

Ordinary Differential Equations Answers

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Answer 1 (2)

Solve

$$\frac{dy}{dx} + 3y = 6e^{-2x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int 3 dx} = e^{3x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}(ye^{3x}) = 6e^x$$

$$ye^{3x} = 6e^x + A$$

$$y(x) = 6e^{-2x} + Ae^{-3x}$$

Answer 2 (2)

Solve

$$\frac{dy}{dx} - 2y = e^{3x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int -2 dx} = e^{-2x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}(ye^{-2x}) = e^x$$

$$ye^{-2x} = e^x + A$$

$$y(x) = e^{3x} + Ae^{2x}$$

Answer 3 (3)

Solve

$$y'' - 4y' + 4y = 0$$

Step 1. Characteristic equation:

$$r^2 - 4r + 4 = 0 \implies (r - 2)^2 = 0$$

$$y(x) = (A + Bx)e^{2x}$$

Answer 4 (3)

Solve

$$\frac{dy}{dx} + 2y = y^3 e^{-x}$$

where $f(x) = 2$, $g(x) = e^{-x}$, $n = 3$.

Step 1. Divide by y^3 :

$$y^{-3} \frac{dy}{dx} + 2y^{-2} = e^{-x}$$

Step 2. Let

$$\begin{aligned} z &= y^{-2}, & \frac{dz}{dx} &= -2y^{-3} \frac{dy}{dx} \\ y^{-3} \frac{dy}{dx} &= -\frac{1}{2} \frac{dz}{dx} \end{aligned}$$

Step 3. Substitute:

$$-\frac{1}{2} \frac{dz}{dx} + 2z = e^{-x}$$

$$\frac{dz}{dx} - 4z = -2e^{-x}$$

Step 4. Integrating factor:

$$w(x) = e^{\int -4 dx} = e^{-4x}$$

$$\begin{aligned} \frac{d}{dx}(ze^{-4x}) &= -2e^{-5x} \\ ze^{-4x} &= \frac{2}{5}e^{-5x} + A \\ z &= Ae^{4x} + \frac{2}{5}e^{-x} \end{aligned}$$

Step 5. Back-substitute $z = y^{-2}$:

$$y^{-2} = Ae^{4x} + \frac{2}{5}e^{-x}$$

$$y(x) = \frac{1}{\sqrt{Ae^{4x} + \frac{2}{5}e^{-x}}}$$

Answer 5 (3)

Solve

$$y'' + 9y = 0$$

Step 1. Characteristic equation:

$$r^2 + 9 = 0 \implies r = \pm 3i$$

$$y(x) = A \cos(3x) + B \sin(3x)$$

Answer 6 (4)

Solve

$$y'' - y = e^x$$

Step 1. Homogeneous solution:

$$r^2 - 1 = 0 \implies r = \pm 1$$

$$y_h = Ae^x + Be^{-x}$$

Step 2. Particular solution (resonance with e^x): Try $y_p = Cxe^x$.

$$\begin{aligned} y'_p &= Ce^x + Cxe^x, & y''_p &= 2Ce^x + Cxe^x \\ y''_p - y_p &= 2Ce^x = e^x \quad \Rightarrow \quad C = \frac{1}{2} \end{aligned}$$

$$y(x) = Ae^x + Be^{-x} + \frac{1}{2}xe^x$$

Answer 7 (4)

Solve

$$y'' + 4y = \cos(2x)$$

Step 1. Homogeneous solution:

$$r^2 + 4 = 0 \implies r = \pm 2i$$

$$y_h = A \cos(2x) + B \sin(2x)$$

Step 2. Particular solution (resonance): try

$$y_p = x(C \cos(2x) + D \sin(2x))$$

After substitution one obtains

$$4D \cos(2x) - 4C \sin(2x) = \cos(2x)$$

so

$$C = 0, \quad D = \frac{1}{4}$$

$$y(x) = A \cos(2x) + B \sin(2x) + \frac{1}{4}x \sin(2x)$$

Answer 8 (4)

Solve

$$\frac{dy}{dx} + y = y^2 e^x$$

where $f(x) = 1$, $g(x) = e^x$, $n = 2$.

Step 1. Divide by y^2 :

$$y^{-2} \frac{dy}{dx} + y^{-1} = e^x$$

Step 2. Let

$$\begin{aligned} z &= y^{-1}, & \frac{dz}{dx} &= -y^{-2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} &= \frac{dz}{dx} \end{aligned}$$

Step 3. Substitute:

$$\begin{aligned} -\frac{dz}{dx} + z &= e^x \\ \frac{dz}{dx} - z &= -e^x \end{aligned}$$

Step 4. Integrating factor:

$$w(x) = e^{\int -1 dx} = e^{-x}$$

$$\begin{aligned} \frac{d}{dx}(ze^{-x}) &= -1 \\ ze^{-x} &= -x + A \\ z &= (-x + A)e^x \end{aligned}$$

Step 5. Back-substitute $z = y^{-1}$:

$$y = \frac{1}{z} = \frac{1}{(-x + A)e^x} = \frac{e^{-x}}{A - x}$$

$$y(x) = \frac{e^{-x}}{A - x}$$

Answer 9 (5)

Solve

$$y'' - 3y' + 2y = e^{2x}$$

Step 1. Homogeneous solution:

$$r^2 - 3r + 2 = 0 \implies r = 1, 2$$

$$y_h = Ae^x + Be^{2x}$$

Step 2. Particular solution (resonance with e^{2x}): try

$$y_p = Cxe^{2x}$$

$$y'_p = Ce^{2x} + 2Cxe^{2x}, \quad y''_p = 4Ce^{2x} + 4Cxe^{2x}$$

$$y''_p - 3y'_p + 2y_p = Ce^{2x} = e^{2x} \Rightarrow C = 1$$

$$y(x) = Ae^x + Be^{2x} + xe^{2x}$$

Answer 10 (5)

Solve

$$\frac{dy}{dx} - y = y^3 e^{-2x}$$

where $f(x) = -1$, $g(x) = e^{-2x}$, $n = 3$.

Step 1. Divide by y^3 :

$$y^{-3} \frac{dy}{dx} - y^{-2} = e^{-2x}$$

Step 2. Let

$$\begin{aligned} z &= y^{-2}, & \frac{dz}{dx} &= -2y^{-3} \frac{dy}{dx} \\ y^{-3} \frac{dy}{dx} &= -\frac{1}{2} \frac{dz}{dx} \end{aligned}$$

Step 3. Substitute:

$$-\frac{1}{2} \frac{dz}{dx} - z = e^{-2x}$$

$$\frac{dz}{dx} + 2z = -2e^{-2x}$$

Step 4. Integrating factor:

$$w(x) = e^{\int 2 dx} = e^{2x}$$

$$\frac{d}{dx}(ze^{2x}) = -2$$

$$ze^{2x} = -2x + A$$

$$z = (-2x + A)e^{-2x}$$

Step 5. Back-substitute $z = y^{-2}$:

$$y^{-2} = (-2x + A)e^{-2x}$$

$$y(x) = \frac{e^x}{\sqrt{A - 2x}}$$

Answer 11 (2)

Solve

$$\frac{dy}{dx} - 4y = 8e^{2x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int -4 dx} = e^{-4x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}(ye^{-4x}) = 8e^{-2x}$$

$$ye^{-4x} = -4e^{-2x} + A$$

$$y(x) = -4e^{2x} + Ae^{4x}$$

Answer 12 (2)

Solve

$$\frac{dy}{dx} + 5y = 10e^{-3x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int 5 dx} = e^{5x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}(ye^{5x}) = 10e^{2x}$$

$$ye^{5x} = 5e^{2x} + A$$

$$y(x) = 5e^{-3x} + Ae^{-5x}$$

Answer 13 (3)

Solve

$$y'' + 6y' + 9y = 0$$

Step 1. Characteristic equation:

$$r^2 + 6r + 9 = 0 \implies (r + 3)^2 = 0$$

$$y(x) = (A + Bx)e^{-3x}$$

Answer 14 (3)

Solve

$$\frac{dy}{dx} + 3y = y^2 e^{-x}$$

where $f(x) = 3$, $g(x) = e^{-x}$, $n = 2$.

Step 1. Divide by y^2 :

$$y^{-2} \frac{dy}{dx} + 3y^{-1} = e^{-x}$$

Step 2. Let

$$\begin{aligned} z &= y^{-1} \\ \frac{dz}{dx} &= -y^{-2} \frac{dy}{dx} \\ -y^{-2} \frac{dy}{dx} &= \frac{dz}{dx} \end{aligned}$$

Step 3. Substitute:

$$\begin{aligned} -\frac{dz}{dx} + 3z &= e^{-x} \\ \frac{dz}{dx} - 3z &= -e^{-x} \end{aligned}$$

Step 4. Integrating factor:

$$\begin{aligned} w(x) &= e^{\int -3 dx} = e^{-3x} \\ \frac{d}{dx}(ze^{-3x}) &= -e^{-4x} \\ ze^{-3x} &= \frac{1}{4}e^{-4x} + A \\ z &= Ae^{3x} + \frac{1}{4}e^{-x} \end{aligned}$$

Step 5. Back-substitute $z = y^{-1}$:

$$y(x) = \frac{1}{Ae^{3x} + \frac{1}{4}e^{-x}}$$

Answer 15 (3)

Solve

$$y'' - 2y' + 5y = 0$$

Step 1. Characteristic equation:

$$r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i$$

$$y(x) = e^x (A \cos(2x) + B \sin(2x))$$

Answer 16 (4)

Solve

$$y'' + y = \sin x$$

Step 1. Homogeneous solution:

$$y_h = A \cos x + B \sin x$$

Step 2. Particular solution (resonance): try

$$y_p = x(C \cos x + D \sin x)$$

Substitution yields

$$-2C \sin x + 2D \cos x = \sin x$$

so

$$C = -\frac{1}{2}, \quad D = 0$$

$$y(x) = A \cos x + B \sin x - \frac{1}{2}x \cos x$$

Answer 17 (4)

Solve

$$y'' - 4y = 8e^{2x}$$

Step 1. Homogeneous solution:

$$r^2 - 4 = 0 \implies r = \pm 2$$

$$y_h = Ae^{2x} + Be^{-2x}$$

Step 2. Particular solution (resonance): try

$$y_p = Cxe^{2x}$$

$$y_p'' - 4y_p = 2Ce^{2x} = 8e^{2x} \Rightarrow C = 4$$

$$y(x) = Ae^{2x} + Be^{-2x} + 4xe^{2x}$$

Answer 18 (4)

Solve

$$\frac{dy}{dx} - 2y = y^3 e^x$$

where $f(x) = -2$, $g(x) = e^x$, $n = 3$.

Step 1. Divide by y^3 :

$$y^{-3} \frac{dy}{dx} - 2y^{-2} = e^x$$

Step 2. Let

$$z = y^{-2}$$

$$\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dz}{dx}$$

Step 3. Substitute:

$$-\frac{1}{2} \frac{dz}{dx} - 2z = e^x$$

$$\frac{dz}{dx} + 4z = -2e^x$$

Step 4. Integrating factor:

$$w(x) = e^{\int 4 dx} = e^{4x}$$

$$\frac{d}{dx}(ze^{4x}) = -2e^{5x}$$

$$ze^{4x} = -\frac{2}{5}e^{5x} + A$$

$$z = Ae^{-4x} - \frac{2}{5}e^x$$

Step 5. Back-substitute $z = y^{-2}$:

$$y(x) = \frac{1}{\sqrt{Ae^{-4x} - \frac{2}{5}e^x}}$$

Answer 19 (5)

Solve

$$y'' + 4y' + 4y = e^{-2x}$$

Step 1. Homogeneous solution:

$$r^2 + 4r + 4 = 0 \implies (r + 2)^2 = 0$$

$$y_h = (A + Bx)e^{-2x}$$

Step 2. Particular solution (double root, multiply by x^2): Try

$$y_p = Cx^2e^{-2x}$$

$$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} = e^{-2x} \Rightarrow C = \frac{1}{2}$$

$$y(x) = (A + Bx)e^{-2x} + \frac{1}{2}x^2e^{-2x}$$

Answer 20 (5)

Solve

$$\frac{dy}{dx} + y = y^4 e^{-x}$$

where $f(x) = 1$, $g(x) = e^{-x}$, $n = 4$.

Step 1. Divide by y^4 :

$$y^{-4} \frac{dy}{dx} + y^{-3} = e^{-x}$$

Step 2. Let

$$\begin{aligned} z &= y^{-3} \\ \frac{dz}{dx} &= -3y^{-4} \frac{dy}{dx} \\ y^{-4} \frac{dy}{dx} &= -\frac{1}{3} \frac{dz}{dx} \end{aligned}$$

Step 3. Substitute:

$$\begin{aligned} -\frac{1}{3} \frac{dz}{dx} + z &= e^{-x} \\ \frac{dz}{dx} - 3z &= -3e^{-x} \end{aligned}$$

Step 4. Integrating factor:

$$\begin{aligned} w(x) &= e^{\int -3 dx} = e^{-3x} \\ \frac{d}{dx}(ze^{-3x}) &= -3e^{-4x} \\ ze^{-3x} &= \frac{3}{4}e^{-4x} + A \\ z &= Ae^{3x} + \frac{3}{4}e^{-x} \end{aligned}$$

Step 5. Back-substitute $z = y^{-3}$:

$y(x) = (Ae^{3x} + \frac{3}{4}e^{-x})^{-1/3}$