Computational Statistics
Homework 2

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Exercise 5.4. Simulated Annealing Algorithm

(a) Reproducing simulations from Example 5.5

With the code in the jupyter notebook, we reproduce Example 5.5., i.e. Simulated Annealing algorithm with :

$$\begin{aligned}
h(x) &= [\cos(50x) + \sin(20x)]^2 \\
a_t &= max(x^{(t)} - r, 0) \\
b_t &= min(x^{(t)} + r, 1) \\
u &\sim \mathcal{U}(a_t, b_t) \\
\rho^{(t)} &= \min\left\{\exp\left(\frac{h(u) - h(x^{(t)})}{T_t}\right), 1\right\} \\
T_t &= \frac{1}{\log(t)}
\end{aligned}$$

The algorithm is, at each time t:

1. Simulate $u \sim \mathcal{U}(a_t, b_t)$.
2. Accept $x^{(t+1)}$ with probability $\rho^{(t)}$, take$x^{(t+1)} = x^{(t)}$ otherwise.
3. Update T_t to T_{t+1}.

And we start a sequence of 2500 simulations with $x^{(0)} = 0$. As in Example 5.5, four sequences are simulated and represented in Figure 1.

(b) Changing parameters in r and T_t

We now define $T_t = \frac{c}{\log(t)}$. We show in Figure 2 simulated sequences for a range of r and c values. We see that lower values of c seem to give a faster convergence. Regarding r, $r = 0.75$ seems to be the value that fastens convergence the most.

Figure 1: Reproduction of Example 5.5 : Simulated Annealing algorithm

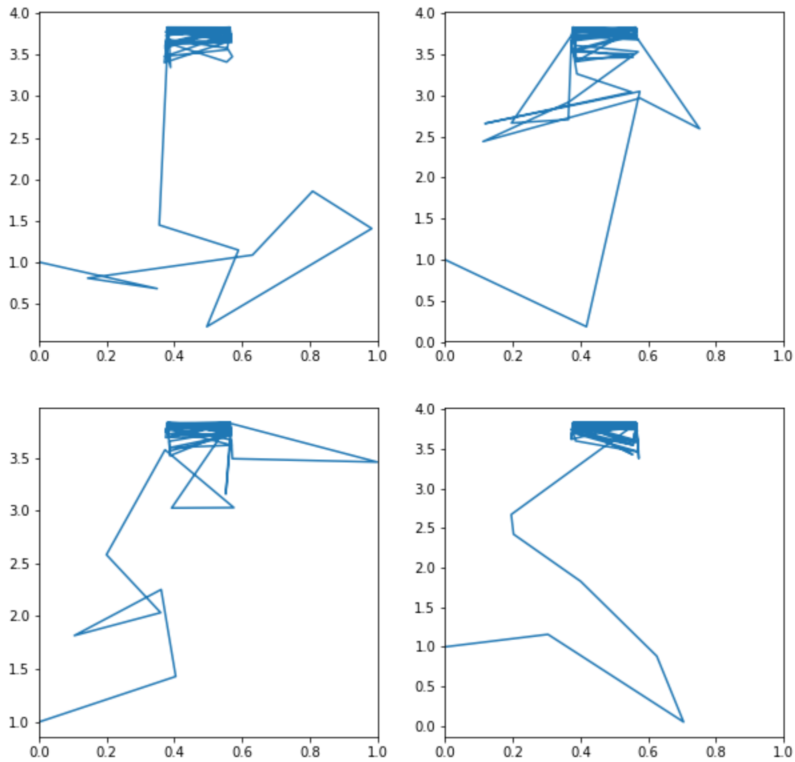
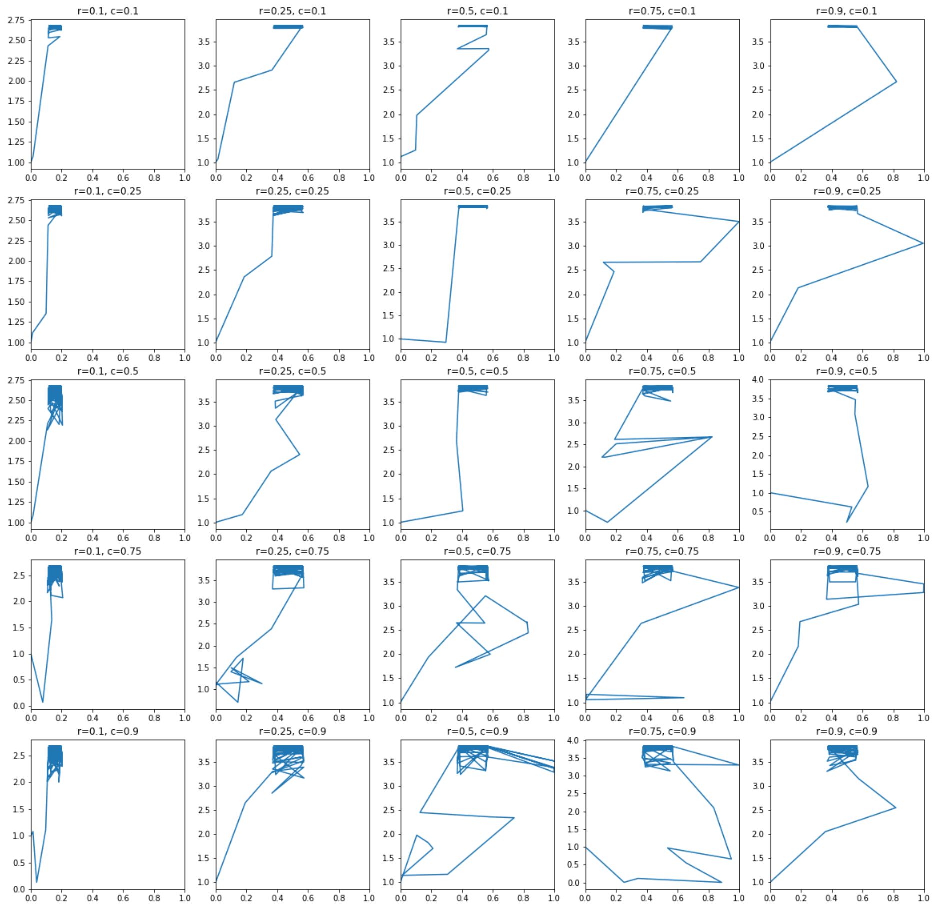


Figure 2: Simulated Annealing for various values of r and c



Exercise 5.22. EM Algorithm on Bernouilli Variables

Here, we observe $X_1, ..., X_n$ i.i.d. depending on $Z_1, ..., Z_n$ independently distributed as $\mathcal{N}(\zeta, \sigma^2)$. We have, for a known threshold u :

$$X_i = \begin{cases} 0 & \text{if} \quad Z_i \leq u \\ 1 & \text{if} \quad Z_i > u \end{cases}$$

Our aim is to obtain MLE estimates for ζ, σ^2.

(a) Likelihood function

We want to compute $\mathcal{L}(x; (\zeta, \sigma))$. We have :

$$\begin{aligned} \mathcal{L}(x; (\zeta, \sigma)) &= \prod_{i=1}^{n} \mathbb{P}(X_i = x_i | \zeta, \sigma) \\ &= p^S (1-p)^{n-S} \end{aligned}$$

As the X_i are drawn independently from a Bernouilli. p is the probability for X_i to be equal to 1, thus:

$$\begin{aligned} p &= \mathbb{P}(Z_i > u) \\ &= \mathbb{P}\left(\frac{Z_i - \zeta}{\sigma} > \frac{u - \zeta}{\sigma}\right) \\ &= 1 - \mathbb{P}\left(\frac{Z_i - \zeta}{\sigma} \leq \frac{u - \zeta}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{u - \zeta}{\sigma}\right) \\ &= \Phi\left(\frac{\zeta - u}{\sigma}\right) \end{aligned}$$

S is the number of x_i equal to one, thus we can write :

$$S = \sum_{i=1}^{n} x_i$$

(b) Complete Likelihood

We are now interested in the complete likelihood function $\mathcal{L}((z); (\zeta, \sigma))$. We have :

$$\begin{aligned} \mathcal{L}(z; (\zeta, \sigma)) &= \prod_{i=1}^{n} \mathbb{P}(Z_i = z_i \mid \zeta, \sigma) \\ &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z_i - \zeta)^2}{2\sigma^2}\right) \end{aligned}$$

Thus,

$$\begin{aligned}\log \mathcal{L}((z);(\zeta,\sigma)) &= \sum_{i=1}^{n} -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(z_i-\zeta)^2}{2\sigma^2} \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{n}\frac{(z_i-\zeta)^2}{2\sigma^2}\end{aligned}$$

Then, we take the expectation of this function, on the observed data, i.e. the (x_i), regarding the random variables Z_i :

$$\begin{aligned}\mathbb{E}\left[\log \mathcal{L}((Z_i)_i;(\zeta,\sigma))|x_i\right] &= -\frac{n}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{n}\mathbb{E}\left[\frac{(Z_i-\zeta)^2}{2\sigma^2}\middle| x_i\right] \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\mathbb{E}[Z_i^2 - 2\zeta Z_i + \zeta^2|x_i] \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\mathbb{E}[Z_i^2|x_i] - 2\zeta\mathbb{E}[Z_i|x_i] + \zeta^2)\end{aligned}$$

(c) EM Sequence

We now want to give the EM sequence. We first treat $\zeta^{(t)}$ sequence :

$$\begin{aligned}\frac{\partial \mathbb{E}\left[\log \mathcal{L}((Z_i)_i;\zeta=\zeta^{(t)},\sigma=\sigma^{(t)})|x_i\right]}{\partial \zeta^{(t)}} &= -\frac{1}{2(\sigma^{(t)})^2}\sum_{i=1}^{n}(-2\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}] + 2\zeta^{(t)}) \\ &= -\frac{n}{(\sigma^{(t)})^2}\zeta^{(t)} + \frac{1}{(\sigma^{(t)})^2}\sum_{i=1}^{n}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}]\end{aligned}$$

We chose $\zeta^{(t+1)}$ to minimize the expected log-likelihood of the complete data, thus:

$$\zeta^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}]$$

We are now interested in $\sigma^{(t)}$. We compute :

$$\begin{aligned}\frac{\partial \mathbb{E}\left[\log \mathcal{L}((Z_i)_i;\zeta=\zeta^{(t)},\sigma=\sigma^{(t)})|x_i\right]}{\partial (\sigma^{(t)})^2} &= -\frac{n}{2}\frac{2\pi}{2\pi(\sigma^{(t)})^2} \\ &+ \frac{1}{2((\sigma^{(t)})^2)^2}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - 2\zeta^{(t)}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}] + (\zeta^{(t)})^2\right]\end{aligned}$$

Minimization of the expected log-likehood of the complete data gives:

$$
\begin{aligned}
(\sigma^{(t+1)})^2 &= \frac{1}{n}\left[\sum_{i=1}^{n}\left(\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - 2\zeta^{(t+1)}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}] + (\zeta^{(t+1)})^2\right)\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - 2\zeta^{(t+1)}\sum_{i=1}^{n}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}] + n(\zeta^{(t+1)})^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - 2\zeta^{(t+1)}\sum_{i=1}^{n}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}] + n(\zeta^{(t+1)})^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - 2\zeta^{(t+1)}n\zeta^{(t+1)} + n(\zeta^{(t+1)})^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - n(\zeta^{(t+1)})^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - n(\zeta^{(t+1)})^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}\mathbb{E}[Z_i^2|x_i,\zeta^{(t)},\sigma^{(t)}] - \frac{1}{n}\left(\sum_{i=1}^{n}\mathbb{E}[Z_i|x_i,\zeta^{(t)},\sigma^{(t)}]\right)^2\right]
\end{aligned}
$$

(d) Expectations Computation

We want to compute $\mathbb{E}[Z_i|x_i,\zeta,\sigma]$. Let's first suppose that $X_i = 1$. Then :

$$
\begin{aligned}
\mathbb{E}[Z_i|X_i = 1\zeta,\sigma]] &= \frac{1}{\mathbb{P}(X_i = 1)}\int_u^\infty z\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(z-\zeta)^2}{2\sigma^2}}dz \\
&= \frac{1}{\mathbb{P}(Z_i > u)}\int_{\frac{u-\zeta}{\sigma}}^\infty(\zeta+\sigma x)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2}}\sigma dx \qquad \leftarrow x = \frac{z-\zeta}{\sigma} \\
&= \frac{1}{1-\Phi(\frac{u-\zeta}{\sigma})}\int_{\frac{u-\zeta}{\sigma}}^\infty(\zeta+\sigma x)\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \\
&= \frac{1}{1-\Phi(\frac{u-\zeta}{\sigma})}\left[\zeta\int_{\frac{u-\zeta}{\sigma}}^\infty\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx - \sigma\frac{1}{\sqrt{2\pi}}\int_{\frac{u-\zeta}{\sigma}}^\infty(-x)e^{\frac{-x^2}{2}}dx\right] \\
&= \frac{1}{1-\Phi(\frac{u-\zeta}{\sigma})}\left[\zeta(1-\Phi\left(\frac{u-\zeta}{\sigma}\right)) - \sigma\frac{1}{\sqrt{2\pi}}\left[e^{\frac{-x^2}{2}}\right]_{\frac{u-\zeta}{\sigma}}^\infty\right] \\
&= \frac{1}{1-\Phi(\frac{u-\zeta}{\sigma})}\left[\zeta(1-\Phi\left(\frac{u-\zeta}{\sigma}\right)) + \sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{(u-\zeta)^2}{2\sigma}}\right] \\
&= \zeta + \sigma\frac{\varphi\left(\frac{u-\zeta}{\sigma}\right)}{1-\Phi\left(\frac{u-\zeta}{\sigma}\right)}
\end{aligned}
$$

Now, if $X_i = 0$:

$$\mathbb{E}[Z_i|X_i = 0, \zeta, \sigma] = \frac{1}{\mathbb{P}(X_i = 0)} \int_{-\infty}^{u} z \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(z-\zeta)^2}{2\sigma^2}} dz$$

Using the same substitution $x = \frac{z-\zeta}{\sigma}$ as in the first case, we have:

$$\begin{aligned}\mathbb{E}[Z_i|X_i = 0, \zeta, \sigma] &= \frac{1}{\Phi(\frac{u-\zeta}{\sigma})} \left[\zeta\Phi\left(\frac{u-\zeta}{\sigma}\right) - \sigma\frac{1}{\sqrt{2\pi}} e^{-\frac{(u-\zeta)^2}{2\sigma}}\right] \\ &= \zeta - \sigma\frac{\varphi\left(\frac{u-\zeta}{\sigma}\right)}{\Phi\left(\frac{u-\zeta}{\sigma}\right)}\end{aligned}$$

Thus, we generally have :

$$\mathbb{E}[Z_i|x_i, \zeta, \sigma] = \zeta + \sigma H_i\left(\frac{u-\zeta}{\sigma}\right)$$

With H_i as described in the exercice. Let's compute the expectation $\mathbb{E}[Z_i^2|x_i, \zeta, \sigma]$, in the same way. First :

$$\begin{aligned}\mathbb{E}[Z_i^2|X_i = 1, \zeta, \sigma] &= \frac{1}{\mathbb{P}(X_i = 1)} \int_{u}^{\infty} z^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(z-\zeta)^2}{2\sigma^2}} dz \\ &= \frac{1}{\mathbb{P}(Z_i > u)} \int_{\frac{u-\zeta}{\sigma}}^{\infty} (\zeta + \sigma x)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2}} \sigma dx \qquad \leftarrow x = \frac{z-\zeta}{\sigma} \\ \mathbb{E}[Z_i^2|X_i = 1, \zeta, \sigma](1 - \Phi\left(\frac{u-\zeta}{\sigma}\right)) &= \zeta^2 \int_{\frac{u-\zeta}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\ &\quad - 2\zeta\sigma \int_{\frac{u-\zeta}{\sigma}}^{\infty} (-x) \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\ &\quad + \sigma^2 \int_{\frac{u-\zeta}{\sigma}}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\ &= \zeta^2(1 - \Phi\left(\frac{u-\zeta}{\sigma}\right)) + 2\zeta\sigma\varphi\left(\frac{u-\zeta}{\sigma}\right) + A\end{aligned}$$

Where $A = \sigma^2 \int_{\frac{u-\zeta}{\sigma}}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$. We use integration by parts to compute A.

$$\begin{aligned}A &= -\frac{\sigma^2}{\sqrt{2\pi}} \left(\left[xe^{-\frac{x^2}{2}}\right]_{\frac{u-\zeta}{\sigma}}^{\infty} - \int_{\frac{u-\zeta}{\sigma}}^{\infty} e^{-\frac{x^2}{2}} dx\right) \\ &= \sigma(u-\zeta)\varphi\left(\frac{u-\zeta}{\sigma}\right) + \sigma^2(1 - \Phi\left(\frac{u-\zeta}{\sigma}\right))\end{aligned}$$

Thus, we have:

$$\mathbb{E}[Z_i^2|X_i = 1, \zeta, \sigma] = \zeta^2 + \sigma^2 + \sigma(u+\zeta)\frac{\varphi\left(\frac{u-\zeta}{\sigma}\right)}{1 - \Phi\left(\frac{u-\zeta}{\sigma}\right)}$$

By re-using the same substitution and integration by parts for $\mathbb{E}[Z_i^2|X_i = 0]$, we find that $\mathbb{E}[Z_i^2|X_i = 0, \zeta, \sigma] = \zeta^2 + \sigma^2 - \sigma(u + \zeta)\frac{\varphi\left(\frac{u-\zeta}{\sigma}\right)}{\Phi\left(\frac{u-\zeta}{\sigma}\right)}$, so we can conclude that :

$$\mathbb{E}[Z_i^2|x_i, \zeta, \sigma] = \zeta^2 + \sigma^2 + \sigma(u + \zeta)H_i\left(\frac{u-\zeta}{\sigma}\right)$$

(e) Convergence

Let's return to the expected complete-data log-likelihood, which is equal to :

$$\begin{aligned}
&\mathbb{E}\left[\log \mathcal{L}(Z|x_i, \zeta^*, \sigma^*)\right] = \\
&-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\mathbb{E}[Z_i^2|x_i, \zeta^*, \sigma^*] - 2\zeta\mathbb{E}[Z_i|x_i, \zeta^*, \sigma^*] + \zeta^2) \\
&= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[\zeta^* + \sigma^* H_i\left(\frac{u-\zeta^*}{\sigma^*}\right) - 2\left[(\zeta^*)^2 + (\sigma^*)^2 + \sigma^*(u+\zeta^*)H_i\left(\frac{u-\zeta^*}{\sigma^*}\right)\right] + \zeta^2\right]
\end{aligned}$$

This expression is continuous in ζ, σ and ζ^*, σ^* We can thus use Theorem 5.16. Every limit point of the EM sequence is a stationnary point of the log-likehood. As the log-likelihood is log-concave, is only has one maximum. The EM sequence converges to the points of this maximum, and thus maximize the log-likelihood.

Exercise 5.33. EM on Bayesian Hierarchical Models

We have a hierarchical bayesian model, i.e. :

$$\begin{aligned} X|\theta &\sim f(x|\theta) \\ \theta|\lambda &\sim \pi(\theta|\ \lambda) \\ \lambda &\sim \gamma(\lambda) \end{aligned}$$

We want to estimate the posterior $\pi(\theta|x)$, and in order to do so, we use EM Algorithm.

(a) Log-Likelihood

Our aim is to compute $\log \pi(\theta|x)$. Let θ^* be any value defined in the same set as θ. We use the fact that :

$$\begin{aligned} &\pi(\theta|x) = \frac{\pi(\theta, \lambda|x)}{k(\lambda|\theta, x)} \\ &\Leftrightarrow \log \pi(\theta|x) = \log \pi(\theta, \lambda\ |x) - \log k(\lambda|\theta, x) \\ &\Leftrightarrow \log \pi(\theta|x) k(\lambda|\theta^*, x) = \log \pi(\theta, \lambda|x) k(\lambda|\theta^*, x) - \log k(\lambda|\theta, x) k(\lambda|\theta^*, x) \\ &\Leftrightarrow \int \log \pi(\theta|x) k(\lambda|\theta^*, x) d\lambda = \int \log \pi(\theta|\lambda, x) k(\lambda|\theta^*, x) d\lambda - \int \log k(\lambda|\theta, x) k(\lambda|\theta^*, x) d\lambda \\ &\Leftrightarrow \log \pi(\theta|x)) \underbrace{\int k(\lambda|\theta^*, x) d\lambda}_{=1} = \int \log \pi(\theta|\lambda, x) k(\lambda|\theta^*, x) d\lambda - \int \log k(\lambda|\theta, x) k(\lambda|\theta^*, x) d\lambda \end{aligned}$$

Thus, we have, for any θ^*:

$$\log \pi(\theta|x) = \int \log \pi(\theta|\lambda, x) k(\lambda|\theta^*, x) d\lambda - \int \log k(\lambda|\theta, x) k(\lambda|\theta^*, x) d\lambda \tag{1}$$

(b) EM Sequence

We want to show that the EM Sequence improves $\log \pi(\theta^{(j)}|x)$ at each step j, i.e. $\log \pi(\theta^{(j+1)}|x) \geq \log \pi(\theta^{(j)}|x), \forall j \in \mathbb{N}$. We re-write Equation 1 as:

$$\log \pi(\theta|x) := Q(\theta|\theta^*, x) - \mathbb{E}_{\theta^*}[\log k(\lambda|\theta, x)] \tag{2}$$

Where the expectation is taken with respect to $k(\lambda|\theta^*, x)$. Let's define the EM Sequence $(\theta^{(j)})_{j\in\mathbb{N}}$ for each step with :

$$\begin{aligned} \theta^{(j+1)} &= \arg\max_\theta Q(\theta|\theta^{(j)}, x) \\ &= \arg\max_\theta \int \log \pi(\theta|\lambda, x) k(\lambda|\theta^{(j)}, x) d\lambda \end{aligned}$$

By definition of $\theta^{(j+1)}$:

$$Q(\theta^{(j+1)}|\theta^{(j)}, x) \geq Q(\theta^{(j)}|\theta^{(j)}, x) \tag{3}$$

We finally want to show that :

$$\mathbb{E}_{\theta^{(j)}}[\log k(\lambda|\theta^{(j+1)}, x)] \leq \mathbb{E}_{\theta^{(j)}}[\log k(\lambda|\theta^{(j)}, x)]$$

Jensen's inequality for concave functions is the following: for any concave function f,

$$f(\mathbb{E}[X]) \geq \mathbb{E}[f(X)]$$

Taking $f = \log$ and $X = \frac{k(\lambda|\theta^{(j+1)},x)}{k(\lambda|\theta^{(j)},x)}$, we have :

$$\begin{aligned}
\mathbb{E}_{\theta^{(j)}}\left[\log\left(\frac{k(\lambda|\theta^{(j+1)}, x)}{k(\lambda|\theta^{(j)}, x)}\right)\right] &\leq \log \mathbb{E}_{\theta^{(j)}}\left[\frac{k(\lambda|\theta^{(j+1)}, x)}{k(\lambda|\theta^{(j)}, x)}\right] \\
&= \log \int \frac{k(\lambda|\theta^{(j+1)}, x)}{k(\lambda|\theta^{(j)}, x)} k(\lambda|\theta^{(j)}, x) d\lambda \\
&= 0
\end{aligned}$$

Which directly implies that :

$$\mathbb{E}_{\theta^{(j)}}[\log k(\lambda|\theta^{(j+1)}, x)] \leq \mathbb{E}_{\theta^{(j)}}[\log k(\lambda|\theta^{(j)}, x)] \tag{4}$$

Thus, by taking $\theta^* = \theta^{(j)}$ in (2), we showed (3) and (4), directly giving :

$$\log \pi(\theta^{(j+1)}|x) \geq \log \pi(\theta^{(j)}|x)$$

According to Theorem 5.16., if $Q(\theta|\theta^*, x)$ is continuous in both θ and θ^*, every limit point of $(\theta^{(j)})_{j\in\mathbb{N}}$ is a stationnary point of $\log \pi(\theta|x)$, and $\log \pi(\theta^{(j)}|x)$ converges monotonically to $\log \pi(\hat{\theta}|x)$ for some stationnary point $\hat{\theta}$.

(c) Application

We apply this EM strategy to the hierarchical model :

$$\begin{aligned}
X|\theta &\sim \mathcal{N}(\theta, 1) \\
\theta|\lambda &\sim \mathcal{N}(\lambda, 1)
\end{aligned}$$

With $\pi(\lambda) = 1$. At a given step j, we want to compute :

$$\begin{aligned}
\theta^{(j+1)} &= \arg\max_{\theta} Q(\theta|\theta^{(j)}, x) \\
&= \arg\max_{\theta} \int \log \pi(\theta|\lambda, x) k(\lambda|\theta^{(j)}, x) d\lambda
\end{aligned}$$

Following Baye's rule, we have:

$$\pi(\theta, \lambda|x) = \frac{\pi(x|\theta)\pi(\theta\ |\lambda)\pi(\lambda)}{\pi(x)}$$

We know $\pi(x|\theta), \pi(\theta\ |\lambda)$ and $\pi(\lambda)$. $\pi(x)$ is a constant only depending on the observed data. Q can be re-wrote as:

$$Q(\theta|\theta^{(j)}, x) = R(\theta) + \underbrace{\int \log \frac{\pi(\lambda)}{\pi(x)} k(\lambda|\theta^{(j)}, x) d\lambda}_{=C}$$

Thus, $\arg\max_\theta Q(\theta|\theta^{(j)}, x) = \arg\max_\theta R(\theta)$. We have :

$$R(\theta) = \int \underbrace{\log[\pi(x|\theta)\pi(\theta\ |\lambda)]}_{A(\theta,\lambda)}\underbrace{k(\lambda|\theta^{(j)}, x)}_{B(\lambda,\theta^{(j)})}d\lambda$$

Compute :

$$\begin{aligned}
A(\theta,\lambda) &= \log[\pi(x|\theta)] + \log[\pi(\theta\ |\lambda)] \\
&= -\frac{1}{2}\log(2\pi) - \frac{(x-\theta)^2}{2} - \frac{1}{2}\log(2\pi) - \frac{(\theta-\lambda)^2}{2}, \\
\frac{\partial A(\theta,\lambda)}{\partial\theta} &= \theta(x-\theta) - \theta(\theta-\lambda) \\
&= \theta(x - 2\theta + \lambda)
\end{aligned}$$

Finally, as $\int B(\lambda, \theta^{(j)})d\lambda = 1$,

$$\begin{aligned}
\frac{\partial R(\theta)}{\partial\theta} &= \int B(\lambda,\theta^{(j)})\frac{\partial A(\theta,\lambda)}{\partial\theta}d\lambda \\
&= \theta(x-2\theta) + \theta\int \lambda B(\lambda,\theta^{(j)})d\lambda \\
&= \theta(x - 2\theta + \mathbb{E}[\lambda|\theta^{(j)}, x])
\end{aligned}$$

This leads us to :

$$\theta^{(j+1)} = \frac{1}{2}(x + \mathbb{E}[\lambda|\theta^{(j)}, x])) \tag{5}$$

However, we haven't computed $\mathbb{E}[\lambda|\theta^{(j)}, x])$ yet. Following Bayes's rule :

$$k(\lambda|\theta^{(j)}, x) = \frac{\pi(x|\theta^{(j)})\pi(\theta^{(j)}|\lambda)\pi(\lambda)}{\pi(x)\pi(\theta^{(j)}|x)}$$

We know $\pi(\theta^{(j)}|x)$ as we have calculated it during the following step. $\pi(x)$ is a constant that we can estimate by other means. We also know $\pi(x|\theta^{(j)}) = \frac{1}{\sqrt{2\pi}}e^{\frac{(x-\theta^{(j)})^2}{2}}$ and $\pi(\theta^{(j)}\lambda) = \frac{1}{\sqrt{2\pi}}e^{\frac{(\theta^{(j)}-\lambda)^2}{2}}$. Using $\pi(\lambda) = 1$ seems strange (as $\pi(\lambda)$ should be a p.d.f), but we will stick to the problem's rules. Then,

$$\begin{aligned}
\int \lambda k(\lambda|\theta^{(j)}, x)d\lambda &= \frac{e^{\frac{(x-\theta^{(j)})^2}{2}}}{\sqrt{2\pi}\pi(x)\pi(\theta^{(j)}|x)}\int \lambda\frac{1}{\sqrt{2\pi}}e^{\frac{(\theta^{(j)}-\lambda)^2}{2}}\pi(\lambda)d\lambda &(6) \\
&= \frac{e^{\frac{(x-\theta^{(j)})^2}{2}}}{\sqrt{2\pi}\pi(x)\pi(\theta^{(j)}|x)}\theta^{(j)},\ \text{if } \pi(\lambda) = 1 &(7)
\end{aligned}$$

Using (5) and (7), we can apply EM Algorithm to this bayesian hierarchical model.