Revision Notes by Sally Yang

MICROECONOMIC PRINCIPLES

LENT TERM

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PURE STRATEGIES

Set of players	$N = \{1, \dots, n\}$
Action set for player i	$A_i = \{$Action 1, Action 2, $\dots\}$
Action profile	$\underline{a} = (a_1, \dots, a_n)$ — a_1: player 1's action
Action profile of all players other than i	$\underline{a}_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$
Utility of player i	$u_i(\underline{a}) =$ scalar payoff

STRATEGY A plan of action; maps info to action
In perfect info simul game, the player's (pure) strategy is just $a_i \in A_i$

BEST RESPONSE
Best response correspondence (BR) of i is $b_i(\underline{a}_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, \underline{a}_{-i})$ for any $\underline{a}_{-i}$

DOMINATION

Strategy a_i strictly dominates a_i' if $u(a_i, \underline{a}_{-i}) > u_i(a_i', \underline{a}_{-i})$ $\forall \underline{a}_{-i}$
Strategy a_i weakly dominates a_i' if $u(a_i, \underline{a}_{-i}) \geq u_i(a_i', \underline{a}_{-i})$ $\forall \underline{a}_{-i}$
$u(a_i, \underline{a}_{-i}) > u_i(a_i', \underline{a}_{-i})$ for some $\underline{a}_{-i}$

a_i is strictly dominant if it strictly dominates any other a_i' — BR
a_i is strictly undominated if no strategy strictly dominates a_i
a_i is strictly dominated if a strategy strictly dominates a_i — never selected

A strict dominant strategy equilibrium of a game is a strategy profile $\underline{a}$ st $\forall \underline{a}'_{-i}$ $\forall i \in N$ (★ not the payoffs of said profile (beware in exams!))
$$u_i(a_i, \underline{a}'_{-i}) > u_i(a_i', \underline{a}'_{-i}) \quad \forall a_i' \in A_i$$
$\geq$ for weak DSE

$$\underline{a} \text{ is a DSE} \Leftrightarrow a_i \in b_i(\underline{a}'_{-i}) \ \forall \underline{a}'_{-i} \ \forall i \in N$$

- DSE may be inefficient (Pareto-dominated) – may exist an outcome with higher payoff for all
- DSE may not exist!
- DSE $\Rightarrow$ NE

RATIONAL Has beliefs about what other players do and maximises own payoff given beliefs
- Assumes each player knows their own payoff
- Rationality $\Rightarrow$ DSE (if $\exists$)
  $\Rightarrow$ Never choose dominated strategy

ITERATED DOMINANCE
Assume, in a non-cooperative game,
- rationality
- common knowledge of players' rationality
  - Players only employ strategies that survive iterative elimination of strictly dominated strategies
- common knowledge of game structure

- You can eliminate dominated strategies repeatedly to find DSE.
- But people aren't 100% rational in reality so you'll never be 100% sure of other players' rationality

★ Eliminate only strictly (not weakly) dominated strategies.
Can eliminate weakly too but may exclude some NEs

(Pure strategy)
$\boxed{\text{NASH EQUILIBRIUM}}$ is a strategy profile $\underline{a} = (a_i, \underline{a}_{-i})$ s.t. $\forall i \in N$

$$u_i(\underline{a}) \geq u_i(a_i', \underline{a}_{-i}) \quad \forall a_i' \in A_i$$

$$\underline{a} \text{ is NE} \iff \exists i \in N \text{ s.t. } a_i \in b_i(\underline{a}_{-i})$$

- Strategies profiles are independent, common knowledge
- May have many/no pure strategy NEs
- NE may not be Pareto efficient; some NEs may be more risky (re miscommunication)
- No incentive to deviate (holding constant other players' strategies)

MIXED STRATEGIES

Pure strategy NE may not exist due to non-convexities in choice sets and discontinuities of BR
But mixed strategy NE always exists!

Each player's randomisation is independent. Maximise each player's expected payoff holding other players' strategies (choice of probabilities) to get BR. Then find the intersections of those BRs to get NE.

DEFINITION In a complete information static game $\Gamma = \{\overset{\text{players}}{N}, \overset{\text{actions}}{\{A_i\}_{i\in N}}, \overset{\text{payoffs / utilities}}{\{u_i\}_{i\in N}}\}$
A mixed strategy for $i \in N$, σ_i, is a probability distribution over actions in A_i, s.t.

$$\boxed{0 \leq \sigma_i(a_i) \leq 1 \quad \forall a_i \in A_i \quad \text{and} \quad \sum_{a_i \in A_i} \sigma_i(a_i) = 1}$$

This implies $\sigma_i(a_i)$ is the probability that player i chooses action a_i

$\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ is the mixed strategy profile chosen by all but player i

SIMPLEX $\Delta(A_i)$ is the set of possible probability distributions on a given set A_i

- Closed, bounded, convex

PAYOFFS FROM MIXED STRATEGIES

Assuming player i chooses σ_i independently from other players' choices, we can say

Payoff to player i from choosing $\sigma_i \in \Delta(A_i)$ when others follow σ_{-i} is

probability that $1, \ldots, N$ chooses $a_1, \ldots, a_N$

$$\boxed{u_i(\sigma_i, \sigma_{-i}) = \sum_{\underline{a} \in A} \Pr(\underline{a}) u_i(\underline{a}) = \sum_{\underline{a} \in A} \left(\prod_{j \in N} \sigma_j(a_j) \right) u_i(\underline{a})}$$

e.g. If $\overset{\text{possible actions}}{A_i} = \{B, C\}$ and $N=2$, then Player 1's expected payoff is
$u_1(\sigma_1, \sigma_2) = \sigma_1(B)\sigma_2(B)\, u_1(B,B) + \sigma_1(B)\sigma_2(C)\, u_1(B,C) + \sigma_1(C)\sigma_2(B)\, u_1(C,B) + \sigma_1(C)\sigma_2(C)\, u_1(C,C)$

BEST REPLY of i, $b_i(\sigma_{-i}) = \underset{\sigma_i \in \Delta(A_i)}{\arg\max}\, u_i(\sigma_i, \sigma_{-i})$

One example is $\forall \sigma_1 \in b_1(\sigma_2)$, $\begin{cases} \sigma_1 = 0 & \text{if } \sigma_2 < \ldots \\ \sigma_1 \in [0,1] & \text{if } \sigma_2 = \ldots \\ \sigma_1 = 1 & \text{if } \sigma_2 > \ldots \end{cases}$

DOMINATION

Strategy σ_i strictly dominates a_i if $u(\sigma_i, \underline{a}_{-i}) > u_i(a_i, \underline{a}_{-i}) \quad \forall \underline{a}_{-i}$
Strategy σ_i weakly dominates a_i if $u(\sigma_i, \underline{a}_{-i}) \geq u_i(a_i, \underline{a}_{-i}) \; \forall \underline{a}_{-i}$
$u(\sigma_i, \underline{a}_{-i}) > u_i(a_i, \underline{a}_{-i})$ for some $\underline{a}_{-i}$

We may be able to find a strictly/weakly dominant mixed strategy when there are no strictly/weakly dominant pure strategies!

NASH EQUILIBRIUM is a strategy profile $\underline{\sigma} = (\sigma_i, \sigma_{i-1})$ s.t. $\forall i \in N$

can be a continuum (infinite!)

$$\boxed{u_i(\underline{\sigma}) \geq u_i(a_i, \underline{\sigma}_{-i}) \quad \forall a_i \in A_i}$$

MSNE

Assumes
- Strategy profiles are independently chosen
- Each agent knows and believes which strategies others adopt
- Each agent is rational (maximises expected utility given his beliefs)

If a_i is strictly dominated then $\sigma_i(a_i) = 0$ in any NE
If a_i is weakly dominated then $\sigma_i(a_i) > 0$ in NE only if the a_{-i} for which a_i is strictly worse cannot occur

$$\sigma \text{ is an NE} \iff \begin{array}{ll} u_i(\sigma_i, \sigma_{-i}) = u_i(a_i, \sigma_{-i}) & \forall a_i \text{ s.t. } \sigma_i(a_i) > 0 \\ u_i(\sigma_i, \sigma_{-i}) \geq u_i(a_i, \sigma_{-i}) & \forall a_i \text{ s.t. } \sigma_i(a_i) = 0 \end{array}$$

Maximise each player's expected payoff holding other players' strategies (choice of probabilities) to get BR.
Then find the intersections of those BRs to get NE.

SOLVING NE - INDIFFERENCE CONDITION

We find σ_1 s.t. player 2 is indifferent between every pure strategy played, vice versa

At σ_1^*, $EU_2(B, \sigma_1) = EU_2(C, \sigma_1)$
At σ_2^*, $EU_1(B, \sigma_2) = EU_1(C, \sigma_2)$
Then solve simultaneous equations to find NE!
☆ if $EU_2(B, \sigma_1) > EU_2(C, \sigma_1)$, player 2 will just pick B as best response
Not always the case though! Safest to plot BRs to different probilities

☆ Pure strategies are just special case of mixed strategy
When asked to find mixed NE, don't forget to find pure NEs too

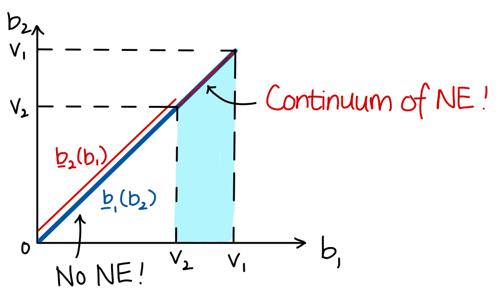
FIRST PRICE AUCTION

$N = 2$, $A_i = \{ \underset{\text{bid (\$)}}{b_i} \geq 0 \}$, assume $v_1 > v_2$

Higher bid wins (then pay own bid); if $b_1 = b_2$, 1 wins

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & \text{if } b_1 \geq b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases} \qquad u_2(b_1, b_2) = \begin{cases} v_2 - b_2 & \text{if } b_1 < b_2 \\ 0 & \text{if } b_1 \geq b_2 \end{cases}$$

$$\underline{b}_1(b_2) = \begin{cases} b_1 < b_2 & \text{if } b_2 > v_1 \\ b_1 \leq b_2 & \text{if } b_2 = v_1 \\ b_1 = b_2 & \text{if } b_2 < v_1 \end{cases} \qquad \underline{b}_2(b_1) = \begin{cases} b_2 \leq b_1 & \text{if } b_1 \geq v_2 \\ b_2 = b_1 + \varepsilon & \text{if } b_1 < v_2 \end{cases}$$



The compelling NE is $b_1 = b_2 = v_2$
Bidder 1 wins, pays v_2, gets payoff $v_1 - v_2$

The other (not-compelling) NE are $b_1 = b_2 \in (v_2, v_1]$
Bidder 1 wins, pays $b_1 = b_2 = v_2$, gets payoff $v_1 - b_1$
BUT bidder 2 realistically don't always match b_1 even though they know they'll lose ...

DUOPOLY - COURNOT

Two firms $N=\{1,2\}$, each produces q_i with cost $c_i(q_i)$
Both face inverse demand $p(q)$, where aggregate output $q=q_1+q_2$
price taker

The payoff (profits) of each firm
Take as given!
$$u_i(q_i, q_{-i}) = p(q)q_i - c_i(q_i)$$
depends on production decision of both firms

In Cournot, firms choose quantities simultaneously to maximise profits

$$\boxed{\max_{q_i} u_i(q_i, q_{-i}) = \max_{q_i} p\left(\sum_{k\in N} q_k\right) q_i - c_i(q_i)}$$

FOC: $$\frac{\partial u_i}{\partial q_i} = \underbrace{p(q) + \frac{\partial p(q)}{\partial q_i} q_i}_{MR} - \underbrace{\frac{\partial c_i(q_i)}{\partial q_i}}_{MC} \leq 0$$

MC = MR if $q_i > 0$

MC ≥ MR if $q_i = 0$

CARTEL

Rather than holding q_{-i} constant and find q_i^*, all firms maximise joint profits

$$\boxed{\max_{q_1,\dots,q_n} p(q) - \sum_{i\in N} c_i(q_i)}$$

Profits for each firm are higher in the cartel

DEFECTION Cartels are hard to sustain since it's prisoner's dilemma

If firm j produces cartel quantity $\hat{q}_j$, firm i has incentive to deviate since $b_i(\hat{q}_j) > \hat{q}_i$

DUOPOLY - BERTRAND

Two firms $N=\{1,2\}$, each sets p_i with marginal cost c

Both face demand $q_i(p)_i = \begin{cases} q(p_i) & \text{if } p_i<p_j \\ \frac{q(p_i)}{2} & \text{if } p_i=p_j \\ 0 & \text{if } p_i>p_j \end{cases}$

A profit maximising monopolist chooses price to

$$\boxed{\max_p u(p) = \max_p q(p)(\overset{\pi \text{ per unit}}{p-c})}$$

to get price $\bar{p}$

Two firms face this problem

$$\boxed{\max_{p_i} u_i(p_i,p_j) = \max_{p_i} q_i(p_i,p_j)(\overset{\pi \text{ per unit}}{p_i - c_i(q_i)})}$$

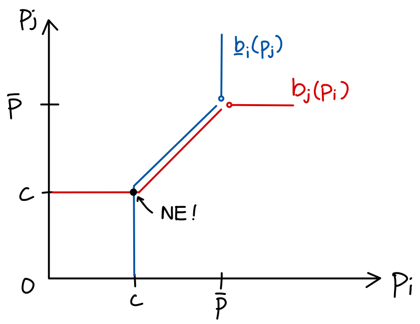
Which is equal to maximising sales

$p_i = b_i(p_j) = \begin{cases} \bar{p} & \text{if } p_j > \bar{p} \\ p_j - \varepsilon & \text{if } \bar{p} \geq p_j > c \\ c & \text{if } p_j \leq c \end{cases}$

capture whole mkt at monopoly price
undercut j and capture whole mkt
no benefit to pricing below MC

Unique NE: $p_1 = p_2 = c$

- Perfect competition with just 2 firms!



$p=MC$ weakly dom $p<MC$
$p=\bar{p}$ weakly dom $p>\bar{p}$
$p\in(MC,\bar{p})$ weakly dom $p=MC$

BERTRAND VS COURNOT

BERTRAND	COURNOT
Duopoly is enough to get perfect comp ($p=MC$)	Few producers don't eliminate markups ($P>MC$) Gov must intervene
✓ Incentive to form cartel	✓ Incentive to form cartel
More likely when capacity is easy to adjust	More likely when capacity is hard to adjust

EXTENSIVE FORM GAME

A detailed description of the sequential structure of players' decision (with branches and nodes) in which time plays an explicit role. It has:

HISTORIES

For a finite sequence $(a^1, a^2, \dots, a^M)$, subhistories are $\overset{\text{"You're the first to go!"}}{\varnothing}$ and all sequences $(a^1, \dots, a^m)$ for $1 \leq m \leq M$ (same for infinite)

e.g. $\{\varnothing, a_1, (a_1, a_2), \dots, (a_1, \dots, a_M)\}$

Terminal histories describes, at every end node, what has happened so far

A history is a subhistory of some terminal history (describes every node in the tree)

A set of players $N = \{A, B\}$

A set of terminal histories (at end nodes!) $\mathcal{Z} = \{[(2,0),y], [(2,0),n], [(1,1),y]\}$

A player function that maps (labels) any non-terminal node to the player making the decision

e.g. $P(\varnothing) = A$ (Player A makes the first choice) $\quad P(\text{A's Choice}_1) = P(\text{B's Choice}_2) = 2$

The player's payoffs from each terminal history labels all terminal nodes $\quad$ e.g. $u([(2,0),y]) = 2$

STRATEGY

A fully contingent plan specifying the player's decision for every history after which the player is called upon to play. The set $\mathcal{P}_i = P^{-1}(i)$ for player i.

- Assumes perfect info and plan is complete (others can play on your behalf)

  e.g. $s_A = s(\varnothing) \in \{\text{A's Choice}_1, \text{A's Choice}_2\}$ since $\mathcal{P}_1 = \{\varnothing\}$.

  $s_B = \{s_B(\text{A's Choice}_1), s_B(\text{A's Choice}_2)\}$, $s_B(\text{A's Choice}_1) \in \{yes_1, no_1\}$, $s_B(\text{A's Choice}_2) \in \{yes_2, no_2\}$

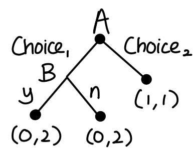
- Mixed strategies don't add much due to the nature of sequential games and perfect info

OUTCOMES

$o(s) \in \mathcal{Z}$ is the outcome (terminal history) from players following strategies s_i

- S and S' are outcome equivalent iff they lead to the same terminal node.
- e.g. $u_A[o(\underline{\text{Choice } 2, y})] = u_A[o(\underline{\text{Choice } 2, n})] = 1$

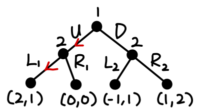
☆ Can never happen, but must specify!
Aids in backwards induction

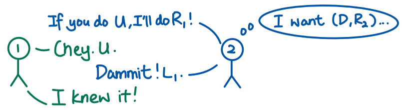


NASH EQUILIBRIUM

A strategy profile s^* s.t. $\forall i \in N$, $u_i(o(s_i^*, s_{-i}^*)) \geq u_i(o(s_i, s_{-i}^*))\ \forall s_i$

- $\forall i \in N$, $b(s^*_{-i}) = s^*_i$
- But there exists NE built on non credible threats! Eliminate by backward induction.





☆ Easiest way to find is to put all strategies in normal form (a matrix) (Q4/2018)

BACKWARD INDUCTION

NORMAL FORM associated with an extensive form game is $\Gamma = \{N, S, u_i(o(\cdot))\}$

Then NE are $(U,(L_1,L_2))$, $(U,(L_1,R_2))$, $(D(R_1,R_2))$

SUBGAME A part of the game tree that begins in a specific node. Follows from a terminal history (perfect)

- "self-contained"; rest of tree doesn't affect outcome/preferences

An NE s_i^* of a game is a subgame perfect equilibrium iff it is an NE for every proper subgame

- Eliminates NE built on non credible threats!

KUHN THEOREM Every finite extensive form game with perfect info has at least 1 subgame perfect equilibrium

- Use backward induction — start from subgames closest to terminal node, find NE, add that to SPE
- Gives whole set of SPE; Uniqueness not guaranteed
- But SPE may not be efficient

COURNOT-STACKELBERG

Initially (assuming $MC=c$, dd is $p=a-(q_1+q_2)$), NE is $\left(\frac{a-c}{3}, \frac{a-c}{3}\right)$, profit for each firm $\frac{(a-c)^2}{9}$

Now, firm 1 (leader) sets quantity q_1 first, firm 2 (follower) observes q_1 and chooses q_2

Using backward induction, firm 2 takes q_1 as given, $\max\limits_{q_2} q_2(a-(q_1+q_2)-c) \overset{\text{FOCs}}{\Rightarrow} q_2 = \frac{1}{2}(a-c-q_1)$

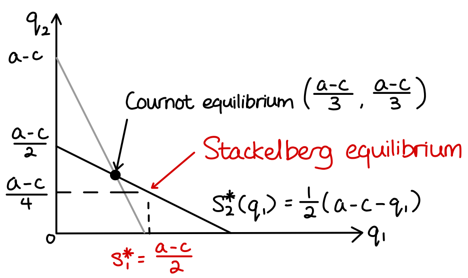
$$S_2^*(q_1) = \begin{cases} \frac{1}{2}(a-q_1-c) & \text{if } q_1 \leq a-c \\ 0 & \text{if } q_1 \geq a-c \end{cases}$$

Then, firm 1 $\max\limits_{q_1} q_1(a-(q_1+s_2^*(q_1))-c) \Leftrightarrow \max\limits_{q_1} q_1\left(a-\left(q_1+\frac{1}{2}(a-c-q_1)\right)-c\right) \overset{\text{FOC}}{\Rightarrow} s_1^* = q_1^* = \frac{a-c}{2}$

Thus, the SPE outcome is $\left(q_1^* = \frac{a-c}{2},\ q_2^* = \frac{a-c}{4}\right)$

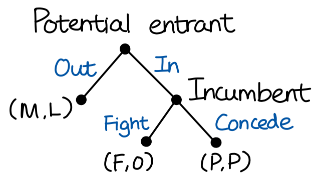
Profits are $\left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{16}\right)$

leader has competitive advantage (bigger market share, profits)



First mover advantage (get to choose point on follower's BR curve)

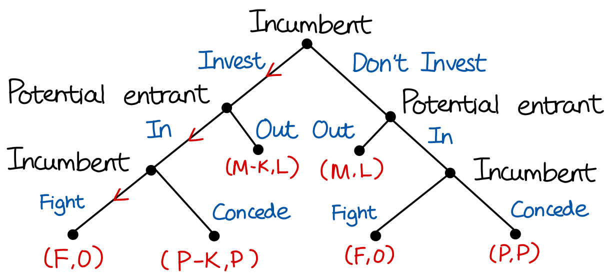
MARKET ENTRY



P > L > 0
M > P > F

Ability to fight doesn't deter entry

Now, add ability of incumbent to "invest" K — lowers all of incumbent's payoffs except (Invest, In, Fight)



If incumbent doesn't invest (by backward induction) outcome is (P,P)

For (Invest, In, Fight) to happen, F > P-K

⇒ Then potential entrant chooses Out since L > 0

⇒ If incumbent invests, outcome is (M-K, L)

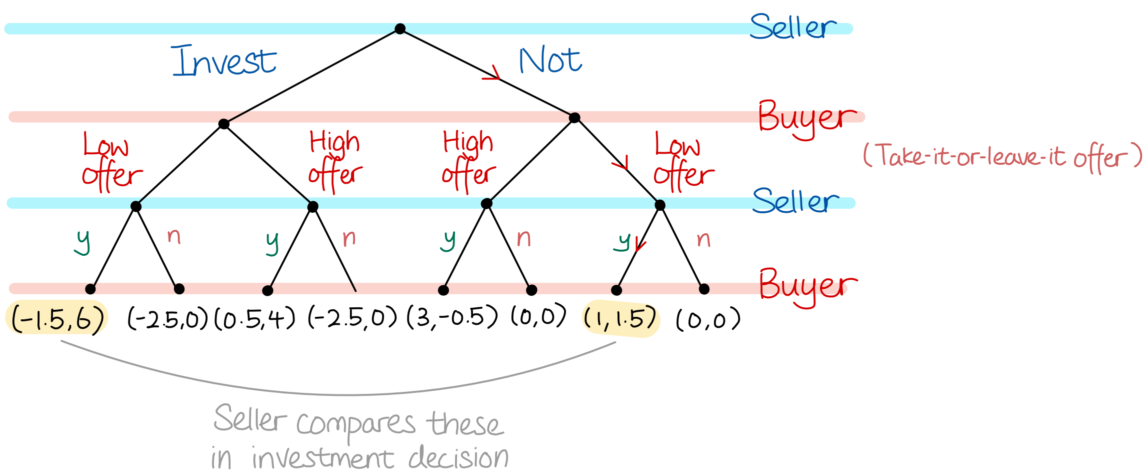
⇒ Incumbent invests if M-K > P

⇒ There exists an SPE in which entry is deterred if M-P > K > P-F

HOLD UP PROBLEM

When downstream buyer has too much bargaining power,
upstream seller (who knows they'll be squeezed) will underinvest

- Seller reaps no increase in social surplus from investment



REPEATED GAMES

Players' overall payoffs depend on present and future stage game payoffs
Threat of a lower future payoff may induce a player now to choose a strategy different from the stage game NE strategies (strategies players will choose if the same game is played only once)

or "average discounted payoff"

DISCOUNTED UTILITY If utility is u_τ at time τ, then utility at time t is

bc $1+\delta+\delta^2+\ldots=\frac{1}{1-\delta}$

$$U_t[(u_\tau)_{\tau=0}^{\infty}] = (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u_\tau \quad \text{with no set end}$$

make gross → average

$$U_t[(u_\tau)_{\tau=0}^{T}] = \frac{1-\delta}{1-\delta^{T-t}}\sum_{\tau=t}^{T}\delta^{\tau-t}u_\tau \quad \text{with } T+1 \text{ total periods}$$

because $1+\delta+\delta^2+\ldots+\delta^{T-t}=\frac{1-\delta^{T-t}}{1-\delta}$

PROPERTIES

When $T=\infty$, U_t is average of payoff today and from tomorrow onwards:

$$U_t=(1-\delta)u_t+\delta U_{t+1}$$

Players are time consistent: if $u'_\tau=u_\tau$ $\forall\tau\le t$ and

value or function

$$U_0[(u_1,u_2,\ldots)]\ge U_0[(u'_1,u'_2,\ldots)]$$
$$\Rightarrow U_t[(u_1,u_2,\ldots)]\ge U_t[(u'_1,u'_2,\ldots)]$$

If you prefer (500, 0, 0, ...) to (0, 505, 0, 0, ...)
you prefer $(\underbrace{0,\ldots,0}_{365},500)$ to $(\underbrace{0,\ldots,0,0}_{366},505)$

But not necessarily for $t'>t$
You may prefer utilities (100, 100, 2, 0, 0, ...) to (100, 100, 1, 1, 1, ...)
at $t=0$ and $t=1$, but not $t=2$

THREAT OF RETALIATION

$T=2$

	L	C	R
T	1,1	5,0	0,0
M	0,5	4,4	0,0
B	0,0	0,0	3,3

Player 1
Play M in $t=0$
Play B in $t=1$ if (M,C) happened Play T in $t=1$ if (M,C) did not happen

Player 2
Play C in $t=0$
Play R in $t=1$ if (M,C) happened Play L in $t=1$ if (M,C) did not happen

For this strategy to be SPE:
By backward induction, at $t=0$, (B,R) is NE
At $t=1$, (M,C) is NE if $\underbrace{\frac{4+\delta 3}{1+\delta}}_{\text{payoff if you don't cheat}} \ge \underbrace{\frac{5+\delta}{1+\delta}}_{\text{payoff if you cheat}} \Rightarrow \delta\ge\frac{1}{2}$

INFINITELY REPEATED GAMES

No certainty of a final stage (e.g. lifetime consumption). Cannot use BI
δ can be seen as probability that game doesn't end at next round (0 payoff)

ONE-SHOT DEVIATION PRINCIPLE

A strategy satisfies BIP if no player can increase his payoff in any subgame in which
- he is the first mover
- he takes as given all other players' strategies and the rest of his own strategy

A strategy profile in a discounted, infinitely repeated game with perfect information is a SPE iff it satisfies ISDP

REPEATED (FINITE) PRISONERS' DILEMMA

- Per period utility of player i from playing a^t is $g_i(a^t)$
- Both players discount with δ
- If $T=2$, Player 1's utility is $\Pi_i = \frac{1-\delta}{1-\delta^2}\left[g_i(a^0) + \delta g_i(a^1)\right]$
  - If he plays the same strategy $a^0 = a^1 = a$, $\Pi_i = g_i(a)$
  - If $g_i(a^t) = 1\ \forall t$, $\Pi_i = 1$

BACKWARD INDUCTION

At the last game, players face the strategic form game

1 \ 2	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

Unique NE is (D,D) with payoffs (0,0)

At the second last game, players face the strategic form game

$A = \frac{\delta^{T-1}(1-\delta)}{1-\delta^{T-1}}$

1 \ 2	C	D
C	A, A	-A, 2A
D	2A, -A	0, 0

Unique NE is still (D,D) with payoffs (0,0)

INFINITELY REPEATED PRISONERS' DILEMMA

ALWAYS CHEAT

(D,D) $\forall t$ is SPE because payoff by deviating to C is $-\frac{(1-\delta)}{A} < 0$ (payoff by sticking to D)
depends on t

GRIM TRIGGER

If $\delta \geq \frac{1}{2}$ then everyone playing this strategy is SPE:

- Play C at $t=1$
- Play C if no one chose D previously
- Always play D if someone chose D previously ← play the NE that minimaxs the one-time deviant (punishment)

Comparing payoffs, $\underbrace{(1-\delta)\sum_{t=0}^{\infty}\delta^t}_{\text{Conform}} \geq \underbrace{(1-\delta)(2+0+\ldots)}_{\text{Defect today (continuation payoff)}}$

$\Rightarrow 1 \geq (1-\delta)2$

$\Rightarrow \delta \geq \frac{1}{2}$

Also check that in the subgame in which both play D ("punishment phase"), neither wants to deviate

GENERAL REPEATED GAME

DEFINITIONS

G be a given stage game; a strategic form game $G = \{N, A_i, g_i(a^t)\}$

We can then derive the infinitely repeated game associated with G

$$G^\infty = \{N, \mathcal{H}, P, U_i(\sigma)\}$$

- Set of histories. $\mathcal{H} = \bigcup_{t=0}^{\infty} A^t$, $A^0 = \emptyset$
  all possible subhistories
- Player function maps set of all non-terminal nodes to the player making the decision
  $P(h) = N \quad \forall h \in \mathcal{H} - \mathcal{Z}$ terminal histories

PAYOFF

the mixed strategy chosen at time t, given a determinate history

Player i's payoff for G^∞ when $\delta < 1$ is $U_i(\sigma) = (1-\delta)\sum_{t=0}^{\infty} \delta^t g_i(\sigma^t(h^t))$

$h^t = \{a^0, a^1, \dots, a^{t-1}\}$ is history known to players at the start of time t

$\mathcal{H}^t = A^{t-1}$ is space of all possible period t histories

STRATEGY

A pure strategy for player i in game G^∞ is an infinite sequence of mappings $\{s_i^t\}_{t=0}^{\infty}$

Player i's pure strategy for the stage game at time t maps all possible period t histories to all possible actions ("do this if that happened") $s_i^t : \mathcal{H}^t \to A_i$.

In general, player can mix strategies in every possible stage game, but note that mixed strategies cannot depend on past mixed strategies, only on their realisations

A behavioural mixed strategy is an infinite sequence of mappings $\{\sigma_i^t\}_{t=0}^{\infty}$

$\sigma_i^t : \mathcal{H}^t \to \Delta(A_i)$ $\quad \Delta(A_i)$ is the set of possible probability distributions on A_i

THEOREMS

Both easily verified by ISD

1. If there's a unique NE, then the strategy that each player plays the NE independently of history is an SPE of $G^\infty(\delta)$
2. If G has two NE $\{\alpha^{1*}, \alpha^{2*}\}$, then for all functions $j : T \to \{1,2\}$ (e.g. $j(0)=1, j(1)=2, j(2)=2, \dots$)
   time periods
   The strategy "each player plays $\alpha_i^{j(t),*}$ in period t" is SPE of G^∞
   i.e. doesn't matter what NE as long as it's an NE

NASH THREAT FOLK THEOREM

SET OF FEASIBLE PAYOFFS

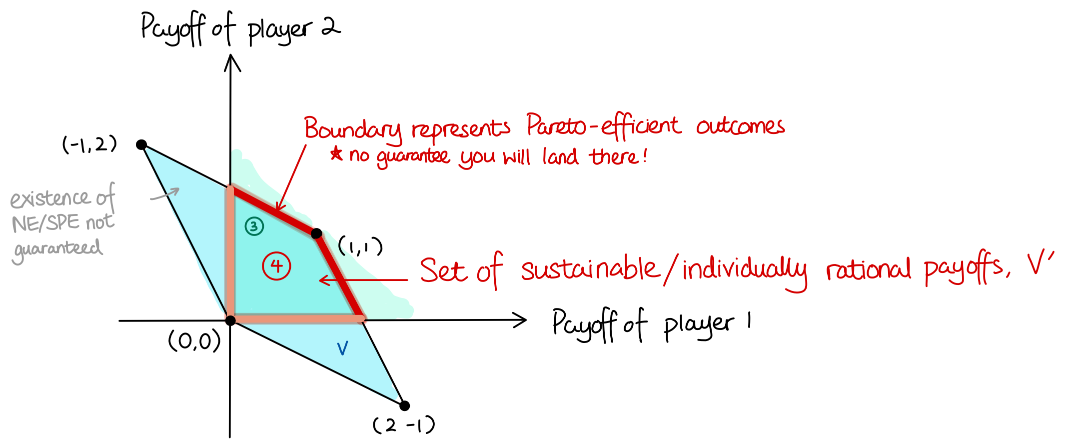
Includes every payoff achievable by pure strategies and every weighted sum of those payoffs

NASH THREAT FOLK THEOREM

Let α^* be a NE of the stage game G with payoff vector $e^* = \{e_1^*, e_2^*, \ldots,\}$ (each player's payoff; players)

Then for all feasible $\pi \in V$ (set of feasible vectors) s.t. $\pi_i \geq e_i^* \; \forall i \in N$, (Pareto improvement for all)

$\exists\, \delta_\pi$ (least amt of patience needed) s.t. $\forall\, \delta_\pi < \delta < 1$, there is an SPE of $G^\infty(\delta)$ with payoff π



1. Draw feasible set V (plot pure strategy outcomes and connect them)
2. Pick any NE you know (in particular, the NE with lowest payoff for all players)
3. Highlight all payoffs that would be Pareto improvements over the NE
4. Find intersection. Any payoffs in that set can be achieved in an SPE if players are patient enough

Gives a "lower bound" on payoffs achievable in some equilibrium

PROOF Grim trigger strategy.

Play to achieve an outcome with payoff weakly better than a given NE but switch to play the NE ("threat point") once someone deviates. Punishment is subgame perfect. Will always find some δ s.t. payoff lies in intersection

MINMAX

The worst payoffs a rational player can get is the minmax payoff: minmax payoff to player 1 is the lowest payoff that player 2 can impose on player 1 when player 1 is best replying (rational!) i.e. choosing a point on player 1's BR curve with lowest payoffs possible

- In general, the minmax payoff of a player is lower when focusing on minmax rather than general mixed strategies

1\2	B	S
B	1,2	0,0
S	0,0	2,1

Under pure strategies, the minmax payoff for both players is 1

q is 2's probability of playing B

Under mixed strategies, player 2 $\underset{q}{\min}\left(\underbrace{\max\{q, 2(1-q)\}}_{\text{player 1 plays to maximise payoffs}}\right)$

player 2 mins 1's BR payoffs; player 1's payoff from player 2's mixed strat

$\Rightarrow q^* = \frac{2}{3} \Rightarrow$ minmax payoff for player 2 is $\frac{2}{3}$

everyone else picks the strategy that screws i over

In general, the payoff player i gets when he is minmaxed, $\underline{\pi}_i = \underset{\sigma_{-i}}{\min}\left(\underbrace{\underset{\sigma_i}{\max}\, g_i(\sigma_i, \sigma_{-i})}_{\text{i is best responding}}\right)$

INDIVIDUALLY RATIONAL A payoff π_i for player i is individually rational iff $\pi_i \geq \underline{\pi}_i$ (minmax payoff)

Set of individually rational payoffs is $I = \{(\pi_i, \pi_{-i}) \mid \pi_i > \underline{\pi}_i\}$ (for each player)

Set of feasible and individually rational payoffs is $\mathcal{V} = I \cap V$

SUBGAME PERFECT FOLK THEOREM Consider a stage game s.t. $\dim(\mathcal{V}) = \underbrace{|N|}_{\text{no. of players}}$

$\forall v \in \mathcal{V}, \exists \underline{\delta}_v$ s.t. $\forall \delta > \underline{\delta}_v$, there exists a SPE of $G^\infty(\delta)$ with payoff v

"lower bound" on patience

- Supports a wider range of feasible payoffs with SPE since minmax payoff doesn't have to be NE
- Proof: grim trigger, but punishment only for a few periods

DESIGN A STRATEGY

Designing a strategy for a finite repeated game that makes a certain outcome SPE

1. Last stage must be an NE(s)
2. Play ideal outcome in stage 1
3. Subsequent stage strategy conditional on history (has punishment)
4. After specifying strategy, verify SPE by verifying no deviation at each stage
5. If your "ideal stage outcome" is itself an NE, just play it in every stage regardless of history

CARTELS: REPEATED GAMES

USUAL SETUP

Two firms, $c(q_i) = cq_i$ (constant MC),
Inverse demand (linear) $P(q_1+q_2) = a-(q_1+q_2)$, $c<a$

SINGLE-STAGE SOLUTIONS

Compete $q_1^c = q_2^c = \frac{a-c}{3}$ $\pi_1^c = \pi_2^c = \frac{(a-c)^2}{9}$

Monopoly $Q^m = \frac{a-c}{2}$, $\Pi^m = \frac{(a-c)^2}{4}$

Cartel $q_1^m = q_2^m = \frac{a-c}{4}$, $\pi_1^m = \pi_2^m = \frac{(a-c)^2}{8}$

But firm 2 can deviate from $q_2^m = \frac{a-c}{4}$ to $\bar{q}_2 = \frac{3(a-c)}{8}$ and get $\bar{\pi}_2 = \frac{9(a-c)^2}{64}$

INFINITELY REPEATED

Average discounted profits of each firm $(1-\delta)\sum_{t=0}^{\infty} \delta^t \pi_i(t)$

Cartel (q_1^m, q_2^m) can be SPE!

Prove by grim trigger strategy:
Firm i chooses q_i^m, but always chooses q_i^c if any other firm deviates

Firm i doesn't want to deviate iff $\pi_i^m \geq (1-\delta)\bar{\pi}_i + \delta \pi_i^c \Rightarrow \delta \geq \frac{9}{17}$
And no firm want to deviate from punishment since (q_1^c, q_2^c) is stage game NE

INCOMPLETE INFORMATION GAMES

STRATEGIC FORM

N Players in the game

A_i Player i's action set

$\mathbb{T}_i$ Player i's set of possible types ← Player i's private info – e.g. i's cards
often has same strategies but different payoffs (thus utilities)

μ Distribution over possible types ← Beliefs $\mu_i(t_{-i}) = P(T_{-i} = t_{-i})$

$u_i : A \times T \to \mathbb{R}$ Player i's utility function, $u_i(a|t)$ ← utility of calling when my cards are better than yours

INFORMATION TYPES

Type is a random variable T_i which realisations $t_i \in \mathbb{T}_i$.

- $T_{-i} = (T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n)$ is profile of types for all players other than i

  Player i observes only t_i; doesn't know T_{-i} but knows μ_i (has beliefs over t_{-i})

- Assume types are independent $\mu(t) = \prod_{j \in N} \mu_j(t_j)$ $T_i \perp\!\!\!\perp T_{-i}$ knowing one doesn't give info on another

  $\mu(t_1, t_2) = \mu_1(t_1) \times \mu_2(t_2)$

STRATEGY

$s_i : \mathbb{T}_i \to A_i$ maps available info to actions

Strategy profile is a strategy for every player $s(T) = (s_1(T_1), \dots, s_N(T_N))$

$s_{-i}(T_{-i}) = (s_1(T_1), \dots, s_{i-1}(T_{i-1}), s_{i+1}(T_{i+1}), \dots, s_N(T_N))$

EX POST DOMINANCE

Strategy s_i ex post weakly dominates s_i' if $\forall a_{-i}\ \forall t \in \mathbb{T}$ (no matter what everyone else does; no matter what type of game happens)

$u_i(s_i(t_i), a_{-i}|t) \geq u_i(s_i'(t_i), a_{-i}|t)$ with $>$ for some a_{-i}

- s_i is ex post dominant if it ex post weakly dominates any other s_i'
- cannot do better than play s_i if learned everything about game
- e.g. second price sealed auction
- Strong and rare
- Ex post DSE: s s.t. $\forall i \in N\ \forall t \in \mathbb{T}\ \forall a_{-i} \in A_{-i}$

$$u_i(s_i(t_i), a_{-i}|t) \geq u_i(s_i'(t_i), a_{-i}|t) \qquad \forall s_i'$$

INTERIM EXPECTED UTILITY

Interim stage occurs just after a player knows his type $T_i = t_i$, when strategies are chosen (in a Bayesian game)

Interim expected utility of a (pure) strategy profile s is defined by

$$u_i(s|t_i) = \sum_{\mathbb{T}_{-i}} u_i(s(t)|t)\mu(t_{-i}|t_i) : \mathbb{T}_i \to \mathbb{R}$$

Strategy is a function of type
I may call if I have a good hand but fold otherwise

$u(s_1(t_1), s_2(t_2)|t_1, t_2)$

Probability of them having certain cards given my own hand (Updating my beliefs)

★ $s_i(t_i)$ not in equation
If I have a flush, my utility is unaffected by what I'll do if I were to have a pair

Interim expected utility from player i's POV:

$$u_i(a_i, s_{-i}|t_i) = \sum_{\mathbb{T}_{-i}} u_i(a_i, s_{-i}(t_{-i})|t)\mu(t_{-i}|t_i)$$

BEST REPLY $b_i(s_{-i}|t_i) = \underset{a_i \in A_i}{\arg\max}\, U_i(a_i, s_{-i}|t_i)$

Optimal action given other players' strategies and my own type

BAYES NASH EQUILIBRIUM A pure strategy $s(t)$ is an (interim) BNE if

$$\forall i \in N\ \forall t_i \in T_i \quad s_i(t_i) \in b_i(s_{-i}|t_i)$$
$$\Longleftrightarrow \forall i \in N\ \forall t_i \in T_i \quad U_i(s|t_i) \geq U_i(a_i, s_{-i}|t_i) \quad \forall a_i \in A_i$$

Everyone's best responding given what they know (interim optimality)

$\boxed{\text{Strategy is ex post DSE} \Rightarrow \text{BNE}}$ proof in W5 slides. Converse not true

Corollary: if $\exists$ an ex post strict DSE, they will play it

BAYESIAN STATIC GAME EG

Player 1's type $T_1 = \{A, B\}$
Player 2's type $T_2 = \{C\}$ unimportant

Player 1 has perfect info; knows exactly which game he is playing; can choose a different action in each game

Player 2 has "small uncertainty" over whether she's playing game A or B
- Doesn't discover which game she's playing until after strategy is played
- Must choose same strategy in both games
- She thinks she is playing game A with probability $\frac{9}{10}$; B with probability $\frac{1}{10}$ $\mu(C, A) = 0.9$

Game A

1 \ 2	L	R
U_A	1,1	0,0
D_A	0,0	6,6

Game B

1 \ 2	L	R
U_B	1,0	0,10
D_B	0,0	6,6

☆ Analyse both game matrices at the same time

Strategy of player 1 $s_1: \{A, B\} \rightarrow \{U, D\}$
Strategy of player 2 $s_2: s_2 \in \{L, R\}$ cannot act upon 1's private info. Has to pick a constant action

$$b_1(s_2|A) = \begin{cases} U_A & \text{if } s_2 = L \\ D_A & \text{if } s_2 = R \end{cases} \qquad b_1(s_2|B) = \begin{cases} U_B & \text{if } s_2 = L \\ D_B & \text{if } s_2 = R \end{cases}$$

$$b_2(s_1) = \begin{cases} L & \text{if } s_1 = (U_A, D_B) \because \mu(C,A) = 0.9 \\ R & \text{otherwise} \end{cases}$$

To find b_2, player 2 chooses a_2 to maximise $U_2(a_2, s_1|t_2)$
$= \mu(A|C) u_2(a_2, s_1(A)|A, C) + \mu(B|C) u_2(a_2, s_1(B)|B, C)$

e.g. if player 2 expects player 1 to play U_A, U_B, her two possible expected payoffs are

$$\mu(A|C) u_2(L; U_A, U_B|A, C) + \mu(B|C) u_2(L; U_A, U_B|B, C) = 0.9(1) + 0.1(0) = 0.9$$
$$\mu(A|C) u_2(R; U_A, U_B|A, C) + \mu(B|C) u_2(R; U_A, U_B|B, C) = 0.9(0) + 0.1(10) = 1$$
$$\Rightarrow b_2((U_A, U_B)|C) = R$$

The unique (pure strategy) BNE is $s_1(A) = D_A$, $s_1(B) = D_B$, $s_2 = R$

EX ANTE BAYES NASH EQUILIBRIUM

We can also find the ex ante BNE by rewriting in normal form

	L	R
U_A, U_B	$1, \mu$	$0, 10(1-\mu)$
U_A, D_B	μ, μ	$6(1-\mu), 6(1-\mu)$
D_A, U_B	$1-\mu, 0$	$6\mu, 6\mu+10(1-\mu)$
D_A, D_B	$0, 0$	$6, 6$

"Weakest" but easiest to analyse

$\boxed{\text{Ex post BNE} \Rightarrow \text{Interim BNE} \Rightarrow \text{Ex ante BNE}}$

Ex ante BNE $\Rightarrow$ Interim BNE if $\mu(t) > 0 \quad \forall t$
each type occurs with $Pr > 0$

INCOMPLETE INFORMATION GAMES (MIXED STRATEGY)

MIXED STRATEGY $\sigma(T) = (\sigma_1(T_1), \dots, \sigma_N(T_N))$ maps information T to probability dist over actions a_i

INTERIM EXPECTED UTILITY

Interim expected utility of a (pure) strategy profile s is defined by

$$U_i(\sigma|t_i) = \sum_{T_{-i}}\left(\sum_{a\in A} u_i(a|t)\prod_{j\in N}\sigma_j(a_j|t_j)\right)\mu(t_{-i}|t_i) : T_i \to \mathbb{R}$$

Interim expected utility from player i's POV:

$$U_i(a_i, \sigma_{-i}|t_i) = \sum_{T_{-i}}\sum_{a_{-i}\in A_{-i}} u_i(a|t)\prod_{i\neq j}\sigma_j(a_j|t_j)\mu(t_{-i}|t_i)$$

BAYES NASH EQUILIBRIUM A mixed strategy σ is an (interim) BNE if

$$\forall i \in N \, \forall t_i \in T_i \qquad \sigma_i(t_i) \in b_i(\sigma_{-i}|t_i)$$

$$\Leftrightarrow \forall i \in N \, \forall t_i \in T_i \qquad U_i(\sigma|t_i) \geq U_i(a_i, \sigma_{-i}|t_i) \quad \forall a_i \in A_i$$

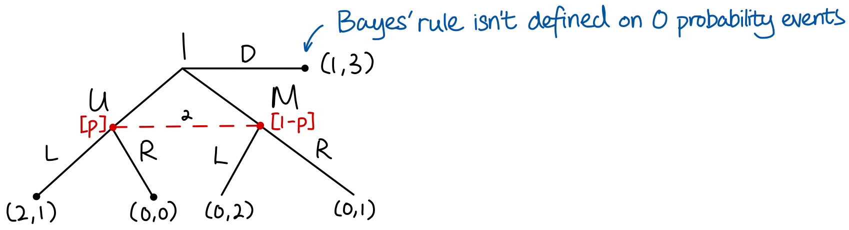
Everyone's best responding given what they know (interim optimality)

COMPUTE BNE

$$\sigma \text{ is BNE} \Leftrightarrow \begin{cases} U_i(\sigma|t_i) = U_i(a_i, \sigma_{-i}|t_i) & \forall a_i \text{ s.t. } \sigma_i(a_i|t_i) > 0 \\ U_i(\sigma|t_i) \geq U_i(a_i, \sigma_{-i}|t_i) & \forall a_i \text{ s.t. } \sigma_i(a_i|t_i) = 0 \end{cases}$$

- This is the indifference condition (e.g. set $U_2(\sigma_1, X|L) = U_2(\sigma_1, Y|L)$ to find σ_1)
- Strictly dominated strategies are never chosen in a BNE
- Weakly dominated strategies are chosen only if the scenario in which they are dominated never happens ($Pr = 0$) in the BNE

INCOMPLETE INFORMATION GAMES (DYNAMIC)



Player 2 uncertain if player 1 played U or M; forms beliefs about which node he's at

- Subgames cannot split up information sets, so game only has one subgame (can't do SPE!!)

PERFECT BAYESIAN COMPETITION

A profile of strategies and of beliefs at each info set of the game such that

1. Each player's strategy maximises EU at every node given the beliefs and strategies of others
2. Beliefs are updated using Bayes rule at each info set reached with positive probability

$S_1 = \{U, M, R\}$
$S_2 = \{L, R\}$
$\forall p \, \forall a_1, \; U_2(a_1, L) > U_2(a_1, R)$
Thus, $[(D,R); p=q]$ is not PBE for any q, even though it is NE (incredible threat)

$[(U,L); p=1]$ is the unique PBE

COURNOT DUOPOLY (ASYMMETRIC INFORMATION)

COST TYPE

Firm 1's cost: $c(q_1) = cq_1$

Firm 2's cost is either high $C_H(q_2) = C_H q_2$ or low $C_L(q_2) = C_L q_2$ $\quad C_L < C < C_H$

INVERSE DEMAND

$p(q_1+q_2) = a-(q_1+q_2)$

STRATEGY

Firm 2 knows its cost type

- will choose a different quantity (q_2^H, q_2^L) depending

Firm 1 doesn't know but believes that there's equal probability

- will choose a unique quantity

(EXPECTED) UTILITY

Firm 1's expected profit: $\frac{1}{2}q_1(a-(q_1+q_2^H)-c)+\frac{1}{2}q_1(a-(q_1+q_2^L)-c)$

Firm 2's profit: $q_2^H(a-(q_1+q_2^H)-C_H)$ or $q_2^L(a-(q_1+q_2^L)-C_L)$

BEST REPLY

the only difference!

works due to cost/dd linearity

$$q_1 = \frac{1}{2}\left(\frac{a-q_2^H-c}{2}+\frac{a-q_2^H-c}{2}\right) = \frac{a-E(q_2)-c}{2} \qquad E(q_2) = \frac{q_2^H+q_2^L}{2}$$

$$q_2^H = \frac{a-q_1-C_H}{2} \quad \text{or} \quad q_2^L = \frac{a-q_1-C_L}{2}$$

INTERIM BNE

★ unique

Intersections of BRs. Solve with algebra:

$$q_1 = \frac{a-2c+\frac{C_H+C_L}{2}}{3}$$

$$q_2^H = \frac{a-2C_H+c}{3}+\frac{C_H-C_L}{12}$$

$$q_2^L = \frac{a-2C_L+c}{3}-\frac{C_H-C_L}{12}$$

$$\boxed{q_2^H < q_2^L}$$

if $C_H = C_L = C$ then back to usual stage game

COURNOT DUOPOLY (PERFECT INFORMATION)

Firm 1 knows if firm 2's cost is C_H or C_L, will choose 2 different quantities depending on scenario

UTILITY

Firm 1's profit: $\hat{q}_1^H(a-(\hat{q}_1^H+\hat{q}_2^H)-c)$ or $\hat{q}_1^L(a-(\hat{q}_1^L+\hat{q}_2^L)-c)$

Firm 2's profit: $\hat{q}_2^H(a-(\hat{q}_1^H+\hat{q}_2^H)-c_H)$ or $q_2^L(a-(\hat{q}_1^L+\hat{q}_2^L)-c_L)$

Same as before but q depends on L/H

BEST REPLY

$$\hat{q}_1^H = \frac{a-\hat{q}_2^H-c}{2} \qquad \hat{q}_1^L = \frac{a-\hat{q}_2^L-c}{2}$$

$$\hat{q}_2^H = \frac{a-\hat{q}_1^H-c_H}{2} \quad \text{or} \quad \hat{q}_2^L = \frac{a-\hat{q}_1^L-c_L}{2}$$

INTERIM BNE

Intersections of BRs. Solve with algebra:

$$\hat{q}_1^H = \frac{a-2c+c_H}{3} \qquad \hat{q}_2^H = \frac{a-2c_H+c}{3}$$

and

$$\hat{q}_1^L = \frac{a-2c+c_L}{3} \qquad \hat{q}_2^L = \frac{a-2c_L+c}{3}$$

COMPARISON

In the presence of perfect vs asymmetric info:

$$\boxed{\begin{gathered}\hat{q}_1^L < q_1 < \hat{q}_1^H \\ \hat{q}_2^H < q_2^H < q_2^L < \hat{q}_2^L\end{gathered}}$$

Feedback effect: Firm 1 knows firm 2 isn't producing much (C_H), so it $\uparrow q_1$
This $\downarrow p$ and causes firm 2 to produce even less

MECHANISM DESIGN

Goal : implement an outcome that one of the actors (designer/principals) like by designing a game played by other actors (agents)

Problem : Asymmetric info (agents have more)
Hidden actions : Designer may not be able to observe/verify all choices agents have

Key principles : Agents are free not to interact with designer (contract must benefit agents)
Agents only take actions or give info if they benefit from it

GIBBARD - SATTERTHWAITE THEOREM

Related to Arrow's Impossibility Theorem

The only non-manipulable rule that maps society's preference relations to outcomes is dictatorship. i.e. always making the principal the first mover in a sequential game

- Games must be designed with particular context and goal in mind
- We must work with special classes of preference

AUCTIONS

FIRST PRICE SEALED BID Each player i submits secret bid $b_i > 0$. Highest bidder j wins, pay b_j

$$\text{Payoff} = \begin{cases} v_i - b_i & \text{if } i \text{ wins } (b_i > b_j) \\ 0 & \text{if } i \text{ loses } (b_i < b_j) \end{cases}$$

- Bidding $b_i \geq v_i$ is weakly dominated by $b_i = \frac{v_i}{2}$
- But any $b_i < v_i$ is not dominated by any other $b_i' < v_i$
  - Any $b_i' > b_i$ does worse when all opponents bid $< b_i$
  - Any $b_i' < b_i$ does worse when all opponents bid $\in (b_i', b_i)$
  - So the bottom line is bid less than v_i

EQUILIBRIUM v_1, v_2 independently distributed with CDF F, PDF f

Expected payoff of 1 with strategy $\beta(v_1)$,

player 2's bid; 1 and 2 have same strategy! Implies you must be very confident about opponents' strategy

$$\begin{aligned}\Pi_1 = (\underline{b}, \underline{v}; \beta) &= Pr(1 \text{ wins} \mid b_1, \beta)(v_1 - b_1) \\ &= Pr(v_2 \text{ s.t. } \beta(v_2) < b_1)(v_1 - b_1) \\ &= Pr(v_2 \leq \beta^{-1}(b_1))(v_1 - b_1) \\ &= F(\beta^{-1}(b_1))(v_1 - b_1)\end{aligned}$$

$$\text{FOC}: \underbrace{\frac{f(v_1)}{\beta'(v_1)}}_{\Delta Pr(\text{win})} \times \underbrace{(v_1 - \beta(v_1))}_{\text{Payoff from winning}} + \underbrace{F(v_1)}_{Pr(\text{win})} \times \underbrace{(-1)}_{\Delta \text{payment if win}} = 0$$

Tradeoff: $\uparrow$ bid
- $\Rightarrow \uparrow Pr(\text{win}) \Rightarrow \uparrow$ benefit when win
- $\Rightarrow \uparrow$ payment when win

In NE, best to bid $b = \beta(v_1) = \dfrac{\int_0^{v_1} f(v)\,dv}{F(v_1)} = E(v_2 \mid v_2 \leq v_1)$

Solve FOCs, take ODEs (not tested)

DUTCH AUCTION Price starts high and is continuously lowered; first bidder who accepts quote wins

Player's strategy is q_i, the first quote accepted

$$\text{Payoff} = \begin{cases} v_i - q_i & \text{if } i \text{ wins } (q_i > q_j) \\ 0 & \text{if } i \text{ loses } (q_i < q_j) \end{cases}$$

- Payoff of i given q_1, q_2 is the same as payoff of i when $b_1 = q_1, b_2 = q_2$ in a 1st price sealed bid
- Dutch and 1st Price essentially the same — strategic equivalence

SECOND PRICE SEALED BID

Winner pays second highest bid (Vickrey auction)

DOMINANT STRATEGY

Highest bid among opponents: $\max_{1,\dots,n} b_i$

$b_i = v_i$ weakly dominates $b_i' < v_i$

If $\max_{1,\dots,n} b_i > v_i$	both strategies get 0
If $b_i' < \max_{1,\dots,n} b_i < v_i$	b_i yields $v_i - b_i$, b_i' yields 0
If $\max_{1,\dots,n} b_i < v_i$	both strategies get $v_i - \max_{1,\dots,n} b_i$

$b_i = v_i$ weakly dominates $b_i' > v_i$

If $\max_{1,\dots,n} b_i > b_i'$	both strategies get 0
If $v_i < \max_{1,\dots,n} b_i < b_i'$	b_i yields 0, b_i' yields $v_i - b_i' < 0$
If $\max_{1,\dots,n} b_i < v_i$	both strategies get $v_i - \max_{1,\dots,n} b_i$

DSE: $b_i = v_i \; \forall i$. Bidder with highest value wins

EFFICIENT

All bidders truthfully reveal their value
Bidder with highest value will win
Auction allocates prize efficiently ... with some assumptions:

EXTERNALITY

Bidder i gets negative payoff if j wins
- e.g. auctioning off nuclear weapons

Then that bidder has no dominant strategy
- May even bid $> v_i$ if thinks j will win

COMMON VALUE

Rather than having "private" valuations, bidders have unique private info about value of item but no bidder knows exact worth
- e.g. auctioning off an oil well to oil companies
- When you win, you infer that others had info that value was low
- Winner's curse (overpay)
- Fear of winner's curse can make bidders too cautious – mkt breakdown!

ENGLISH AUCTION

Price continuously raised till last person standing wins
- Should drop out at $p = v_i$
- Essentially the same as the second price auction
- Widely used

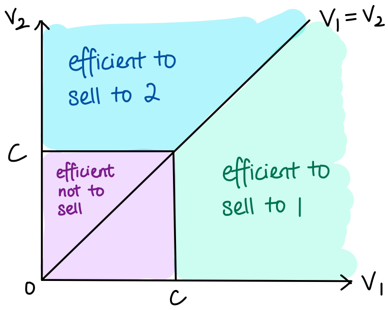
PROFIT-MAXIMISATION PROBLEM

Seller incurs a cost c in selling and doesn't know bidders' values $v_1, \dots, v_n$
Wants to design a mechanism that
gives sellers an incentive to tell the seller their values AND maximises profit
- has no direct reason to care about efficient allocation

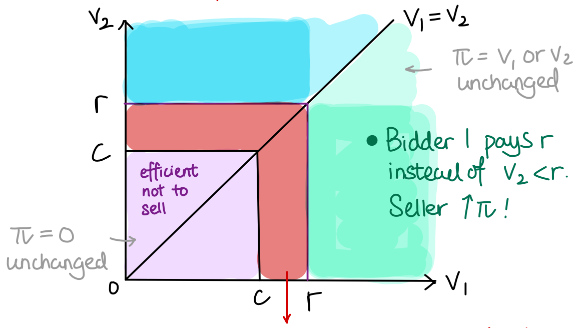
ENGLISH AUCTION

Seller chooses reserve price, r
Winner pays second highest bid or r, whichever higher
Losers pay nothing

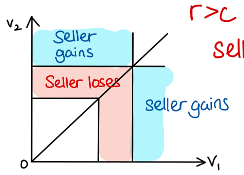
For efficiency, simply set $r = c$



If $r > c$



$r > c$ costs seller a good sale.
seller $\downarrow \pi$



☆ Bidders always worse off with reserve price

PROFIT-MAX r

Seller chooses r without knowing v, but attaches probabilities to the v's

$$\max_r \left[Pr(\max\{v_1, v_2\} > r)\, r + E_v \max\{0, \min\{v_1, v_2\} - r\} \right]$$

alt form: $\max_r \int_r^{\infty} \left[\int_0^x \max\{r, y\} f(y)\,dy \right] f(x)\,dx$ not tested

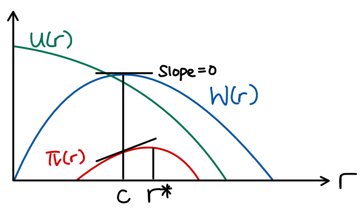
PROFIT-MAX r: SIMPLER APPROACH

Total welfare (seller's profit + buyers' utility) is $W(r)$
- Maximised at $r = c$

Buyers' welfare, $U(r)$, is strictly decreasing in r

Seller's profits, $\pi(r) = W(r) - U(r)$

$\pi'(r) = W'(r) - U'(r)$
$\pi'(c) = -U'(c) > 0$
Slope of $\pi(r)$ is always increasing at $r = c$



Seller always sets $r^* > c \implies$ inefficient!

ADVERSE SELECTION

Uniform price: $P(q) = pq$
Two-part tariff: $P(q) = T + pq$
Multipart tariff: $P(q) = T + p_1 q$
$T+$

CONSUMER

Utility: $U(q, P \mid \theta_i) = \begin{cases} \theta_i u(q) - P(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}$ (price: $P(q)$)

$u(q)$ is such that $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u(0) = 0$
θ_i is buyer's type, which affects marginal valuation of q

There are two types of consumers:
High valuation: $\theta_i = \theta_H$ $\quad 0 < \theta_L < \theta_H$
Low valuation: $\theta_i = \theta_L$

MONOPOLIST

Economy has two commodities q, m, q supplied only by monopolist
Cost function $c(q) = cq$ CRS!
Profit: $\Pi = P(q^*) - cq^*$

Aims to maximise profit but often hurts efficiency (max profit + CS)
bc it's profitable to restrict q_L or even exclude θ_L consumers (to protect IC_H)

PERFECT PRICE DISCRIMINATION

If monopolist is perfectly informed of each consumer's θ_i,
solve $\max\limits_{P_i, q_i} P_i - cq_i$ s.t. $\theta_i u(q_i) - P_i \geq 0$ (IR) Pick best P&q to me s.t. consumer still willing to buy

Substitute $P_i = \theta_i u(q_i)$ in and take FOC: $c = \theta_i u'(q^*_i)$ $\quad \forall i \in \{C, H\}$

solve for q_i^* then put in

Solution is take-it-or-leave-it offer:
$(0,0)$ or $(P_i(q_i^*), q_i^*)$, $i \in \{L, H\}$ s.t. $P_i^* = \theta_i u(q_i^*)$
which is where marginal cost = marginal revenue $\frac{dP_i^*(q_i^*)}{dq} = c$, this is efficient
yet consumer surplus has been completely extracted:
$P_i^*(q_i^*) - cq_i^* = \theta_i u(q_i^*) - cq_i^*$, $\forall i \in \{L, H\}$.
That's why antitrust looks at CS not efficiency

Can also make it a two-part tariff: $P_i^*(q_i^*) = \underset{\text{fixed fee}}{T_i^*} + \underset{\text{slope of pricing schedule}}{p_i^*} q_i^*$, $T_i^* = \theta_i u(q_i^*) - cq_i^*$ $\forall i \in \{L, H\}$

- $p_i^* = \frac{\partial P}{\partial q_i}$ $\quad \forall i \in \{C, H\}$
- $T_i^* = U_i(q_i^*) - cq_i^*$
  - Equivalent to CS under perfect comp $P_i^*(q_i^*) - cq_i^* = \theta_i u(q_i^*) - cq_i^*$
  - In perf comp, $p = mc$, no fixed fee as rivals will undercut (cannot extract CS)

ASYMMETRIC INFORMATION

Monopolist cannot observe buyer's type/valuation — must offer the same contract/menu of offers to all
$M = (P_H, q_H; P_L, q_L)$ No other price/quantity should be feasible

☆ This is a screening problem (with one informed, one uninformed party) and uninformed party moves first to make offer. Can also solve with BI. If the informed moves first, it's "signalling"

Monopolist maximises the expected profits associated with menu M

$$\max_{P_L, q_L, P_H, q_H} \Pi = \lambda(P_L - cq_L) + (1-\lambda)(P_H - cq_H), \quad \lambda = \Pr(\theta_i = \theta_L)$$

Subject to below constraints:

INDIVIDUAL RATIONALITY/PARTICIPATION CONSTRAINT (IR)

Sometimes the guaranteed "outside option" doesn't give just 0 payoff tho

Given $P(q)$, consumer will buy $q > 0$ (want to play the game) iff $U(q, P|\theta_i) \geq U(\text{not participating}) = 0$

$$\Leftrightarrow \begin{array}{l} \theta_H u(q_H) - P(q_H) \geq 0 \\ \theta_L u(q_L) - P(q_L) \geq 0 \end{array}$$

may not be in this form

Utility of participant at least as high as they would have been if didn't participate

INCENTIVE COMPATIBILITY CONSTRAINT (IC)

only needed if mechanism designer can't know the types

For both types of consumers to pick the right item that the monopolist picked for them (so agent of type θ_H doesn't do what θ_L does, vice versa)

$$\theta_H u(q_H) - P_H \geq \theta_H u(q_L) - P_L \quad (\text{for consumer } \theta_H)$$
$$\theta_L u(q_L) - P_L \geq \theta_L u(q_H) - P_H \quad (\text{for consumer } \theta_L)$$

REVELATION PRINCIPLE

Any social choice function that can be implemented by any mechanism (arbitrary/non-truthful/indirect) can be implemented by a truthful, direct mechanism (in which each agent reveals their true type)
IC and IR are all you need to solve for contracts in which agents of type θ get outcome $x(\theta)$

- No need to prove something $\forall$ mechanisms ... just $\forall$ truthful mechanisms will do

DIRECT MECHANISM

A mechanism that makes agents just state their types outright to the designer/auctioneer
No need to signal

SPENCE-MIRRLEES SINGLE CROSSING CONDITION

$$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u/\partial p}\right] = u'(q) > 0$$

payment to seller

Marginal utility of consumption (relative to marginal utility of money) rises with agent's type θ

- Thus, usually IC of only one type of agent binds
- Can usually omit IC_L: θ_L agents don't want to mimic higher-type agents

MONOPOLIST'S RELAXED PROBLEM

IR_H doesn't bind — can just drop it

$$\underbrace{\theta_H u(q_H) - P_H \geq \theta_H u(q_L) - P_L}_{IC_H} \underbrace{>}_{\theta_H > \theta_L} \overbrace{\theta_L u(q_L) - P(q_L)}^{IR_L} \geq 0 \Rightarrow \theta_H u(q_H) - P_H > 0$$

IC_L doesn't bind — can just drop it

At perfect price discrimination solution, $\theta_L u(q^*_L) - P^*_L(q^*_L) = 0$ & $\theta_L u(q^*_H) - P^*_H(q^*_H) = (\theta_L - \theta_H)u(q^*_H) < 0$

$$\Rightarrow \theta_L u(q^*_L) - P^*_L(q^*_L) > \theta_L u(q^*_H) - P^*_H(q^*_H)$$

The problem now becomes $\max_{P_L, q_L, P_H, q_H} \Pi = \lambda(P_L - cq_L) + (1-\lambda)(P_H - cq_H)$, $\lambda = Pr(\theta_i = \theta_L)$

s.t. $\theta_H u(q_H) - P_H \geq \theta_H u(q_L) - P_L$ IC_H

$\theta_L u(q_L) - P(q_L) \geq 0$ IR_L

If IC_H / IR_L doesn't bind (i.e. always holds), can always add $\varepsilon > 0$ to P_H / P_L and $\uparrow E(\Pi)$ without affecting the other constraint, so both constraints must stay. Substitute constraints in to get unconstrained maximisation problem

$$\max_{q_L, q_H} \Pi = \lambda(\theta_L u(q_L) - cq_L) + (1-\lambda)(\theta_H u(q_H) - (\theta_H - \theta_L)u(q_L) - cq_H)$$

To see if solution is optimal or corner, check derivative

$$\frac{\partial \Pi}{\partial q_L} = [\lambda\theta_L - (1-\lambda)(\theta_H - \theta_L)]u'(q_L) - \lambda c$$

i.e. Number of low-types rel. to high-types

$\boxed{\text{CASE I}}$ $\lambda\theta_L - (1-\lambda)(\theta_H - \theta_L) \leq 0 \Rightarrow \frac{\partial \Pi}{\partial q_L} < 0 \quad \forall q_L \geq 0$

Monopolist chooses q_L at a corner: $q^{**}_L = 0$, $P^{**}_L = 0$. Specialises in high-end market, serves only θ_H: $q^{**}_H = q^*_H$, $P^{**}_H = P^*_H(q^*_H)$. Inefficient but more profitable

from perfect info/dis. case

Also check that IC_L omitted is satisfied: $0 = \theta_L u(q^{**}_L) - P^{**}_L > \theta_L u(q^{**}_H) - P^{**}_H < 0$

$\boxed{\text{CASE I}}$ $\lambda\theta_L - (1-\lambda)(\theta_H - \theta_L) > 0 \Rightarrow \exists q_L > 0$ s.t. $\frac{\partial \Pi}{\partial q_L} = 0$

Monopolist serves both types of consumers with optimal menu $\mathcal{M}^{**} = (P^{**}_H, q^{**}_H, P^{**}_L, q^{**}_L)$

PROPERTIES

- Efficient at the top: $\theta_H u'(q^{**}_H) = \theta_H u'(q^*_H) = c$
- Inefficient at the bottom: $q^{**}_L < q^*_L$, $\theta_L u'(q^{**}_L) = \frac{\lambda\theta_L}{\lambda\theta_L - (1-\lambda)(\theta_H - \theta_L)} c > c$ MC was distorted due to IC_H
- Doesn't extract all CS from θ_H consumer: $P^{**}_H < P^*_H(q^*_H)$ $P^{**}_H = \theta_H u(q^{**}_H) - (\theta_H - \theta_L)u(q^{**}_L)$ "info rent" – giving θ_L higher utility would give θ_H bigger discount
- Extracts all CS from θ_L consumer: $P^{**}_L < P^*_L(q^*_L)$, $P^{**}_L = \theta_L u(q^{**}_L)$
  at the cost of leaving the θ_H-type some CS/giving θ_H-type a discount

Also check that IC_L omitted is satisfied: $\overbrace{P^*_H - P^*_L = \theta_H [u(q^{**}_H) - u(q^{**}_L)]}^{IC_H} > \theta_L [u(q^{**}_H) - u(q^{**}_L)]$

$$\Rightarrow \theta_L u(q^{**}_L) - P^*_L > \theta_H u(q^{**}_H) - P^*_H$$

Can be implemented as two-part tariff / discount scheme (W7 slides)

INSURANCE-SYMMETRIC INFO

Individual of type t has accident with probability σ_t.

Insurance market is competitive with free entry, sells insurance that pays $z_t = 0$ or K (full insurance or none) in accident at unit price p_t so total premium is $p_t z_t$

No accident: Income $= Y$ consumption $= y_t = Y - p_t z_t$

Accident: Income $= Y - K$ consumption $= x_t = Y - K - p_t z_t + z_t = Y - K + (1-p_t) z_t$

Expected utility: $\sigma_t(x_t) + (1-\sigma_t)u(y_t)$

Risk averse (u increasing and concave): $u' > 0,\ u'' < 0$

$\boxed{\text{BOTH DON'T KNOW}}$ Both insurer and driver don't observe σ_t; believe it's uniformly distributed in $[0,1]$

Consumer's problem: $\max\limits_{z \in \{0,K\}} \int_0^1 [\sigma_t u(x) + (1-\sigma_t)u(y)]\, d\sigma_t \Leftrightarrow \max\limits_{z \in \{0,K\}} \bar{\sigma}_t u(x) + (1-\bar{\sigma}_t)u(y)$

Risk neutrality and perfect competition imply $p = \bar{\sigma}$ ← MC

By Jensen's inequality, all consumers buy insurance

$\boxed{\text{BOTH KNOW}}$ Both insurer and driver observe σ_t

Consumer's problem: (replace expectations with actual) $\max\limits_{z_t \in \{0,K\}} \sigma_t u(x_t) + (1-\sigma_t)u(y_t) = \sigma_t(Y - K + (1-p_t)z_t) + (1-\sigma_t)u(Y - p_t z_t)$

By Jensen's inequality $E(u(x)) \leq u E(x)$ when u concave

Risk neutrality and perfect competition imply $p = \bar{\sigma}$, $p_t = AC_t = MC_t$

Consumer will buy insurance $z_t = K$ whenever $\underbrace{u(Y - pK)}_{\text{insured}} \geq \underbrace{\sigma_t u(Y-K) + (1-\sigma_t)u(Y)}_{\text{expected } u \text{ from not insured}}$

Willingness to pay is $p(\sigma_t)K$, where $p(\sigma_t)$ is price at which a consumer of type t is indifferent between buying or not, i.e. solution to $u(Y - pK) = \sigma_t u(Y-K) + (1-\sigma_t)u(Y)$

If $\sigma_t = p$ then $u(Y - pK) = u(Y - \sigma_t K) = u(E(\text{consumption})) \overset{\text{Jensen's inequality}}{\geq} E(u(\text{consumption}))$

so consumer will take the offer which implies

$\boxed{\text{ONLY DRIVER KNOWS}}$ Only driver knows σ_t, and insurer must offer same price to all drivers

Type t purchases whenever $p \leq \overset{\text{increases in } \sigma_t}{p(\sigma_t)}$ so higher risk types buy first

If $u = -e^{-x}$ (CARA) then $u(Y - pK) = \sigma_t u(Y-K) + (1-\sigma_t)u(Y) \Leftrightarrow -e^{-Y} e^{p(\sigma_t)K} = -(\sigma_t e^{-Y} e^{K} + (1-\sigma_t)e^{-Y})$

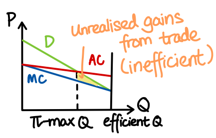
Divide by $-e^{-Y}$ and take log, we have $p(\sigma_t) = \log\left(\frac{\sigma_t(e^K - 1) + 1}{K}\right) \approx \frac{\sigma_t(e^K - 1)}{K} \equiv (K^*)^{-1}\sigma_t > \sigma_t$

first order approx around $\sigma_t = 0$ <1

Thus, buy iff $\overset{\approx p(\sigma_t)}{(K^*)^{-1}\sigma_t} > p \Leftrightarrow \sigma_t > pK^*$

If $\sigma_t \sim U[0,1]$, mkt dd is $q(p) = 1 - pK^*$ or $p(q) = \frac{1-q}{K^*}$

$AC(q) = \underbrace{E}_{\text{Expected accident pr}}(\sigma_t \,|\, \underbrace{\sigma_t > 1-q}_{\text{among the ppl whose accident prob > than the ppl I sell to}}) = \frac{1}{q}\int_{1-q}^{1} t\,dt = \frac{1}{q}\left(\frac{1}{2}t^2\right)\Big|_{1-q}^{1} = 1 - \frac{q}{2} \Rightarrow AC(p) = \frac{1}{2} + \frac{1}{2}K^* p$



For there to be pooling perfectly competitive equilibrium, $p = AC$

SIGNALLING

Informed player moves first to signal private info;
uninformed then observes and acts accordingly

WORKER EDUCATION

★ ignores human capital theory that ↑edu ⇒ ↑pdy. Edu is a pure signal

① Nature distributes pdy: e.g. π of workers have H pdy

② Worker of type θ_i decides to get a level $e \in [0,1]$ of education at cost $c(e|\theta_i)$
- Cost is strictly increasing: $c(0|\theta_i)=0$, $c_e(e|\theta_i)>0$
- MC of education is higher for θ_L than θ_H: $c_e(e|\theta_H)<c_e(e|\theta_L)$

③ Employer at first know only prior probability π_i that worker is of type θ_i but now updates beliefs based on observed e and makes wage offer $w(e)$
using a function, explained below
- Betrand competition
- Technology is s.t. pdy of worker is $f(\theta_i)$; $f(\theta_H)>f(\theta_L)$
- If employer knew worker's type, $w_i=f(\theta_i)$

④ Worker accepts the offer that pays the most (if equal wages offered, chooses randomly)

SEPARATING EQUILIBRIA

- There is a multiplicity of separating perfect Bayesian equilibria with varying e/w characterised by $e_H \in [\underline{e}, \bar{e}]$, $e_L=0$, and workers receive the efficient wage $w_i=f(\theta_i)$
- Pareto dominant equilibrium is $e_H=\underline{e}$, minimises cost of acquiring edu
- Multiplicity/existence of other PBE arises from our freedom to specify arbitrary off-the-eqm-path (paths of 0 probability) beliefs
- But no eqm is efficient since θ_H workers waste resources to signal their type by investing in edu

BACKWARD INDUCTION

(3b) Employer makes offer to max payoff
$$\Pi(w_h(e_i)) = \begin{cases} f(\theta_i)-w_h(e_i) & \text{if } w_h(e_i)\geq w_{-h}(e_i) \\ \frac{1}{2}[f(\theta_i)-w_h(e_i)] & \text{if } w_h(e_i)=w_{-h}(e_i) \\ 0 & \text{if } w_h(e_i)<w_{-h}(e_i) \end{cases}$$

Best reply is $\begin{cases} w_h(e_i)=w_{-h}(e_i)+\varepsilon & \text{if } w_{-h}(e_i)<f(\theta_i) \\ w_h(e_i)\leq w_{-h}(e_i) & \text{if } w_{-h}(e_i)=f(\theta_i) \\ w_h(e_i)<w_{-h}(e_i) & \text{if } w_{-h}(e_i)>f(\theta_i) \end{cases}$

The unique Bayesian NE of the Bertrand Competition subgame is
$w_h(e_i)=w_{-h}(e_i)=f(\theta_i)$, $\forall i \in \{H,L\}$

(3a) Employer updates updates beliefs upon observing e
must specify for all levels of e — chosen & not chosen by workers in eqm

Assuming two types of workers separate in previous stage (θ_H chooses e_H, θ_L chooses e_L), Bayes rule implies that only equilibrium posterior beliefs compatible with a separating eqm are s.t. $\Pr\{\theta=\theta_i|e_i\}=1$

Can impose $\Pr\{\theta=\theta_H|e\}=\begin{cases} 1 & \text{if } e\geq e_H \\ 0 & \text{if } e<e_H \end{cases}$ to ensure all workers pick e_H or e_L

or even $\Pr(\theta=\theta_H|e)=\begin{cases} 1 & \text{if } e=e_H \\ 0 & \text{if } e\neq e_H \end{cases}$ i.e. θ_H-types all get MScs. If you get a PhD you're stupid

- By Bayes rule $\Pr(A\cap B)=\Pr(A|B)\Pr(B)$, so since $\Pr(e\notin\{e_L,e_H\})=0$ in eqm, the off-the-eqm path $\Pr(\theta=\theta_H|e\notin\{e_L,e_H\})$ can be anything

② $IC_H: \; w(e_H) - c(e_H|\theta_H) \geq w(e_L) - c(e_L|\theta_H) \Rightarrow f(\theta_H) - c(e_H|\theta_H) \geq f(\theta_L) - c(e_L|\theta_H)$ given the eqm

$IC_L: \; w(e_L) - c(e_L|\theta_L) \geq w(e_H) - c(e_H|\theta_L) \quad\;\; f(\theta_L) - c(e_L|\theta_L) \geq f(\theta_H) - c(e_H|\theta_L)$ wage offer $w(e_i)$

Since edu is a pure signal, the only Bayesian eqm has $e_L^* = 0$. PROOF

Let $e_L^* > 0$. IC_H imply $e_H > e_L$.
θ_L worker has incentive to deviate with $e_L < e_L^*$ as it gives a higher payoff
$f(\theta_L) > f(\theta_L) - c(e_L|\theta_L)$
"the worst type I can be identified as is θ_L anyway"

$$\Rightarrow \begin{matrix} f(\theta_H) - c(e_H|\theta_H) \geq f(\theta_L) \\ f(\theta_L) \geq f(\theta_H) - c(e_H|\theta_L) \end{matrix} \Rightarrow \boxed{c(e_H|\theta_L) \geq f(\theta_H) - f(\theta_L) \geq c(e_H|\theta_H)}$$

"would I like to ↑ my wage by $w_H - w_L$ by ↑e?"
H-type says yes, but L-type finds it too costly

Thus, all Bayesian NE are s.t. $e_H = [\underline{e}, \bar{e}]$, $e_L = 0$ $\quad c(\underline{e}|\theta_L) = f(\theta_H) - f(\theta_L) = c(\bar{e}|\theta_H)$

With beliefs $\Pr(\theta = \theta_H | e) = \begin{cases} 1 & \text{if } e = e_H \\ 0 & \text{if } e = e_L \end{cases}$ as well as beliefs for other levels of e (see previous page) that will not be chosen in eqm (beliefs never actly applied)

POOLING EQUILIBRIA

All workers choose the same level of education e^*

$$w(e^*) = \pi_H f(\theta_H) + (1 - \pi_H) f(\theta_L)$$

BACKWARD INDUCTION

③ Employers can't use wages to learn pdy of workers, so offer same wage = expected pdy to all workers

$$w(e^*) = \begin{cases} \pi_H f(\theta_H) + (1 - \pi_H) f(\theta_L) & \text{if } e = e^* \\ f(\theta_L) & \text{if } e \neq e^* \end{cases}$$ ← won't happen at eqm

Must have beliefs $\Pr(\theta = \theta_H | e_H) = \begin{cases} \pi_H & \text{if } e = e^* \\ 0 & \text{if } e \neq e^* \end{cases}$

② L-type workers mustn't choose to reveal themselves at $w(e^*)$. For that to happen

$$w(e^*) - c(e^*|\theta_L) \geq f(\theta_L) - c(e|\theta_L) \quad \forall e \neq e^*$$

H-type workers mustn't choose to reveal themselves at $w(e^*)$. For that to happen

$$w(e^*) - c(e^*|\theta_H) \geq f(\theta_L) - c(e|\theta_H) \quad \forall e \neq e^*$$

* usually either IC-H/L won't bind

get by equalising ICs above and find lowest maximum

Conclusion: multiplicity of pooling perfect Bayesian equilibria: $e^* \in [0, \hat{e}]$
due to off-the-eqm-path beliefs

- Misallocation since $w \neq$ pdy: θ_H workers paid too little, θ_L paid too much (unfair)
- Pareto dominant pooling eqm: $e^* = 0$. Efficient (kinda) Not wasting resources to signal
- Inefficient whenever $e^* > 0$ (or $f(\theta_L) < 0$)

MORAL HAZARD

Principal wants agent to perform some task; Principal cannot observe agent's effort
Can only motivate by rewarding based on outcome

GAME

1. Principal chooses wage schedule with IC and IR as a function of outcome
2. Agent picks effort
3. Effort and chance determine the outcome
4. Payments are made according to wage schedule

SETUP

The outcome of the task $q \in \{0,1\}$
- Success: $q=1$
- Failure: $q=0$

Agent chooses effort $e \in [0,1]$

Probability of success $P(q=1|e)=e$ more effort, higher prob of success

Cost of effort $\emptyset(e)=\frac{e^2}{2}$

Wage $w(q)$ depends on outcome

Principal is risk neutral and maximises $E(q-w)$

Agent's preferences $U(w)-\emptyset(e)$,
- $U'(\cdot)>0$, $U''(\cdot)\leq 0$ increasing, concave
- Separable in wage and effort
- Utility if he resigns is 0 (outside option)

FIRST BEST CONTRACT

If Principal can observe agent's effort

agent's IR – binds at optimal contract

$$\boxed{\max_{e,w_i} \; e(1-w_1)-(1-e)w_0 \quad \text{s.t.} \quad eU(w_1)+(1-e)U(w_0)-\frac{e^2}{2}\geq 0}$$

no need for IC: just set $w_0=w_1=-\infty$ when $e\neq e^*$

Take $\mathcal{L}$, FOCs: $\frac{U'(w_1^*)}{w_1^*}=\frac{U'(w_0^*)}{w_0^*} \Rightarrow w_1^*=w_0^*=w^*$

- Agent receives full insurance s.t. w^* is independent of output (as agent is risk-averse)
- Agent is indifferent btwn accepting/rejecting as IR binds

$$eU(w_1^*)+(1-e)U(w_0^*)-\frac{e^2}{2} = U(w^*)-\frac{(e^*)^2}{2}=0 \Rightarrow \boxed{e^*=U'(w^*)}$$

If agent is risk neutral: $U(x)=x$, $U'(x)=1$, first best contract is $e^*=1$

★ Always solve for wage schedules for if principal wants different levels of e

SECOND BEST CONTRACT
If Principal cannot observe agent's effort

The Agency Problem: principal has to incentivise agent to exert the effort that maximises principal's expected profits but still maximises agent's expected utility exclusively via wages

$e(w_1,w_0) \in \underset{e\in[0,1]}{\arg\max}\ eU(w_1)+(1-e)U(w_0)-\frac{e^2}{2}$ maximises agent's expected utility

FOC: $e(w_1,w_0) = U(w_1)-U(w_0)$ becomes agent's IC

Second best contract is the solution to

$w_1^* = w_0^*$ no longer works with IC since agent will exert no effort

$$\boxed{\max_{w_0,w_1} e(w_1,w_0)(1-w_1)-(1-e(w_1,w_0))w_0 \quad \text{s.t.} \begin{cases} e(w_1,w_0) = U(w_1)-U(w_0) & \text{agent's IC} \\ e(w_1,w_0)U(w_1)+(1-e(w_1,w_0))U(w_0) \geq \frac{(e(w_1,w_0))^2}{2} \end{cases}}$$ agent's IR

If agent is risk neutral: $U(x)=x$, IC becomes $e(w_1,w_0)=w_1-w_0$

For $e=1$, set $w_1-w_0=1$ and sub into IR $\Rightarrow w_0=-\frac{1}{2}$, $w_0=\frac{1}{2}$
Principal "sells" activity to agent at $-w_0$ and agent keeps proceeds

Usually we assume limited liability constraint: $w_0 \geq 0$, $w_1 \geq 0$ (will be binding for w_0)
Can still get $e=1$ with $w_1=0$, but then expected profits $=0$ (not optimal).

Sub $w_0=0$, $U(x)=x$ into IC to get $e(w_1,0)=w_1$
Sub $e(w_1,0)=w_1$, $w_0=0$, $U(x)=x$ into objective fn and take FOC:

$w_0^*=0$, $w_1^*=\frac{1}{2} \Rightarrow e^*=\frac{1}{2}$, Principal's $\pi^{**}=\frac{1}{4}$, Agent's expected utility $=\frac{1}{8}$
Principal trades off the lower effort against higher compensation that Agent needs to exert effort

If Agent is risk-averse, full insurance gives no incentives to work
Taking $\mathcal{L}$, (λ-IR, μ-IC)

$$\frac{1}{U'(w_1)} = \lambda+\mu\frac{1}{e(w_1,w_0)}$$
$$\frac{1}{U'(w_0)} = \lambda-\mu\frac{1}{1-e(w_1,w_0)}$$

If $\mu=0$, $U'(w_1)=U'(w_0) \Rightarrow w_1=w_0$ (full insurance, no incentives) so $\mu>0 \Rightarrow \lambda>0$
So we distort optimal (full) insurance by setting $w_1^{**}>w_0^{**}$
to create incentives $e^{**}=U(w_1^{**})-U(w_0^{**})>0$ ($\lambda>0$)
and extract all surplus $e^{**}U(w_1^{**})+(1-e^{**})U(w_0^{**})=\frac{(e^{**})^2}{2}$

- insures the agent just enough to minimise principal's payments to agent
- trades off need to provide incentives to agent with profits from insuring agent

FREE RIDER

A good is rivalrous if consumption by an agent reduces possibilities of consumption by others
A good is excludable if you have to pay to consume the good

	Rivalrous	Non-rivalrous
Excludable	Private good	Public good e.g. patented research
Non-excludable	e.g. free parking	Pure public good e.g. national defence

COORDINATION FAILURE

An expresso machine (pure public good) costs c pounds

$n=2, \; v_1 = v_2 = v$

$Pr(\text{a player buys machine}) = p_i = c$

Payoff to i if i buys machine : $v - c$

Pr(at least one other player buys) or $[1-(1-p_i)^n]$

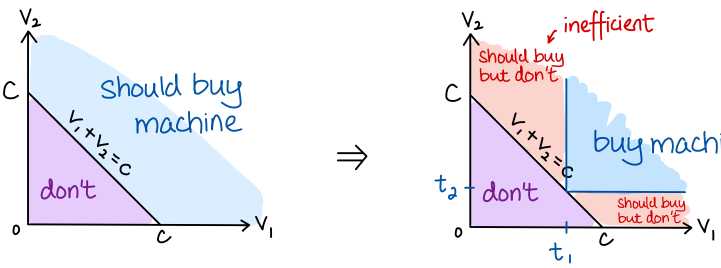
Payoff to i if i doesn't buy machine : $p_{-i} v$

$$b_i(p_{-i}) = \begin{cases} \overset{\text{buy}}{\{c\}} & \text{if } v-c > p_{-i}v \\ \{c, 0\} & \text{if } v-c = p_{-i}v \\ \underset{\text{don't buy}}{\{0\}} & \text{if } v-c < p_{-i}v \end{cases}$$

Mixed strategy equilibrium requires indifference $\Rightarrow p_i = p_{-i} = \frac{v-c}{v}$ $\Rightarrow$ Coordination failure! Sometimes both buy/don't

SPLIT THE COST

Agree on payments t_1, t_2 s.t. $t_1 + t_2 = c$



FREE RIDER PROBLEM

Economic agents that consume more than their fair share of a public resource
bear less than a fair share of the costs of its production

May cause the good to be underproduced (Pareto inefficient) / overused

Two-good economy with n consumers, private good x and pure public good z

- Endowment: $(\omega_x, \omega_z) = (X, 0)$
- Public good must be produced using good x: $z = f(x)$ (f is monotonic, f^{-1} exists)

The Pareto optimal allocation x, z, x_i, z_i is the solution to the problem

$$\max_{x,z,x_i,z_i} \sum_{i=1}^{n} \underset{\text{Pareto weight}}{\alpha_i} \overset{\text{Utility of consumer } i}{U^i(x_i, z_i)} \quad \text{s.t.} \begin{cases} z = f(x) \\ \sum_{i=1}^{n} x_i \leq X - x & \text{Private good feasibility constraint} \\ z_i \leq z \quad \forall i \in \{1, \dots, n\} & \text{Public good feasibility constraint (due to nonrivalry)} \end{cases}$$

Binding as U is monotonic

reduces to just $\sum_{i=1}^{n} x_i = X - f^{-1}(z)$

PARETO OPTIMALITY CONDITION

Competitive eqm

FOCs: $\begin{cases} \alpha_i U^i_x(x_i, z) = \lambda \\ \sum_{i=1}^{n} \alpha_i U^i_z(x_i, z) = \lambda f^{(-1)\prime}(z) \end{cases} \Rightarrow$ Social MRS = MRT (MC) $\boxed{\sum_{i=1}^{n} \frac{U^i_z(x_i, z)}{U^i_x(x_i, z)} = \frac{1}{f'(x)}}$

PRIVATE CONTRIBUTION NE

Each individual i donates d_i to produce z .

Best reply for i given d_{-i} is $\max_{x_i, d_i} U^i(x_i, z)$ s.t. $\begin{cases} z \le f(\sum_{i=1}^n d_i) & \text{pdn fn of z} \\ x_i + d_i \le X_i & \text{i's budget constraint} \end{cases}$ } binding

$$\Rightarrow \max_{d_i} U^i\left(X_i - d_i, f(\textstyle\sum_{i=1}^n d_i)\right)$$

$$\overset{FOC}{\Rightarrow} \boxed{\frac{U_z^i(x_i,z)}{U_x^i(x_i,z)} = \frac{1}{f'(\sum_{i=1}^n d_i)}}$$

Underproduction!! Only care about private benefit from investment

Private MRS = MRT

LINDAHL EQUILIBRIUM

Assume personal prices p_i can be established for i

Producer sells z units to each individual i at p_i to max profits $\max_z z\sum_{i=1}^n p_i - f^{-1}(z)$ and produces till $\sum_{i=1}^n p_i = \underbrace{f^{-1}(z)}_{MC}$

Consumer will $\max_{x_i, z_i} U^i(x_i, z_i)$ s.t. $x_i + p_i z_i = X_i \overset{FOC}{\Rightarrow} \frac{U_z^i(x_i,z)}{U_x^i(x_i,z)} = p_i \Rightarrow \sum_{i=1}^n \frac{U_z^i(x_i,z)}{U_x^i(x_i,z)} = \frac{1}{f'(x)}$

Pareto optimal!

☆ Difficult to implement if consumer's valuation/dd for z are private info

PERSONALISED TAXATION

Set $t_i(z_i) = p_i z_i$ tax rate = Lindahl prices

Budget constraint becomes $x_i + t_i(z_i) = X_i \Rightarrow \sum_{i=1}^n \frac{U_z^i(x_i,z)}{U_x^i(x_i,z)} = \frac{1}{f'(x)}$

Individuals have incentive to underestimate their demand since it $\uparrow\uparrow x_i$ and not $\downarrow z$ by much

Usually economists argue for public provision, but Coase bargaing can give private solutions

MORAL HAZARD IN TEAMS

Risk-neutral agents choose e_i simultaneously and independently. Cost of effort $\phi(e_i) = \frac{e_i^2}{2}$
Output is strictly increasing and concave: $x(e_1+e_2)$, $x'(\cdot)>0, x''(\cdot)<0$

If effort is observable, specify $e_i^* = \arg\max_{e_i} x(e_1+e_2) - \phi(e_1) - \phi(e_2)$

cost of exerting effort

Full info contract: $(e_1^*, e_2^*, s_1^*, s_2^*)$

PARTNERSHIP

Agent i takes $s_i \in [0,1]$ of total o/p, $\sum_{i=1}^{n} s_i = 1$

Best reply, given e_{-i} is $\hat{e}_i = \arg\max_{e_i} s_i x(e_i + e_{-i}) - \phi(e_i)$

$\sum \hat{e}_i < \sum e_i^*$, unless $\exists s_i = 1$ (sole proprietorship)

- Each agent has incentive to free ride as he only receives returns from his investment in proportion to s_i, not full returns

BUDGET BREAKER

A third party who takes all output away if it is too low

Partnership share of i is $S_i(x) = \begin{cases} s_i^* & \text{if } x \geq x(e_1^*+e_2^*) \\ 0 & \text{if } x < x(e_1^*+e_2^*) \end{cases}$. s_i^* s.t. $s_i^* x(e_1^*+e_2^*) > \phi(e_i^*)$

$$\text{Agent's payoff} = \begin{cases} s_i^* x(e_i^*+e_{-i}^*) - \phi(e_i) > 0 & \text{if } e_i \geq e_i^* \\ -\phi(e_i^*) \leq 0 & \text{if } e_i < e_i^* \end{cases}$$

← Best reply!

Makes the first best effort (e_1^*, e_2^*) an NE

$$^{\star}\text{Budget breaker's payoff} = \begin{cases} 0 & \text{if } x \geq x(e_i^*+e_{-i}^*) \\ x & \text{if } x < x(e_i^*+e_{-i}^*) \end{cases}$$

← no bribery allowed!

However, multiple NEs exist: if partner slacks and chooses $\hat{e}_{-i} < e_{-i}^*$

Then if $\exists \bar{e}_i = e_i^* + e_{-i}^* - \hat{e}_{-i}$ s.t. $s_i^* x(e_i^*+e_{-i}^*) - \phi(\bar{e}_i) \geq 0$

i's BR is $\bar{e}_i$. i feels pivotal. ← small change in action causes large, discontinuous change in outcome

Asymmetric NE $(\bar{e}_i, \hat{e}_{-i})$

I'll make up for it by working more!

---

if $\forall \bar{e}_i = e_i^* + e_{-i}^* - \hat{e}_{-i}$, $s_i^* x(e_i^*+e_{-i}^*) - \phi(\bar{e}_i) < 0$

i's BR is $\hat{e}_i = 0$

The NE $(0,0)$ always exists

No matter how much I work, we can't make the cutoff... I give up

- Team contribution is enhanced in situations where there is a large prize only for the winning team

- Empirically, people game prices but less than theorised
- Contribute less if the game is repeated ("realises they can freeride")
- Contribute more if allowed to communicate

EFFICIENT MECHANISM

GENERAL FRAMEWORK

Any social choice problem :

usually "outcomes"

- n individuals
- set of alternatives A from which to choose e.g. give good to X, Y or Z in auction
- value to i from alternative $x \in A$ being chosen. $v_i(x)$
- monetary transfer scheme $t = (t_1, t_2, \dots, t_n)$

Individual i's contribution to the rest of society :

often negative

total welfare of others

$$\sum_{j \neq i} v_j(x^*) - \sum_{j \neq i} v_j(x^*_{-i})$$

the alternative that max total utility
How others are doing when I'm in

the alternative that max total utility if i didn't exist in it
How everyone is doing without me

VICKREY-CLARKE-GROVES MECHANISM

Each individual makes a claim about his valuation function (type)

- Can lie by announcing $\hat{v}_i \neq v_i$

  Must construct transfers that give everyone incentive to report truthfully

The utilitarian alternative that max sum of the announced valuation is (social utility)

$$x^*(\hat{v}) \text{ s.t. } \sum_{j=1}^{n} \hat{v}_j(x^*(\hat{v})) \geq \sum_{j=1}^{n} \hat{v}_j(x) \ \forall x \in A$$

The utilitarian alternative that max sum of the announced valuation when i does not value good is $x^*_{-i}(\hat{v}_{-i})$ s.t. $\sum_{j \neq i}^{n} \hat{v}_j(x^*_{-i}(\hat{v}_{-i})) \geq \sum_{j \neq i}^{n} \hat{v}_j(x) \ \forall x \in A$

VCG Transfer rule : $\boxed{t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*_{-i}(\hat{v}_{-i}))}$ value that i adds to society

$t_i > 0$ VCG mechanism pays i i is pivotal

$t_i < 0$ i pays VCG mechanism . i is pivotal ← usually this, due to resource scarcity

$t_i = 0$ i doesn't affect outcome i is non-pivotal ← usually this, if i doesn't get to participate in the first place

PROOF : VCG EFFICIENT

To prove : announcing true value is a dominant strategy $\forall i$ (incentive compatible)
When they do, the utilitarian alt $x^*_i(v_i, v_{-i})$ is automatically picked

If i has the worst type $\underline{v}_i$ to society (e.g. lowest value buyer/highest cost seller)
then utilitarian alternative to society $x^{**}_{-i}(v_{-i})$ "ignores" i (i contributes least)

$$\sum_{j \neq i}^{n} \hat{v}_j(x^*_{-i}(\hat{v}_{-i})) + \underline{v}_i(x^{**}_{-i}(v_{-i})) \geq \sum_{j \neq i}^{n} \hat{v}_j(x) + \underline{v}_i(x)$$

Then $t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - V(\underline{v}_i, \hat{v}_{-i})$, where $V(\underline{v}_i, \hat{v}_{-i}) = \hat{v}_j(x^*_{-i}(\hat{v}_{-i})) + \underline{v}_i(x^{**}_{-i}(v_{-i}))$

Taking $\hat{v}_{-i}$ as given, i $\max_{\hat{v}_i} v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + t_i(\hat{v}_i, \hat{v}_{-i}) = v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i})) - V(\underline{v}_i, \hat{v}_{-i})$
$= v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i}))$

unaffected by $\hat{v}_i$, can ignore like a "sunk cost"

By definition, best alternative is $x^*(v_i, \hat{v}_{-i})$, achieved by truthfully announcing $\hat{v}_i = v_i$

↯ □

APPLICATION

When allocating a prize with private values, VCG is a second price auction

$$t_i(\hat{v}) = \begin{cases} \sum_{j\neq i} \hat{v}_j(\underbrace{x^*(\hat{v})}_{\text{"give it to i"}}) - \sum_{j\neq i} \hat{v}_j(\underbrace{x^*_{-i}(\hat{v}_{-i})}_{\text{"give it to some guy k w highest value"}}) = 0 - v_k & \text{if i reports highest valuation} \\ \sum_{j\neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j\neq i} \hat{v}_j(x^*_{-i}(\hat{v}_{-i})) = v_k - v_k = 0 & \text{if i doesn't} \end{cases}$$

"give it to some guy k w highest value"

Auction with externalities :

	X	Y	Z
x	v_x	0	0
y	0	v_y	-5
z	0	0	v_z

y doesn't want z to win good

sum of x & y's welfare

If efficient allocation is Z, i.e. $x^*(\hat{v}) = Z$, z pays $t_z(Z) = -5 - v_{XY}$

sum of x & y's welfare if X/Y won

DEFICIT WITH PUBLIC GOODS

Expresso machine problem: let $\hat{v}_1 + \hat{v}_2 > 50$, $\hat{v}_1, \hat{v}_2 < 50$ Efficient to buy.

Must also consider machine owner (player 3)

By VCG, $t_1 = \hat{v}_2 - 50 < 0$ ← welfare of current owner if don't buy

$t_2 = \hat{v}_1 - 50 < 0$

But $t_1 + t_2 > -50$! VCG causes a deficit. Need to subsidise.

$v_1 + v_2 > 50$

proof not tested

As VCG is the mechanism that maximises revenue that implements an efficient rule

Since even VCG yields a deficit, there does not exist a budget balanced, efficient mechanism

The second best mechanism is split the cost, obtained from solving the public good problem

PUBLIC GOOD PROBLEM

A mechanism is described by two functions

① Decision rule $\alpha(\hat{v})$ linking valuations (info obtained) to the alternatives (good is bought, not bought)

② Transfer rule $t(\hat{v})$ which specifies transfer scheme

Mechanisms must be feasible: $t_1(\hat{v}) + t_2(\hat{v}) + 50 \leq 0 \quad \forall \hat{v}$

incentive compatible: truthtelling as dominant strategy

See W10 slides

No improvement on split the cost mechanism possible