Revision Notes by Sally Yang

INDUSTRIAL ECONOMICS

EC313

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REVISION TIPS

1. Master the problem sets and past papers

2. Try EC411 past papers

3. Exam questions are often extensions of concepts/models learnt. Check their Wikipedia pages for generalised solutions. Stay inquisitive.

4. If a certain question setup keeps reappearing but you haven't learnt it in lectures (at least in detail), ask for the model/concept name and learn the generalised solution. Examples include double auctions, Benoît and Krishna-style finite Bertrand with discrete prices, sequential bargaining.

5. Ask and answer lots of questions.

6. Certain parts of the lecture may in fact be non-examinable, sometimes because it's extra and sometimes because it's regarded as revision for second-year material (!) you won't know until you asked.

REVISION: MATH & STATS

Non-Exhaustive

Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$F(b) = F(a) + \int_a^b f(x)dx$$

Chain Rule

$$\left.\frac{dg(y(x))}{dx}\right|_x = \left.\frac{dg(y)}{dy}\right|_{y(x)} \cdot \left.\frac{dy}{dx}\right|_x$$

Differentiate Inverse

$$f(x) = y \qquad \frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(f^{-1}(y))}{dx}} = \frac{1}{\frac{df(x)}{dx}}$$

$$b(x) = b_i \qquad \frac{\partial b^{-1}(b_i)}{\partial b_i} = \frac{1}{b'(b^{-1}(b_i))} = \frac{1}{b'(v_i)}$$

Envelope Theorem

$$\left.\frac{dU(x,y(x))}{dx}\right|_{y=y^*} = \left.\frac{\partial U(y,x)}{\partial x}\right|_{y=y^*} + \underbrace{\left.\frac{\partial U(y,x)}{\partial y}\right|_{y=y^*}}_{=0} \underbrace{\left.\frac{dy}{dx}\right|_{y=y^*}}_{\text{Must exist by assumption}}$$

i.e. optimal y^* must be differentiable in x

Integrate by Parts

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

$$\text{or } \int u\,dv = uv - \int v\,du$$

Integrate by Substitution

$$\text{Let } x = 1-p, \quad dx = -dp$$

$$\int_0^1 p \cdot n(1-p)^{n-1}dp = \int_1^0 (1-x)nx^{n-1}(-dx) = \int_0^1 n(x^n - x^{n-1})dx$$

↑ If $p=0$, $x=1$ ↑ flip again!!

Hessian

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \qquad \left.\begin{aligned} |H|_1 &= \frac{\partial^2 f}{\partial x^2} < 0 \\ |H|_2 &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \end{aligned}\right\} \text{Maximum}$$

REVISION: MATH & STATS

Non-Exhaustive

Conditional Expectations

$$E(Y|X=x) = \sum_y y f_{Y|X}(y|x) = \frac{\sum_y y f_{X,Y}(x,y)}{f_X(x)}$$ by Baye's theorem

can take out from Σ

If x is observed, $E(Y|X)$ is just a number as usual.
If not, $E(Y|X)$ is itself a random variable.
If Y and X are independent, $E(Y|X)=E(Y)$, $Var(Y|X)=Var(Y)$

$$E(Y|Y<a) = \frac{\int_{-\infty}^{a} y f(y) dy}{\int_{-\infty}^{a} f(y) dy} = \frac{\int_{-\infty}^{a} y f(y) dy}{F(a)}$$

Distribution of the Maximum

$$x_i \overset{iid}{\sim} F$$

$$Pr(\max\{x_1,\dots,x_n\} \leq y) = F(y)^n$$

$$Pr(\max\{x_1,\dots,x_n\} = y) = nF(y)^{n-1} f(y)$$

$\max\{\cdot\}$ is convex as

$$\max\{\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda) y_2\} < \lambda \max\{x_1, y_1\} + (1-\lambda)\max\{x_2, y_2\}$$

Distribution of the Minimum

$$x_i \overset{iid}{\sim} F$$

$$Pr(\min\{x_1,\dots,x_n\} \leq y) = 1-(1-F(y))^n$$

$$Pr(\min\{x_1,\dots,x_n\} = y) = n(1-F(y))^{n-1} f(y)$$

Bernoulli Distribution

$$Pr(Y=k) = p^k (1-p)^k$$

$$E(Y) = p$$

Binomial Distribution

number of trials

$$Pr(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$$ number of successes

$$E(Y) = np$$

Geometric Distribution

$$Pr(Y=k) = (1-p)^{k-1} p$$

$$E(Y) = \frac{1}{p}$$

Jensen's Inequality

$E(f(\theta)) \geq f(E(\theta))$ if f convex

$E(f(\theta)) \leq f(E(\theta))$ if f concave

LERNER INDEX

$-\frac{1}{e} > 0 \longrightarrow \infty$ as $e \to -\infty$

ELASTICITY OF DEMAND $\quad e = \frac{\partial Q}{Q} \div \frac{\partial P}{P}$

Inelastic DD: $e \in (-1, 0)$
Elastic DD: $e < -1$

Comes from monopoly FOC: $P(1+\frac{1}{e}) = MC$ Optimal to produce at $e \in (-1, 0)$

BERTRAND

Symmetric Solution: $p_i^* = c$

Goods are strategic complements as BR is upward-sloping $\rightarrow$ Second mover advantage

$\uparrow p_1 \Rightarrow \uparrow p_2(p_1)$ (marginal payoff of P2)

DISCRETE PRICE WAR Integer prices (Benoît and Krishna)

If $P_{-i} = 0$, BR: any P_i
If $P_{-i} = 1$, BR: $P_i = 1$.
If $P_{-i} \geq 2$, BR: $P_i = \min\{\underbrace{P_{-i} - 1}_{\text{undercut}}, \underbrace{P^m}_{\text{monopoly price}}\}$

(if rival price is too high)

can't undercut anymore

PSNE are $\underbrace{(0,0), (1,1)}$.

Discrete pricing enables multiple NE in Bertrand model

COURNOT

$$q_i^* \in \arg\max_{q_i} q_i\left(a - b(q_1 + \dots + q_N) - c\right)$$

Trick: Assume FOC satisfied $\forall i$, solve $\sum_{i=1}^{N} [a - 2bq_i - Q - c] = 0$

Symmetric Solution: $q_i^* = \frac{a-c}{b(N+1)}$, $p_i^* = \frac{a+Nc}{N+1} \xrightarrow{N\to\infty} c$

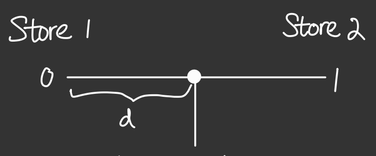
MERGER Implies that without cost synergies (c unchanged), only merger for monopoly is profitable

$$2 \cdot \Pi_{N=2} \leq \Pi_{N=1} \text{ iff } N \leq 2$$

TRADEOFF Model a zero stage where firms decide to enter
Relaxing price competition in first stage encourages firm entry

HOTELLING MODEL

$$U = s - td^2 - p$$



Find Marginal consumer

$$s - td^2 - p_1 = s - t(1-d)^2 - p_2$$

$$D_1(p_1,p_2) = d = \left(\frac{1}{2} + \frac{p_2-p_1}{2}\right)N$$
$$D_2(p_1,p_2) = 1-d = \left(\frac{1}{2} + \frac{p_1-p_2}{2t}\right)N$$

Symmetric $p_i^* = \underset{p_i}{\operatorname{argmax}}\ D_i(p_i, p_{-i}) \cdot (p_i - c) = c + t$

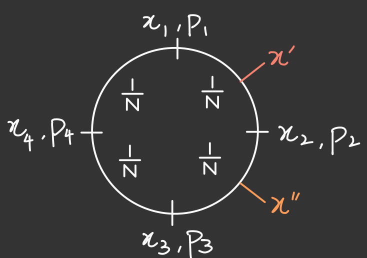
$\uparrow$ degree of product differentiation

$\boxed{\text{SPATIAL CHOICE}}$ Add zero stage, choose d

Two-firm NE : $(\frac{1}{2}, \frac{1}{2})$ given exogenous price 'median voter'
BUT with endogenous price, $(\frac{1}{2}, \frac{1}{2})$ may no longer be NE as
No pdt diff $\Rightarrow p^* = c \Rightarrow \pi^* = 0$:(

Three-firm : No NE

SALOP CIRCULAR CITY



Consumers distributed uniformly around circle,
travel $|x_i - x|$ distance at unit cost t to buy good at x_i

$$U_i(x) = s - t|x_i - x| - p_i$$

Marginal consumer : $s - t(x' - x_1) - p_1 = s - t(x_2 - x') - p_2$

$$\Rightarrow x' = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2t} \quad \text{Similarly} \quad x'' = \frac{x_2 + x_3}{2} + \frac{p_3 - p_2}{2t}$$

$$D_i(p_{i-1}, p_i, p_{i+1}) = x'' - x' = \frac{x_{i+1} - x_{i-1}}{2} + \frac{p_{i+1} - 2p_i + p_{i-1}}{2t} = \frac{1}{N} + \frac{p_{i+1} - 2p_i + p_{i-1}}{2t}$$

differentiation

$$\Rightarrow p_i^* = c + \frac{t}{N}$$

N(firms)

When N↑, consumer welfare↑ 1. Travel less on average 2. Prices fall

VERTICAL PRODUCT DIFFERENTIATION

Different quality instead of features (horizontal)
If all prices were the same, there'd be a universal winner

QUALITIES

N products with qualities (WLOG) $u_0 < u_1 < \dots < u_N$, $c = 0$, $p_k \geq 0$
(u_0: exit)

PREFERENCE FOR QUALITY $t \sim U[a,b]$ interpreted as income distribution

UTILITY $U_k(t) = u_k(t - p_k)$

Marginal consumer between k−1 and k : $U_{k-1}(t_k) = U_k(t_k)$

$$t_k = p_{k-1}(1 - c_k) + p_k c_k \qquad c_k = \frac{u_k}{u_k - u_{k-1}} > 1$$

p_{k-1}: decreasing in; p_k: increasing in (less people below k buy)

BOUND ON N(FIRMS)

There is a bound on N in the NE

- independent of prices, but affected by width of income distribution

Assume more than 1 good survives. $t_k > a$

"Top quality" firm N $\max_{p_N} p_N(b - t_N)\frac{1}{b-a} \Rightarrow t_N < \frac{b}{2}$

If $a > \frac{b}{2}$, above leads to contradiction.
$N = 1$, everyone strictly prefers one product independent of its quality

If $a < \frac{b}{2}$, firm k $\max_{p_k} p_k(t_{k+1} - t_k)\frac{1}{b-a} \Rightarrow t_{k+1} - 2t_k - \dots = 0 \Rightarrow t_k < \frac{1}{2}t_{k+1}$
(t_{k+1}, t_k: functions of p_k)

$t_{N-1} < \frac{b}{4}$, $t_{N-2} < \frac{b}{8}$, … So possible N is decided by if $a > \frac{b}{4}$, $a > \frac{b}{8}$, …

PRODUCT DIFFERENTIATION: LIMITATION

Applicable in industrial markets, when product design is lengthy

- firms <u>take x_i/u_k as given</u>, compete in p_i
  If sequential (set x_i before p_i), see spatial choice
- If set (x_i, p_i) simultaneously, no NE due to undercutting

WAR OF ATTRITION (DRILLING GAME)

Two firms own two adjacent leases with perfectly correlated but unknown value $v \sim F$
Drilling cost c is known, $\mathbb{E}(v) > c$. Discount at rate β.

HIGH VALUE TRACTS Drill now if $\mathbb{E}(\text{payoff} \mid \text{drill now}) > \mathbb{E}(\text{payoff} \mid \text{wait now})$

I drill in next period knowing I'll profit for sure — I don't drill in next period if I did, make losses for sure — I drill, having learnt nothing about v

$$\mathbb{E}(v-c) > \beta\left[q\left(\Pr(v\ge c)\,\mathbb{E}(v-c \mid v\ge c) + \Pr(v<c)\cdot 0\right) + (1-q)\,\underbrace{\mathbb{E}(v-c)}_{>0}\right] \quad \forall q\in[0,1]$$

q = Pr(rival drills) — rival drills finds it is worth drilling — rival drills finds it is not worth drilling — Rival doesn't drill

Because $\mathbb{E}(v-c \mid v\ge c) \ge \mathbb{E}(v-c)$, the RHS is maximised at $q=1$. Rewrite condition as

$$\mathbb{E}(v-c) > \beta \Pr(v\ge c)\,\mathbb{E}(v-c \mid v\ge c)$$

If $v = \begin{cases} 1 & \Pr = p \\ 0 & \Pr = 1-p \end{cases}$, condition becomes $p - c \ge \beta p(1-c)$

LOW VALUE TRACTS Prefer rival to drill first if $\mathbb{E}(v-c) \le \beta \Pr(v\ge c)\,\mathbb{E}(v-c \mid v\ge c)$

Backwards Induction — In time T, both drill for sure, expected payoff $= \mathbb{E}(v-c) > 0$

MSNE — In time $T-1$, solve for a symmetric MSNE $q^* = \Pr(\text{I drill}) = \Pr(\text{rival drills})$

Indifference condition: $\mathbb{E}(\text{payoff} \mid \text{drill now}) = \mathbb{E}(\text{payoff} \mid \text{wait now})$

$$\mathbb{E}(v-c) = \beta\left[q^* \Pr(v\ge c)\,\mathbb{E}(v-c \mid v\ge c) + (1-q^*)\,\mathbb{E}(v-c)\right]$$

Induction — In time $T-2, \ldots, 1$, will have same q^* as

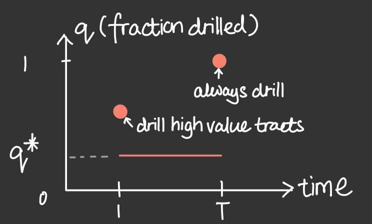
$$\mathbb{E}(\text{payoff} \mid \text{drill now}) = p - c,$$
$$\mathbb{E}(\text{payoff} \mid \text{wait now}) = \beta\left[q\Pr(v\ge c)\,\mathbb{E}(v-c \mid v\ge c) + (1-q)\,\mathbb{E}(v-c)\right]$$

payoff from adopting MSNE strategy in next period (indifference + induction) different reason, but same payoff!

PSNE
2 periods

In time 2, both drill for sure, expected payoff $= \mathbb{E}(v-c)$
In time 1, check incentive to deviate from PSNE, taking rival's action as given

EMPIRICAL SUPPORT



Prediction

Empirically,
- $\frac{\%\text{ drilled}}{\text{year}}$ is U-shaped
- % productive (i.e. tract value) declined over time

STACKELBERG GAME

① Firm 1 $\max_{q_1} [a-b(q_1+q_2)-c]q_1$ ② Firm 2 $\max_{q_2} [a-b(q_1+q_2)-c]q_2$

$$q_1^* = 2q_2^* = \frac{a-c}{2b}$$

$\Pi_1^S > \Pi_1^{Cournot}$ $\Pi_2^S < \Pi_2^{Cournot}$

First mover advantage — downward sloping BR

Generalisation: The n^{th} mover produces twice that of $n+1^{th}$ mover

BERTRAND COLLUSION

Finite T + Continuous p : Never collude, $p^* = c$ by BI

Consider infinite T

GRIM TRIGGER

Play p^m in T=1 and if no one deviates; play c otherwise

SPNE for β large enough ($\beta > 1 - \frac{1}{N}$ in canonical eg)

FOLK THEOREM

All symmetric payoff divisions from 0 to $\frac{\Pi^M}{N}$ sustainable in an SPNE for β large enough

- Further claim: Any payoff satisfying $\pi_1 + \pi_2 \leq \Pi^M$ works!
- Theory can't predict much about price! Many equilibria !!

FOLK THEOREM (FINITE T)

Benoît and Krishna 1986

If prices are discrete and stage game has multiple NE, can punish towards end of game while still ensuring subgame perfection

- Enables collusion!
- Write out payoff (profit) matrix and identify NEs!
- Should have at least two NEs: "bad" and "worse" (the usual competitive NE)

  !! reminder "both collude" is not NE

- Strategy template:

  Stage 1: Collude

  Stage t: Collude if rival never deviated; else pick "worse" NE

  Stage T: Play "bad" NE if rival never deviated; else pick "worse" NE

- Threshold for β, if needed, should be solved using stage T-1, when incentive to deviate is greatest.
- "Bad" NE is used as reward in final stages

COLLUSION WITH PRICE WAR

Stochastic demand shocks explain price wars' existence

$$\tilde{D}(p) = \begin{cases} D(p) & \text{with } Pr = \alpha \\ 0 & \text{with } Pr = 1-\alpha \end{cases}$$

$$\tilde{\pi}_i(p_i, p_{-i}) = (p_i - c)\tilde{D}(p)\frac{1}{N}$$

Demand is split (rationed) equally as

We now only observe own price history $\{p_{it}, q_{it}\}_t$

- If you deviate when everyone else collude, you get $N\pi$

TRIGGER STRATEGY

$$p_{it} = \begin{cases} p \in [c, p^m] & \text{if } t=1 \\ & \text{or } \pi_{jt-1} > 0 \ \forall j \\ & \text{or } \pi_{jt-T} = 0 \text{ and } \pi_{jt-\tau} = 0 \ \forall j \forall \tau < T \\ c & \text{otherwise} \end{cases}$$

triggered punishment T periods ago

punishment lasted for previous T-1 periods, now ended

CONDITION FOR SPNE

Coordinate

no need to solve

$$V = \alpha(\pi(p) + \beta V) + (1-\alpha)(\beta^T V) = \frac{\alpha \pi(p)}{1 - \alpha\beta - (1-\alpha)\beta^T}$$

Cheat

$$V^c = \alpha(N\pi(p) + \beta^T V) + (1-\alpha)(\beta^T V)$$

IC CONSTRAINT $V \geq V^c \Rightarrow \pi(p) + \beta V \geq N\pi(p) + \beta^T V$

$$\Rightarrow 1 \geq N(1 - \beta\alpha) + (1 - (1-\alpha)N)\beta^T$$

PROFIT MAX $p = p^m$ as usual (price unaffected by IC)

Noisy demand $1 - \alpha > \frac{1}{N}$

$T^* = 1$ (corner), violates IC, no collusion

Little noise $1 - \alpha < \frac{1}{N}$

T^* is smallest T s.t. IC binds.

Condition for SPNE: low demand noise, $\beta > \beta^*(N) + c$

DOUBLE AUCTION

SETUP

Seller: Sells (supplies) one unit good at cost $c \in [0,1]$. $U(\text{trade}) = p - c$.
Buyer: WTP $= v \in [0,1]$. $U(\text{trade}) = v - p$

Seller offers "ask price" a; buyer offers "bid" b.
If $b \geq a$, trade at $p = \frac{a+b}{2}$
If $b < a$, no trade

EFFICIENCY

Always efficient to trade when $v \geq c$. Total surplus $= v - p + p - c = v - c$
No trade equilibrium is inefficient when $v \geq c$

SIMULTANEOUS GAME

If c, v are privately known (incomplete information game)

Any price $x \in \mathbb{R}$ can be supported by this general NE strategy

$$b(v) = \begin{cases} x & \text{if } x \leq v \\ 0 & \text{if } x > v \end{cases} \qquad a(c) = \begin{cases} x & \text{if } x \geq c \\ 1 & \text{if } x < c \end{cases}$$

"I don't wanna trade" "I don't wanna trade"

$x \in [c, v]$ implies NE outcome will be to trade at $p = x$;
Otherwise, no trade.

If c, v are known: just collapse the above strategy depending on where x is

Trade NE: $a = b = x$ is an NE $\forall x \in [c, v]$ Only when $c \leq v$

No-Trade NE:
$a = x, b = 0$ is an NE $\forall x > v$ Seller asks above v, buyer rejects all asks
$a = 1, b = x$ is an NE $\forall x < c$ buyer bids below c, seller rejects all bids

SEQUENTIAL GAME

Either buyer bids first or seller asks first

NE: Same as above, but not SPNE due to non-credible threats

SPNE: Using backwards induction:

If seller asks first, buyer will bid $\underset{\text{accept ask, trade}}{b = a}$ if $a \leq v$, $\underset{\text{reject ask, no trade}}{b < a}$ if $a > v$

$\Rightarrow$ Seller asks $a = v$. Trade occurs in SPNE at $p = v$. Seller gets all surplus.

If buyer bids first, ..., trade occurs in SPNE at $p = c$. Buyer gets all surplus.

RUBINSTEIN'S BARGAINING

Stage 0, 2, 4, ... Player 1 offers $(x, 1-x)$, game ends iff Player 2 accepts

Stage 1, 3, 5, ... Player 2 offers $(y, 1-y)$, game ends iff Player 1 accepts

PAYOFF $(\beta^t x, \beta^t(1-x))$ if $(x, 1-x)$ accepted at time t

NE There is one for every x'!

$\begin{cases} \text{P1: Always demand } x = x'\text{; always accept } y \text{ iff } y \geq x' \\ \text{P2: Always demand } y = x'\text{; always accept } x \text{ iff } x \leq x' \end{cases}$

- Not SPNE (non-credible threat)
- P2 can reject and offer P1 $1 < x < \beta$ next.

SPNE Offer $\underset{\text{own payoff}}{x^*} = \frac{1}{1+\beta}$, accept any share $\geq \frac{\beta}{1+\beta}$. Unique.

Check incentive to deviate, assuming opponent is playing this already:

If offer $x > x^*$, Opponent rejects.

Payoff from next period $= \beta\left[1 - \frac{\beta}{1+\beta}\right] = \frac{\beta}{1+\beta} < x^*$

Should have offered x^*, get $\frac{1}{1+\beta}$.

If offer $x < x^*$, opponent does accept but own payoff is strictly lower

If reject x^* that opponent offers

Payoff from next period $\leq \beta\left(\frac{1}{1+\beta}\right) = \frac{\beta}{1+\beta}$

SUMMARY
- No haggling in equilibrium!
- First mover advantage, but disappears with patience $\beta \to 1$
- Unique

EXTENSION If player 1 discounts at β_1, player 2 at β_2,

SPNE has player 1 receiving $\frac{1-\beta_2}{1-\beta_1\beta_2} = \frac{1}{1+\beta}$ if $\beta_1 = \beta_2 = \beta$

DEMAND ESTIMATION: HOMOGENEOUS PRODUCTS

$$\ln Q = X'\delta + \beta \ln P + \varepsilon$$

$$\frac{\partial \ln Q}{\partial \ln P} = \frac{\partial \ln Q}{\partial Q}\frac{\partial Q}{\partial P}\frac{\partial P}{\partial \ln P} = \frac{\partial Q}{\partial P}\frac{P}{Q} = e$$

DEMAND ESTIMATION: HETEROGENEOUS PRODUCTS

Multinomial logit

Product k: $U_{ik} = \beta z_{ik} - \alpha p_k + \varepsilon_{ik} = \delta_{ik} + \varepsilon_{ik}$

product k and individual i characteristics

mean utility

Outside good: $U_{i0} = 0$

LOG WEIBULL $f(\varepsilon_i) = e^{-\varepsilon_i - e^{-\varepsilon_i}}$ $\quad F(\varepsilon_i) = e^{-e^{-\varepsilon_i}}$

MARKET SHARE $S_k = Pr(\text{product of firm k is chosen over all others})$

McFadden $\quad S_k = \dfrac{e^{\delta_k}}{\sum_j e^{\delta_j}}, \quad \delta_{ik} = \beta z_{ik} - \alpha p_k$

DERIVATION $S_k = Pr(\delta_k + \varepsilon_k > \delta_j + \varepsilon_j \ \forall j)$

Full derivation not tested

$$\overset{i.i.d.}{=} \int_{-\infty}^{\infty} f(\varepsilon_k) \prod_j F(\delta_k + \varepsilon_k - \delta_j)\, d\varepsilon_k$$

$$= \int_{-\infty}^{\infty} e^{-\varepsilon_k} e^{-e^{-\varepsilon_k}} \prod_j e^{-e^{-(\delta_k+\varepsilon_k-\delta_j)}}\, d\varepsilon_k$$

$$-e^{-\varepsilon_k} - \sum_{j\neq k} e^{-(\delta_k+\varepsilon_k-\delta_j)}$$

$$= \int_{-\infty}^{\infty} e^{-\varepsilon_k} e^{-e^{-\varepsilon_k}(1+\sum_j e^{\delta_j-\delta_k})}\, d\varepsilon_k$$

$$-e^{-\varepsilon_k}(1 - \sum_{j\neq k} e^{\delta_j - \delta_k}$$

$$= \frac{1}{1+\sum_j e^{\delta_j-\delta_k}} \left[e^{-e^{-\varepsilon_k}(1+\sum_j e^{\delta_j-\delta_k})} \right]_{-\infty}^{\infty}$$

$$= \frac{e^{\delta_k}}{\sum_j e^{\delta_j}}$$

$$\int_{-\infty}^{\varepsilon} e^{-\varepsilon_1} e^{-ae^{-\varepsilon_1}}\, d\varepsilon_1 = \frac{1}{a} e^{-ae^{-\varepsilon}}$$

DEMAND ESTIMATION: HETEROGENEOUS PRODUCTS

Individual choice probability $P_{ik} = \dfrac{e^{\delta_{ik}}}{\sum_j e^{\delta_{ij}}}$, $\delta_{ik} = \beta z_{ik} - \alpha p_k$

Market share $S_k = \dfrac{e^{\delta_k}}{\sum_j e^{\delta_j}}$, $\delta_k = \beta z_k - \alpha p_k$

$\boxed{\text{ESTIMATION}}$ With consumers' purchase data $\{(\overset{\text{i's choice of product}}{c_i}, p_1, \dots, p_m, x_{ik})\}_{i=1}^N$

do MLE to pick β, α to maximise likelihood of observing actual choice k (i.e. P_{ik})

$$\mathcal{L} \overset{\text{i.i.d.}}{=} \prod_{j=1}^{m} \prod_{i=1}^{N} P_{ij}^{\mathbb{1}(c_i = j)}$$

Heterogeneity can let β_i vary by consumer, do MLE/GMM as above

But with random coefficient β_i and time dimension,
Correlation of observations of the same individual over time violates iid

$$\mathcal{L} = \prod_{j \in \mathcal{J}(t)} \prod_{i=1}^{N} \int \prod_t P_{ijt}^{\mathbb{1}(c_{it} = j)} f(\beta_i) d\beta_i$$

$\boxed{\text{ESTIMATION}}$ With aggregate (firm level) data $\{(s_k, p_k, x_k)\}_k$

‼ $\ln(S_k) - \ln(S_0) = \delta_k = \beta z_k - \alpha p_k$

OLS on $\ln(S_k) - \ln(S_0) = \beta x_k - \alpha p_k + \xi_k$

unobserved product characteristics, e.g. design (OVB, unless firms are price takers)

Find an IV for this
e.g. rival product characteristics

OVB In reality, $\ln(S_k) - \ln(S_0)$ never a perfect fit. Implies the presence of ξ_k
It is likely that $\uparrow \xi_k \Rightarrow \uparrow \delta_k \Rightarrow \downarrow \text{PED} \Rightarrow \uparrow p - mc \Rightarrow \uparrow p$. Always instrument for price.

Assumes $x_{ij} = x_j$, $\varepsilon_{ij} = \varepsilon_j$,
i.e. observed and unobserved product characteristics don't depend on i (✓ horsepower ✗ distance to airport)
$\Rightarrow \delta_{ij} = \delta_j$, $P_{ij} = s_j$

Heterogeneity Mixed logit assuming $\beta_i \overset{\text{iid}}{\sim} N$, find $E(\beta_i)$, $Var(\beta_i)$

LIMITATIONS Own-price elasticity $\frac{\partial s_k}{\partial p_k} = -\alpha s_k(1-s_k)$ increases in price
but people who buy more expensive products may be less price sensitive

Cross elasticity $\frac{\partial s_k}{\partial p_\ell} = \alpha s_k s_\ell$ depends only on market shares and prices
but not similarities in goods

- i.i.d (mutually uncorrelated) errors impose unrealistic IIA
- Drop non-chosen alternatives, test equality of coefficients across models

NESTED LOGIT $$\Pr(\text{choice } 1) = \frac{e^{\delta_1}}{e^{\delta_1} + [e^{\frac{\delta_2}{\rho}} + e^{\frac{\delta_3}{\rho}}]^\rho}$$

test $\rho = 1$
correlation in taste between two nested goods

$$\Pr(\text{choice } 2 \mid \text{choice } 1) = \frac{e^{\frac{\delta_2}{\rho}}}{e^{\frac{\delta_2}{\rho}} + e^{\frac{\delta}{\rho}}}$$

FIRM COST ESTIMATION $$mc_k = \gamma w_k + u_k$$
cost shifter

$$\pi_f = \sum_{k \in J(f)} (p_k - mc_k) s_k N$$
all products produced by firm f

$$\xrightarrow[p_k]{\text{FOC wrt}} s_k + \sum_{j \in J(f)} (p_j - mc_j) \frac{\partial s_j}{\partial p_k} = 0$$

Solve for mc_j and regress!

EMPIRICAL TESTS OF OLIGOPOLY

Test which model is applicable to the industry in focus

DEMAND MODEL $\quad p_n = z_n\beta_z - Q_n\beta_Q + \varepsilon_n$ $\quad$ IV using cost shifters w_n

SUPPLY MODEL $\quad MC_n(Q_n) = w_n\gamma + Q_n\lambda + u_n$ ← unobserved supply shifter

(w_n: observed supply shifter; λ: structural cost parameter)

$$MR_n(Q_n) = \frac{\partial}{\partial Q_n} P(Q_n)Q_n = P(Q_n) + Q_n\frac{\partial P(Q_n)}{\partial Q_n}$$

Set equal to MC

$$MC_n(Q_n) = P(Q_n) + Q_n\frac{\partial P(Q_n)}{\partial Q_n}$$

Bertrand and Cournot are consistent with linear supply equation

BERTRAND $\quad$ Implies $p = mc$.

Structural supply: $\quad p_n(Q_n) = w_n\gamma + Q_n\lambda + u_n$

COURNOT

$$\max_{q_i} (p_n(Q_n) - c_i)q_i \Rightarrow p_n(Q_n) - c_i + q_i \underbrace{\frac{\partial p_n(Q_n)}{\partial Q_n}}_{= -\beta_Q} \underbrace{\frac{\partial Q_n}{\partial q_i}}_{= 1 \text{ in Cournot}} = 0.$$

! doesn't explain supply variation between firms

Structural supply: (aggregate data, assume $q = \frac{Q}{n}$)

$$\begin{aligned} p_n(Q_n) &= mc(Q_n) - \frac{Q_n}{N}\beta_Q \\ &= w_n\gamma + Q_n\lambda + u_n - \frac{Q_n}{N}\beta_Q \\ &= Q_n\left(\lambda - \frac{\beta_Q}{N}\right) + w_n\gamma + u_n \end{aligned}$$

If demands and costs are linear, then cannot tell Bertrand from Cournot!!!

BUT if demand shifters change the slope of demand:

$$p_n = z_n\beta_z - Q_n\beta_Q + x_nQ_n\beta_{Qx} + \varepsilon_n \Rightarrow \frac{\partial p_n(Q_n)}{\partial Q_n} = -\beta_Q + x_n\beta_{Qx}$$

Structural supply (Cournot): $\quad p_n(Q_n) = Q_n\left(\lambda + \frac{x_n\beta_{Qx} - \beta_Q}{N}\right) + w_n\gamma + u_n$

Structural supply (Bertrand): $\quad$ Unchanged

TEST for interactions of demand shifters and Q_n!

GENERALISED SOLUTION

Conjectural variation

Take usual Cournot FOC: $p_n(Q_n) - c_i + q_i\frac{\partial p_n(Q_n)}{\partial q_i} = 0$

$$\hookrightarrow = \frac{\partial p_n(Q_n)}{\partial Q_n}\frac{\partial Q_n}{\partial q_i}$$

Firm data $\quad p_n(Q_n) = -q_i\frac{\partial p_n(Q_n)}{\partial Q_n}\frac{\partial Q_n}{\partial q_i} + w_n\gamma + \lambda q_i + u_i$

($\frac{\partial Q_n}{\partial q_i}$ = 1: Cournot; = 0: Bertrand)

Industry data (nested model) $\quad p_n(Q_n) = -Q_n\frac{\partial p_n(Q_n)}{\partial Q_n}\theta + w_n\gamma + \lambda Q_n + u_n$

(θ = 1: Monopoly/Collusion; $= \frac{1}{N}$: Cournot; = 0: Bertrand)

PED $\frac{\partial P(Q)}{\partial Q} = \beta_Q$, separately identified using demand

BUTTERS ADVERTISING MODEL

2 firms with $MC=0$ choose p_1, p_2

Due to exogenous adv'ing, each firm reaches $\alpha \in (0,1)$ of consumers

$\alpha(1-\alpha)$ see offer from firm i, α^2 see both

Consumer buys from lower-priced firm

$Pr(A \cap B) = Pr(A)Pr(B) = \alpha^2$
$Pr(\neg A \cap \neg B) = (1-\alpha)^2$
$Pr((A \cap \neg B) \cup (\neg A \cap B)) = \alpha(1-\alpha)$

$$D(p_i) = \begin{cases} \alpha(1-\alpha) & \text{if } p_i > p_{-i} \\ \alpha^2 + \alpha(1-\alpha) & \text{if } p_i < p_{-i} \\ \frac{\alpha^2 + \alpha(1-\alpha)}{2} & \text{if } p_i = p_{-i} \end{cases}$$

NO PSNE

A pure strategy involves prices $p_1, p_2 \geq 0$

$p_i \leq v$: as can set $p_i = v$ to guarantee $\Pi_i > 0$

$p_i < p_{-i}$: Can $\uparrow p_i$ to $\uparrow \Pi_i$.

$p_i = p_{-i}$: Can $\downarrow p_i$ to $\uparrow \Pi_i$.

$p_i = p_{-i} = 0$: Can deviate to v.

SYMMETRIC MSNE

$F(p): [\underline{p}, \bar{p}] \to [0,1]$

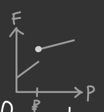
$$\mathbb{E}(\Pi_i) = p_i \alpha(1-\alpha) + \alpha^2 [1 - F(p_i)]$$

↑ given rival is using F
$Pr(p_{-i} > p_i)$

DERIVATION

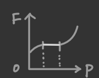
Shape of F

① No mass points in F on $(\underline{p}, \bar{p})$



If there is a mass point at $\underline{p}$, firm I could shift mass from $\underline{p}$ to $\underline{p} - \varepsilon$ to $\uparrow E(\Pi_i)$

② No gaps in support of F



If there is gap (p', p'') in support (i.e. rival never plays)

I can shift the mass $(p' - \varepsilon, p')$ to p'' s.t. demand is (almost) not affected but $\uparrow E(\Pi_i)$

Finding support $[\underline{p}, \bar{p}]$

③ $\bar{p} = v$. If $\bar{p} < v$, then $\bar{p}\alpha(1-\alpha) < v\alpha(1-\alpha)$

☆ $\mathbb{E}(\pi(p))$ is the same $\forall p \in [\underline{p}, \bar{p}]$. If $\exists p \notin [\underline{p}, \bar{p}]$ s.t. $\mathbb{E}(\pi(p)) > \mathbb{E}(\pi(\bar{p}))$, deviate

↑ or any other p in support of G

④ Find $\underline{p}$ using trick:

Rival's F must make me indifferent between $\underline{p}, \bar{p} = v$

$$\underbrace{v\alpha(1-\alpha)}_{F(v)=1} = \underbrace{\underline{p}\alpha(1-\alpha)}_{F(\underline{p})=0} \Rightarrow \underline{p} = v(1-\alpha)$$

⑤ Find F using trick:

Rival's F must make me indifferent between v and any $p_i \in [v(1-\alpha), v)$:

$$\alpha(1-\alpha)p_i + \alpha^2 [1 - F(p_i)] = v(1-\alpha)$$

$$F(p_i) = \frac{1}{\alpha}\left[1 - (1-\alpha)\frac{v}{p}\right]$$

$E(\Pi_i)$ must be same on $[v(1-\alpha), v]$ to make everyone mix

Conclusion

Explains price dispersion

Forces firms to disguise prices using randomisation

NELSON INFORMATIVE ADVERTISING

Assume quality is exogenous, discount future with β

$\boxed{\text{SETUP}}$

$$\pi_{LH} > \pi_{HH} > \pi_{LL}$$

low quality product
disguised as high

Consumers begin by presuming all low quality
Can spend A to signal high quality
Regardless, they learn true quality after period 1

$$\pi_{LH} + \beta\pi_{LL} + \beta^2\pi_{LL} + \dots > \pi_{LL} + \beta\pi_{LL} + \beta^2\pi_{LL} + \dots$$

Also, $\pi_{HH} + \beta\pi_{HH} > \pi_{HL} + \beta\pi_{HH}$

$\boxed{\text{SEPARATING EQUILIBRIUM}}$ Only high quality firm pay A to advertise

advertising expenditure

$$IC_L: \quad \pi_{LL} + \beta\pi_{LL} \geq \pi_{LH} + \beta\pi_{LL} - A$$

$$IC_H: \quad \pi_{HH} + \beta\pi_{HH} - A \geq \pi_{HL} + \beta\pi_{HH}$$

Advertisement here has no intrinsic value (burning money)
but does solve informational problem

STIGLER FIXED SEARCH

WTP v, choose to visit (sample) k firms at $MC = c$, $p_k \overset{iid}{\sim} F$

Let $p_i \overset{iid}{\sim} F$, then let $M_k = \min\{p_1, \dots, p_k\} \sim G$

$$G(m_k) \overset{iid}{=} 1 - [1 - F(m_k)]^k \Rightarrow g(m_k) = k f(m_k) [1 - F(m_k)]^{k-1}$$

$$\max_k E(v - m_k | k) = \max_k v - \int_{\substack{\text{range} \\ \text{of } x_i}} m_k g(m_k) \, dm_k$$

$\boxed{\text{SOLVE}}$ find k s.t.

$$E(v - m_k | k) \geq E(v - m_k | k-1)$$
$$E(v - m_k | k) \geq E(v - m_k | k+1)$$

k is discrete don't take FOC!!

Either solve inequalities or just plug in values of k to check

SEQUENTIAL SEARCH

Each search costs c, stop at any time t

SEARCH WITH RECALL

Can choose between all prices surveyed

$$\Pi_t = y_t - ct, \quad y_t = \overset{\text{WTP}}{v} - \min(p_1, \dots, p_t)$$

SEARCH WITHOUT RECALL

Exploding offer

$$\Pi_t = y_t - ct, \quad y_t = v - p_t$$

PROBLEM

$$\max_S E(\Pi_{N(S)})$$

where S is a stopping rule $\{p_1, \dots, p_n\} \to \{\text{stop}, \text{continue}\}$
$N(S)$ is period after which you stop

Stationarity of problem (p_i iid) and monotonicity of decision implies we just need to find...

RESERVATION PRICE

Stop when $p_t \leq r$

To find r, let $MB = MC$: $\quad E[\underbrace{(v-p_t)}_{\text{benefit from one more draw}} - \underbrace{(v-r)}_{\text{benefit from stopping now}} \mid \overbrace{p_t \leq r}^{\text{given that the current price is already below } r}] = E[c - 0 \mid p_t \leq r]$

i.e. optimal r should make me indifferent between stopping and continuing when I've already found a price $p_{t-1} \leq r$

$$\Rightarrow \int_{-\infty}^{r} (r - p_t) f(p_t)\, dp_t = c$$

STOPPING TIME

$E(N(r)) = \frac{1}{F(r)}$ as $N(r)$ is geometric, $Pr(N(r) = 1) = F(r)$

EXPECTED PAYOFF

$$V(r) = E(v - p_t \mid p_t \leq r) - c\, E(N(r)) = \frac{\int_{-\infty}^{r} (v - p_t) f(p_t)\, dp_t}{F(r)} - \frac{c}{F(r)}$$

$$\int_{-\infty}^{r} (v - p_t) f(p_t)\, dp_t - v(r) \int_{-\infty}^{r} f(p_t)\, dp_t = c = \int_{-\infty}^{r} (r - p_t) f(p_t)\, dp_t \quad \text{at optimum } r$$

$$V(r) = v - r$$

SOLUTIONS

1 Solve $\int_{-\infty}^{r} (r-p) f(p)\, dp = c$

2 Solve $v - r = V = E(\max\{v-p, V\}) - c$

$$= \int_{-\infty}^{v-V} (v-p) f(p)\, dp + V \underset{\substack{\uparrow \\ \text{when } v-p<V}}{\int_{v-V}^{\infty}} f(p)\, dp - c$$

ALTERNATE PROBLEM

Search for a prize

$$\Pi_t = y_t - ct, \quad y_t = p_t$$

$$E(p_t - r \mid p_t \geq r) = E(c - 0 \mid p_t \geq r) \Rightarrow \int_r^{\infty} (p_t - r) f(p_t)\, dp_t = c$$

$$V = E(\max\{p, V\}) - c = \int_{-\infty}^{V} V f(p)\, dp + \int_V^{\infty} p f(p)\, dp - c$$

$$E(N(r)) = \frac{1}{1 - F(r)}$$

$$V(r) = r$$

DIAMOND'S SEARCH

n firms, $MC=0$, iid consumers with $WTP=v$

A fraction $\frac{1}{n}$ of consumers learn p_1, another $\frac{1}{n}$ learn p_2...

Consumer can get new quotes at c per quote

market friction!

If $n > 1 + \frac{v}{c}$, then unique SPNE is $p_i = v \ \forall i$

① Firms guarantee $\pi > 0$ by setting $p > c$.

② No consumer searches. Consider firm with maximal price which makes positive profits by ①. If some of its consumer searches, can lower price a bit to increase profits. So no buyers search. Move on to argue for second highest priced firm... all firms.

③ Cannot have $p_i < v$. Lowest priced firm can increase to $p_i + \varepsilon < v$ and no one will leave (no search).

④ $p_i = v$ is SPNE : Check for deviation. If one firm lowers price to induce search,

$Pr(\text{finding that firm in one search}) = \frac{1}{n-1}$ and $E(N(\text{searches})) = n-1$ geometric dist

Consumer's expected cost $= (n-1)c > v$ by assumption, won't search.

No search cost $c=0$: $\underline{n} \to \infty$ $p_i = 0$ (Bertrand)

With search cost $c>0$ and $n > 1 + \frac{v}{c}$, $p_i = p^m$ (Monopoly)

No price dispersion despite search costs.

Not robust to heterogeneous c; also, internet $\downarrow c$

VARIAN'S SEARCH MODEL

α informed consumers (buy from min-priced firms)
$1-\alpha$ uninformed (buy at random)
WTP = v

NO PSNE

guaranteed demand

$$\pi_i(p_i, p_{-i}) = \begin{cases} p_i\left(\frac{\alpha}{L} + \frac{1-\alpha}{n}\right) & \text{if } p = p_{min} \\ p_i\left(\frac{1-\alpha}{n}\right) & \text{if } p > p_{min} \end{cases}$$

L: N(firms with p_{min})

$\pi^* > 0$ as 'worst case scenario' $p_i = v$ still yields $v\left(\frac{1-\alpha}{n}\right) > 0$
If $L = 1$, p_{min} firm can $\uparrow p$ to $\uparrow \pi$
If $L > 1$, can $\downarrow p$ to $\uparrow \pi$

MSNE $G: [\underline{p}, \bar{p}] \to [0,1]$ s.t. $[\underline{p}, \bar{p}] \subseteq [0, v]$

Assuming rivals play G

$$\mathbb{E}(\pi_i(p_i)) = \begin{cases} p_i\left(\alpha + \frac{1-\alpha}{n}\right) & \text{if } p_i < \underline{p} \\ p_i\left[\alpha(1-G(p_i))^{n-1} + \frac{1-\alpha}{n}\right] & \text{if } p_i \in [\underline{p}, \bar{p}] \\ p_i\left(\frac{1-\alpha}{n}\right) & \text{if } p_i > \bar{p} \end{cases}$$

1. G has no gaps and mass points (see Butters model)

2. $\bar{p} = v$. If $\bar{p} < v$, $\bar{p}\left(\frac{1-\alpha}{n}\right) < v\left(\frac{1-\alpha}{n}\right)$

3. Find $\underline{p}$ using indifference trick:

$$v\left(\frac{1-\alpha}{n}\right) = \underline{p}\left(\alpha + \frac{1-\alpha}{n}\right)$$

$$\underline{p} = \frac{v(1-\alpha)}{\alpha n + 1 - \alpha}$$

4. Find $G(p_i)$ using indifference trick: $\forall p_i \in [\underline{p}, \bar{p}]$,

$$p_i\left[\alpha(1-G(p_i))^{n-1} + \frac{1-\alpha}{n}\right] = v\left(\frac{1-\alpha}{n}\right)$$

$$G(p_i) = 1 - \left\{\frac{1}{\alpha}\left[\frac{v(1-\alpha)}{p \cdot n} - \frac{1-\alpha}{n}\right]\right\}^{\frac{1}{n-1}}$$

SURPLUS As $\alpha \to 1$, $\underline{p} \to \overset{MC=0}{0}$ and $G(p_i) \to 1$ for $p_i > 0 \Rightarrow$ mass point at 0
CS $\to$ v, PS $\to$ 0

As $\alpha \to 0$, $\underline{p} \to$ V $\Rightarrow$ Mass point at v
CS $\to$ 0, PS $\to$ 1

INCOMPLETE INFORMATION GAMES

Types: $(\theta_1,\dots,\theta_N) \in \Theta = \Theta_1 \times \dots \times \Theta_N$ distributed by $\underbrace{f(\theta_1,\dots,\theta_N)\ (\text{prior})}_{\text{common knowledge!}}$

Action profile: $(a_1,\dots,a_N) \in A = A_1 \times \dots \times A_N$

Payoff: $u_i : A \times \Theta \to \mathbb{R}$

Ex ante: Players don't even know own type θ_i

Interim: Learnt θ_i but not θ_{-i}

Ex post: Learnt all $\theta_1,\dots,\theta_N$

CONDITIONAL BELIEF

$$f(\theta_{-i} \mid \theta_i) = \frac{f(\theta_1,\dots,\theta_N)}{f(\theta_i)} = \prod_{j \neq i} f(\theta_j) \text{ if types independent}$$

$$\underbrace{f(\theta_i) = \int_{y_{-i} \in \Theta_{-i}} f(y_{-i}, \theta_i)\, dy_{-i}}_{\text{Integrate over all possible realisations of rival types}}$$

BAYESIAN STRATEGY

pure strategies $\quad s_i : \Theta_i \to A_i \qquad (s_1,\dots,s_N) \in S = S_1 \times \dots \times S_N$

mixed strategies $\quad s_i(a_i \mid \theta_i),\ s_i : \Theta_i \to \Sigma_i A_i$

the set of probability measures defined over A_i

INTERIM EXPECTED PAYOFF under $s = (s_1,\dots,s_N)$,

pure strategies $\quad U_i(s \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} u(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) f(\theta_{-i} \mid \theta_i)$

mixed strategies $\quad U_i(s \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{a \in A} u(a, \theta_i, \theta_{-i}) \prod_{j=1}^{N} s_j(a_j \mid \theta_j)\, f(\theta_{-i} \mid \theta_i)$

integrate over possible actions drawn by other players

BEST REPLY $\quad b_i(s_{-i} \mid \theta_i) \in \arg\max_{a_i \in A_i} u_i(a_i, s_{-i} \mid \theta_i)$ (given)

BAYESIAN NE

pure/mixed strategies $\quad (s_1,\dots,s_N)$ s.t. $\forall i, \theta_i, \quad U(s \mid \theta_i) \geq U(a_i' \mid \theta_i)\ \forall a_i' \in A_i$

SECOND PRICE AUCTION

$v_i \overset{iid}{\sim} F$ on $[0, V]$ privately known

Reserve price R

Payoff: $v_i - b_{(2)}$ if win; 0 if lose

SEPARATING EQUILIBRIUM $b_i = v_i \ \forall i$ Weakly DSE; efficient

If $v_i < R$: Bid v_i, else pay at least R, $v_i - R < 0$

If $v_i \geq R$: If $b_i > v_i$, if $b_i > b_{(2)} > v_i$, make loss; if $b_{(2)} < v_i$ same payoff

If $b_i < v_i$, if $b_i \leq b_{(2)} < v_i$, lose auction, if $b_i > b_{(2)}$ same payoff

POOLING EQUILIBRIUM $b_1 = V$, all others bid $b_i = R$ Not DSE

If values are public, bidding opponent's value is an NE too!

ENGLISH AUCTION Bidders call out successively higher bids until one bidder remains

Strategic equivalence: $b_i = v_i$

NE - PUBLIC VALUATIONS Many NE. Sufficient conditions

Winner i: $v_i \geq b_{(2)}$ weakly positive payoffs from winning

Losers j: $v_j \leq b_{(1)}$ negative payoffs from winning

Eliminating weakly dominated strategies yields $b_i = v_i$

see argument above

FIRST PRICE AUCTION

$v_i \sim F$ on $[0, V]$

Reserve price R

Payoff: $v_i - b_i$ if win; 0 if lose

Interim expected payoff $U_i(v_i, b_i) = (v_i - b_i) Pr(b_i > b_j \ \forall j \neq i)$

$\boxed{\text{BAYESIAN NE}}$ Strategy $b_i : [0, v_i] \to \mathbb{R}$

Assume there is <u>symmetric</u> $b(v_i) = b_i(v_i)$ b <u>differentiable</u>, <u>strict monotone</u>

Approach 1
Envelope Theorem

$$dU_i(v_i, b_i) = \frac{\partial U_i(v_i,b_i)}{\partial v_i} dv_i + \frac{\partial U_i(v_i,b_i)}{\partial b_i} db_i$$

$$\frac{dU_i(v_i,b_i)}{dv_i} = \frac{\partial U_i(v_i,b_i)}{\partial v_i} + \frac{\partial U_i(v_i,b_i)}{\partial b_i}\frac{db_i}{dv_i}$$

exists by assumption that $b(v_i)$ is differentiable

$$\left.\frac{dU_i(v_i,b_i)}{dv_i}\right|_{b_i = b(v_i)} = \left.\frac{\partial U_i(v_i,b_i)}{\partial v_i}\right|_{b_i=b(v_i)} + \underbrace{\left.\frac{\partial U_i(v_i,b_i)}{\partial b_i}\right|_{b_i=b(v_i)}}_{=0} \left.\frac{db_i(v_i)}{dv_i}\right|_{b_i=b(v_i)}$$

optimal bid

$$= \left.\frac{\partial U_i(v_i,b_i)}{\partial v_i}\right|_{b_i=b(v_i)}$$

$$= Pr(b(v_i) > b(v_j) \ \forall j \neq i)$$

$$= Pr(v_i > v_j \ \forall j \neq i) \quad b \text{ is strict monotone}$$

$$= F(v_i)^{N-1}$$

$$U_i(v_i, b(v_i)) = U(0, b(0)) + \int_0^{v_i} \frac{dU_i(x, b(x))}{dx} dx$$

$$(v_i - b(v_i)) F(v_i)^{N-1} = 0 + \int_0^{v_i} F(x)^{N-1} dx$$

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{N-1} dx}{F(v_i)^{N-1}}$$

$$= \frac{v_i F(v_i)^{N-1} - \int_0^{v_i} F(x)^{N-1} dx}{F(v_i)^{N-1}} = \frac{\int_0^{v_i} x \frac{dF(x)^{N-1}}{dx} dx}{F(v_i)^{N-1}} \quad \text{integration by parts}$$

BAYESIAN NE

Approach 2

Assume all others are playing $b(v_i)$ already

$$\begin{aligned} Pr(b_i > b_j \; \forall j \neq i) &= Pr(b_i > b(v_j) \; \forall j \neq i) \\ &= Pr(b^{-1}(b_i) > v_j \; \forall j \neq i) \\ &\overset{iid}{=} Pr(b^{-1}(b_i) > v_1) \ldots Pr(b^{-1}(b_i) > v_i) \\ &= F(b^{-1}(b_i))^{N-1} \end{aligned}$$

$$U_i(v_i, b_i) = (v_i - b_i) F(b^{-1}(b_i))^{N-1}$$

$$\frac{\partial U_i(v_i, b_i)}{\partial b_i} = -F(b^{-1}(b_i))^{N-1} + (v_i - b_i) \frac{\partial F(b^{-1}(b_i))^{N-1}}{\partial b^{-1}(b_i)} \frac{\partial b^{-1}(b_i)}{\partial b_i} = 0$$

Assume $b_i = b(v_i)$ at optimum $\quad b^{-1}(b_i) = v_i, \quad \frac{\partial b^{-1}(b_i)}{\partial b_i} = \frac{1}{b'(b^{-1}(b_i))} = \frac{1}{b'(v_i)}$

$$-F(v_i)^{N-1} + (v_i - b(v_i)) \frac{\partial F(v_i)^{N-1}}{\partial v_i} \frac{1}{b'(v_i)} = 0$$

$$v_i \frac{\partial F(v_i)^{N-1}}{\partial v_i} = b(v_i) \frac{\partial F(v_i)^{N-1}}{\partial v_i} + F(v_i)^{N-1} b'(v_i)$$

$$\int_0^{v_i} x \frac{\partial F(x)^{N-1}}{\partial x} dx = b(v_i) F(v_i)^{N-1} - b(0) F(0)^{N-1}$$

$$b(v_i) = \frac{\int_0^{v_i} x \frac{\partial F(x)^{N-1}}{\partial x} dx}{F(v_i)^{N-1}} = \mathbb{E}(v_{(2)} \mid v_{(2)} < v_i)$$

cdf of the highest of N-1 valuations

REVENUE

$$\mathbb{E}\{\max(b_1, \ldots, b_n)\} = \int_{-\infty}^{\infty} b(x) f_{\max(v_1, \ldots, v_n)}(x) dx$$

☆ On average, seller's expected revenue $= \max_{j \neq i} b_j$. Same as 2nd price!

REVENUE EQUIVALENCE

Under independent private values and risk neutral bidders, first price and second price auctions give same expected revenue

first price: winner has highest v_i, payment: expected second highest v

second price: winner has highest v_i, payment: actual second highest v

⇒ Same expected utility

ROBUSTNESS

Risk aversion

second price auction: unchanged

first price auction: bid more aggressively because

$$U_i = \overset{V(\cdot) \text{ concave (risk averse)}}{V}(\underbrace{v_i - b_i}_{\text{falls, but magnitude is dampened by risk aversion}}) \underbrace{Pr(b_i > b_j \; \forall j \neq i)}_{\text{rises, but magnitude is unaffected by risk attitudes}}$$

⇒ higher expected revenue from first price auction

EMPIRICS OF AUCTIONS

Seller doesn't know $F(v_i)$,
but wish to infer it from past bid data $\{b_i^t\}_{i=1,\dots,N;t}$ — number of auctions (for the same item)

INDIRECT INFERENCE

Let $H(b_i)$ be the (observed) bid distribution
Assume rival bids are all drawn iid from H

Bidder solves $\max_{b_i} (v_i - b_i) H(b_i)^{N-1}$ ← $Pr(b_j < b_i \ \forall j \neq i)$

$$v_i = b_i + \frac{H(b_i)}{(N-1)H'(b_i)}$$

This assumes bidders are playing DSE (best responding);
if they are actually at an NE that is not DSE
(e.g pooling equilibrium of second price auction), then TROUBLE

☆ Since we assume everyone uses $H(b)$, if H is strictly monotone
$F(v_i) = Pr(v_{it} \leq v_i) = Pr(b(v_{it}) \leq b(v_i)) = H(b(v_i))$
Feed support of H into $v(b)$ to derive support of F

1. Assume $b_i(v_i)$ at NE is invertible (separating equilibria)
2. Find empirical $H(b)$: $\hat{H}(b) = \sum_{i,t} \frac{1}{NT} \mathbb{1}(b_{it} \leq b)$ ← step function, can also use kernels, etc.

   Consistent. $\lim_{NT\to\infty} \sum_{i,t} \frac{1}{NT} \mathbb{1}(b_{it} \leq b) = E[\mathbb{1}(b_{it} \leq b)] = Pr(b_{it} \leq b) = H(b)$

   Can also use LOO $\hat{H}_i(b_i) = \sum_{j\neq i,t} \frac{1}{(N-1)T} \mathbb{1}(b_{jt} \leq b)$ fit using i's rival's bids
   to detect heterogeneity, i.e. $F_i(v_i) \neq F_j(v_j)$
3. Find pseudovalues $\hat{v}_i = b_i + \frac{\hat{H}(b_i)}{(N-1)\hat{H}'(b_i)}$ that rationalise observed bids

   If $\hat{H}$ is discrete, $\hat{H}'(b_i)$ should be $\hat{H}(b_i) - \hat{H}(b_i - 1)$
4. $\hat{F}(v_i) = \sum_{i,t} \frac{1}{NT} \mathbb{1}(\hat{v}_{it} \leq v_i)$

   $\lim_{NT\to\infty} \hat{F}(v_i) = E(\mathbb{1}(v_{it} \leq v_i)) = Pr(v_{it} \leq v_i) = F(v_i)$

ASSUMPTIONS

1. Bids are generated from equilibrium play
2. v_i are iid across bidders i and auctions t

WINNER'S CURSE = 'Bad news effect'

Bidders risk-neutral
Assume true value v of good is exogenous
θ_i is private and unbiased : $E(\theta_i | v) = v$

Interpret: Common value to all bidders, but noisy signal ($\theta_i = v + \varepsilon_i$, $E(\varepsilon_i) = 0$)

Highest bidder overestimates v: $E(\max_i\{\theta_i\} | v) > \max_i E(\theta_i | v) = v$

Proof: Jensen's inequality. $\max\{\cdot\}$ is convex as
$\max\{\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\} < \lambda \max\{x_1, y_1\} + (1-\lambda)\max\{x_2, y_2\}$
$\forall \lambda \in [0,1]$, $x, y \in \mathbb{R}$, $x \neq y$

Implication: Winners make losses

Rational bidders shade bid down due to winners' curse
in addition to strategic effect (e.g. markdown strategy in 1st price)

CONDITIONAL ESTIMATE

$E(v | \theta_i, i \text{ wins}) = E(v | \theta_i, \theta_i \geq \theta_j \, \forall j)$

As $N \to \infty$,

conflict

Competitive effect	Smaller markup, bid more aggressively
Bad news effect	$\uparrow Var(\theta_i)$, $\max\{\theta_i\} \gg v$, bid less aggressively

MONEY LEFT ON TABLE

$\frac{b_{(1)} - b_{(2)}}{b_{(2)}}$ How much winners overpaid (uncertainty)

- Empirically, declines with N, seems like winner's curse dominates
  - But $\pi > 0$ regardless of N — no one is making losses, no evidence
- Claims that $b_{(1)}$ increases with N, but the true v is an OVB increasing both

BEST RESPONSE TEST

$$\Pi(\gamma) = \sum_i \underbrace{\sum_{k \in K_i}}_{\text{goods}} (\underbrace{\pi_k}_{\text{actual revenue}} - \gamma b_{ik}) \underbrace{\mathbb{1}(\gamma b_{ik} - \underbrace{b_k}_{\text{max bid for } k} > 0)}_{=1 \text{ iff } i \text{ wins } k}$$

$\gamma = 1$: $\Pi(1)$ = Total sum of profits across bidders and goods.
Bidders are ex post rational ($\Rightarrow$ interim rational)
$\gamma > 1$: Win more often, less returns per good (could've
$\gamma < 1$: Win less often, more returns per good (winner's curse)

Assumes bidders know what rival bidders bid (ex post)
Better formulation would be interim

COLLUSION IN AUCTIONS

Independent Private Values

ENGLISH AUCTION Susceptible to collusion; deviants can be punished immediately

FIRST PRICE AUCTION Enforcement only possible with repeat auctions (trigger)

PRE-AUCTION KNOCKOUT

Cartel holds (English) auction prior; winner gets item at R
Winner pays $b_2 - R$ to all other members (split evenly)

- Efficient; maximise collusive surplus,
- Requires side payment

Strategy: bid R if $\exists$ member i s.t. $v_i - R \geq 0$

SELLER RESPONSE

Pr(designated bidder has valuation $> R$)
$= 1 - \Pr$(all cartel members have valuation $< R$)
$= 1 - F(R)^N$

$$\max_R \; R[1 - F(R)^N]$$

$$R = \frac{1-F(R)^N}{NF(R)^{N-1}f(R)} \;>\; \arg\max_R \; R[1-F(R)] \text{ under competition}$$

NO SIDE PAYMENT

Each bidder $\max_{Q_i} (v_i - R)Q_i(v_i)$
↑ probability that upon announcing bid, i becomes designated bidder

IC: Cartel wants truthful reporting of v_i. But this implies $Q_i(v_i) = \bar{Q}_i \; \forall i$

- Q_i constant in v_i — if changing v_i can $\uparrow Q_i$, will misreport!
- Symmetry implies $\bar{Q}_i = \frac{1}{N}$
- Inefficient, doesn't maximise collusive surplus

RANKING MECHANISM

Each cartel member ranks k goods by preference
For every good, cartel bidder with highest ranking is designated

- Almost efficient with $k \to \infty$ (reflects ranking of valuations accurately)
- $\downarrow$ var(mkt %) compared to side payments structure (have to assign some goods to every member instead of just ££)

TESTING

If cartel is efficient, bid lower than non-cartel bids on average
Formally, cdf of non-cartel bids stochastically dominates cartel bids

If cartel/non-cartel members defer in some X
Regress bids on that X and compare residuals instead

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COLLUSION IN AUCTIONS

Public valuations

Assume: bidders don't use weakly dominated strategies, i.e. $b_i = v_i$ under competition

BIDDER SELECTION Member with highest v_i becomes designated bidder

SIDE PAYMENT

If cartel member wins:
- Highest valuation member: $v_h - \max\{\underset{\text{non-members' bids}}{v_o}, R\} - \text{Side payment} \geq v_h - \max\{v_o, \underset{\text{low valuation members' bids}}{v_\ell}, R\}$
- Low valuation member: Side payment ≥ 0 ← loses in competition anyway — no enforcement needed

$\max(v_\ell, v_o) - v_o \geq s \geq 0$

If outside member has higher valuation than low valuation member, then no side-payment will be given (cartel either loses, or wins but pay the same $b_{(2)}$ to auctioneer)

If cartel members lose: Payoff $= 0$ (no incentive to deviate)

OUTCOME

Still efficient (bidder with highest v_i wins)
If $s > 0$, cartel captures rent from auctioneer
- Always weakly more rent as cartels make bidding less aggressive
(rent = valuation − price paid)

COLLUSION IN AUCTIONS

Common value auctions

$v_i = v$ but only get unbiased signal θ_i

BIDDER SELECTION Any member can be picked (no need to reveal) as $v_i = v$, same ex post payoff
Pools information: Optimal bid is R if cartel thinks $E(v|\theta_1,\theta_2) - R \geq 0$
$= \frac{\theta_1+\theta_2}{2} = \bar{\theta}$

↓ Probability of inefficiency $(v - b < 0)$ $\Pr(b(\theta_{(1)}) > v) < \Pr(R\mathbb{1}\{\bar{\theta} \geq R\} > v)$

RESERVE PRICE Increases.

Competition: $\underset{r}{\operatorname{argmax}} \int_{b^{-1}(r)}^{\infty} b(x) f_{\theta_{(1)}}(x)\,dx \qquad \theta_{(1)} \sim F$

Collusion: $\underset{r}{\operatorname{argmax}} \int_{r}^{\infty} R\, g_{\bar{\theta}}(x)\,dx \qquad \bar{\theta} \sim G$

ENFORCEMENT Deviation is profitable if $\bar{x} - R > 0$ (just set $b_i = R + \varepsilon$)
Must enforce through grim trigger/'cement shoes'

MORAL HAZARD

SETUP

Agent

Effort $e = 0, 1$, wage w

Utility $= U(w) - e$

concave

Principal

Observe only S/F

$$\text{Payoff} = \begin{cases} S & Pr = p_e \\ F & Pr = 1 - p_e \end{cases} \qquad p_1 > p_0$$

Offers wage w_S, w_F to induce effort

HIGH EFFORT

IR $\quad p_1 U(w_S) + (1-p_1) U(w_F) - 1 \geq \underline{U}$

IC $\quad p_1 U(w_S) + (1-p_1) U(w_F) - 1 \geq p_0 U(w_S) + (1-p_0) U(w_F)$

$$\Rightarrow \underbrace{(p_1 - p_0)[U(w_S) - U(w_F)]}_{\text{MB of effort}} \geq \underbrace{1}_{\text{MC of effort}}$$

Principal $\max_{w_S, w_F} \; p_1 (S - w_S) + (1-p_1)(F - w_F)$ s.t. IC, IR

IR binds (if not, $\downarrow w$ to $\uparrow \pi$)

IC binds (if not, $\downarrow w_S$ $\uparrow w_F$ to $\uparrow \pi$)

Just do simultaneous equations on IC, IR $\Rightarrow \begin{cases} U(w_F^*) = \underline{U} - \frac{p_0}{p_1 - p_0} \\ U(w_S^*) = \underline{U} + \frac{1 - p_0}{p_1 - p_0} \end{cases}$

LOW EFFORT

$w = w_S = w_F$ s.t. $U(w) = \underline{U}$ (IR binds)

OPTIMAL EFFORT

Prefer to induce high effort if $\pi_1 - \pi_0 > 0$

$$\Rightarrow (p_1 - p_0)\underbrace{(S-F)}_{\substack{\text{Induce high effort} \\ \text{if } S-F \text{ is large}}} - w - p_1 w_S^* - (1-p_1) w_F^* > 0$$

don't depend on S, F

(Big bonus for CEOs but flat rate for janitors)