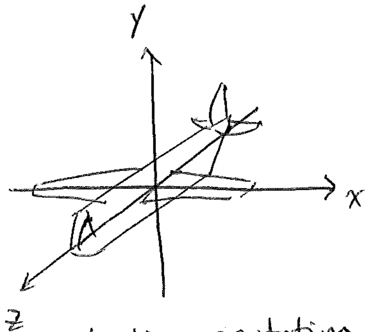
**Problem:** <u>Gimbal lock</u>

- when two axes align
- difficult to get certain ~~axes~~ configurations of rotations



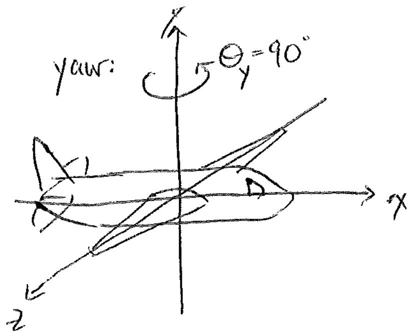

starting orientation

want result: yaw 90°, roll 10°, pitch 0°

$\Rightarrow \underset{\text{roll}}{R_z(\theta_z = 10°)} \Rightarrow \underset{\text{yaw}}{R_y(\theta_y = 90°)} \Rightarrow R_x(\theta_x = 0°)$

OR $\Rightarrow \underset{\text{yaw}}{R_y(\theta_y = 90°)} \Rightarrow R_x(\theta_x = 10°) \Rightarrow R_z(\theta_z = 0°)$

what if order of rotations must be $x \to y \to z$ ?




1. $\theta_x = 0$ since we want pitch $= 0°$
2. $\theta_y = 90°$ to get yaw $90°$
3. $\theta_z = ?$ now z-axis is pitch rotation $\Rightarrow$ lost d.o.f. for <u>roll</u> rotation

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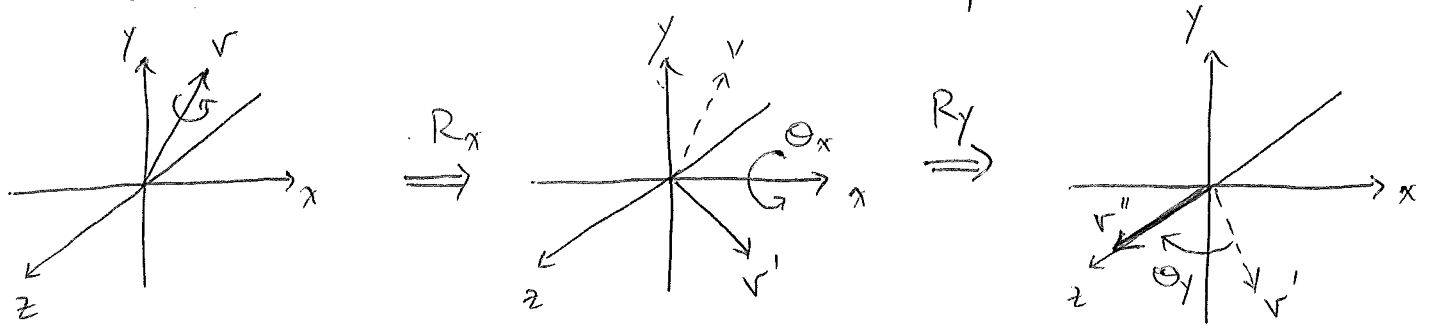
To avoid Gimbal lock, need to carefully consider order of rotations

<u>Another issue</u>: rotation about <u>arbitrary</u> axis $v$




($v'$ in x-z plane) ($v''$ aligned with $R_z$)

$v'' = z\text{-axis}$
$= R_y R_x v$

$$p' = \underbrace{(R_y R_x)^{-1}}_{\text{rotate back to } v} \underbrace{R_z}_{\text{rotate around } z} \underbrace{(R_y R_x)}_{v'' \text{ align with } z}$$

# Quaternion representation

Alternative to matrix representation with Euler angles

$\Rightarrow$ extension to complex numbers

## Aside: review of complex numbers

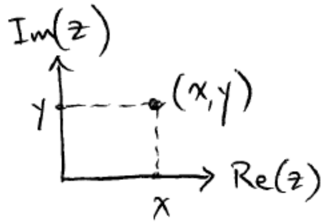
imaginary number: $i = \sqrt{-1}$ $\Rightarrow$ $x^2 + 1 = 0$

$x^2 = -1$

$x = \pm i$

complex numbers take form:

$$\boxed{z = \underbrace{x}_{\text{real}} + i\underbrace{y}_{\text{imaginary}}}$$




can be written $z = (x, y)$ e.g. $z = (x, 0)$ pure real

$z = (0, y)$ pure imaginary

## Operations:

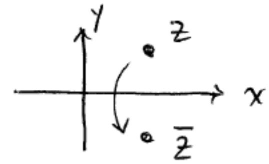
Addition: $z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$

Multiply: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$ $\leftarrow$ distributive rule

$= (x_1 x_2 + i^2 y_1 y_2 + i(x_1 y_2 + x_2 y_1))$

$= (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

Complex conjugate: $z = (x, y)$ $\bar{z} = (x, -y)$ $\Rightarrow$




Mirror$_x$ $S(1,-1)$

Magnitude: $|z| = \sqrt{x^2 + y^2}$

$= \sqrt{z\bar{z}}$ $\Rightarrow$ $z\bar{z} = |z|^2$

Inverse: $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ $\Rightarrow$ $z\left(\dfrac{\bar{z}}{|z|^2}\right) = \dfrac{z\bar{z}}{|z|^2} = \dfrac{|z|^2}{|z|^2} = 1$

Example $z_1 = (x, y)$

$z_2 = (\cos\theta, \sin\theta)$

$z_1 z_2 = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$ equivalent

Recall 2D CCW rotation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$

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## Quaternions: extension to complex numbers

$$q = \underset{\text{scalar}}{s} + \underbrace{ia + jb + kc}_{\text{imaginary}}$$

3 imaginary dimensions

$i^2 = j^2 = k^2 = -1$

$ijk = -1$

$ij = -ji = k$

$jk = -kj = i$

$ki = -ik = j$

$q$ can be written: $q = (s, v)$   $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

4D entity

**Operations:**

Addition: $q_1 + q_2 = (s_1 + s_2,\ a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2)$

$= (s_1 + s_2,\ v_1 + v_2)$

Multiply: $q_1 q_2 = (\underbrace{s_1 s_2 - v_1 \cdot v_2}_{\text{real}},\ \underbrace{s_1 v_2 + s_2 v_1 + v_1 \times v_2}_{\text{imaginary}})$

negative since $i^2 a_1 a_2 + j^2 b_1 b_2 + k^2 c_1 c_2$

Magnitude: $|q|^2 = s^2 + v \cdot v$

Inverse: $q^{-1} = \frac{1}{|q|^2}(s, -v) = \frac{\bar{q}}{|q|^2}$   if normalized $q^{-1} = \bar{q}$

Identity: $q^{-1} q = (1, 0)$ (unit quaternion)

$|q^{-1} q| = 1$

# 3D Rotations using quaternions

$$q = (s, v) \quad \text{where} \quad v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\boxed{q = \left(\cos\left(\frac{\theta}{2}\right), u\sin\left(\frac{\theta}{2}\right)\right)}$$

where $u$: unit vector in direction of axis of rotation

$\theta$: angle of rotation

always norm 1

$$|q|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\underbrace{(\cancel{u \cdot u})}_{1}$$
$$= 1$$

$$p = (0, p)$$

point to be rotated: $p = ip_x + jp_y + kp_z$

$$\boxed{\begin{aligned} p' &= qpq^{-1} \\ &= (s,v)(0,p)(s,-v) \end{aligned}}$$

$= qp\bar{q}$ for $|q| = 1$

**Example** $\theta = 90°$ ccw

$u = (1,0,0)$ rotation about x-axis (in y-z plane)

$$q = (\cos\underbrace{45°}_{\theta/2}, \ \sin 45°\underbrace{(1,0,0)}_{u})$$

$$p' = qpq^{-1}$$
$$= qp\bar{q} \leftarrow \text{if } q \text{ normalized? yes}$$

$$p' = (s,v)(0,p)(s,-v)$$
$$= s^2p + v(p \cdot v) + 2s(v \times p) + v \times (v \times p)$$