P3

a). $\theta_{t|t} = \theta_{t-1|t-1} + c_\infty (y_t - \theta_{t-1|t-1})$

$$= c_\infty y_t + (1-c_\infty)\theta_{t-1|t-1}$$

$$= c_\infty y_t + (1-c_\infty)(c_\infty y_{t-1} + (1-c_\infty)\theta_{t-2|t-2})$$

....

Suppose. $c_\infty = \alpha$ $\quad 1 - c_\infty = \beta$

$$= \alpha y_t + \beta(\alpha y_{t-1} + \beta\theta_{t-2|t-2})$$

recursively $= \alpha(y_t + \beta y_{t-1} + \beta^2 y_{t-2} + \cdots \beta^{t-1} y_1)$

$$= \alpha(\beta^{t-1} y_1 + \beta^{t-2} y_2 + \cdots \beta^0 y_t) = \alpha \sum_{k=1}^{t} \beta^{t-k} y_k.$$

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b) $\vec{y} = (y_1 \ldots, y_T)$ $\quad H_t = \{y_{t-1}, \ldots, y_1\}$ $\quad \theta_{t|t} = E[X_t | y_t, H_t]$

$\because$ $\theta_{t|t}$ is weighted linear combination of $y_{t-1} \ldots y_1$ in a forward direction, for backward, $\theta_{t|T}$ should also the linear combination of $y_T, y_{T-1} \cdots y_1$ with a reverse order. (symmetric)

Since only $y_{t_1}, \ldots, y_{t_R}$, $R \ll T$ are non-zero and $\theta_{t|T}$ is a linear combination of y's, other points ($y=0$) won't contribute to θ. Therefore, taking all non-zero y's into consideration, $\theta_{t|T} = \alpha \sum_{r=1}^{R} \beta^{|t-t_r|} y_{t_r}$.

c). $k(t_i, t_j) = e^{-\gamma |t_i - t_j|}, \quad r > 0$

$$\underbrace{\hat{\theta}(t) = \underset{\theta(t) \in H}{\arg\min}\ \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta(t_i))^2 + \lambda \|\theta(t)\|_H^2}_{J}$$

$$\nabla_\theta J = (\vec{y} - k(\cdot,t)\vec{\theta})^T (\vec{y} - k(\cdot,t)\vec{\theta}) + 2\sigma^2\lambda\, \vec{\theta}^T\vec{\theta} = 0$$

$$\hat{\theta} = \left((k(\cdot,t)^T k(\cdot,t) + 2\sigma^2\lambda I)^{-1}\right) k(\cdot,t)^T \vec{y} = (K + 2\sigma^2\lambda I)^{-1} k(\cdot,t)^T \vec{y}$$

$(k(\cdot,t) = [k(t,t_1), k(t,t_2) \cdots k(t,t_n)]^T$

$$K = \begin{bmatrix} k(t_1,t_1) & \cdots & k(t_1,t_n) \\ \vdots & & \\ k(t_n,t_1) & \cdots & k(t_n,t_n) \end{bmatrix}$$

$$\Rightarrow \vec{\alpha} = (K + CI)^{-1}\vec{y}. \qquad C = 2\sigma^2.$$

d). $\theta_{t|T} = \vec{\alpha} \sum_{r=1}^{R} \beta^{|t-t_r|} y_{t_r}. \qquad \vec{\alpha} = \underbrace{(K + CI)^{-1}}_{\text{symmetry}}\vec{y}.$

continuous

$$\hat{\theta}(t) = \sum_{i=1}^{n} \alpha_i k(t_i, t_j)$$

$$= \vec{\alpha}^T \begin{bmatrix} k(t,t_1) \\ k(t,t_2) \\ \vdots \\ k(t,t_n) \end{bmatrix}$$

$$= (k(t,t_1), \cdots k(t,t_n)] (K + CI)^{-1}\vec{y}$$

$$= \underbrace{(e^{-\gamma|t-t_1|} \cdots e^{-r|t-t_n|}] (K + CI)^{-1}\vec{y}}_{\text{counterpart of } \theta_{t|T}} \quad \text{— continuous-time}$$