact, any invariant set different from ρ_d alone could not have it as average. We provide next a characterization of the stabilizable manifold.

Proposition 9: Assume CHC. For any measurement operator M, there exist a feedback Hamiltonian F and a Hamiltonian compensation H_c able to stabilize an arbitrary desired pure state ρ_d for the FME (37) iff

$$[\rho_d, (M + M^\dagger)] \neq 0. \qquad (44)$$

Proof: Consider as before a basis where $\rho_d = \text{diag}(1,0)$, and let M^H and M^A denote the Hermitian and anti-Hermitian part of M, respectively. By (44), M^H cannot be diagonal in the chosen basis. In fact, assume M^H to be diagonal, then, by Proposition 7, $M^S - F$ must be brought to diagonal form to ensure invariance of ρ_d. Hence, by the same result, it follows that ρ_d cannot be made attractive. However, if M^H is not diagonal, we can always find an appropriate F in order to get an upper diagonal $L = M^H + i(M^S - F)$, and $H' = H + (FM + M^\dagger F)/2$. To conclude, it suffices to devise a compensation Hamiltonian H_c such that the condition $i(H' + H_c)_P - \frac{1}{2}l_S^* l_P = 0$ is satisfied. ∎

The above proof naturally suggests a constructive algorithm for designing the feedback and correction Hamiltonian required to stabilize the desired state. From our analysis, we also recover the results of [45] recalled before. For example, the states that are never stabilizable within the control assumptions of [45] are the ones commuting with the Hermitian part of $M = \sigma_+$, that is, $M^H = \sigma_x$. On the xz plane in the Bloch's representation, the latter correspond precisely to the equatorial points. The following example serves to illustrate the basic ideas we shall extend to the d-level case.

Example 4: The simplest choice to obtain an attractive generator is to engineer a dissipative part determined by $L = \sigma_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let $H = n_0\mathbb{1}_2 + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, with $n_0, n_x, n_y, n_z \in \mathbb{R}$. Consider e.g., $M = \frac{1}{2}\sigma_x$ and $F = -\frac{1}{2}\sigma_y$. Notice that in this case $\frac{1}{2}(FM + M^\dagger F) = 0$, thus $H' = H$. Substituting in the FME (37), one clearly obtain the desired result, provided that $H_c = -n_x\sigma_x - n_y\sigma_y$.

The spin measurement models considered above have been already exploited for stabilization problems (see e.g., [43]), although in the context of strategies necessitating a real-time estimate of the state [12]—so-called *Bayesian feedback* techniques in the physics literature [46]. Assume that it is possible to continuously monitor a single observable, e.g., σ_x in the above example. Since the choice of the reference frame for the spin axis

is conventional, by suitably adjusting the relative orientation of the measurement apparatus and the sample, it is then in principle possible to prepare and stabilize any desired pure state with the same control strategy.

D. Extension To Multi-Level Systems

The previous two-dimensional results naturally extend to generic d-level systems. This will also provide an example of an attractive state, which does not satisfy the sufficient condition of Proposition 6. Let the pure state to be FME-stabilized be written as $\rho_d = \text{diag}(1, 0, \ldots, 0)$. Under CHC, we may without loss of generality assume H to be diagonal in this basis.

Proposition 9: The pure state ρ_d is a globally attractive, invariant state for the FME (37) conditioned over the continuous measurement of the operator:

$$M = \frac{1}{2} \begin{pmatrix} 0 & m_1 & & 0 \\ m_1 & 0 & \ddots & \\ & \ddots & \ddots & m_{d-1} \\ 0 & & m_{d-1} & 0 \end{pmatrix}$$

and a Markovian feedback Hamiltonian

$$F = \frac{i}{2} \begin{pmatrix} 0 & m_1 & & 0 \\ -m_1 & 0 & \ddots & \\ & \ddots & \ddots & m_{d-1} \\ 0 & & -m_{d-1} & 0 \end{pmatrix}$$

with $m_i \neq 0$, for $i = 1, \ldots, (d-1)$.

Proof: First, observe that $L = (l_{i,j}) = M - iF$, the only elements different from zero are $l_{i,i+1} = m_i$. By writing $\rho = (\rho_{ij})_{i,j=1,\ldots,d}$, one gets

$$\mathcal{D}(L, \rho) = 4 \begin{pmatrix} m_1^* m_1 \rho_{22} & \cdots & m_{d-1}^* m_1 \rho_{2d} & 0 \\ \vdots & \ddots & & \vdots \\ m_1^* m_{d-1} \rho_{d2} & \cdots & m_{d-1}^* m_{d-1} \rho_{dd} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} -$$

$$2 \begin{pmatrix} 0 & |m_1|^2 \rho_{12} & \cdots & |m_{d-1}|^2 \rho_{1d} \\ |m_1|^2 \rho_{21} & 2|m_1|^2 \rho_{22} & \cdots & (|m_1|^2+|m_{d-1}|^2)\rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ |m_{d-1}|^2 \rho_{d1} & (|m_1|^2+|m_{d-1}|^2)\rho_{d2} & \cdots & 2|m_{d-1}|^2 \rho_{dd} \end{pmatrix}.$$

(45)

Define $\hat{H} = \text{diag}[0, 1, \ldots, (d-1)]$. Thus, the function $V_d(\rho) = \text{tr}(\hat{H}\rho)$ is a valid global Lyapunov function for the target state ρ_d in $\mathfrak{D}(\mathcal{H}_I)$ [34]. Indeed, $V(\rho) \geq 0$ and $V(\rho) = 0$ iff $\rho = \rho_d$. Computing $\dot{V}_d(\rho) = \text{tr}(\hat{H}\mathcal{D}(L, \rho))$ using (45), one obtains

$$\dot{V}_d(\rho) = 4 \left[\sum_{i=2}^{d-1} (i-1)|m_{ii}|^2 \rho_{i+1,i+1} - \sum_{j=1}^{d-1} j|m_{jj}|^2 \rho_{j+1,j+1} \right]$$
$$- 4 \sum_{i=1}^{d-1} |m_{ii}|^2 \rho_{i+1,i+1}.$$

We conclude by applying Lyapunov stability theorem. Since $|m_{ii}|^2 > 0$ and $\rho_{i,i} \geq 0$, the derivative is always non-positive and can be zero {iff} $\rho_{i,i} = 0$, $i = 2, \ldots, d$, i.e., $\rho = \rho_d$. ∎

The matrix $\hat{H}$ in the above proof is essentially a Hamiltonian with energy gaps renormalized to one, whereas F and M play a role analogous to the σ_y and σ_x observables of the $d = 2$

case. Notice that their form is not different from that of standard, higher-dimensional spin observables.

E. On Markovian-Feedback State Preparation

The feedback strategies we consider preserve the Markovian character of the open-system evolution. Thus, in a sense, the corresponding control problem may then be seen as a "Markovian environment design" problem [45]—implying that we may write the QDS generator *independently* of the system state. This, along with the remark in Section III-C, ensures the desired convergence feature for *each* initial state (in other words, robustness with respect to errors in the initial state estimation is guaranteed).

The main advantage with respect to other feedback-design strategies is represented by the potential ease in practical implementations, since virtually no signal-processing stage is required in the realization of the feedback loop. This should be contrasted with Bayesian feedback strategies [43], [46], whereby an updated state estimate has to be obtained through real-time integration of (33), and used to tailor a state-dependent feedback action on the underlying evolution. Such a task becomes rapidly prohibitive as the dimensionality of the target system grows.

As a potential disadvantage, however, the Markovian output-feedback we use requires strong control capabilities and perfect detection. On one hand, an infinite bandwidth is needed to feed back the measurement output in real time. On the other hand, both the feedback and measurement parameters have to be accurately tuned, along with both the system Hamiltonian and its control compensation, if needed. Nevertheless, for state stabilization problems, one may assess the role of the perfect-detection hypothesis and the possibility to relax it. If $\eta < 1$, the FME is modified as follows [41]:

$$\frac{d}{dt}\rho_t = \mathcal{F}(H + 1/2(FM + M^\dagger F), \rho_t)$$
$$+ \mathcal{D}(M - iF, \rho_t) + \varepsilon\mathcal{D}(F, \rho_t) \quad (46)$$

where we defined $\varepsilon = (1 - \eta)/\eta$.

In [10], generators of the form (1)–(2) are rewritten in a convenient way by choosing a suitable Hermitian basis in $\mathfrak{B}(\mathcal{H}_i) \approx \mathbb{C}^{d \times d}$. In fact, endowing $\mathbb{C}^{d \times d}$ with the inner product $\langle X, Y \rangle := \text{trace}(X^\dagger Y)$ (Hilbert-Schmidt), we may use a basis where the first element is $\frac{1}{\sqrt{d}}\mathbb{I}_d$, and complete it with a orthonormal set of Hermitian, traceless operators. This can always be done for finite d, for example by employing the natural d-dimensional extension of the Pauli matrices [10], [49]. In such a basis, all density operators are represented by d^2-dimensional vectors $\bar{\rho} = (\rho_0, \rho_1, \ldots, \rho_{d^2-1})^T$, where the first component ρ_0, relative to $\frac{1}{\sqrt{d}}\mathbb{I}_d$, is invariant and equal to $\frac{1}{\sqrt{d}}$ for TP-dynamics. Let $\rho_v = (\rho_1, \ldots, \rho_{d^2-1})^T$. Hence, any QDS generator $\mathcal{L}(\rho)$ must take the form

$$\frac{d}{dt}\bar{\rho} = \left(\begin{array}{c|c} 0 & 0 \\ \hline C & D \end{array} \right) \begin{pmatrix} 1/\sqrt{d} \\ \rho_v \end{pmatrix}. \quad (47)$$

Assume that the dynamics has a unique attractive state $\bar{\rho}^{(0)}$. Thus D must be invertible and we obtain

$$\bar{\rho}^{(0)} = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ -D^{-1}C \end{pmatrix}.$$

Consider now a small perturbation of the generator depending on the continuous parameter ε, with $1 - \delta < \eta < 1$, and δ

sufficiently small so that $(D + \varepsilon D')$ remains invertible. The generator becomes

$$\frac{d\bar{\rho}}{dt} = \left[\left(\begin{array}{c|c} 0 & 0 \\ \hline C & D \end{array} \right) + \varepsilon \left(\begin{array}{c|c} 0 & 0 \\ \hline C' & D' \end{array} \right) \right] \left(\begin{array}{c} 1/\sqrt{d} \\ \rho_v \end{array} \right) \quad (48)$$

and the new attractive, unique equilibrium state is:

$$\bar{\rho}^{(\varepsilon)} = \frac{1}{\sqrt{d}} \left(\begin{array}{c} 1 \\ -(D + \varepsilon D')^{-1}(C + \varepsilon C') \end{array} \right).$$

Because $\bar{\rho}^{(\varepsilon)}$ is a continuous function of ε, we are guaranteed that for a sufficiently high detection efficiency the perturbed attractive state will be arbitrarily close to the desired one in trace norm. Therefore, if we relax our control task to a state preparation problem with sufficiently high fidelity, this may be accomplished with a sufficiently high detection efficiency, yet strictly less than 1.

Insofar as noise suppression is the intended task, we see how monitoring a perfectly dissipative environment may be useful for control process. However, the ability to suppress the noise source via feedback is necessarily limited by the form of (37). It is apparent that the feedback action is only able to *modify the skew-Hermitian part of the noise operator L*, and even in cases where this may suffice, (nearly) perfect detection is needed. Nonetheless, Markovian feedback may prove to be extremely interesting when only partial noise suppression is considered, for instance in order to achieve longer coherence times. In this spirit, we turn to analyze how our techniques may be employed to synthesize DFSs or NSs in the Markovian limit where open-loop control is not an option.

F. DFS Synthesis With Markovian Feedback

As remarked, the feedback loop can only modify the skew-Hermitian part of the measurement operator M, which imposes strict constraints on the non-unitary generators that are able to be synthesized. A natural question is to what extent we might be able to generate DFSs or NSs by closed-loop control. In the single-observable feedback setting under examination, the DFS notion turns out to be appropriate.

Theorem 5: Let $p = d/2$, if d is even, $p = (d+1)/2$, if d is odd, and assume CHC for (37). Then a DFS of (at least) dimension p can be generated by Markovian feedback for every measurement operator M.

Proof: A DFS for (37) can be generated, under CHC hypotesis, iff there exist a choice of basis such that

$$L = M - iF = \left(\begin{array}{c|c} c\mathbb{1}_{\text{DFS}} & P \\ \hline 0 & R \end{array} \right)$$

$$= \left(\begin{array}{c|c} \text{Re}(c)\mathbb{1}_{\text{DFS}} & P/2 \\ \hline P^\dagger/2 & R^H \end{array} \right) + i \left(\begin{array}{c|c} \text{Im}(c)\mathbb{1}_{\text{DFS}} & -iP/2 \\ \hline iP^\dagger/2 & R^A \end{array} \right)$$

where we have decomposed L into Hermitian (H) and skew-Hermitian (A) parts as before. The skew-Hermitian part can be arbitrarily modified under CHC hypothesis, by choosing the appropriate F. Thus, it remains to prove that there exists a basis where M^H has the form of L^H above, and the block proportional to the identity is (at least) p-dimensional. M^H is indeed Hermitian, and can be diagonalized. Let D^H be the diagonal

matrix of the (real) eigenvalues of M^H. Then we are looking for a U and a Hilbert space decomposition such that

$$\left(\begin{array}{c|c} \text{Real}(c)\mathbb{1}_{\text{DFS}} & P/2 \\ \hline P^\dagger/2 & R^H \end{array} \right) = U D^H U^\dagger$$

$$= \left(\begin{array}{c|c} U_{\text{DFS}} & U_P \\ \hline U_Q & U_R \end{array} \right) \left(\begin{array}{c|c} D_{\text{DFS}}^H & 0 \\ \hline 0 & D_R^H \end{array} \right) \left(\begin{array}{c|c} U_{\text{DFS}}^\dagger & U_Q^\dagger \\ \hline U_P^\dagger & U_R^\dagger \end{array} \right).$$

Hence, we want to find p orthonormal vectors to stack in $(U_{\text{DFS}} U_P)^\dagger$ such that

$$(U_{\text{DFS}}|U_P) \left(\begin{array}{c|c} D_{\text{DFS}}^H & 0 \\ \hline 0 & D_R^H \end{array} \right) \left(\begin{array}{c} U_{\text{DFS}}^\dagger \\ \hline U_P^\dagger \end{array} \right) = c'\mathbb{1}_{\text{DFS}}$$

with $c' \in \mathbb{R}$. This is equivalent to ask that the compression of M^H to some subspace is equivalent to a scalar matrix. Let u_1, u_2 be normalized eigenvectors of M^H of eigenvalues d_1, d_2, respectively. Then by taking $u_3 = \alpha u_1 + \beta u_2$, such that $|\alpha|^2 + |\beta|^2 = 1$, we can construct a vector that is an eigenvector of M^H restricted to the one-dimensional subspace generated by u_3 itself. u_3 has then eigenvalue $|\alpha|^2 d_1 + |\beta|^2 d_2$, a convex combination of the former eigenvalues d_1, d_2. Take the diagonal elements of D^H in descending order, and pair the first with the last, the second to last but one, and so on. If d is odd, the eigenvalue in the middle will remain unpaired. Thus, from these pairs and the respective eigenvectors we can then obtain p new vectors as illustrated above, which are all eigenvectors of M^H restricted to their linear span, with the same eigenvalue (in general, c' will be a convex combination of the two middle eigenvalues. For odd d, it will be the middle eigenvalue). From the resulting linear span, we can then obtain the desired DFS, by choosing a feedback Hamiltonian F such that the Q block of $M - iF$ is zero. ∎

Remarks: Note that the above proof provides a constructive *algorithm for generating the DFS*. The result is potentially useful in light of ongoing efforts for efficiently finding quantum information-preserving structures [4], [23], [51]. Both the CHC assumption and the ability to perfectly monitor the noise channel are demanding for present experimental capabilities, however the promise of a new technique to generate DFSs may prompt further developments in this direction. In parallel, further study is needed in order to weaken the above requirements, as it is likely to be possible in specific contexts.

From another perspective, it is intriguing to compare Theorem 5 to the analysis of continuous-time quantum error correction presented in [52]. In that case, the target system is assumed to be a quantum register, and under the assumption that *independent errors* are occurring on different qubits, a Markovian feedback strategy is identified such that the closed-loop behavior implements continuous-time quantum error correction for a so-called "stabilizer code" [2]. This may be seen as equivalent to the generation of a DFS able, in particular, to encode $(n-1)$ logical qubits in a 2^n dimensional space. Even if our analysis follows different lines, the setting for our DFS-generation problem is similar, and our result consistently leads to the same encoding efficiency for $d = 2^n$—provided we can compound the noise effect in a single measurement operator. Interestingly, no assumption is made at this stage on the structure of

the Hilbert space, neither do we impose any constraint on the form of the "error", that is, the measurement or noise operator in our case.

Before concluding, we present a simple example of generation of attractive DFS for coupled qubits via Markovian feedback, which further illustrates some of our results.

Example 4: Consider $\mathcal{H} = \mathcal{H}_q \otimes \mathcal{H}_q$, $\mathcal{H}_q = \mathrm{span}\{|0\rangle, |1\rangle\}$, and a controlled closed-loop evolution driven by (37), with H diagonal and $M = \sigma_x \otimes \sigma_x$. Assume that we are able to monitor M and actuate the feedback Hamiltonian $F = -\sigma_y \otimes \sigma_x$. Then $L = M - iF = 2\sigma_- \otimes \sigma_x$. If we consider $\mathcal{H}_{\mathrm{DFS}} = \mathrm{span}\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle\}$, we obtain block-decomposition of the form (4), where L is such that $L_{\mathrm{DFS}} = 0$, $L_Q = 0$, $L_P = \sigma_x$. Hence, by Corollary 3 $\mathcal{H}_{\mathrm{DFS}}$ is a DFS and by Corollary 6, we can prove it is attractive. Notice that the noise operator $M = \sigma_x \otimes \sigma_x$ does admit noiseless subspaces, e.g., $\mathcal{H}'_{\mathrm{DFS}} = \mathrm{span}(|+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle)$, with $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$, but by Proposition 3, none of them can be attractive. This shows how the feedback-generated noiseless structure may offer an advantage with respect to existing ones.

IV. CONCLUSION

We have revisited some fundamental concepts about Markovian dynamics for quantum systems and restated the notion of a general quantum subsystem in *linear-algebraic* terms. A system-theoretic characterization of invariant and noiseless subsystems for Markovian quantum dynamical systems has been provided, with special attention on key model-robustness issues relevant for practical applications. In particular, we have showed that, in order to avoid situations where only fine-tuning of the Hamiltonian and dissipative terms would ensure invariance of a given subsystem, condition (12) must be replaced by the independent conditions (13) and (14). This induces similar modifications on NS invariance conditions. This part of our work both puts on more rigorous mathematical grounds and completes the existing literature on the subject.

When imperfect subsystem initialization is considered, the conditions to be imposed on the Markovian generator become more demanding, which motivates the new notion of asymptotically stable *attractive* subsystem. Interestingly, the possibility that the noise action may be useful or necessary for remaining in the intended subsystem has been independently considered before for specific QIP settings [53]. However, a formal generalization of the idea and an analytical study were still missing. Our linear-algebraic approach, along with Lyapunov's techniques, provides explicit stabilization results which have been illustrated in simple yet paradigmatic examples.

In the second part of the work, the conditions identified for subsystem invariance and attractivity serve as the starting point for designing output-feedback Markovian strategies able to actively achieve the intended quantum stabilization. We have completely characterized the state-stabilization problem for two-level systems described by FMEs of the form (37). While the analysis assumed perfect detection efficiency, a perturbative argument indicated how unique attractive states depend in a continuous fashion on the model parameters. Our suggested DFS generation strategy is also crucially dependent on the perfect detection condition, as otherwise the feedback-corrected FME

would take the form (37), implying an additional error component due to finite efficiency. Nonetheless, the norm of this portion of the noise generator is bounded, and tends to zero for $\eta \to 1$. Therefore, even if the non-unitary dynamics cannot be counteracted exactly, the time-scale of the residual noise action may still be significantly reduced in the desired subspaces for sufficiently high detection efficiency.

The Markovian, output-feedback techniques we employ have also been compared, in terms of robustness features, with the Bayesian-feedback approach. A key advantage of the Markovian approach lies in its intrinsic design simplicity, which makes it possible to avoid a costy real-time integration of the feedback-controlled master equation and thereby paves the way to implementation in higher-dimensional systems.

Further work is needed in order to establish completely general Markovian feedback stabilization results, including finite detection efficiency and *multi-channel continuous monitoring*. From an algorithmic standpoint, it also appears worthwhile to investigate the potential of the linear-algebraic approach in problems related to finding NSs for a given generator, either under perfect or imperfect knowledge. Among the most interesting perspectives, additional investigation is certainly required to establish the full power of Hamiltonian control and Markovian feedback in generating NS structures. This may point to new venues for producing protected realizations of quantum information for physical systems whose dynamics is described by quantum Markovian semigroups.

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Francesco Ticozzi received the "Laurea" degree in management engineering and the Ph.D. degree in automatic control and operations research from the University of Padova, Padova, Italy, in 2002 and 2007, respectively.

Since February 2007, he has been with the Department of Information Engineering, University of Padova, first as a Research Associate, and then as an Assistant Professor (Ricercatore). From 2005 to 2008, he held two visiting appointments at Dartmouth College, Hanover, NH. His research interests include modeling and control of quantum systems, protection of quantum information, and information-theoretic approaches to control systems.

Lorenza Viola received the "Laurea" degree in physics from the University of Trento, Trento, Italy, in 1991, and the Ph.D. degree in physics from the University of Padova, Padova, Italy, in 1996.

After holding a postdoctoral appointment with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, from 1997 to 2000, she was a J. R. Oppenheimer Fellow with the Computer and Computational Sciences Division, Los Alamos National Laboratory. In 2004, she joined the faculty of the Department of Physics and Astronomy, Dartmouth College, Hanover, NH, as an Associate Professor. Over the last ten years, her research has addressed a broad range of issues within theoretical quantum information physics, with emphasis on modeling and control of complex quantum-dynamical systems. In particular, she has been a key contributor to the development of error-control techniques for open quantum systems based on dynamical decoupling methods, and a proposer of the notion of a noiseless subsystem as a general pathway to protected realizations of information in physical systems. She co-authored so far about 70 papers in international journals and peer-reviewed volumes.

Dr. Viola is a member of the editorial board of *Physical Review A* as well as a board member of the International Physics and Control Society. Currently, she is serving as the Chair of the Topical Group on Quantum Information of the American Physical Society.