Roots Of Unity Answers

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Answer 1 (3)

Find all the cube roots of unity ($z^3 = 1$).

Step 1. General formula for nth roots of unity:

$$z_k = e^{2\pi ik/n}, \quad k = 0, 1, \ldots, n-1$$

Step 2. For $n = 3$:

$$z_0 = e^{2\pi i\cdot 0/3} = 1$$
$$z_1 = e^{2\pi i\cdot 1/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_2 = e^{2\pi i\cdot 2/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\boxed{z = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

Answer 2 (3)

Find all the fourth roots of unity ($z^4 = 1$).

Step 1. Using $z_k = e^{2\pi ik/4}$, $k = 0, 1, 2, 3$:

$$
\begin{aligned}
z_0 &= e^0 = 1 \\
z_1 &= e^{2\pi i/4} = e^{i\pi/2} = i \\
z_2 &= e^{2\pi i/2} = e^{i\pi} = -1 \\
z_3 &= e^{6\pi i/4} = e^{3i\pi/2} = -i
\end{aligned}
$$

Step 2. Plot on the complex plane: The points are on the unit circle at $0°, 90°, 180°, 270°$.

$$\boxed{z = 1,\ i,\ -1,\ -i}$$

Answer 3 (4)

Find all the sixth roots of unity ($z^6 = 1$).

Step 1. Use the general formula:

$$z_k = e^{2\pi i k/6}, \quad k = 0, 1, 2, 3, 4, 5$$

Step 2. Compute each root:

$$z_0 = e^0 = 1$$
$$z_1 = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_2 = e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_3 = e^{i\pi} = -1$$
$$z_4 = e^{i4\pi/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$z_5 = e^{i5\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\boxed{z = 1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

Answer 4 (4)

Show that the sum of all fifth roots of unity ($z^5 = 1$) is zero.

Step 1. Let $z_k = e^{2\pi ik/5}$, $k = 0, 1, 2, 3, 4$.

Step 2. The sum:

$$\sum_{k=0}^{4} z_k = 1 + e^{2\pi i/5} + e^{4\pi i/5} + e^{6\pi i/5} + e^{8\pi i/5}$$

Step 3. This is a geometric series with ratio $r = e^{2\pi i/5}$, number of terms $n = 5$:

$$\sum_{k=0}^{4} z_k = \frac{1 - r^5}{1 - r} = \frac{1 - 1}{1 - r} = 0$$

$$\boxed{\sum_{k=0}^{4} z_k = 0}$$

Answer 5 (5)

If z is a seventh root of unity ($z^7 = 1$), find the value of

$$1 + z + z^2 + \cdots + z^6$$

Step 1. Recognize the geometric series:

$$S = 1 + z + z^2 + \cdots + z^6$$

Step 2. Sum of geometric series:

$$S = \frac{1 - z^7}{1 - z} = \frac{1 - 1}{1 - z} = 0$$

$$\boxed{1 + z + z^2 + \cdots + z^6 = 0}$$

Answer 6 (4)

If z is a sixth root of unity ($z^6 = 1$), compute the value of

$$z + z^2 + z^4$$

Step 1. The sixth roots of unity are

$$z_k = e^{2\pi i k/6}, \quad k = 0, 1, 2, 3, 4, 5$$

Step 2. Use symmetry and pairing of roots:

$$z + z^2 + z^4 = z_1 + z_2 + z_4$$

Step 3. Sum the roots using their exponential forms:

$$z_1 = e^{i\pi/3}, \quad z_2 = e^{2i\pi/3}, \quad z_4 = e^{4i\pi/3}$$

Step 4. Convert to rectangular form and sum:

$$z_1 + z_2 + z_4 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\boxed{z + z^2 + z^4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

Answer 7 (4)

If z is a seventh root of unity ($z^7 = 1$), compute

$$\prod_{k=1}^{6}(1 - z^k)$$

Step 1. Factorization of $z^7 - 1$:

$$z^7 - 1 = (z - 1)(1 - z)(1 - z^2)(1 - z^3)\dots(1 - z^6)$$

Step 2. Set $z = 1$:

$$1^7 - 1 = 0 = (1 - 1)\prod_{k=1}^{6}(1 - z^k)$$

Step 3. Compute the product using known formula:

$$\prod_{k=1}^{6}(1 - z^k) = 7$$

$$\boxed{\prod_{k=1}^{6}(1 - z^k) = 7}$$

Answer 8 (5)

Let z_1, z_2, z_3, z_4 be the fourth roots of unity. Compute

$$z_1^2 + z_2^2 + z_3^2 + z_4^2$$

Step 1. Fourth roots of unity:

$$z_0 = 1,\ z_1 = i,\ z_2 = -1,\ z_3 = -i$$

Step 2. Square each root:

$$z_0^2 = 1, \quad z_1^2 = i^2 = -1, \quad z_2^2 = (-1)^2 = 1, \quad z_3^2 = (-i)^2 = -1$$

Step 3. Sum the squares:

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1 + (-1) + 1 + (-1) = 0$$

$$\boxed{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0}$$

Answer 9 (5)

If z is a fifth root of unity ($z^5 = 1$), show that

$$1 + z^2 + z^4 + z^6 + z^8 = 0$$

Step 1. Reduce powers modulo 5:

$$z^5 = 1 \implies z^6 = z, \quad z^8 = z^3$$

Step 2. Rewrite the sum:

$$1 + z^2 + z^4 + z^6 + z^8 = 1 + z^2 + z^4 + z + z^3 = 1 + z + z^2 + z^3 + z^4$$

Step 3. Sum of all fifth roots of unity:

$$1 + z + z^2 + z^3 + z^4 = 0$$

$$\boxed{1 + z^2 + z^4 + z^6 + z^8 = 0}$$

Answer 10 (5)

If z is an eighth root of unity ($z^8 = 1$), compute the sum

$$S = 1 + z + z^3 + z^5 + z^7$$

Step 1. Factor out geometric series:

$$S = 1 + z + z^3 + z^5 + z^7 = (1 + z^4) + z(1 + z^4) + z^3(1 + z^4)$$

Step 2. Note $z^4 = -1$:

$$1 + z^4 = 1 + (-1) = 0$$

Step 3. Multiply terms by 0:

$$S = 0 + 0 + 0 = 0$$

$$\boxed{S = 0}$$

Answer 11 (3)

Point $P(\sqrt{3}, 1)$ is one vertex of an equilateral triangle centered at $(0, 0)$.

Step 1. Rotate P by $120°$ and $240°$ about the origin. Use rotation formula:

$$(x', y') = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

Step 2. Rotate by $120° = 2\pi/3$:

$$x' = \sqrt{3}\cos\frac{2\pi}{3} - 1\sin\frac{2\pi}{3} = -2, \quad y' = \sqrt{3}\sin\frac{2\pi}{3} + 1\cos\frac{2\pi}{3} = 0$$

Step 3. Rotate by $240° = 4\pi/3$:

$$x'' = \sqrt{3}\cos\frac{4\pi}{3} - 1\sin\frac{4\pi}{3} = 1, \quad y'' = \sqrt{3}\sin\frac{4\pi}{3} + 1\cos\frac{4\pi}{3} = -\sqrt{3}$$

$$\boxed{\text{Other vertices: } (-2, 0) \text{ and } (1, -\sqrt{3})}$$

Answer 12 (3)

Square centered at origin with one vertex $(2, 0)$.

Step 1. Distance from origin: $r = 2$. The square has vertices at angles $0°, 90°, 180°, 270°$.

Step 2. Convert to coordinates:

$$(2, 0),\ (0, 2),\ (-2, 0),\ (0, -2)$$

$$\boxed{\text{Vertices: } (2, 0), (0, 2), (-2, 0), (0, -2)}$$

Answer 13 (4)

Rotate point $A(1, 0)$ by $120°$ counterclockwise.

Step 1. Rotation formula:

$$(x', y') = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

Step 2. Compute:

$$x' = 1 \cdot \cos 120° - 0 \cdot \sin 120° = -\frac{1}{2}, \quad y' = 1 \cdot \sin 120° + 0 \cdot \cos 120° = \frac{\sqrt{3}}{2}$$

$$\boxed{\text{Rotated point: } \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$$

Answer 14 (4)

Square with vertices $A(1,2)$ and $B(3,4)$.

Step 1. Vector from A to B:

$$\vec{AB} = (3-1, 4-2) = (2,2)$$

Step 2. Perpendicular vector for square sides:

$$\vec{AD} = (-2,2), \quad \vec{BC} = (-2,2)$$

Step 3. Remaining vertices:

$$D = A + \vec{AD} = (1-2, 2+2) = (-1,4)$$

$$C = B + \vec{BC} = (3-2, 4+2) = (1,6)$$

$$\boxed{\text{Other vertices: } C(1,6), D(-1,4)}$$

Answer 15 (4)

Regular hexagon with vertex $P(1, \sqrt{3})$, center at origin.

Step 1. Distance from center: $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Step 2. Hexagon vertices at angles $\theta + k \cdot 60°$, $k = 0..5$.

Step 3. Compute coordinates (rotating 60° each step):

$$(1, \sqrt{3}),\ (-1, \sqrt{3}),\ (-2, 0),\ (-1, -\sqrt{3}),\ (1, -\sqrt{3}),\ (2, 0)$$

$$\boxed{\text{Remaining vertices: } (-1, \sqrt{3}), (-2, 0), (-1, -\sqrt{3}), (1, -\sqrt{3}), (2, 0)}$$

Answer 16 (5)

Triangle ABC with centroid at origin. $A(3,1)$, $B(-1,2)$, find C.

Step 1. Centroid formula:

$$G = \frac{A+B+C}{3} = (0,0) \implies C = -(A+B)$$

Step 2. Compute:

$$C = -((3,1)+(-1,2)) = -(2,3) = (-2,-3)$$

$$\boxed{C = (-2,-3)}$$

Answer 17 (5)

Regular pentagon, center at origin, vertex $P(2, 0)$.

Step 1. Pentagon vertices at angles $\theta + k \cdot 72°$, $k = 0..4$.

Step 2. Compute coordinates:

$$(2, 0),\ (2\cos 72°, 2\sin 72°),\ (2\cos 144°, 2\sin 144°),$$

$$(2\cos 216°, 2\sin 216°),\ (2\cos 288°, 2\sin 288°)$$

Step 3. Approximate decimal values:

$$(2, 0),\ (0.618, 1.902),\ (-1.618, 1.176),\ (-1.618, -1.176),\ (0.618, -1.902)$$

$$\boxed{\text{Remaining vertices as above}}$$

Answer 18 (5)

Rotate triangle $A(0,0), B(2,0), C(1,\sqrt{3})$ 60° counterclockwise.

Step 1. Rotation formula:

$$(x', y') = (x\cos 60^\circ - y\sin 60^\circ,\ x\sin 60^\circ + y\cos 60^\circ)$$

Step 2. Compute each vertex:

$$A' = (0,0)$$
$$B' = (2\cdot 0.5 - 0\cdot \sqrt{3}/2,\ 2\cdot \sqrt{3}/2 + 0\cdot 0.5) = (1, \sqrt{3})$$
$$C' = (1\cdot 0.5 - \sqrt{3}\cdot \sqrt{3}/2,\ 1\cdot \sqrt{3}/2 + \sqrt{3}\cdot 0.5) = (-1, \sqrt{3})$$

$$\boxed{A'(0,0), B'(1,\sqrt{3}), C'(-1,\sqrt{3})}$$

Answer 19 (5)

Square with vertices $A(1,0), B(0,1), C(-1,0)$. Find the fourth vertex D.

Step 1. Diagonal intersection: midpoint of $AC = ((1+-1)/2, (0+0)/2) = (0,0)$

Step 2. Find D opposite B using midpoint formula:

$$\text{Midpoint of } B \text{ and } D = (0,0) \implies D = 2 \cdot (0,0) - B = -B = (0,-1)$$

$$\boxed{D = (0,-1)}$$

Answer 20 (5)

Regular octagon, center at origin, vertex $(1, 0)$.

Step 1. Vertices at angles $k \cdot 45°$, $k = 0..7$, radius $r = 1$

Step 2. Coordinates:

$$(1, 0),\ (\sqrt{2}/2, \sqrt{2}/2),\ (0, 1),\ (-\sqrt{2}/2, \sqrt{2}/2),$$

$$(-1, 0),\ (-\sqrt{2}/2, -\sqrt{2}/2),\ (0, -1),\ (\sqrt{2}/2, -\sqrt{2}/2)$$

$$\boxed{\text{Remaining vertices as above}}$$