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PROBLEMS OF *applied* ECONOMETRICS

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SETUP · MATCHING

Methodological issues in identification/estimation of policy impact on outcomes

CAUSATION An exogenous change in X changes Y, holding all other (un)observable determinants of Y constant.

ASSUMPTIONS
- Statistical
- Behavioural (Rationality, non-maximising)
- Assumption wrt how people respond to intervention heterogeneously

TREATMENT STATUS $D_i = \begin{cases} 0 & \text{if } i \text{ did not receive treatment} \\ 1 & \text{if } i \text{ received treatment} \end{cases}$

OUTCOMES
Outcome of i without treatment = Y_{i0}
outcome of i with treatment = Y_{i1}
Observed outcome $Y_i = D_i Y_i + (1-D_i) Y_{i0}$

RUBIN-HOLLAND CAUSAL MODEL

TREATMENT EFFECT Effect of treatment on individual i $\Delta_i = Y_{1i} - Y_{0i}$
But you never directly observe both Y_{1i} and Y_{0i} for individual i
i.e. evaluation is a problem of missing data
We need to construct an appropriate counterfactual

AVERAGE TREATMENT EFFECT
$ATE = E(\Delta_i) = E(Y_{1i} - Y_{0i}) = Pr(D_i=1)\, ATET + Pr(D_i=0)\, ATEU$
$ATET = E(Y_{1i} - Y_{0i} | D_i=1)$
$ATEU = E(Y_{1i} - Y_{0i} | D_i=0)$
↳ e.g. estimate the impact of rolling out a programme to a different age group

There are conditional versions, e.g. $ATE(x) = E(\Delta_i | X_i = x)$ (controls)

OTHER EFFECTS
- % of people benefitting from policy $Pr(\Delta_i > 0 | D_i = 1)$
- Distribution of treatment effects $F(\Delta_i | X_i)$
- Selected quantile $\inf \{\Delta_i : F(\Delta_i | D_i = 1) = q\}$

MAIN IDENTIFICATION ISSUES
- Selection into treatment may depend on both Y_{0i} and Y_{1i} (i.e. size of Δ_i) D_i endogenous
- Δ_i heterogeneous

ROY'S MODEL

$Y_{id} - \mu_d$ is absolute advantage in picking d

$$Y_{id} = \mu_d + U_{id}, \quad d = 0, 1$$

mean $E(Y_{id}) / E(Y_{id} | X_i = x)$

SELECTION INTO TREATMENT

i selects into treatment $(D_i = 1)$ if $Y_{i1} > Y_{i0} + C_i$

$Y_{i1} - Y_{i0}$: comparative advantage

C_i: Direct cost of choosing 1 instead of 0. May be heterogeneous, but assume $C_i = 0$ for now

$$\text{ATE} = E(Y_{i1} - Y_{i0}) = E(\mu_1 + U_{i1} - \mu_0 - U_{i0}) = \mu_1 - \mu_0$$

$$\text{ATET} = E(Y_{i1} - Y_{i0} | D_i = 1) = E(\Delta_i | \Delta_i \geq C_i) = \mu_1 - \mu_0 + \boxed{E(U_{i1} - U_{i0} | U_{i1} - U_{i0} \geq \mu_0 - \mu_1 + C_i)} > 0$$

$$\text{ATEU} = E(Y_{i1} - Y_{i0} | D_i = 0) = E(\Delta_i | \Delta_i \leq C_i) = \mu_1 - \mu_0 + \boxed{E(U_{i1} - U_{i0} | U_{i1} - U_{i0} \leq \mu_0 - \mu_1 + C_i)} < 0$$

$$\text{Var}(\Delta_i) = \text{Var}(U_{1i}) + \text{Var}(U_{0i}) - \boxed{2\text{Cov}(U_{0i}, U_{1i})}$$

$\uparrow \text{Cov}(U_{0i}, U_{1i}) \Rightarrow$ ATE/ATET/ATEU diverge less

$\Delta_i = \mu_1 - \mu_0 + U_{i1} - U_{i0}$: individual treatment effects (i-specific)

$\bar{\Delta} = \mu_1 - \mu_0 = \text{ATE}$: mean treatment effects

$v_i = \Delta_i - \bar{\Delta} = U_{i1} - U_{i0}$: deviation of treatment effect around the mean

$v_i = 0$ implies no heterogeneity

$U_{i1} - U_{i0}$: i's comparative advantage, observed by i, unobservable to the analyst

THREE CASES

NO HETEROGENEITY

Δ_i (Coefficient on D_i) is fixed/constant across i given X_i

$U_{1i} = U_{0i} \ \forall i \Rightarrow v_i = 0, \ \Delta_i = \bar{\Delta} \ \forall i$ ATE = ATET

RCT

Δ_i (Coefficient on D_i) is random given X_i

Heterogeneity exists ($v_i \neq 0$) but not acted upon ex ante

Random assignment / i cannot observe $U_{1i} - U_{0i}$. No self selection

$D_i | X_i \perp\!\!\!\perp U_{1i}, U_{0i} \Rightarrow E(U_{1i} - U_{0i} | X_i, D_i = 1) = 0$

SELF SELECTION

Δ_i (Coefficient on D_i) is random given X_i !!

SELECTION EFFECT

causes OLS to no longer consistently estimate ATET

We can write outcome (using the Roy model) as

$$Y_i = D_i Y_{i1} + (1-D_i) Y_{i0} = \ldots = \mu_0 + (\mu_1 - \mu_0 + U_{1i} - U_{0i}) D_i + U_{0i} = \alpha + \Delta_i D_i + \varepsilon_i$$

We know $\hat{\beta}_{OLS} = \frac{\overline{Y_i D_i} - \bar{Y}_i \bar{D}_i}{\overline{D_i^2} - (\bar{D}_i)^2}$ and $\text{plim}(\hat{\beta}_{OLS}) \overset{LLN}{=} \frac{E(Y_i D_i) - E(Y_i)E(D_i)}{E(D_i^2) - [E(D_i)]^2}$

We can show $\text{plim}(\hat{\beta}_{OLS}) = \text{ATET} + E(Y_{i0}|D_i=1) - E(Y_{i0}|D_i=0)$

$= \text{ATE} = \text{ATEU}$ with randomisation or with CIA (conditioning on X)

Selection effect

- Exists even w/o impact heterogeneity ($v_i = U_{1i} - U_{0i} = 0$) if $\text{Cov}(\varepsilon_i, D_i) = \text{Cov}(U_{i0}, D_i) \neq 0$ — some underlying characteristic U_{i0} is related to D_i
- Can occur due to self selection
- Need to balance the observables & unobservables

PROOF

$$\begin{aligned} E(Y_i D_i) &= E(Y_i D_i | D_i=1) \Pr(D_i=1) + E(Y_i D_i | D_i=0) \Pr(D_i=0) \\ &= E(Y_i D_i | D_i=1) \Pr(D_i=1) \\ &= E(Y_i | D_i=1) E(D_i) \end{aligned}$$

$$\begin{aligned} E(Y_i) &= E(Y_i|D_i=1) \Pr(D_i=1) + E(Y_i|D_i=0) \Pr(D_i=0) \\ &= E(Y_i|D_i=1) E(D_i) + E(Y_i|D_i=0)\left[1 - E(D_i)\right] \end{aligned}$$

$$\begin{aligned} \text{plim}\,\hat{\beta}_{OLS} &= \frac{E(Y_i D_i) - E(Y_i)E(D_i)}{E(D_i^2) - [E(D_i)]^2} \\ &= \frac{E(Y_i D_i) - E(Y_i)E(D_i)}{E(D_i) - [E(D_i)]^2} \\ &= \frac{E(Y_i|D_i=1)E(D_i) - \left\{E(Y_i|D_i=1)E(D_i) - E(Y_i|D_i=0)\left[1-E(D_i)\right]\right\}E(D_i)}{\left[1-E(D_i)\right]E(D_i)} \\ &= \frac{E(Y_i|D_i=1) - E(Y_i|D_i=1)E(D_i) - E(Y_i|D_i=0)\left[1-E(D_i)\right]}{1-E(D_i)} \\ &= \frac{E(Y_i|D_i=1)\left[1-E(D_i)\right] - E(Y_i|D_i=0)\left[1-E(D_i)\right]}{1-E(D_i)} \\ &= E(Y_i|D_i=1) - E(Y_i|D_i=0) \\ &= E(Y_{i1}|D_i=1) - E(Y_{i0}|D_i=0) \\ &= E(Y_{i1}|D_i=1) - E(Y_{i0}|D_i=1) + E(Y_{i0}|D_i=1) - E(Y_{i0}|D_i=0) \\ &= \text{ATET} + E(Y_{i0}|D_i=1) - E(Y_{i0}|D_i=0) \end{aligned}$$

Selection effect
Exists even without heterogeneity
e.g. can occur due to Xs

MATCHING

Pair treated individuals ($D_i = 1$) with untreated individuals ($D_i = 1$) who are observably similar (X_i)
With more controls X_i, it makes us more confident about the CIA

CONDITIONAL INDEPENDENCE ASSUMPTION

$$(Y_{i1}, Y_{i0}) \perp\!\!\!\perp D_i | X_i, \text{ i.e. } Pr(D_i | X_i, Y_{i0}, Y_{i1}) = Pr(D_i | X_i)$$

Treatment status independent of the unobservables (that affect outcomes)

Individuals cannot <u>self-select</u> into treatment based on <u>anticipated treatment impact</u>
CIA will make matching as good as random... but have I included enough observables? 🤔
CIA makes ATE = ATET = ATEU at a given X_i value, but the distribution of X_i may differ for the treated and untreated (D_i and X_i may be correlated! Some ppl may be outside common support)
So ATET and ATEU may differ wrt the entire treated and untreated populations

COMMON SUPPORT ASSUMPTION

$$0 < Pr(D_i = 1 | X_i) < 1 \quad \forall X_i$$
$$\nexists x \in X_i \text{ s.t. } D_i = 0 \text{ or } D_i = 1$$

e.g. if every girl is treated, there's no untreated girl out there who can be the control group

In practice, if $P(x) = 0$ or 1 for some x, treated units with this x-value should be dropped.
Estimation only takes place over (and the $\widehat{ATET}$ only relevant for people in) the common support of X.

SUTVA/NO SPILLOVERS

D_i has no effect on the outcome Y_j of individual $j \neq i$.

- No general equilibrium effects / social interactions
- Particularly problematic if treated i spillover to untreated j
  - We can't even tell who's treated!
  - Then, the estimate = direct effect (ATET) − indirect effect
  - Can understate direct effect (if both effects same direction)
  - Can massively understate social benefit (= direct + indirect effect)

Taking expectations over X

Vanilla LIE: $E_X[E(Y|x)] = E(Y)$

ADVANCED LIE: $E_{x|z}[E(Y|x,z)|z] = E(Y|z)$

Collapses along X-dimension, but not Z (which we keep constant)

PROPENSITY SCORE MATCHING

When there are many controls X_i, difficult to match exactly (curse of dimensionality)
Reduce the multidimensional X_i into a unidimensional propensity score:

$$P(x) = P(D_i = 1 \mid X_i = x) \quad \forall x \in X_i$$

Theorem: $CIA \Rightarrow Y_{i0}, Y_{i1} \perp\!\!\!\perp D_i \mid P(X_i)$

- If we assume CIA, we can just condition (match) on $P(X_i)$ instead of X_i

$E(Y_{0i} \mid P(X_j) = P(X_i), D_i = 0) \overset{CIA}{=} E(Y_{0i} \mid P(X_i), D_i = 1)$

outcome of untreated js that share propensity scores with the treated — counterfactual untreated outcome of the treatment group

$$\begin{aligned}
ATET &= E(Y_{1i} \mid D_i = 1) - E(Y_{0i} \mid D_i = 1) \\
&\overset{LIE}{=} E(Y_{1i} \mid D_i = 1) - E\left[E(Y_{0i} \mid X_i, D_i = 1) \mid D_i = 1\right] \\
&\overset{CIA}{=} E(Y_{1i} \mid D_i = 1) - E\left[E(Y_{0j} \mid X_j = X_i, D_j = 1) \mid D_i = 1\right] \\
&\overset{\text{Rosenbaum-Rubin}}{=} E(Y_{1i} \mid D_i = 1) - E\left[E(Y_{0j} \mid P(X_j) = P(X_i), D_i = 0) \mid D_i = 1\right] \\
&= E\left[Y_{1i} - E(Y_j \mid P(X_j) = P(X_i), D_i = 0) \mid D_i = 1\right]
\end{aligned}$$

ESTIMATING ATET

① Estimate propensity score $\hat{p}(x_i) := \hat{Pr}(D_i = 1 \mid X_i = x_i)$ for each person using some model of participation (e.g. latent variable model + probit/logit)

② The estimator for $E(Y_j \mid P(X_j) = P(X_i), D_i = 0)$ is $\hat{m}_0(x_i) = \dfrac{\sum_{j:d_j=0} y_j K(\hat{p}(x_j) - \hat{p}(x_i))}{\sum_{j:d_j=0} K(\hat{p}(x_j) - \hat{p}(x_i))}$

③ $\widehat{ATET} = \dfrac{1}{\sum_{i:d_i=1} d_i} \sum_{i:d_i=1} \left[y_i - \hat{m}_0(x_i)\right]$

$\widehat{ATEU} = \dfrac{1}{\sum_i (1 - d_i)} \sum_{i:d_i=0} \left[\hat{m}_1(x_i) - y_i\right]$ $\quad \hat{m}_1(x_i) = \dfrac{\sum_{j:d_j=1} y_j K(\hat{p}(x_i) = \hat{p}(x_j))}{\sum_{j:d_j=1} K(\hat{p}(x_i) = \hat{p}(x_j))}$

$\widehat{ATE} = \dfrac{1}{N} \sum_{i=1}^{N} \dfrac{y_i(d_i - \hat{p}(x_i))}{\hat{p}(x_i)[1 - \hat{p}(x_i)]}$

weight↑ when predicted participation diverges from actual participation

weight↑ when $\hat{p}(x_i)$ gets away from $\frac{1}{2}$

RCT

With random assignment, $R_i \perp\!\!\!\perp Y_{0i}, Y_{1i}$

FULL COMPLIANCE $D_i=1 \Leftrightarrow R_i=1$, $D_i=0 \Leftrightarrow R_i=0$ + random assignment $\Rightarrow D_i \perp\!\!\!\perp Y_{0i}, Y_{1i}$

$$
\begin{aligned}
ITT &= E(Y_i|R_i=1) - E(Y_i|R_i=0) \\
&= E(Y_{1i}|R_i=1) - E(Y_{0i}|R_i=1) \text{ by random assignment} \\
&= E(Y_{1i}|D_i=1) - E(Y_{0i}|D_i=1) \text{ by full compliance} = ATET \\
&= E(Y_{1i}) - E(Y_{0i}) = ATE = ATEU \dots
\end{aligned}
$$

ONE-SIDED NONCOMPLIANCE (selecting out of treatment)

$R_i=0 \Rightarrow D_i=0$ (contrapositive: $D_i=1 \Rightarrow R_i=1$)

$$
\begin{aligned}
ITT &= E(Y_i|R_i=1) - E(Y_i|R_i=0) \\
&= E(Y_{1i}|R_i=1) - E(Y_{0i}|R_i=1) \\
&= E(Y_i|R_i=1, D_i=1)\Pr(D_i=1|R_i=1) + \cancel{E(Y_i|R_i=1, D_i=0)\Pr(D_i=1|R_i=1)} \\
&\quad - E(Y_{0i}|R_i=1, D_i=1)\Pr(D_i=1|R_i=1) - \cancel{E(Y_i|R_i=1, D_i=0)\Pr(D_i=1|R_i=1)} \\
&= E(Y_{1i} - Y_{0i}|R_i=1, D_i=1)\Pr(D_i=1|R_i=1) \\
&= E(Y_{1i} - Y_{0i}|D_i=1)\Pr(D_i=1|R_i=1) \quad \text{since } D_i=1 \Rightarrow R_i=1 \\
&= ATET \times \text{Compliance rate} \\
&= \text{first stage / denominator of Wald estimator} \\
&\quad \Pr(D_i=1|R_i=1) - \underbrace{\Pr(D_i=1|R_i=0)}_{=0}
\end{aligned}
$$

$E(Y_i|R_i=0) = E(Y_{1i}|R_i=0, D_i=1)\Pr(D_i=1|R_i=0) + E(Y_{0i}|R_i=0, D_i=0)\Pr(D_i=0|R_i=0)$
$= E(Y_{0i}|R_i=0, D_i=0)$ since $R_i=0 \Rightarrow D_i=0$
$= E(Y_{0i}|R_i=0) = E(Y_{0i}|R_i=1)$ by random assignment

Conclusion: Wald $\hat{\beta}_{IV}$ consistently estimates ATET under one-sided noncompliance and random assignment... though under one-sided noncompliance, you typically just estimate ITT. Wald estimator is not very necessary.

GENERAL NONCOMPLIANCE (selecting in and out of treatment)

must assume monotonicity: $D_i(1) \geq D_i(0)\ \forall i$
and exclusion restriction: $(Y_{0i}, Y_{1i}, D_i(Z_i)) \perp\!\!\!\perp Z_i$

$$
\begin{aligned}
ITT &= E(Y_i|R_i=1) - E(Y_i|R_i=0) \\
&= E\left[D_i(R_i)Y_{1i} + (1-D_i(R_i))Y_{0i}|R_i=1\right] - E\left[D_i(R_i)Y_{1i} + (1-D_i(R_i))Y_{0i}|R_i=0\right] \\
&= E\left[D_i(1)Y_{1i} + (1-D_i(1))Y_{0i}\right] - E\left[D_i(0)Y_{1i} + (1-D_i(0))Y_{0i}\right] \text{ by exclusion restriction} \\
&= E\left\{(Y_{1i} - Y_{0i})[D_i(1) - D_i(0)]\right\} \\
&= E\left[(Y_{1i} - Y_{0i}) \times 1 \,|\, D_i(1) - D_i(0) = 1\right]\Pr\left[D_i(1) - D_i(0) = 1\right] \\
&\quad + E\left[(Y_{1i} - Y_{0i}) \times 0 \,|\, D_i(1) - D_i(0) = 0\right]\Pr\left[D_i(1) - D_i(0) = 0\right] \\
&\quad + E\left[(Y_{1i} - Y_{0i}) \times -1 \,|\, D_i(1) - D_i(0) = -1\right]\Pr\left[D_i(1) - D_i(0) = -1\right] \\
&= E\left[Y_{1i} - Y_{0i} \,|\, D_i(1) - D_i(0) = 1\right]\Pr\left[D_i(1) - D_i(0) = 1\right] \text{ by monotonicity} \\
&= LATE \times \%\ Compliers \\
&= \text{first stage / denominator of Wald estimator } \Pr\left[D_i(1)=1\right] - \Pr\left[D_i(0)=1\right]
\end{aligned}
$$

because $\Pr\left[D_i(1) - D_i(0) = 1\right] = \Pr\left[D_i(1)=1 \cap D_i(0)=0\right]$
(total probability theorem) $= 1 - \Pr\left[D_i(1)=0\right] - \Pr\left[D_i(0)=1\right]$
$= \Pr\left[D_i(1)=1\right] - \Pr\left[D_i(0)=1\right]$

Conclusion: Wald $\hat{\beta}_{IV}$ consistently estimates LATE under general noncompliance, monotonicity, exclusion restriction and relevance $\left(\Pr\left[D_i(1)=1\right] - \Pr\left[D_i(0)=1\right] \neq 0\right)$

IV

When self selection as per Roy model exists,

$$Y_i = D_i Y_{i1} + (1-D_i) Y_{i0} = \mu_0 + (\mu_1 - \mu_0 + U_{1i} - U_{0i}) D_i + U_{0i}$$
$$= \mu_0 + \left[\mu_1 - \mu_0 + E(U_{1i} - U_{0i} | D_i = 1)\right] D_i + \left[U_{1i} - U_{0i} - E(U_{1i} - U_{0i} | D_i = 1)\right] D_i + U_{0i}$$
$$= \mu_0 + \text{ATET}\, D_i + \eta_i \quad Cov(D_i, \eta_i) \neq 0 !$$

WALD ESTIMATOR (BINARY) $\hat{\beta}_{IV} = \dfrac{\overline{Y_i Z_i} - \overline{Y}_i \overline{Z}_i}{\overline{D_i Z_i} - \overline{D}_i \overline{Z}_i}$

$$\text{plim}\, \hat{\beta}_{IV} = \frac{Cov(Y_i, Z_i)}{Cov(D_i, Z_i)} = \frac{E(Y_i Z_i) - E(Y_i)E(Z_i)}{E(D_i Z_i) - E(D_i)E(Z_i)}$$

$$= \frac{E(Y_i|Z_i=1)\Pr(Z_i=1) - \left[E(Y_i|Z_i=0)\Pr(Z_i=0) + E(Y_i|Z_i=1)\Pr(Z_i=1)\right]\Pr(Z_i=1)}{E(D_i|Z_i=1)\Pr(Z_i=1) - \left[E(D_i|Z_i=0)\Pr(Z_i=0) + E(D_i|Z_i=1)\Pr(Z_i=1)\right]\Pr(Z_i=1)}$$

$$= \frac{E(Y_i|Z_i=1) - \left\{E(Y_i|Z_i=0)\left[1-\Pr(Z_i=1)\right] + E(Y_i|Z_i=1)\Pr(Z_i=1)\right\}}{E(D_i|Z_i=1) - \left\{E(D_i|Z_i=0)\left[1-\Pr(Z_i=1)\right] + E(D_i|Z_i=1)\Pr(Z_i=1)\right\}}$$

$$= \frac{E(Y_i|Z_i=1) - E(Y_i|Z_i=0)}{E(D_i|Z_i=1) - E(D_i|Z_i=0)} \times \frac{\Pr(Z_i=1)\left[1-\Pr(Z_i=1)\right]}{\Pr(Z_i=1)\left[1-\Pr(Z_i=1)\right]}$$

$$= \frac{E(Y_i|Z_i=1) - E(Y_i|Z_i=0)}{E(D_i|Z_i=1) - E(D_i|Z_i=0)}$$

reduced form (Rachael's notation: ITT_Y) / first stage (Rachael's notation: ITT_D)

Condition for $\hat{\beta}_{IV}$ consistency: $Cov(Z_i, \eta_i) = E(Z_i \eta_i) - E(Z_i)E(\eta_i) = 0$

If $U_{0i} = U_{1i}$, $\eta_i = 0$, condition automatically true. In fact, plim $\hat{\beta}_{IV}$ = ATET = ATE

If $U_{0i} \neq U_{1i}$, we need to make <u>assumptions</u> s.t. $Cov(Z_i, \eta_i) = 0$.

Three possible assumptions: $Z_i \perp\!\!\!\perp \eta_i \Rightarrow E(\eta_i|Z_i) = E(\eta_i) \Rightarrow E(Z_i \eta_i) = 0$

(independence — sufficient (strong); mean independence — sufficient (usual); orthogonality — necessary)

And plim $\hat{\beta}_{IV}$ = ATET or LATE or ATE on compliers and defiers (see below).

MONOTONICITY $\forall z_1, z_2 \in Z_i$ s.t. $z_1 \neq z_2$, $D_i(z_1) \geq D_i(z_2)$ or $D_i(z_1) \leq D_i(z_2)$ $\forall i$

EXCLUSION RESTRICTION $(Y_{0i}, Y_{1i}, D_i(Z_i)) \perp\!\!\!\perp Z_i$

RELEVANCE $E(D_i|Z_i)$ should be a nontrivial function of Z_i
Binary case: $E(D_i|Z_i=1) - E(D_i|Z_i=0) \neq 0$

LATE $LATE = E(Y_{1i} - Y_{0i} \mid D_i(1) - D_i(0) > 0)$ is ATE on compliers

- plim $\hat{\beta}_{IV}$ = LATE under monotonicity and exclusion restriction (previous page)

<u>LATE with continuous Z_i is</u>

$$LATE(z, z') = E\left[Y_{1i} - Y_{0i} \mid P(z') - P(z) = 1\right] = \frac{E(Y_i|Z_i=z) - E(Y_i|Z_i=z')}{E(D_i|Z_i=z) - E(D_i|Z_i=z')} \overset{\text{Binary } D_i}{=} \frac{E\left[Y_i|P(Z_i)=P(z)\right] - E\left[Y_i|P(Z_i)=P(z')\right]}{P(z') - P(z)}$$

(Rosenbaum-Rubin theorem)

<u>EC333 definition</u>

$$LATE(P(z), P(z')) = E\left[Y_{1i} - Y_{0i} \mid D_i(P(z')) - D_i(P(z)) = 1\right] = \frac{E\left[Y_i|P(Z_i)=P(z)\right] - E\left[Y_i|P(Z_i)=P(z')\right]}{P(z') - P(z)}$$

ATE on people who'd participate if $Z_i = z'$ ($U_{Di} \leq P(z')$) but not if $Z_i = z$ ($U_{Di} \geq P(z)$)

LOCAL IV Given continuous IV Z_i, $\Delta_{LIV}(p) = \dfrac{\partial E(Y_i|P(Z_i)=p)}{\partial p}$

MARGINAL TREATMENT EFFECT

Roy model: $D_i = \mathbb{1}[\mu_1(Z_i) - \mu_0(Z_i) + U_{1i} - U_{0i} > C_i] = \mathbb{1}[\mu_D(Z_i) > U_{Di}]$

$\mu_D(Z_i) = \mu_1(Z_i) - \mu_0(Z_i)$ observable characteristics making i more likely to opt in

$U_{Di} = C_i - U_{1i} + U_{0i}$ unobservable characteristics making i more likely to opt out (resistance)

Assume U_{Di} is continuous, and CSA is supported

$$D_i = \mathbb{1}\left[\mu_D(Z_i) > U_{Di}\right] = \mathbb{1}\left[F_{U_{Di}}(\mu_D(Z_i)) > F_{U_{Di}}(U_{Di})\right] = \mathbb{1}\left[Pr(\mu_D(Z_i) \geq U_{Di}) > F_{U_{Di}}(U_{Di})\right]$$

$$= \mathbb{1}\left[Pr(D_i = 1 \mid Z_i) > u_{Di}\right], \quad u_{Di} \sim U[0,1]$$

$$F_{U_{Di}}(U_{Di}) \sim U[0,1] \text{ as } \forall x \in [0,1],\ Pr\left[F_{U_{Di}}(U_{Di}) \leq x\right] = Pr\left[U_{Di} \leq F^{-1}_{U_{Di}}(x)\right] = F_{U_{Di}}\left[F^{-1}_{U_{Di}}(x)\right] = x$$

DEFINITION

$MTE(u) = E(\Delta_i \mid U_{Di} = u)$ is the ATE on people with unobserved resistance $U_{Di} = u$.

$MTE(P(Z_i)) = E(\Delta_i \mid U_{Di} = P(Z_i))$ is the ATE on people who are indifferent towards participation.
their unobserved resistance equals propensity to opt in (based on observables)

CONNECTION TO LATE

$$E[Y_i \mid P(Z_i) = P(z)] = P(z)\,E[Y_{i1} \mid P(Z_i) = P(z), D_i = 1] + [1 - P(z)]\,E[Y_{i1} \mid P(Z_i) = P(z), D_i = 0]$$

$$= P(z)\,\frac{\int_0^{P(z)} E(Y_{i1} \mid U_{Di} = u)\,du}{P(z)} + [1 - P(z)]\,\frac{\int_{P(z)}^1 E(Y_{i0} \mid U_{Di} = u)\,du}{1 - P(z)}$$

Sum of ATE on people whose U_{Di} is below $P(z)$, averaged over the no. of these people

Sum of ATE on people whose U_{Di} is above $P(z)$, averaged over the no. of these people

$$= \int_0^{P(z)} E(Y_{i1} \mid U_{Di} = u)\,du + \int_{P(z)}^1 E(Y_{i0} \mid U_{Di} = u)\,du$$

$$E[Y_i \mid P(Z_i) = P(z)] - E[Y_i \mid P(Z_i) = P(z')] = \int_0^{P(z)} E(Y_{i1} \mid U_{Di} = u)\,du + \int_{P(z)}^1 E(Y_{i0} \mid U_{Di} = u)\,du - \int_0^{P(z')} E(Y_{i1} \mid U_{Di} = u)\,du - \int_{P(z')}^1 E(Y_{i0} \mid U_{Di} = u)\,du$$

$$= \int_{P(z')}^{P(z)} E(Y_{i1} \mid U_{Di} = u)\,du - \int_{P(z)}^{P(z')} E(Y_{i0} \mid U_{Di} = u)\,du$$

$$= \int_{P(z)}^{P(z')} E(Y_{i1} - Y_{i0} \mid U_{Di} = u)\,du$$

$$LATE(P(z), P(z')) = \frac{E[Y_i \mid P(Z_i) = P(z)] - E[Y_i \mid P(Z_i) = P(z')]}{P(z') - P(z)} = \frac{\int_{P(z)}^{P(z')} E(\Delta_i \mid U_{Di} = s)\,ds}{P(z') - P(z)} = \frac{\int_{P(z)}^{P(z')} MTE(s)\,ds}{P(z') - P(z)}$$

$$= E[MTE(U_{Di}) \mid P(z) \leq U_{Di} \leq P(z')]$$

ATE on people who'd participate if $Z_i = z'$ ($U_{Di} \leq P(z')$) but not if $Z_i = z$ ($U_{Di} \geq P(z)$)

$$= \frac{\int_{P(z)}^{P(z')} MTE(u)\,du}{P(z') - P(z)}$$

CONNECTION TO ATE, ATET

Rarely possible as $P(Z_i)$ usually doesn't have full support

$$ATE = \int_0^1 MTE(u)\,du$$

$$ATET = \int_0^1 MTE(u)\,h_{ATET}(u)\,du$$

$$\frac{1 - F_{P(Z_i)}(u)}{\int_0^1 [1 - F_{P(Z_i)}(t)]\,dt} \longrightarrow Pr[P(Z_i) \geq u]$$

weight by probability that someone with $U_{Di} = u$ will be treated

ESTIMATE MTE

$$MTE(P(z)) = \lim_{P(z') \to P(z)} LATE(P(z), P(z')) = \lim_{P(z') \to P(z)} \frac{E[Y_i | P(Z_i) = P(z)] - E[Y_i | P(Z_i) = P(z')]}{P(z') - P(z)}$$

$$= \frac{\partial E[Y_i | P(Z_i) = P(z)]}{\partial P(z)} = \Delta^{LIV}(P(z))$$

① Estimate $P(Z_i)$ function using probit, logit, non-parametric
② Estimate conditional mean $E[Y_i | P(Z_i)]$

③ Estimate slope, $\Delta_{LIV}(P(Z_i)) = \frac{\partial E[Y_i | P(Z_i) = P(z)]}{\partial P(z)} = MTE(P(z))$ at each $P(z)$
④ Integrate using MTE function for ATE, ATET, LATE

DIFFERENCE-IN-DIFFERENCES

MODEL

$$Y_{it} = \alpha + \Delta_i D_{it_1} + \varepsilon_{it}$$

$$\begin{cases} \Delta_i = Y_{i1t_1} - Y_{i0t_1} \\ D_{it} = 0 \text{ if } t = t_0 \\ \varepsilon_{it} = m_t + \eta_i + v_{it} \end{cases}$$

m_t: "macro" time trends common to everyone; η_i: individual fixed effects constant across time; v_{it}: random shocks/pure noise

IDENTIFYING ASSUMPTION

$$E(\varepsilon_{it} | D_{it}) = m_t + E(\eta_i | D_{it}) + 0 \quad \text{i.e. } E(v_{it} | D_{it}) = 0$$

m_t: allowed to vary with time but must be common to treatment/control

$E(\eta_i | D_{it})$: allowed to vary by treatment status but fixed across time

$E(v_{it} | D_{it}) = 0$: No selection on individual-specific time-varying unobservables

No selection on trends: $E(Y_{i0t_1} - Y_{i0t_0} | D_{it}) = E(Y_{i0t_1} - Y_{i0t_0})$ (parallel trends !!!)

- This allows identification of ATET. For ATE we also need $\Delta_i \perp\!\!\!\perp D_{it} \Leftrightarrow (Y_{i1t_1} - Y_{i0t_0}) \perp\!\!\!\perp D_{it}$
- Requires $E(v_{it} | D_{it}) = 0$.
  - Time-varying shocks (macro trends) must have same average effect on treated and control.
  - Entity-specific shocks must not be time-varying
  - Ashenfelter's dip: If enrolment in training is more likely when earnings temporarily fall, treatment group will experience faster earnings growth even absent the treatment
  - Can check pretrends, but you can only assume post-treatment underlying trends are parallel

Selection on levels is allowed: $E(Y_{i0t_0} | D_{it_1})$ can depend on D_{it_1}

- $E(\eta_i | D_{it})$ can be a function of D_{it}
- e.g. people who enroll in job training usually have lower earnings historically

ESTIMATOR

$$\begin{aligned} ATET &= E(Y_{i1t_1} - Y_{i0t_1} | D_{it_1} = 1) \\ &= E(Y_{i1t_1} - Y_{i0t_0} | D_{it} = 1) - E(Y_{i0t_1} - Y_{i0t_0} | D_{it_1} = 1) \\ &= E(Y_{i1t_1} - Y_{i0t_0} | D_{it} = 1) - E(Y_{i0t_1} - Y_{i0t_0} | D_{it_1} = 0) \quad \text{by parallel trends} \\ &= E(Y_{it_1} - Y_{it_0} | D_{it_1} = 1) - E(Y_{it_1} - Y_{it_0} | D_{it_1} = 0) \end{aligned}$$

$$\widehat{ATET} = \frac{\sum_i d_{it_1}(y_{it_1} - y_{it_0})}{\sum_i d_{it_1}} - \frac{\sum_i (1 - d_{it_1})(y_{it_1} - y_{it_0})}{\sum_i (1 - d_{it_1})}$$

Can use repeated cross section if treatment and control group can be separated before the treatment and composition of the groups (wrt entity FEs) didn't change

1st "diff" within group removes entity FE

$$E(Y_{it_1} - Y_{it_0} | D_{it} = 1) = \alpha + E(\Delta_i | D_{it_1} = 1) + m_{t_1} + E(\eta_i | D_{it_1} = 1) - [\alpha + m_{t_0} + E(\eta_i | D_{it_1} = 1)] = m_{t_1} - m_{t_0} + E(\Delta_i | D_{it} = 1)$$

$$E(Y_{it_1} - Y_{it_0} | D_{it} = 1) = \alpha + m_{t_1} + E(\eta_i | D_{it_1} = 0) - [\alpha + m_{t_0} + E(\eta_i | D_{it_1} = 0)] = m_{t_1} - m_{t_0}$$

2nd "diff" across group removes time FE

$$E(Y_{it_1} - Y_{it_0} | D_{it_1} = 1) - E(Y_{it_1} - Y_{it_0} | D_{it_1} = 0) = E(\Delta_i | D_{it} = 1)$$

DID SPECIFICATION

BASELINE $y_{it} = \alpha + X_{it}'\beta + \delta \text{Treat}_i \times \text{Post}_t + \gamma \text{Treat}_i + \lambda \text{Post}_t + v_{it}$

$$\alpha = E(y_{it} \mid D_i = 0, Z_i = 0)$$
$$\alpha + \gamma = E(y_{it} \mid D_i = 1, Z_i = 0)$$
$$\alpha + \lambda = E(y_{it} \mid D_i = 0, Z_i = 1)$$
$$\alpha + \delta + \gamma + \lambda = E(y_{it} \mid D_i = 1, Z_i = 1)$$

$$\alpha = E(\eta_i \mid D_i = 0) + m_{t_0}$$

$\gamma = E(y_{it} \mid D_i = 1, Z_i = 0) - E(y_{it} \mid D_i = 0, Z_i = 0) = E(\eta_i \mid D_i = 1) - E(\eta_i \mid D_i = 0)$ entity FE time-invariant

$\lambda = E(y_{it} \mid D_i = 0, Z_i = 1) - E(y_{it} \mid D_i = 0, Z_i = 0) = m_{t_1} - m_{t_0}$ time FE common to all entities

$$\delta = \text{ATET} = E(y_{it} \mid D_i = 1, Z_i = 1) - E(y_{it} \mid D_i = 1, Z_i = 0) - \{E(y_{it} \mid D_i = 0, Z_i = 1) - E(y_{it} \mid D_i = 0, Z_i = 0)\}$$

FIXED EFFECTS $y_{it} = X_{it}'\beta + \delta \text{Treat}_i \times \text{Post}_t + \gamma_j \sum_j D_{ij} + \lambda_s \sum_s Z_{st} + v_{it}$

Entity FE $D_{ij} = \mathbb{1}(i=j)$

Year FE $Z_{st} = \mathbb{1}(s=t)$

LEADS & LAGS $y_{it} = X_{it}'\beta + \sum_{s=0}^{t_0-1} \gamma_s \text{Treat}_i \times Z_{st} + \sum_{s=t_0}^{T} \delta_s \text{Treat}_i \times Z_{st} + \gamma_j \sum_j D_{ij} + \lambda_s \sum_s Z_{st} + v_{it}$

leads (anticipation); contemporaneous effect + lags (diffusion of TEs)

Test $H_0: \sum_{s=0}^{t_0-1} \gamma_s = 0$ for pretrends.

Rejection need not invalidate study if it's some interesting anticipatory effect

STAGGERED DID $y_{it} = X_{it}'\beta + \delta \text{Treat}_{it} + \gamma_j \sum_j D_{ij} + \lambda_s \sum_s Z_{st} + v_{it}$

LINEAR TRENDS $y_{it} = X_{it}'\beta + \delta \text{Treat}_{it} + \gamma_j \sum_j D_{ij} + \lambda_s \sum_s Z_{st} + \eta_i t \sum_j D_{ij} + v_{it}$

$\eta_i t \sum_j D_{ij}$ control for entity-specific linear trends (not the same as FE!)

EVENT STUDY $y_{it} = X_{it}'\beta + \delta_t \text{Treat}_{it} + \gamma_j \sum_j D_{ij} + \lambda_s \sum_s Z_{st} + v_{it}$

☆ Technically one D_{ij}, one Z_{st} and one δ_t should be omitted as the reference group — link to incidental parameters problem (LT)

REGRESSION DISCONTINUITY DESIGN

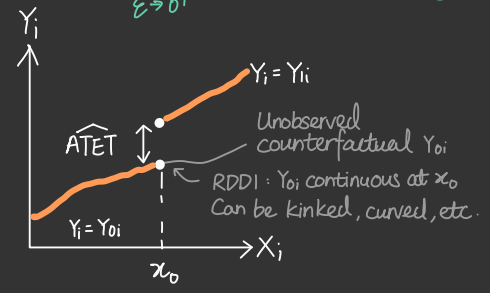
$$Y_i = D_i Y_{i1} + (1 - D_i) Y_0 = Y_{i0} + \Delta_i D_i$$

IDENTIFYING ASSUMPTION

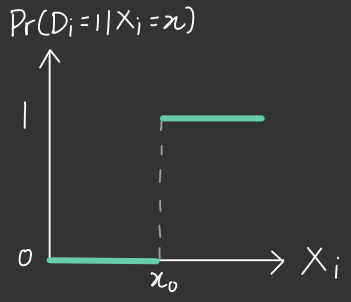
(RDD1) $E(Y_{0i} | X_i = x)$ is continuous at x_0

- Y_{0i} may be affected by many individual characteristics α_i. Those α_i should also be continuous at x_0.
- Precludes manipulation around threshold — always check for balance of observable baseline characteristics for people just above and below x_0
- If not balanced, it suggests that unobserved parts of α_i may also be unbalanced
- Don't need to know the precise function linking Y_i to X_i (can always approximate) Just need to assume that the function is continuous at x_0

(RDD2) $\lim_{\varepsilon \to 0^+} E(\Delta_i | X_i = x_0 + \varepsilon)$ is well-defined



SHARP DESIGN $D_i = f(X_i)$



- D_i is fixed given X_i
- Selection on observables only
- Point x_0 at which f is discontinuous is known
- $Pr(D_i = 1 | X_i = x) \in \{0, 1\}$, violating CSA
- Analogous to simple linear regression with truly exogenous treatment / a valid IV with full compliance, since the determinant of D_i is fully observed

$$\lim_{\varepsilon \to 0^+} [E(Y_i | X_i = x_0 + \varepsilon) - E(Y_i | X_i = x_0 - \varepsilon)]$$

$$= \lim_{\varepsilon \to 0^+} E(\Delta_i | X_i = x_0 + \varepsilon) \lim_{\varepsilon \to 0^+} E(D_i | X_i = x_0 + \varepsilon) - \lim_{\varepsilon \to 0^+} E(\Delta_i | X_i = x_0 - \varepsilon) \lim_{\varepsilon \to 0^+} E(D_i | X_i = x_0 - \varepsilon)$$

(well-defined (RDD2); $\lim E(D_i | X_i = x_0 + \varepsilon) = 1$; $\lim E(D_i | X_i = x_0 - \varepsilon) = 0$)

$$+ \lim_{\varepsilon \to 0^+} E(Y_{i0} | X_i = x_0 + \varepsilon) - \lim_{\varepsilon \to 0^+} E(Y_{i0} | X_i = x_0 - \varepsilon)$$

($= 0$ by RDD1)

$$= E(\Delta_i | X_i = x_0)$$ ATE on people with $X_i = x_0$

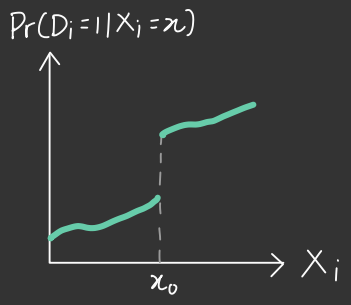
RDD1 and 2 are implicitly used in all following proofs

FUZZY DESIGN

D_i is random given X_i

$\mathbb{1}(X_i > x_0)$ is like an IV for $Pr(D_i=1|X_i=x)$

$Pr(D_i=1|X_i=x)$ is discontinuous at x_0 ('Relevance')



① Homogeneous TE $\Delta_i = \Delta$

$$\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)]$$
$$= \Delta \lim_{\varepsilon\to 0^+}[E(D_i|X_i=x_0+\varepsilon) - E(D_i|X_i=x_0-\varepsilon)]$$

$$\Delta = \frac{\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)]}{\lim_{\varepsilon\to 0^+}[E(D_i|X_i=x_0+\varepsilon) - E(D_i|X_i=x_0-\varepsilon)]}$$

② Heterogeneous TE, but $D_i \perp\!\!\!\perp \Delta_i | X_i$ $\forall i$ s.t. $X_i \in \mathcal{N}(x_0)$ (in the neighbourhood of x_0)

- $\mathbb{1}(X_i > x_0)$ is basically an IV with one-sided noncompliance

$$\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)]$$
$$= \lim_{\varepsilon\to 0^+} E(\Delta_i|X_i=x_0+\varepsilon) \lim_{\varepsilon\to 0^+} E(D_i|X_i=x_0+\varepsilon) - \lim_{\varepsilon\to 0^+} E(\Delta_i|X_i=x_0-\varepsilon) \lim_{\varepsilon\to 0^+} E(D_i|X_i=x_0-\varepsilon)$$
$$= E(\Delta_i|X_i=x_0) \lim_{\varepsilon\to 0^+}[E(D_i|X_i=x_0+\varepsilon) - E(D_i|X_i=x_0-\varepsilon)]$$

ATE at x_0 $E(\Delta_i|X_i=x_0) = \frac{\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)]}{\lim_{\varepsilon\to 0^+}[E(D_i|X_i=x_0+\varepsilon) - E(D_i|X_i=x_0-\varepsilon)]}$

③ $D_i \perp\!\!\!\perp \Delta_i | X_i$ not satisfied (due to self selection, etc.), but

$(\Delta_i, D_i(x)) \perp\!\!\!\perp X_i$ $\forall i$ s.t. $X_i \in \mathcal{N}(x_0)$ ('exclusion restriction')

$\forall \varepsilon > 0$ s.t. $D_i(x_0+\varepsilon) \geq D_i(x_0-\varepsilon)$ $\forall i$ (monotonicity)

- $\mathbb{1}(X_i > x_0)$ is basically an IV with general noncompliance

'Local' LATE (Compliers near cutoff) $\lim_{\varepsilon\to 0^+} E(\Delta_i|X_i=x_0+\varepsilon, D_i(x_0+\varepsilon) - D_i(x_0-\varepsilon) = 1) = \frac{\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)]}{\lim_{\varepsilon\to 0^+}[E(D_i|X_i=x_0+\varepsilon) - E(D_i|X_i=x_0-\varepsilon)]}$ Similar working as IV estimator

Summary

The same estimator identifies different TEs under different assumptions

External validity concern: ATE identified only for people with X_i near x_0 (unless $\Delta_i = \Delta$)

NONPARAMETRIC ESTIMATOR

Can estimate each limit using one-sided kernel regressions

$$\hat{y}^+ = \frac{\sum_{i:x_i>x_0} y_i K(x_i>x_0)}{\sum_{i:x_i>x_0} K(x_i>x_0)} \qquad \hat{d}^+ = \frac{\sum_{i:x_i>x_0} d_i K(x_i>x_0)}{\sum_{i:x_i>x_0} K(x_i>x_0)}$$

Weighted average of y_i/d_i for i above cutoff
Similar for $\hat{y}^-/\hat{d}^-$

- Basically comparing local averages just above & below cutoff
  - Imposes no structure, thus has lower precision than parametric
- Kernel need not be the same/symmetric for above and below. Kernel need not be smooth
- Can e.g. apply equal weight to obs within self-defined range and zero weight outside

! Only doable when running variable is continuous.

If discrete (e.g. whole numbers $\ldots, x_{-1}, x_0, \ldots$), then the sharp RD

$$\lim_{\varepsilon\to 0^+}[E(Y_i|X_i=x_0+\varepsilon) - E(Y_i|X_i=x_0-\varepsilon)] = \lim_{\varepsilon\to 0^+}[E(Y_{i1}|X_i=x_0+\varepsilon) - E(Y_{i0}|X_i=x_0-\varepsilon)]$$

$$= \lim_{\varepsilon\to 0^+}[E(\Delta_i|X_i=x_0+\varepsilon) + E(Y_{i0}|X_i=x_0+\varepsilon) - E(Y_{i0}|X_i=x_0-\varepsilon)]$$

$$= E(\Delta_i|X_i=x_0) + \underline{E(Y_{i0}|X_i=x_0) - E(Y_{i0}|X_i=x_{-1})} \neq 0 \text{!! Bias!}$$

PARAMETRIC ESTIMATOR

Necessary when running variable is discrete (e.g. whole nos)

$$Y_i = \alpha + \beta \Pr(D_i=1|X_i) + f(X_i) + \varepsilon_i$$

$f(X_i)$: flexible function of running variable
approximates relationship between Y_i and X_i with a continuous function
↑flexibility ⇒ ↓power

Sharp RD example of $\Pr(D_i=1|X_i) = \begin{cases} 1 & \text{if } X_i \geq x_0 \\ 0 & \text{if } X_i < x_0 \end{cases}$

Generally, $\Pr(D_i=1|X_i)$ just needs a large jump at x_0 (not necessarily $0 \to 1$)
and it can be anything when not around x_0

SELECTIVITY BIAS

Tradeoff between bandwidth (bias) and precision/power (sample size)

- Check for robustness to bandwidth (choice of kernel)

OTHER THREATS

Spillovers

- Confounding unobservables also change around cutoff (i.e. RDD!)
  - Cutoff ideally should be arbitrary rather than e.g. land borders
  - Visualise covariates against running variable (should be smooth around x_0)
  - Visualise density of running variable (should be smooth around x_0)
- Mistaking a non-linear relationship around cutoff as discontinuity
  - Try more flexible specifications for $f(X_i)$, e.g. quadratic spline

Revision Notes by Sally Yang

PROBLEMS OF *applied* ECONOMETRICS

LENT TERM

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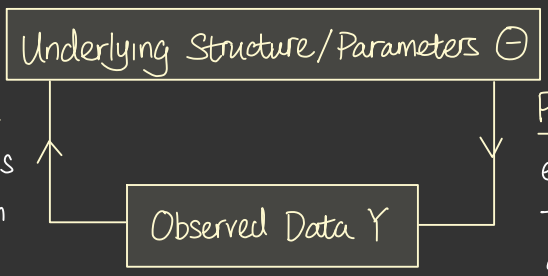
INTRODUCTION



Statistical Models and Inference

A structure writing Y as functions of some unknown Θ e.g. Gaussian

Parametric: $\dim(\Theta) < \infty$
Non parametric: $\dim(\Theta) = \infty$
Semi parametric: Nonparametrics as nuisance parameters

Probability models (CDFs)

e.g. for Gaussian, the value of $\theta = (\mu, \sigma^2)$ affects distribution / observed realisations of Y

Weaker (not fewer) assumptions = Less structure
But less structure is not always better (possibly more noise)

Random variable Y has a set of possible outcomes $\mathcal{Y}$ ('event space'). $y \in \mathcal{Y}$ is a particular outcome.

Structure / Assumption	Asymptotic Result / Property
Draws $Y_1, .., Y_N$ from population Y are random (each unit has equal chance to be selected) and i.i.d. with mean μ	LLN $\bar{Y} \equiv \frac{1}{N}\sum_{i=1}^{N} Y_i \to \mu$ as $N \to \infty$
+ Y_i has finite variance $\sigma^2 < \infty$ (counterexample: profits; if sample var is too large, pop var may not exist)	CLT $\sqrt{N}(\bar{Y} - \mu) \xrightarrow{N\to\infty} N(0, \sigma^2)$

EXTERNAL VALIDITY

Sample may not be representative of population
Economic research doesn't insist on high R^2

- Fitting the relationship in the sample too well may cause overfitting wrt pop^n

PROBABILITY MOMENT $E_{f(Y)}(Y^i) = \int_y y^i f(y)\,dy$

$E(Y) = \mu_Y$	Population mean
$E(Y^2) = \sigma_Y^2 + \mu^2$	since $\text{var}(Y) = E[Y - E(Y)]^2 = E(Y^2) - [E(Y)]^2$
$E(Y^3)$	Skewness / asymmetry
$E(Y^4)$	Kurtosis, how heavy the distribution's tails are (relative to Gaussian)

High kurtosis ⇒ extreme events more common. Mean/median no longer useful!

LAW OF ITERATED EXPECTATIONS

$$E[E(Y|X)] = E_{f(x)}[E_{f(Y|x)}(Y|X)] = \int_x \left[\int_y y f(y|x)\,dy\right] f(x)\,dx$$

$$= \int_x \left[\int_y y f(y|x) f(x)\,dy\right] dx \overset{\star}{=} \int_x \left[\int_y y f(x|y) f(y)\,dy\right] dx$$

$$= \int_y y\, f(y) \left[\int_x f(x|y)\,dx\right] dy = \int_y y\, f(y)\,dy = E(Y)$$

By Bayes' Theorem, $f(y|x)f(x) = f(x|y)f(y) = f(x,y)$

POINT ESTIMATION

Guessing θ, the estimand

The estimate $\hat{\theta}(Y)$ will depend on data Y and have error (e.g. $|\hat{\theta}-\theta|$)

Estimators are functions of random variables (data Y) so they are also random variables and have means, variances (which we also often want to estimate, e.g. for hypothesis testing)

UNBIASEDNESS $\quad E_{f(Y|\theta)}[\hat{\theta}(Y)] = \theta$

CONSISTENCY $\quad \text{plim}[\hat{\theta}(Y)] = \theta \text{ as } N \to \infty$

MINIMUM VARIANCE $\quad \hat{\theta} = \underset{\tilde{\theta}}{\text{argmin}}\ \text{var}(\tilde{\theta})$

"efficient in the class C"

$\tilde{\theta} \in C \leftarrow$ Class of "relevant" estimators being considered

EXAMPLE: OLS

Model relationship between Y and X as linear $E(Y_i|X_i) = X_i'\beta \iff Y_i = X_i'\beta + \varepsilon_i$

Estimator: $\hat{\beta}_{OLS} = \underset{\beta}{\text{arg min}} \sum_{i=1}^{N} (y_i - x_i'\beta)^2 \underset{EC221}{=} (X'X)^{-1}X'y$ is linear in y

GAUSS MARKOV THEOREM If

- A1 $y_i = x_i'\beta + \varepsilon_i$
- A2 $E(\varepsilon_i|X) = 0 \ \forall i$
- A3 $\text{Var}(\varepsilon_i|X) = \sigma^2 \ \forall i$ (Homoscedasticity)
- A4 $\text{Corr}(\varepsilon_i, \varepsilon_j|X) = 0 \ \forall i \neq j$ (No autocorrelation)

NOTE: $E(\varepsilon|X) = 0$ is sufficient for unbiasedness; $E(X'\varepsilon) = 0$ is necessary

Then $\hat{\beta}_{OLS}$ is Best Linear Unbiased Estimator

Best: Min var

Linear: $\hat{\beta}$ is a linear function of Y (NOT referring to A1)

INTERVAL ESTIMATION

✗ "The probability that $\theta \in [a,b]$ is $(1-\alpha)\cdot 100\%$" ☆ θ is not a random variable.

✓ "The probability that, upon repeated sampling of the data, $(1-\alpha)\cdot 100\%$ of interval estimates constructed this way contains the true θ"

$(1-\alpha)\cdot 100\%$ coverage

EXAMPLE: CONFIDENCE INTERVAL

N i.i.d draws from Y, $Y_i \sim N(\theta, \sigma^2)$

By Gaussian property, $\hat{\theta} = \bar{Y} \sim N(\theta, \frac{\sigma^2}{N})$ $\quad se(\hat{\theta}) \equiv \sqrt{\text{var}(\hat{\theta})} = \frac{\sigma}{\sqrt{N}}$.

If we don't know σ, we don't know $se(\hat{\theta})$ and will have to find an estimator for it $\widehat{se}(\hat{\theta})$.

$$\frac{\hat{\theta}-\theta}{\frac{\sigma}{\sqrt{N}}} \sim N(0,1)$$

$$\Pr\left\{ -z_{1-\frac{\alpha}{2}} < \frac{\hat{\theta}-\theta}{\frac{\sigma}{\sqrt{N}}} < z_{1-\frac{\alpha}{2}} \right\} = \Pr\left\{ \hat{\theta} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{N}} < \theta < \hat{\theta} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{N}} \right\} = 1-\alpha$$

$1-\frac{\alpha}{2}$-th quantile of $N(0,1)$

Confidence interval $\left(\hat{\theta} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{N}},\ \hat{\theta} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{N}}\right)$ contains θ $(1-\alpha)\times 100\%$ of the time.

NULL HYPOTHESIS TESTING

$H_0: \theta = \theta_0$ vs $H_1: \neg H_0$ or $\theta \gtrless \theta_0$, ...

SIZE/CONFIDENCE $\Pr(\text{Type I error}) = \Pr(\text{Reject } H_0 \mid H_0) = \alpha$

When H_0 is true, we want to fail to reject it $(1-\alpha)\times 100\%$ of the time

POWER $\Pr(\text{Type II error}) = \Pr(\text{Do not reject } H_0 \mid \neg H_0) = 1-b$ Typically increases with $\theta - \theta_0$

FALSE DISCOVERY RATE In multiple hypothesis testing

$$\Pr(H_0 \mid \text{Reject } H_0) = \frac{\Pr(H_0 \cap \text{reject } H_0)}{\Pr(\text{Reject } H_0)} \neq \frac{\Pr(H_0 \cap \text{reject } H_0)}{\Pr(H_0)} = \alpha$$

Expected no. of rejected null hypotheses that are actually true

People often conflate FDR with size and think FDR = α.

But even if we limit our set of nulls to just the rejected ones / even if all nulls are rejected (i.e. $\Pr(\text{Reject } H_0) = 1$), FDR $\neq \alpha$ unless all nulls in the set are true

Even if $\alpha = 0.05$, FDR may be high if { a large % of tested nulls are true (high $\Pr(H_0)$) and low % of tested nulls are rejected (low $\Pr(\text{reject } H_0)$)

☆ z/t-testing does not tell us $\Pr(H_0)$, $\Pr(H_1)$ or $\frac{\Pr(H_0)}{\Pr(H_1)}$

z/t-stats always conditional on H_0

BUT the LR test does give $\frac{\Pr(\text{observing given data} \mid H_0)}{\Pr(\text{observing given data} \mid H_1)}$

P-VALUE

Probability of observing data Y at least as extreme (i.e. adverse to the null) as the data that was actually observed, y, when H_0 is true ★ always include

Lower p-value $\Rightarrow$ more reasonable to think "no way that y was generated using θ_0!"

1. Assume H_0 true, so $\theta = \theta_0$
2. Observe the implications of this assumption. Usually this means $E(\hat{\theta}) = \theta_0$.
3. Using what you know about $f(\hat{\theta})$ and $\hat{se}(\hat{\theta})$, calculate $\Pr(\hat{\theta}$ is at least as extreme as what you observe $\mid \theta = \theta_0)$

Again, cannot draw conclusions about how likely H_0 is true ($\Pr(H_0)$ or $\Pr(H_1)$), no matter how low p-value is! $\Pr(H_0 \mid \text{reject } H_0) \neq \Pr(\text{reject } H_0 \mid H_0)$

$\Pr(H_0 \mid \text{reject } H_0)$ or $\Pr(H_0 \mid \text{data})$ in general is a posterior probability, requires a prior to calculate (Bayesian inference...). p values are not posteriors.

THREATS TO IDENTIFICATION

- Functional form misspecification (A1)
- OVB, reverse causality, measurement error (A2)
  Data not missing at random (MNAR) — selection bias
  - If Pr(data) is correlated with Y_i, ε_i no longer i.i.d.
  - Selecting on X_i (MAR) affects external, not internal, validity
- Nonstationarity

EXAM ANSWER TEMPLATES

INTERPRET β (structural equation)

The effect of a 1 unit of X increase in treatment on the ☆ outcome for level of observation holding all other determinants of outcome constant (all else constant)

INTERPRET $\hat{\beta}$ (reading regression output)

In sample restriction (e.g. Indian villages), for level of observation, a 1 unit of X increase in treatment increases outcome by ☆ unit of Y, holding all other determinants of outcome constant.

☆
- Y on X: $\Delta Y = \beta \Delta X$
- ln Y on X: $\%\Delta Y = 100\% \times (e^{\beta \Delta X} - 1) \approx 100\% \times \beta \Delta X$ — $\ln Y_{new} - \ln Y = \beta(X+\Delta X) - \beta X \Rightarrow \ln \frac{Y_{new}}{Y} = \beta \Delta X$; $\%\Delta Y = 100\%\left(\frac{Y_{new}}{Y} - 1\right) = 100\% \times (e^{\beta \Delta X} - 1) = e^{\beta \Delta X} - 1$
- Y on ln X: $\Delta Y = \beta \ln(1 + \%\Delta X) \approx \beta \%\Delta X$ — $Y_{new} - Y = \beta(\ln(X(1+\%\Delta X)) - \beta \ln X$; $\Delta Y = \beta \ln(1+\%\Delta X) \approx \beta \%\Delta X$
- ln Y on ln X: $\%\Delta Y = (1+\%\Delta X)^{\beta} - 1 \approx \beta \%\Delta X$ — $\ln Y_{new} - \ln Y = \beta \ln X_{new} - \beta \ln X \Rightarrow \frac{Y_{new}}{Y} = \left(\frac{X_{new}}{X}\right)^{\beta}$; $\%\Delta Y = (1+\%\Delta X)^{\beta} - 1 \approx \beta \%\Delta X$

SIZE VS FALSE DISCOVERY RATE

$$\Pr(\text{Type I error}) = \Pr(\text{Reject } H_0 \mid H_0) = \frac{E(\text{No. rejected and true})}{E(\text{No. true})}$$

Expected number of true hypotheses we reject

$$\Pr(H_0 \mid \text{Reject } H_0) = \frac{E(\text{No. rejected and true})}{E(\text{No. rejected})}$$

Expected number of rejected hypotheses that were true

FDR is unknown and depends on prior probability of any hypothesis being true
For example, if all hypotheses were true, then all rejections are false, but the size is still α.

P-VALUE

Probability of observing data at least as extreme (i.e. adverse to the null) as data that was actually observed when H_0 is true

POWER

$1 - \Pr(\text{Type II error}) = 1 - \Pr(\text{Do not reject } H_0 \mid \neg H_0)$

- Increases when population variance/measurement error is low, sample size is high, $|\theta - \theta_0|$ is large
  ↑size will ↑power (trading off Type I and II errors)

SPILLOVERS

The treatment status D_i of i affects outcome $Y_j(D_j)$ of $j \neq i$. Makes $\hat{\beta}$ biased and inconsistent.
$\hat{\beta}$ will capture (direct effect − indirect effect), which may over- or under-state the direct effect and social effect (direct effect + indirect effect)

Changing level of analysis will protect us from bias, but reduce power as estimating at a higher level of variation introduces variance.

- True model is linear
  For non-linear see other page
- Unobserved entity-specific time-invariant OVB α_i

FIXED EFFECTS

☆ Allows α_i and X_{it} to be correlated
No assumptions on $E(\alpha_i|X_{it})$, unlike RE

Assumptions

FE1 $E(u_{it}|X_{i1},\dots,X_{iT},\alpha_i)=0 \ \forall t$ Strict exogeneity
- to check → Violated by LDV (y_{it-1} on RHS)
- to check → X_{it+1} should not explain y_{it}

FE2 $Rank(\tilde{X}'\tilde{X})=K$ or $N+K$ $\tilde{X}$ is transformed X

FE3 $E(u_i'u_i|X_{i1},\dots,X_{iT})=\sigma^2 I_{T\times T}$
- +FE1 → $Var(u_i|X_{i1},\dots,X_{iT})=\sigma^2 I_{T\times T}$
- → u_{it} cannot be autocorrelated

needs
- Stationarity: $f(\dots X_{t-1},Y_{t-1},X_t,Y_t,\dots)$ doesn't depend on t
- Violation: unit root, time trends, structural breaks
- If violated, do FD, but FE still better than FD if $\exists\theta$ s.t. $Y-\theta X=c$
  i.e. cointegration/stationary residuals

Approach

Same $\hat{\beta}$ different df

Within Regression $\hat{\beta}$ unbiased, consistent

$$y_{it}-\bar{y}_i=(X_{it}-\bar{X}_i)\beta+\varepsilon_{it} \xrightarrow{\text{Stack N,T}} \underset{NT\times 1}{\ddot{Y}}=\underset{K\times 1}{\ddot{X}\beta}+\ddot{\varepsilon} \quad df=NT-K$$

Want $\hat{\alpha}_i$

Dummies Regression $\hat{\beta}$ unbiased, consistent if $\hat{\beta}$ and $\hat{\alpha}$ additively separable (linear)

$$y_{it}=\sum_{i=1}^{N}\alpha_i D_{ni}+X_{it}\beta+\varepsilon_{it} \xrightarrow{\text{Stack N,T}} \underset{NT\times 1}{Y}=\underset{N\times 1}{D\alpha}+\underset{K\times 1}{X\beta}+\varepsilon \quad df=NT-(N+K)$$

($D_{ni}=1$ iff $i=n$)

$\hat{\alpha}_i$ unbiased but inconsistent for small T (only consistent if $T\to\infty$); has high var in short panels

Problems

$\hat{\alpha}_i$ **Incidental Parameters**

Number of entity dummies α_i increases with N, but each α_i is estimated from T obs
$\hat{\alpha}_i$ inconsistent in N for fixed T, though unbiased

$\hat{\beta},\hat{\alpha}_i$ Interactions of α_i with X_{it} $\frac{\partial Y}{\partial X}=f(\alpha_i)$ $\hat{\beta}$ and $\hat{\alpha}$ not additively separable

$\hat{\beta},\hat{\alpha}_i$ Time-varying individual-specific confounders v_{it}

If too different:
- functional form misspecified
- or FE1 violated ⇒ $\hat{\beta}_{FE}$ and $\hat{\beta}_{FD}$ will generally have different plims (wtrg)

FIRST DIFFERENCES

$$\Delta Y_{it}=\Delta X_{it}+\Delta u_{it}$$

☆ Allows α_i and X_{it} to be correlated
No assumptions on $E(\alpha_i|X_{it})$, unlike RE

Assumptions

FD1 $E(u_{it}|X_{i1},\dots,X_{iT},\alpha_i)=0 \ \forall t$
- guarantee → $E(\Delta X'\Delta u|\Delta X)=0$ necessary for consistency
- guarantee → $E(\Delta u_{it}|\Delta x_{it},\dots,\Delta_{iT})=0$ strict exogeneity in the FD sense
- to check → X_i should not explain Δy_{it}

FD2 $Rank(\Delta X'\Delta X)=K$

FD3 $E(\Delta u_i'\Delta u_i|X_{i1},\dots,X_{iT})=\sigma^2 I_{T\times T}$ → u_{it} should be autocorrelated, e.g. random walk $u_{it}=u_{it-1}+v_{it}$
If not, $Cov(\Delta u_{it},\Delta u_{it-1})=-\sigma^2$ regression to the mean

Robustness

Additionally include covariates in levels X_{t-1} in FD regression and do F test. Should be insignificant.
If significant, base model may have omitted a linear trend $X_t t$ ($\Delta X_t t=X_t$)

If FD1 violated

ANDERSON-HSIAO IV

Weak IV
- Different lags may have opposite effects on ΔX_{it} and cancel out
- ↑No. of lags used ⇒ ↓sample size
- If X_{it} and X_{it-1} highly correlated, $Cov(X_{it},X_{it-1})\approx Var(X_{it-1})$

Assume sequential exogeneity $E(u_{it}|X_{i1},\dots,X_{it},\alpha_i)=0$
Use lag X_{it-1} to instrument for ΔX_{it}
Arellano-Bond: all lags valid, so add more!

✓ Validity: $E(X_{it-1}\Delta u_{it})=0$
? Relevance: $Cov(X_{it-1},\Delta X_{it})=Cov(X_{it},X_{it-1})-Var(X_{it-1})$

If T=2 Identical

T>2

Generally different in finite sample:
- FE gives weighted mix of first and lagged diff under a parallel trends assumption
- Note that FE is more efficient than FD under FE3, as FE uses info from more lags

y_i is not linear in β

Censored y_i

e.g. WTP, demand (corners!)

TOBIT $y_i = \max(0, X_i\beta + u_i) \quad u_i \sim N(0, \sigma_u^2)$

OLS is biased and inconsistent since model is non-linear

Ignoring 0s creates sample selection bias $E(y_i|y_i>0, X_i) = X_i\beta + \frac{\sigma\varphi(\frac{X_i\beta}{\sigma_u})}{\Phi(\frac{X_i\beta}{\sigma_u})}$ ↓ omitted variable: "some non-zero term"

$\Pr(y_i = 0|X_i) = \Pr(u_i < -X_i\beta|X_i) = 1 - \Phi(\frac{X_i\beta}{\sigma_u})$

$\Pr(y_i|y_i>0, X_i) = f(u_i|X_i) = \varphi(\frac{y_i - X_i\beta}{\sigma_u}) = \frac{1}{\sigma_u}\varphi(y_i - \frac{X_i\beta}{\sigma_u})$

$L \overset{iid}{=} \prod_i^N \left[\frac{1}{\sigma_u}\varphi(y_i - \frac{X_i\beta}{\sigma_u})\right]^{\mathbb{1}(y_i>0)} \left[1 - \Phi(\frac{X_i\beta}{\sigma_u})\right]^{\mathbb{1}(y_i=0)}$

Binary

Assume y_i is Bernoulli

$E(y_i|X_i) = \Pr(y_i)$

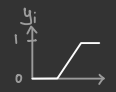
LPM $E(y_i|X_i) = \Pr(y_i = 1|X_i) = X_i\beta$

✓ MTE doesn't depend on X_i value ✓ May be a good approximation

✗ Inconsistent and biased unless the true $X_i\beta \in [0,1]\ \forall i$ beyond that, $\varepsilon_i = 0$ but $\hat{\varepsilon}_i \neq 0$

! At least verify $\hat{y}_i \in [0,1]\ \forall i$

BOUNDED LPM $E(Y_i|X_i) = \Pr(Y_i = 1|X_i) = \min\left[\max(X_i\beta, 0), 1\right]$



Latent variable

$y_i = \mathbb{1}\{y_i^* = X_i\beta + u_i > 0\}$

$u_i|X_i \sim G$ (↑ symmetric) $\forall i.\ E(u_i|X_i) = 0$

$\Pr(y_i = 1) = \Pr(y_i^*|X_i) = \Pr(u_i > -X_i\beta|X_i) = G(X_i\beta)$

Models

LOGIT $\Pr(y_i = 1|X_i) = \frac{e^{X_i\beta}}{1 + e^{X_i\beta}}$

$L \overset{iid}{=} \prod_i^N \left[\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right]^{y_i} \left[\frac{1}{1+e^{X_i\beta}}\right]^{1-y_i}$

✓ In panel, can "delete" $\{\alpha_i\}^N$ by conditioning on entity-specific means $\left\{\sum_{t=1}^T y_{it}\right\}^N$

- $\{\bar{y}_i\}^N$ is a sufficient statistic for $\{\alpha_i\}^N$
- A statistic S is sufficient for a parameter θ if $Y|S$ doesn't depend on θ
- Factorisation criterion: $S(Y)$ sufficient for θ iff $\exists\, g_\theta, h$ s.t. $f_\theta(Y) = g_\theta(S(Y))\,h(Y)$ (f_θ ↑ likelihood function)

PROBIT $\Pr(y_i = 1|X_i) = \Phi(X_i\beta) = \int_{-\infty}^{X_i\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{(X_i\beta)^2}{2}}$

$L \overset{iid}{=} \prod_i^N \left[\Phi(X_i\beta)\right]^{y_i} \left[1 - \Phi(X_i\beta)\right]^{1-y_i}$

TOBIT $\Pr(y_i = 1|X_i) = \max(0, \underbrace{X_i\beta + u_i}_{=y_i^*})\ u_i \sim N(0, \sigma_u^2)$

$L \overset{iid}{=} \prod_i^N \left[\frac{1}{\sigma_u}\varphi(y_i - \frac{X_i\beta}{\sigma_u})\right]^{\mathbb{1}(y_i>0)} \left[1 - \Phi(\frac{X_i\beta}{\sigma_u})\right]^{\mathbb{1}(y_i=0)}$

Problems

Latent heteroscedasticity

$\mathrm{Var}(y_i^*|X_i) = \mathrm{Var}(u_i|X_i) = \sigma_i^2$ must be explicitly modelled in (rescale X_i, y_i by $\frac{1}{\sigma_i}$)

If not, $\hat{\beta}$ may accidentally capture effect of X_i on y_i through σ_i

OVB

Must control for all predictors of y_i, even if uncorrelated with X_i e.g. measurement error in y_i

Any omitted ω_i makes $\hat{\beta}$ inconsistent (ω_i and X_i not additively separable in nonlinear model)

✗ Unobserved heterogeneity α_i in panel (Exception: logit)

- Since FE, FD allow $E(\alpha_i|X_{it})$ to be anything, α_i likely correlated with X_{it}
- Since β now depends on X_{it}, likely also depends on α_i; FE, FD fail by incidental parameters

$Y \sim f(y|\theta)$
$\mathcal{L}(\theta) = f(y|\theta)$
$\ell(\theta) = \log f(y|\theta)$

MAXIMUM LIKELIHOOD $\hat{\theta}_{MLE}$

$\hat{\theta}_{MLE} \equiv \operatorname{argmax} \mathcal{L}(\theta)$
$= \operatorname{argmax} \ell(\theta)$

Conditions

Score function

Univariate

FOC: $\frac{\partial \ell(\hat{\theta}_{MLE})}{\partial \theta} = 0$

SOC: $\frac{\partial^2 \ell(\hat{\theta}_{MLE})}{\partial \theta^2} < 0$

Multivariate

FOC: $\nabla \ell(\hat{\theta}_{MLE}) = J(\theta) = 0$ (Jacobian)

SOC: $\nabla^2 \ell(\hat{\theta}_{MLE}) = H(\theta) = 0$ (Hessian)

Properties

- Consistent
- Efficient — Minimum Variance in class of consistent estimators
- Asymptotically Normal — for doing inference

Under $H_0: \theta = \theta_0$, $\sqrt{N}(\hat{\theta}_{MLE} - \theta_0) \approx N(0, I^{-1}(\theta_0))$

- Inverse Fisher Information: $I^{-1}(\theta_0) = E\left[-\frac{\partial^2 \ell(\hat{\theta}_{MLE})}{\partial \theta^2}\right]^{-1}$
- Large 2nd derivative → f very concave at $\hat{\theta}_{MLE}$
  → More certain that $\hat{\theta}_{MLE}$ is maximiser → Smaller $Var(\hat{\theta}_{MLE})$

Solving

If ℓ is globally smooth and concave → **Calculus**: Solve $\hat{\theta}_{MLE}$ from FOC

If not → **NUMERICAL OPTIMISATION**

Newton-Raphson procedure: easiest to describe when tested

1	Start at some θ_0
2	Calculate $\ell'(\theta_0)$
3	If $\ell'(\theta_0) > 0$, go to $\theta_1 > \theta_0$ where $\theta_1 = \theta_0 + A$ If $\ell'(\theta_0) < 0$, go to $\theta_1 < \theta_0$ where $\theta_1 = \theta_0 - A$
4	Repeat ②-③. Stop when $\lvert\ell'(\theta)\rvert \leq B$

! May not converge (i.e find solution); can get stuck

! Always check for sensitivity to starting point, tuning parameter, tolerance
↳ If sensitive, algorithm may be 'stuck' at a wrong solution (try changing tolerance or algorithm) or solution not unique (calculate L at each solution, choose largest)

GMM

Data Y, model with parameters $\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_K \end{pmatrix}$ characterised by R moment conditions

$E[f(Y,\theta)] = 0 \quad f(Y,\theta) = \begin{pmatrix} f_1(Y,\theta) \\ \vdots \\ f_R(Y,\theta) \end{pmatrix}$

For N draws,

$g_N(\theta) = \frac{1}{N}\sum_{i=1}^{N} f(Y_i,\theta) = 0$

$\hat{\theta}_{GMM} = \underset{\theta}{\operatorname{argmin}}\, g_N(\theta)' W_N g_N(\theta)$

Minimise (weighted) sum of squares

! Minimiser must be unique

↳ Minimises $Var(\hat{\theta}_{GMM})$

Optimal weighting matrix $W_N^* = E[f(Y,\theta)f(Y,\theta)']^{-1}$

- Inverse of covariance of sample moments
- Highly weight/emphasise sample moments with low var (more precisely estimated)
- "Plug-in" estimator: Get $\hat{\theta}$ from unweighted GMM, then calculate $\hat{W}_N^* = \frac{1}{N}\sum_i^N f(Y_i,\hat{\theta})f(Y_i,\hat{\theta})')^-$

Properties

- Consistent
- Asymptotically normal $\sqrt{N}(\hat{\theta}_{GMM} - \theta) \to N(0,V)$ for doing inference
- Efficient (Min var in class of asy. normal estimators)

OVERIDENTIFICATION (SARGAN'S J) TEST

If we have more moments than needed to get a single solution (For linear system, $R > K$)

Test statistic: $ng_N(\hat{\theta}_{GMM})'W_N^* g_N(\hat{\theta}_{GMM}) \sim \chi^2_{R-K}$ under H_0: All population moments are zero

! If the GMM obj fn value is too big to be compatible with H_0, discard suspicious moments

Tests if moments are internally consistent, not if they're valid (they may be invalid in similar ways)

IV ISSUES

$Y = X\beta + \varepsilon$

$X = Z\pi + \epsilon$

BLUNT For a given Y, Z can be a valid instrument for at most one X

violates exclusion

WEAK $\pi \neq 0$ (still valid) but almost 0. Conventional asymptotic approximations do not capture it properly and will generate errors not well-behaved.

$\beta - \hat{\beta} = (Z'X)^{-1}Z'\varepsilon \xrightarrow{N\to\infty} (0)^{-1}0$

! Using many weak IV on RHS may actually make it worse

Homoscedastic / Heteroscedastic: **F>10**

ANDERSON-RUBIN CI $AR(\beta_0) \sim \chi^2_K$ under $H_0: \beta = \beta_0$

Doesn't use π; Robust to weak IV

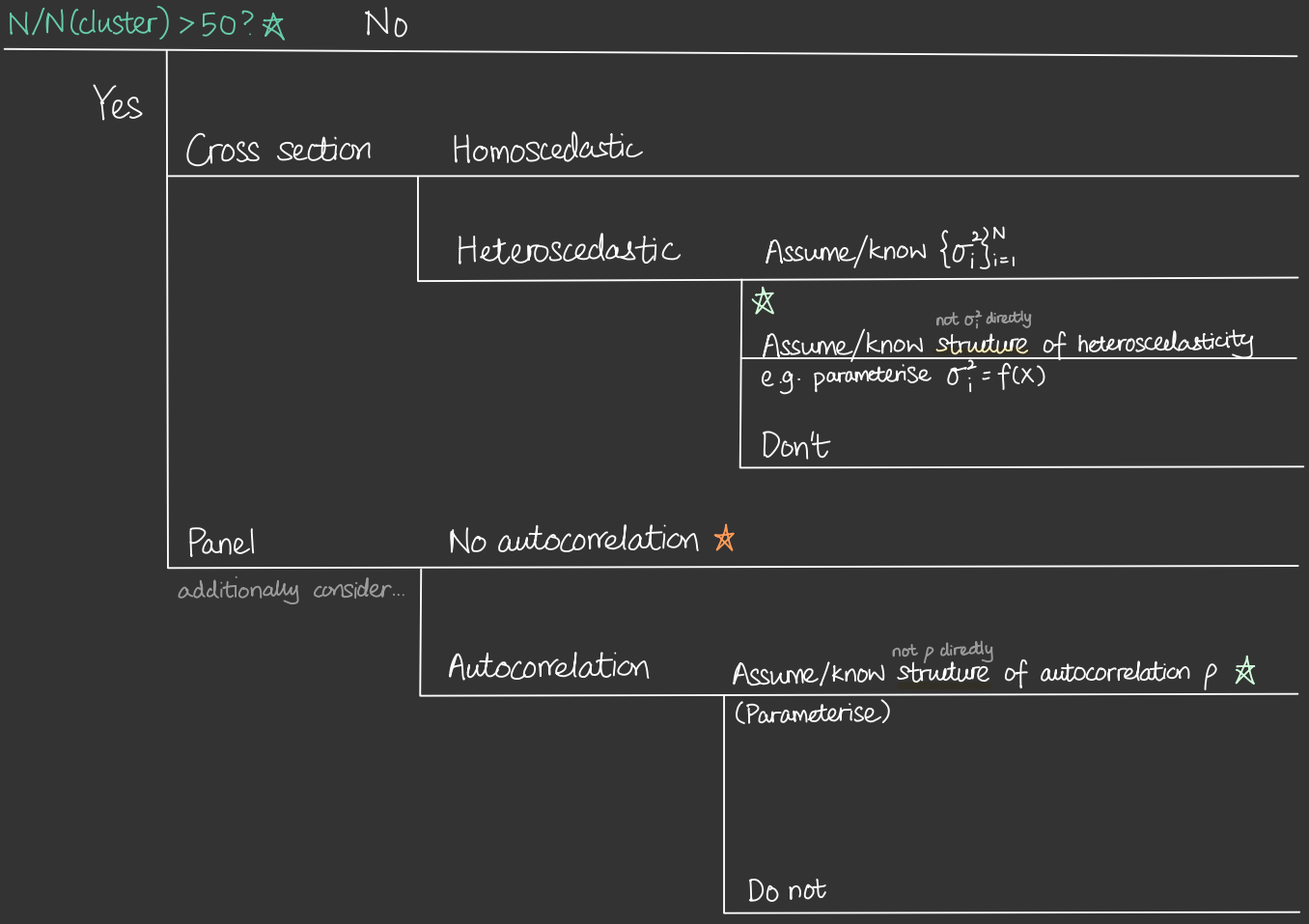
- Test a grid of β_0s to generate CI, keeping β_0s that aren't rejected
- Model may be misspecified if
  - CI is empty or unbounded
  - F stat is low but CI is very tight (nearly empty). Misleading!

Non-linear IV

FORBIDDEN REGRESSION

- Run $X_i = \pi Z_i + \eta_i$ to get $\hat{X}_i$ and plug $\hat{X}_i^2$ into second stage X_i^2 $\left(E(\hat{X}_i^2) \neq [E(\hat{X}_i)]^2\right)$
- Can instrument using Z_i^2 but likely weak
  - No reason why strong first stage for X_i, Z_i should hold for X_i^2, Z_i^2
- Sometimes just fitting the linear IV still gives LATE (even if true model is non-linear)

STANDARD ERRORS



Bootstrap Also works for large N
So even if N is large, consider bootstrapping SEs (in addition to SEs below) to check for robustness

SEs $\frac{1}{N-K}\overset{☆}{\sum_i^N}\hat{u}_i^2$ unbiased, consistent in N

Weighted Least Squares t-stats exact ($\sim N, \chi^2$)

Feasible Weighted Least Squares Fit f, get consistent $\hat{\sigma}_i^2$.
t-stats approximate ($\sim t, F$)

White RSEs $\frac{1}{N-K}\sum_i^N \hat{u}_i^2 x_i x_i'$ consistent in N

Arellano CSE by Entity $\frac{1}{N}\sum_i^N \ddot{X}_i'\hat{u}_i'\hat{u}_i\ddot{X}_i$ consistent in N for fixed T
More conservative (gives larger SEs) if ICC > 0

Newey-West HAC $\frac{1}{T}\frac{\hat{\sigma}_{\ddot{x}}^2}{\hat{\sigma}_{\ddot{x}}^2}\left(1+2\sum_{j=1}^{T-1}\frac{T-j}{T}\hat{\rho}_j\right)$ formula is for TS; for panel just do this person by person
Estimate $\sum_{j=1}^{T}\frac{T-j}{T}\rho_j$ with $\sum_{j=1}^{m-1}\frac{m-j}{m}\hat{\rho}_j$ where m grows slower than T e.g. $m = \frac{3}{4}T^{\frac{1}{3}}$
When $T\to\infty$, $N(\rho_j)\to\infty$ (incidental parameters problem) so use less lags than T
need large T and thin-tailed X_t, u_t for consistency

Arellano CSE by Entity $\frac{1}{N}\sum_i^N \ddot{X}_i'\hat{u}_i'\hat{u}_i\ddot{X}_i$ consistent in N for fixed T
More conservative (gives larger SEs) iff ICC > 0

☆ There is no finite sample unbiased SE estimator except in homoscedasticity / WLS / FWLS ! Everything else works asymptotically (consistency) !

☆ White RSEs $\frac{1}{NT-N-K}\sum_i^N\sum_t^T \ddot{X}_{it}'\ddot{X}_{it}\hat{u}_{it}^2$ inconsistent in N for fixed T even without autocorrelation

☆ Formulae shown are for "core component" of $SE(\hat{\beta})$, not $SE(\hat{\beta})$

☆ The more assumptions a procedure makes, the lower the SE value (e.g. HAC < CSE as HAC leverages the structure of stationarity)

CLUSTERING

Cluster by group : SUTVA violation, Treatment assigned at higher level, Intra cluster correlation
Cluster by time : Spatial spillovers (across entities)
N(Cluster) < 50 : Bootstrap performs better

- Extremum statistic (max, min)
- Stastistic not a smooth function of data (median(X) when X bunches at 0)
- Random variables in the data have unbounded variances kills LLN
- Statistic changes the support of the distribution of data

Yes — Don't bootstrap

No — **BOOTSTRAP**

Given a sample $(X,Y) = \{(X_i, Y_i)\}_{i=1}^{N}$

i.e. N pairs of (X_i, Y_i) — indirectly determines $Y_i = X_i\hat{\beta} + \hat{\varepsilon}_i$

1	Sample <N pairs of $(X_i, \hat{\varepsilon}_i)$ with replacement to make one bootstrapped sample $(X, \hat{\varepsilon})_b$
2	Run estimation on $(X, \hat{\varepsilon})_b$ to get $\hat{\beta}_b$
3	Repeat B times to get $\{\hat{\beta}_b\}_{b=1}^{B}$
4	$\widehat{Var}(\hat{\beta}_{original}) = \frac{1}{B}\sum_{b=1}^{B}(\hat{\beta}_b - \hat{\beta}_{original})^2$

X is fixed e.g. RCT

Sample just $\hat{\varepsilon}_i$

Bootstrapped sample: (X_i, Y_i) where $Y_i = X_i\hat{\beta} + \hat{\varepsilon}_i$

ε_i must be i i d (no heteroscedasticity, etc)

Cluster/Serial correlation

Block/Cluster Bootstrap

Sample clusters with replacement

Can also sample within clusters; usually not done

Blocks must be independent

better account for cluster effects

Wild Cluster Bootstrap

Additionally flip sign of residuals $\hat{\varepsilon}$ with $\frac{1}{2}$ probability for any cluster (then $\hat{Y} = \beta X + \hat{\varepsilon}_{new}$)

$\uparrow \widehat{var}(\hat{\beta})$