Lecture 7

# Information Theory

# Last Times:

- Law of Large Numbers
- Machine Learning
- SGD for minimizing a loss

# Today

- regularization
- logistic log-loss
- KL-Divergence
- entropy and cross-entropy
- maximum entropy distributions
- deviance

# Law of Large numbers (LLN)

- Expectations become sample averages. Convergence for large N.

$$E_f[g] = \int g(x)dF = \int g(x)f(x)dx$$
$$= \lim_{n \to \infty} \frac{1}{N} \sum_{x_i \sim f} g(x_i)$$

- for finite N a sample average
- thus expectations in the replication "dimension" come into play
- mean of sample means and standard error
- this is the sampling distribution
- CLT and all that jazz

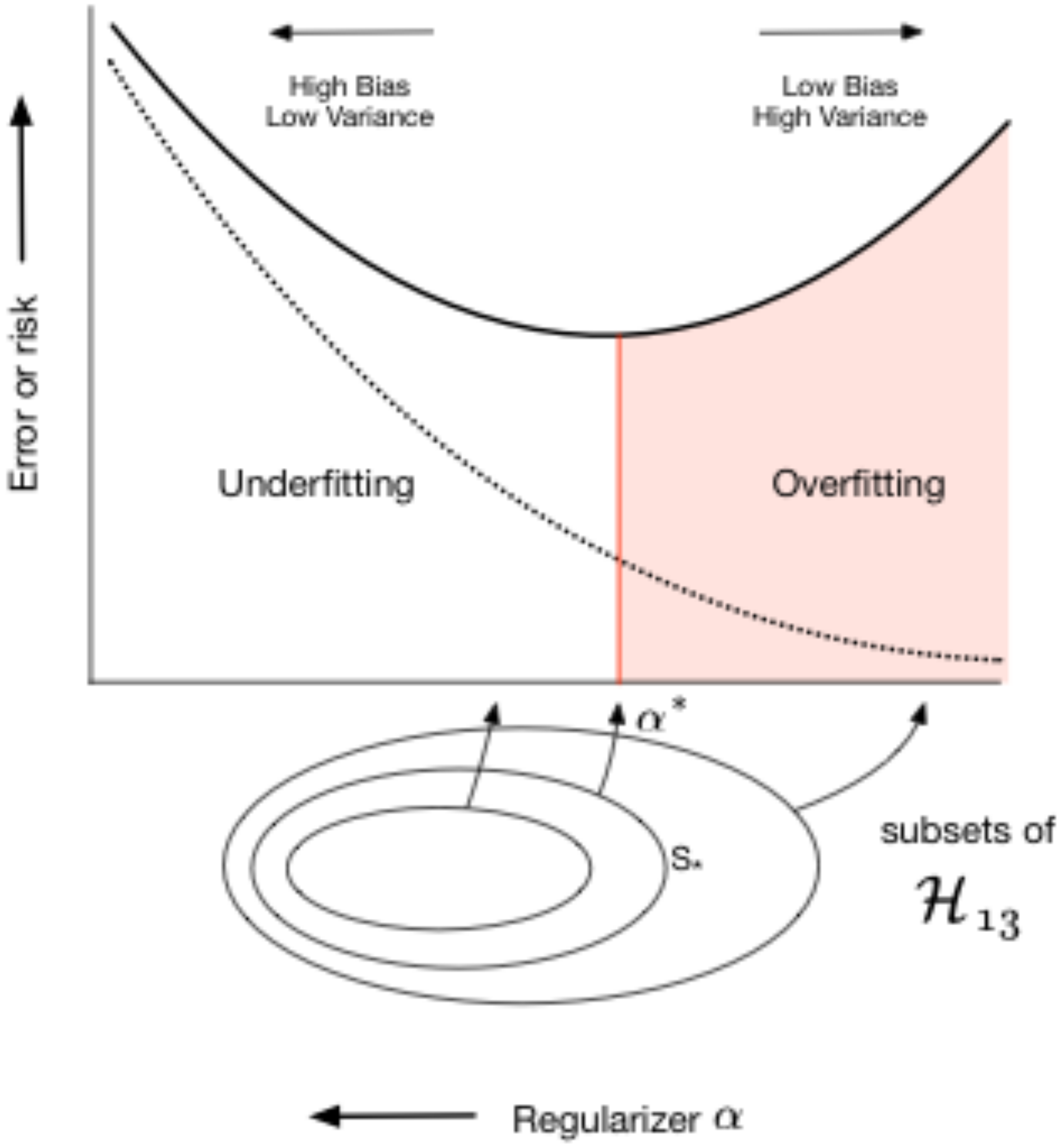
# REGULARIZATION

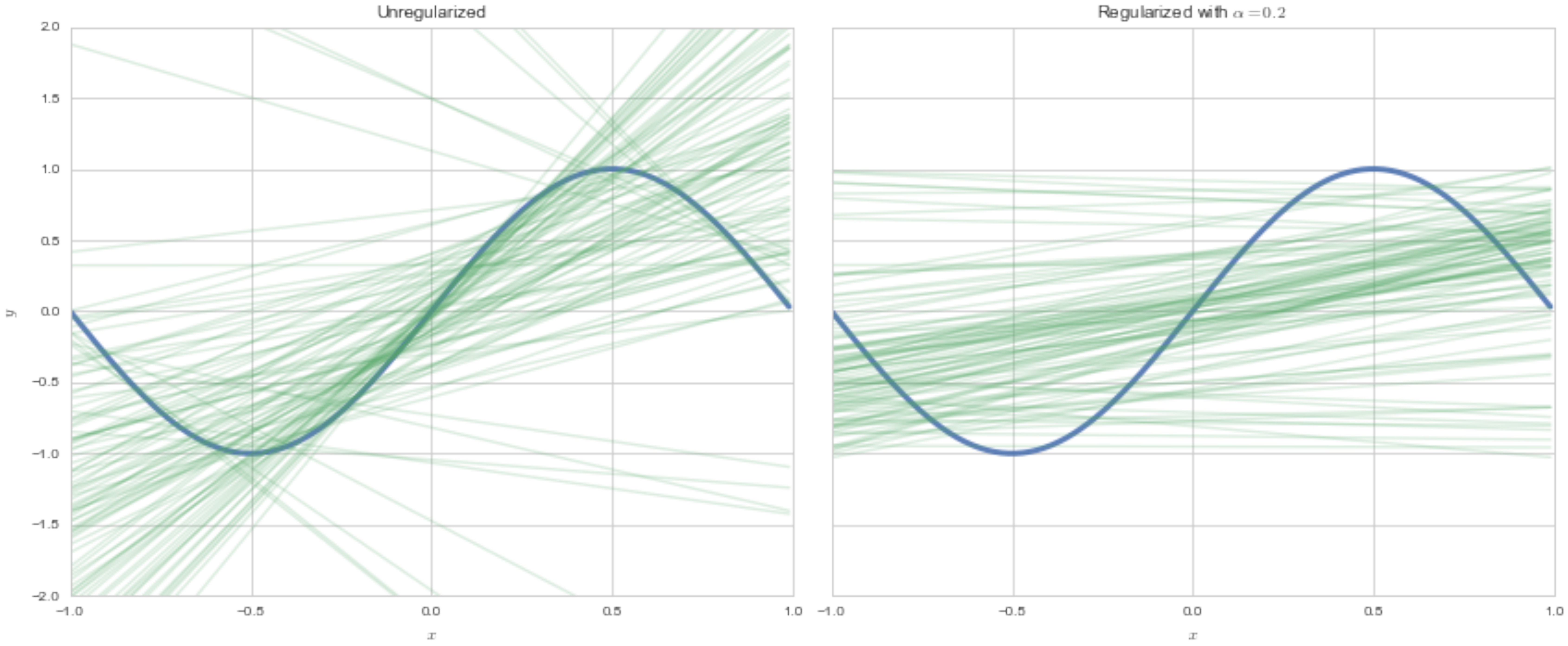
Keep higher a-priori complexity and impose a

complexity penalty

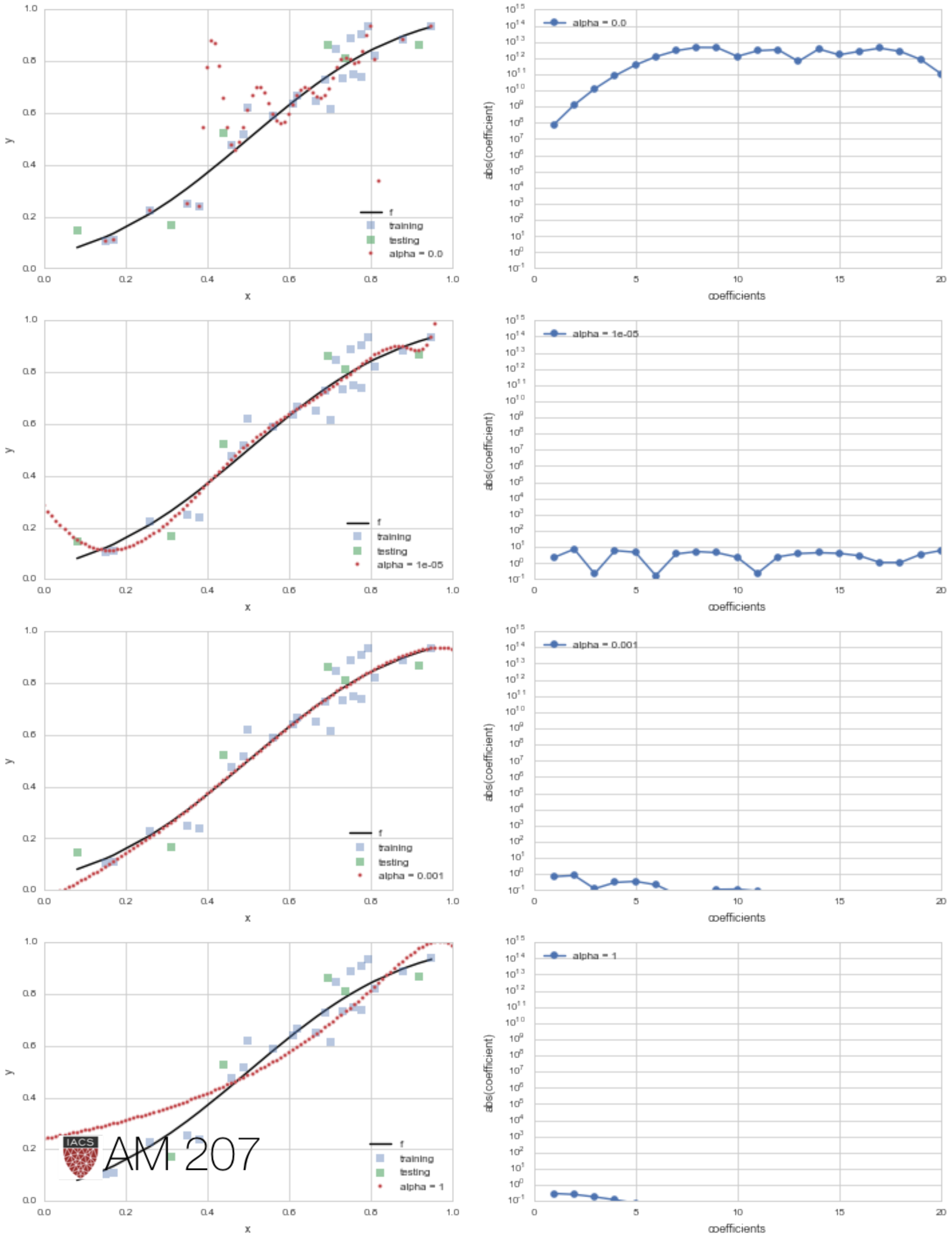
on risk instead, to choose a SUBSET of $\mathcal{H}_{big}$. We'll make the coefficients small:

$$\sum_{i=0}^{j} \theta_i^2 < C.$$





Unregularized
Regularized with $\alpha = 0.2$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
$y$
−1.0
−0.5
0.0
0.5
1.0
$x$

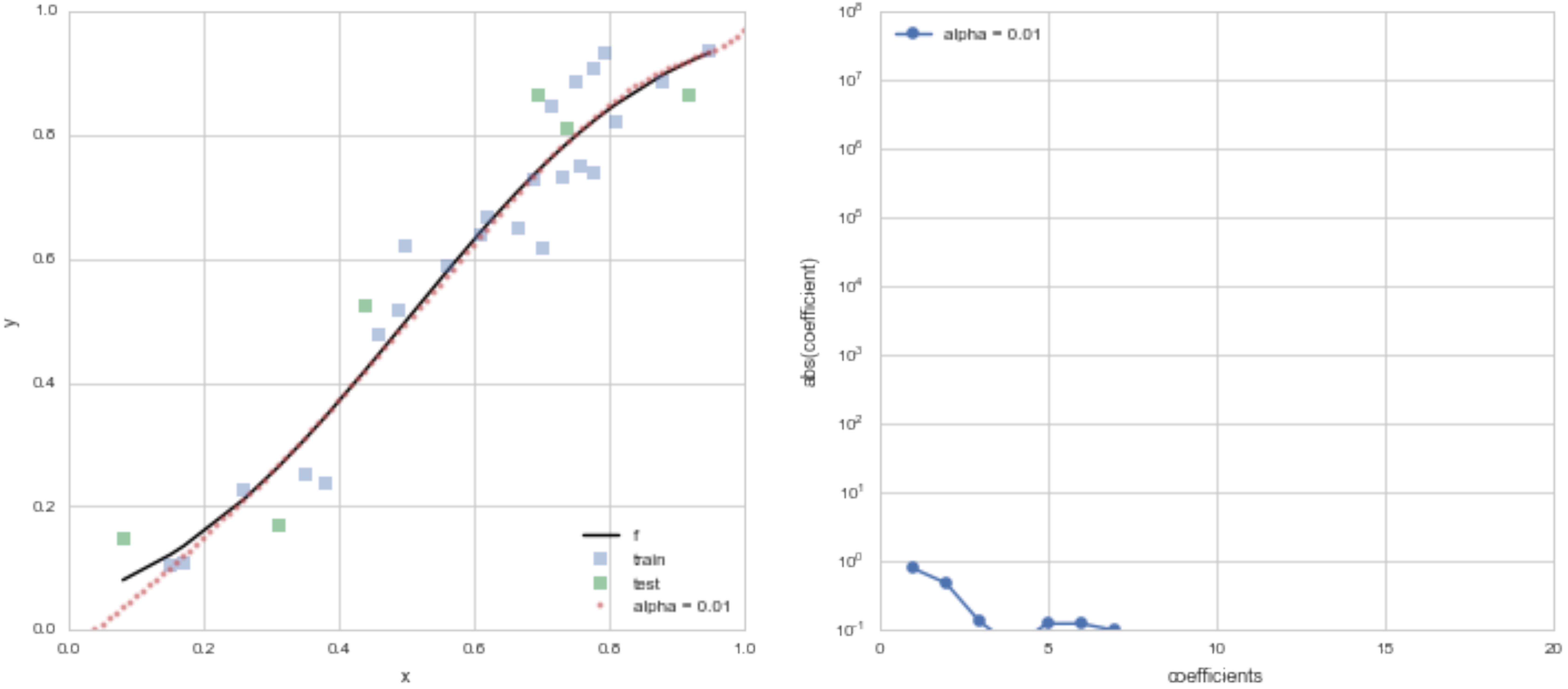



# REGULARIZATION

$$\mathcal{R}(h_j) = \sum_{y_i \in \mathcal{D}} (y_i - h_j(x_i))^2 + \alpha \sum_{i=0}^{j} \theta_i^2.$$

As we increase $\alpha$, coefficients go towards 0.

Lasso uses $\alpha \sum_{i=0}^{j} |\theta_i|$, sets coefficients to exactly 0.


1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
x
f
train
test
alpha = 0.01
alpha = 0.01
abs(coefficient)
0
5
10
15
20
coefficients

# Maximum Likelihood

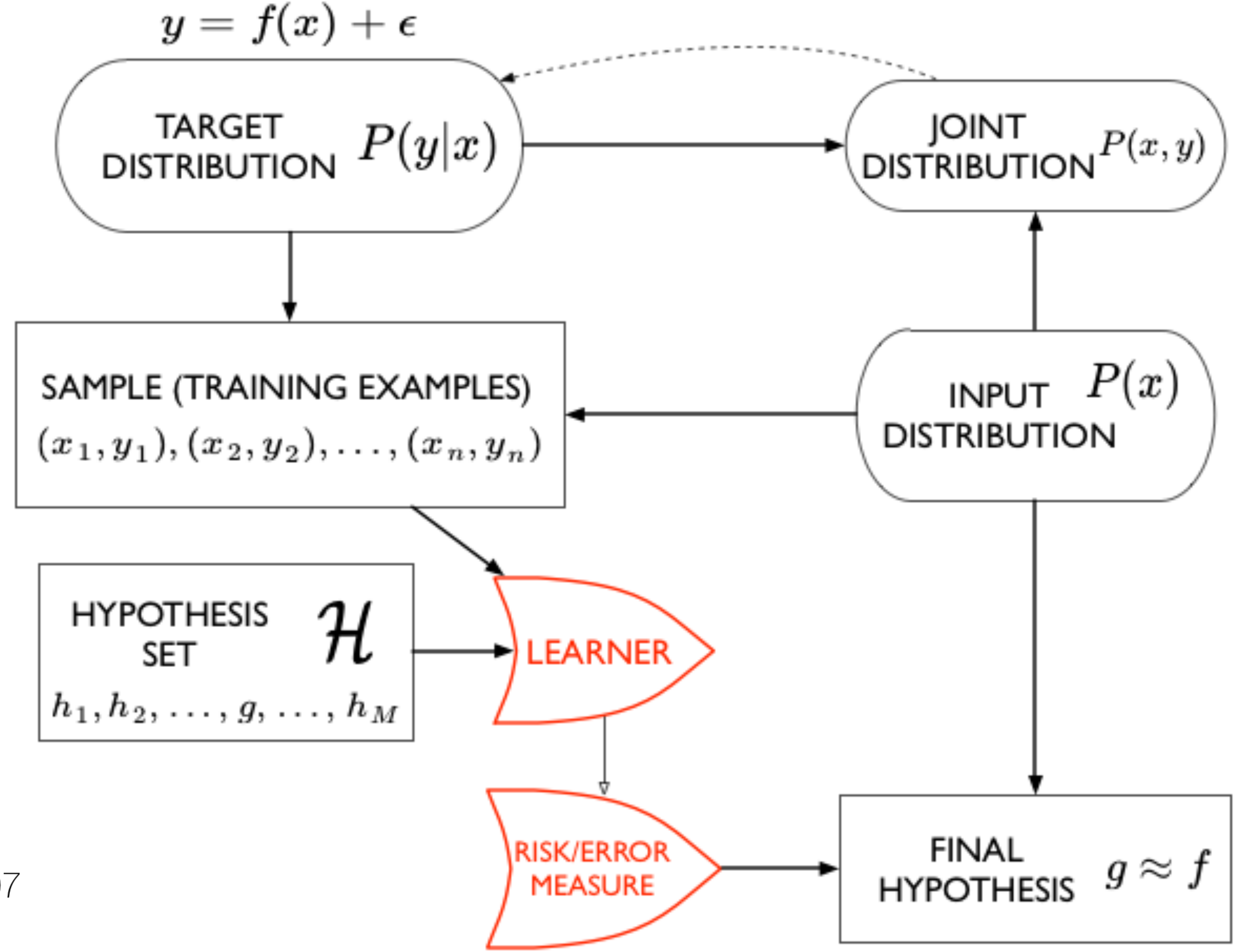
- maximize probability of data given parameters
- $\mathcal{L} = \prod_i p(x_i|\theta)$, instead maximize $\ell = log(\mathcal{L})$
- or minimize a risk -$\ell$
- where do these identifications come from?
- what about overfitting?

# Logistic Regression

Define $h(z) = \dfrac{1}{1+e^{-z}}$.

Then, the conditional probabilities of $y = 1$ or $y = 0$ given a particular sample's features $\mathbf{x}$ are:

$$\begin{aligned} P(y=1|\mathbf{x}) &= h(\mathbf{w}\cdot\mathbf{x}) \\ P(y=0|\mathbf{x}) &= 1 - h(\mathbf{w}\cdot\mathbf{x}). \end{aligned}$$



$y = f(x) + \epsilon$
TARGET DISTRIBUTION $P(y|x)$
JOINT DISTRIBUTION $P(x,y)$
SAMPLE (TRAINING EXAMPLES)
$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
INPUT DISTRIBUTION $P(x)$
HYPOTHESIS SET $\mathcal{H}$
$h_1, h_2, \ldots, g, \ldots, h_M$
LEARNER
RISK/ERROR MEASURE
FINAL HYPOTHESIS $g \approx f$

$$P(y|\mathbf{x},\mathbf{w}) = P(\{y_i\}|\{\mathbf{x}_i\},\mathbf{w}) = \prod_{y_i \in \mathcal{D}} P(y_i|\mathbf{x_i},\mathbf{w}) = \prod_{y_i \in \mathcal{D}} h(\mathbf{w}\cdot\mathbf{x_i})^{y_i}\,(1 - h(\mathbf{w}\cdot\mathbf{x_i}))^{(1-y_i)}$$

$$\begin{aligned}
\ell &= log\left(\prod_{y_i \in \mathcal{D}} h(\mathbf{w}\cdot\mathbf{x_i})^{y_i}\,(1 - h(\mathbf{w}\cdot\mathbf{x_i}))^{(1-y_i)}\right) \\
&= \sum_{y_i \in \mathcal{D}} log\left(h(\mathbf{w}\cdot\mathbf{x_i})^{y_i}\,(1 - h(\mathbf{w}\cdot\mathbf{x_i}))^{(1-y_i)}\right) \\
&= \sum_{y_i \in \mathcal{D}} log\,h(\mathbf{w}\cdot\mathbf{x_i})^{y_i} + log\,(1 - h(\mathbf{w}\cdot\mathbf{x_i}))^{(1-y_i)} \\
&= \sum_{y_i \in \mathcal{D}} (y_i log(h(\mathbf{w}\cdot\mathbf{x})) + (1 - y_i)log(1 - h(\mathbf{w}\cdot\mathbf{x})))
\end{aligned}$$

# What did we learn about learning?

- x-validation: minimizes loss on training, fits hyperparams on validation
- test risk approximates out-of-sample risk
- regularization or complexity selection helps avoid overfitting
- we have seen the context of supervised learning $p(y|x)$

In unsupervised learning, want $p(x)$. Also need to learn these params using MLE or similar.

# KL-Divergence

$$D_{KL}(p,q) = E_p[log(p) - log(q)] = E_p[log(p/q)]$$
$$= \sum_i p_i log(\frac{p_i}{q_i}) \ or \ \int dP log(\frac{p}{q})$$

$$D_{KL}(p,p) = 0$$

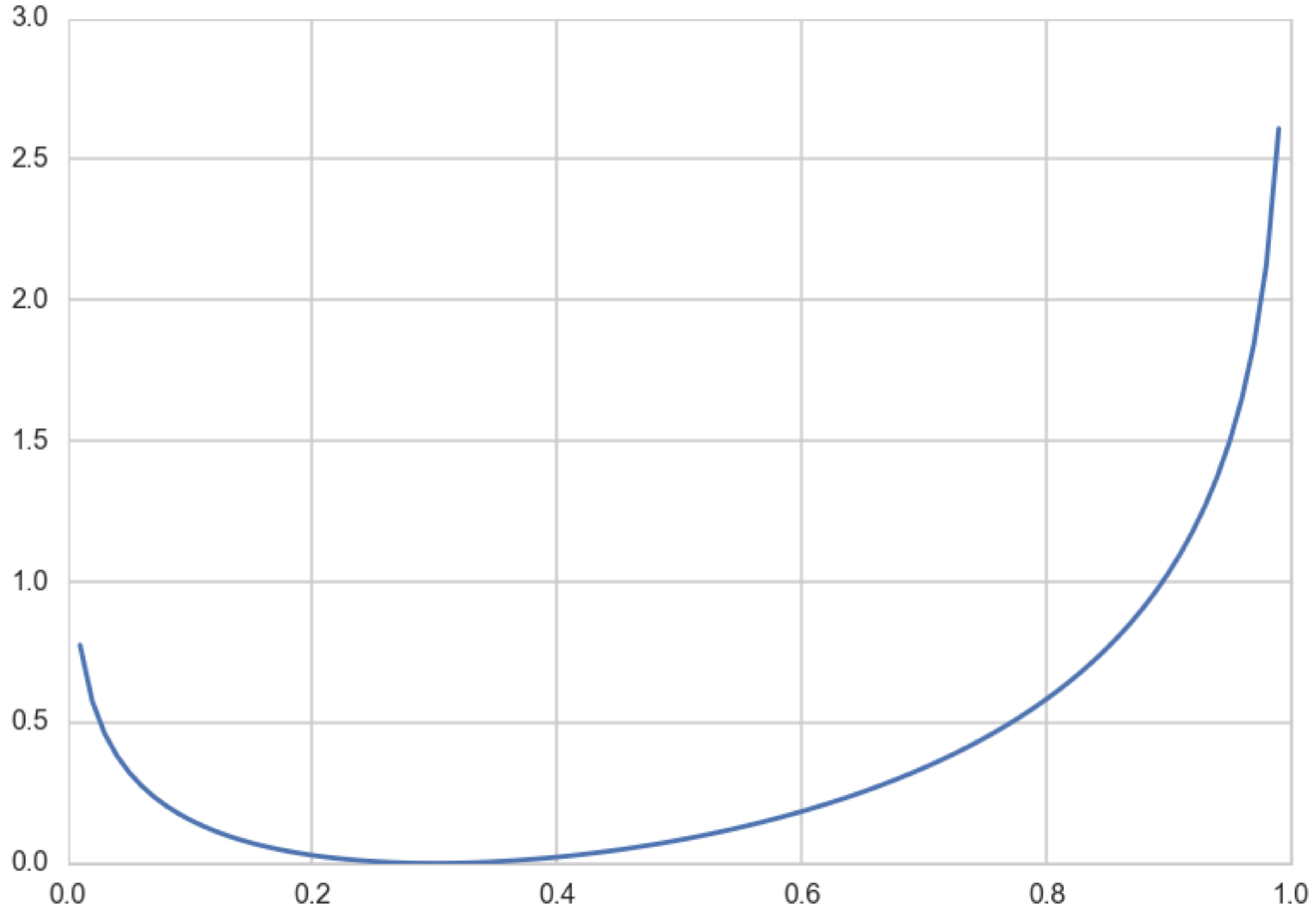
KL divergence measures distance/dissimilarity of the two distributions p(x) and q(x).

# KL example

Bernoulli Distribution p with $p = 0.3$.

Try toapproximate by $q$. What parameter?

```
def kld(p,q):
    return p*np.log(p/q) + (1-p)*np.log((1-p)/(1-q))
```



# KL-Divergence is always non-negative

Jensen's inequality: given a convex function $f(x)$:

$$E[f(X)] \geq f(E[X])$$

$$\implies D_{KL}(p, q) \geq 0 \text{ (0 iff } q = p \, \forall x\text{).}$$

$$D_{KL}(p, q) = E_p[log(p/q)] = E_p[-log(q/p)] \geq -\log(E_p[q/p]) = -\log(\int dQ) = 0$$

PROBLEM: we dont know distribution $p$. If we did, why do inference?

# SOLUTION: Use the empirical distribution

That is, approximate population expectations by sample averages.

So, $E_p[f] \simeq \frac{1}{N} \sum_{i in \mathcal{D}_{train}} f(x_i)$. Go back and see Logistic regression!

# Maximum Likelihood justification

$$D_{KL}(p,q) = E_p[log(p/q)] = \frac{1}{N}\sum_i (log(p_i) - log(q_i))$$

Minimizing KL-divergence $\implies$ maximizing $\sum_i log(q_i)$

Which is exactly the log likelihood! MLE!

# Information and Uncertainty

- coin at 50% odds has maximal uncertainty
- reflects my lack of knowledge of the physics
- many ways for 50% heads.
- an election with $p = 0.99$ has a lot of Information

*information is the reduction in uncertainty from learning an outcome*

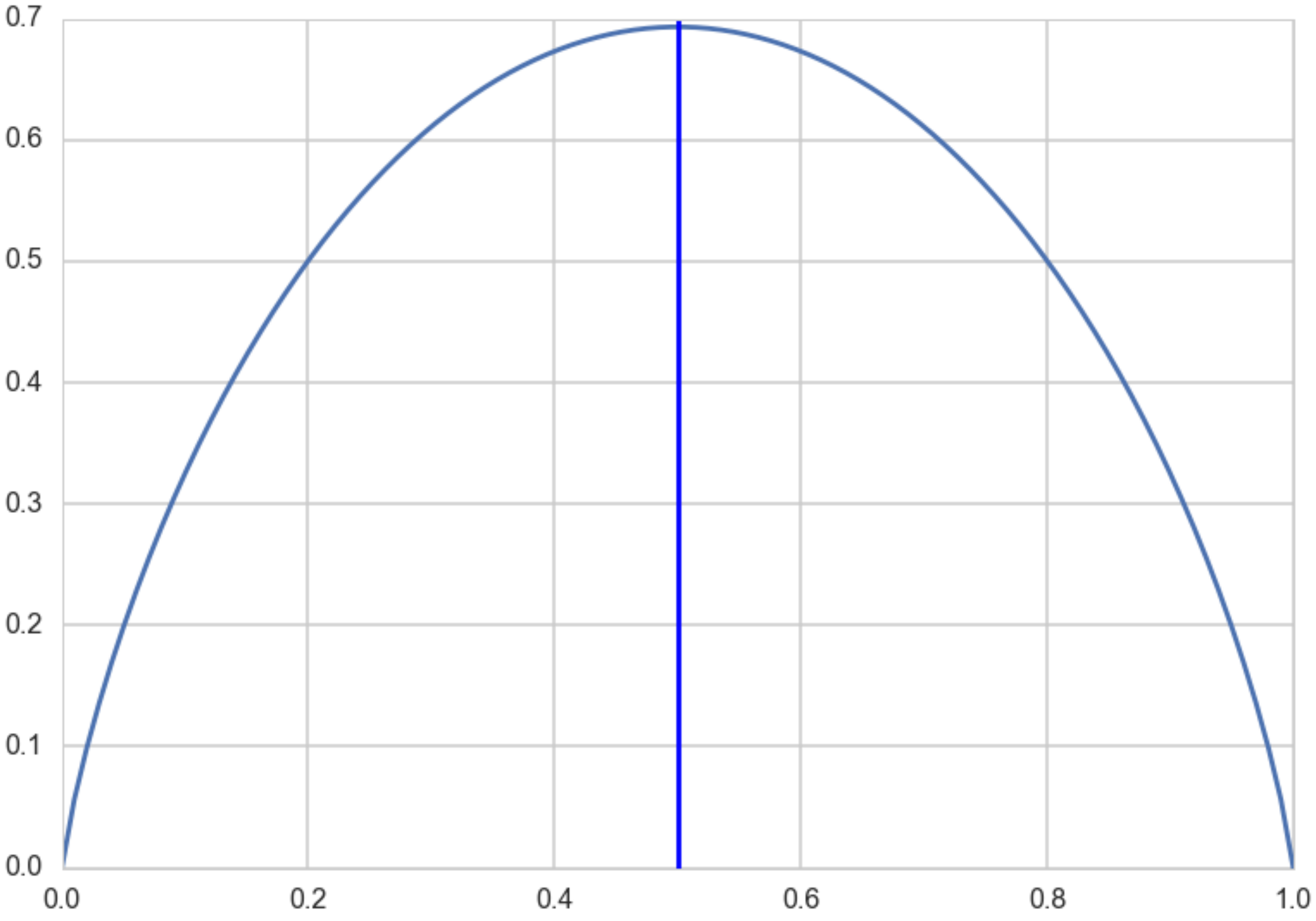
# Information Entropy, a measure of uncertainty

Desiderata:

- must be continuous so that there are no jumps
- must be additive across events or states, and must increase as the number of events/states increases

$$H(p) = -E_p[log(p)] = -\int p(x)log(p(x))dx \;\; OR \; -\sum_i p_i log(p_i)$$

# Entropy for coin fairness




$$H(p) = -E_p[log(p)] = -p * log(p) - (1-p) * log(1-p)$$

```
def h(p):
    if p==1.:
        ent = 0
    elif p==0.:
        ent = 0
    else:
        ent = - (p*math.log(p) + (1-p)* math.log(1-p))
```

# Thermodynamic notion of Entropy

$$P(n_1, n_2, \ldots, n_M) = \frac{N!}{\prod_i n_i!} \prod_i (\frac{1}{M})^{n_i}$$

**Multiplicity:** $W = \frac{N!}{\prod_i n_i!}$

Entropy $H = \frac{1}{N} log(W)$ which is:

$\frac{1}{N} log(P(n_i, n_2, \ldots, n_M))$ **sans constant**

Using Stirling's approximation $log(N!) \sim Nlog(N) - N$ as $N \to \infty$ and where fractions $n_i/N$ are held fixed:

$$H = -\sum_i p_i log(p_i)$$

A particular arrangement $\{n_i\} = (m_1, n_2, n_3, \ldots, n_M)$ is a **microstate** and the overall distribution of $\{p_i\}$, is a **macrostate**.

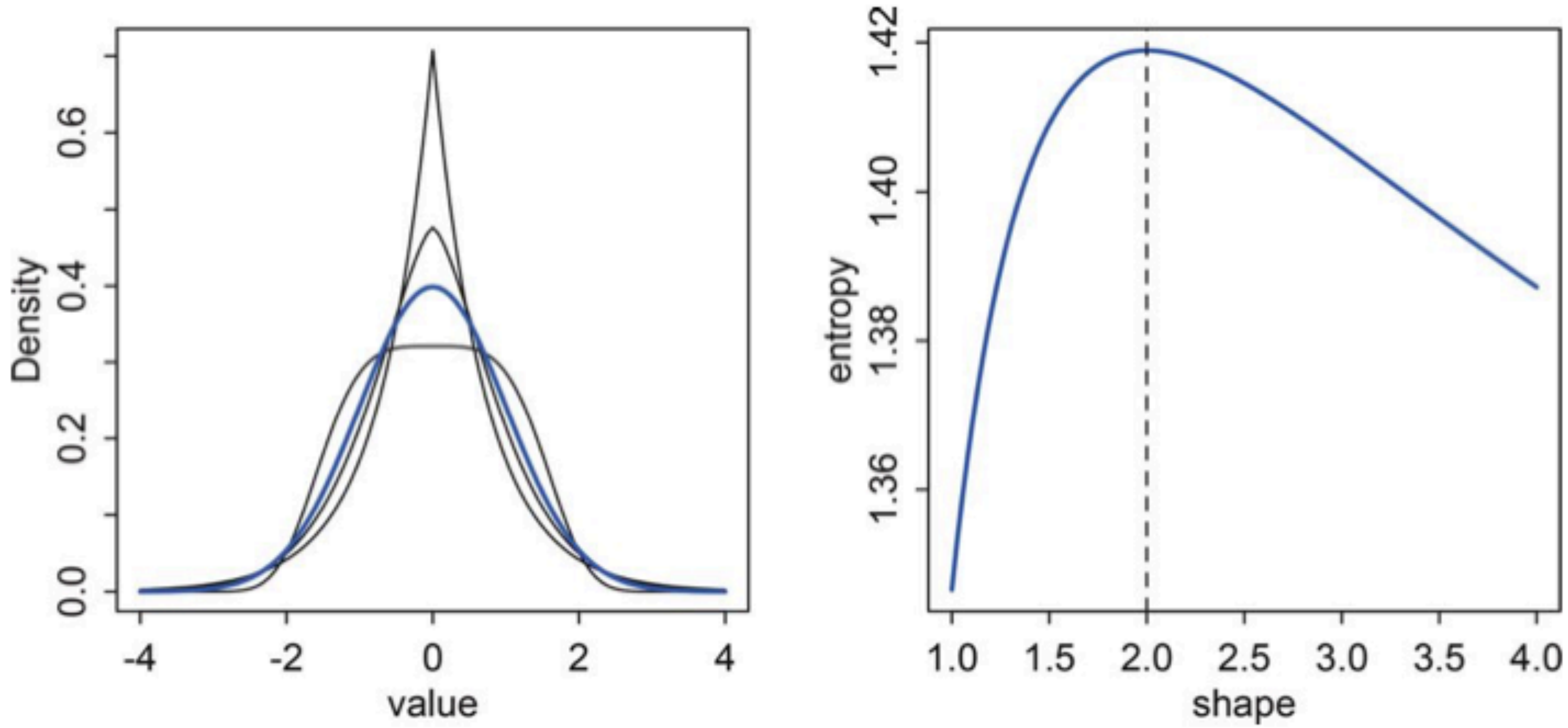
Maximize with Largrange multipliers: $p_j = 1/M$ all equal.

# Maximum Entropy (MAXENT)

- finding distributions consistent with constraints and the current state of our information
- what would be the least surprising distribution?
- The one with the least additional assumptions?

The distribution that can happen in the most ways is the one with the highest entropy

# Normal as MAXENT



# For a gaussian

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$H(p) = E_p[log(p)] = E_p[-\frac{1}{2}log(2\pi\sigma^2) - (x-\mu)^2/2\sigma^2]$$

$$= -\frac{1}{2}log(2\pi\sigma^2) - \frac{1}{2\sigma^2}E_p[(x-\mu)^2] = -\frac{1}{2}log(2\pi\sigma^2) - \frac{1}{2} = \frac{1}{2}log(2\pi e\sigma^2)$$

# Cross Entropy

$$H(p, q) = -E_p[log(q)]$$

Then one can write:

$$D_{KL}(p, q) = H(p, q) - H(p)$$

KL-Divergence is additional entropy introduced by using $q$ instead of $p$.

We saw this for Logistic regression

- $H(p, q)$ and $D_{KL}(p, q)$ are not symmetric.
- if you use a unusual , low entropy distribution to approximate a usual one, you will be more surprised than if you used a high entropy, many choices one to approximate an unusual one.

Corollary: if we use a high entropy distribution to aproximate the true one, we will incur lesser error.

# Back to the gaussian

Consider $D_{KL}(q,p) = E_q[log(q/p)] = H(q,p) - H(q) >= 0$

$$H(q,p) = E_q[log(p)] = -\frac{1}{2}log(2\pi\sigma^2) - \frac{1}{2\sigma^2}E_q[(x-\mu)^2]$$

$E_q[(x-\mu)^2]$ is CONSTRAINED to be $\sigma^2$.

$$H(q,p) = -\frac{1}{2}log(2\pi\sigma^2) - \frac{1}{2} = -\frac{1}{2}log(2\pi e\sigma^2) = H(p) >= H(q)!!!$$

# Importance of MAXENT

- most common distributions used as likelihoods (and priors) are in the exponential family, MAXENT subject to different constraints.
- gamma: MAXENT all distributions with the same mean and same average logarithm.
- exponential: MAXENT all non-negative continuous distributions with the same average inter-event displacement

# Importance of MAXENT

- Information entropy ennumerates the number of ways a distribution can arise, after having fixed some assumptions.
- choosing a maxent distribution as a likelihood means that once the constraints has been met, no additional assumptions.

## The most conservative distribution we could choose consistent with our constraints!

# Model Comparison: Likelihood Ratio

$H(p)$ cancels out!!

$$D_{KL}(p,q) - D_{KL}(p,r) = H(p,q) - H(p,r) = E_p[log(r) - log(q)] = E_p[log(\frac{r}{q})]$$

In the sample approximation we have:

$$D_{KL}(p,q) - D_{KL}(p,r) = \frac{1}{N}\sum_i log(\frac{r_i}{q_i}) = \frac{1}{N} log(\frac{\prod_i r_i}{\prod_i q_i}) = \frac{1}{N} log(\frac{\mathcal{L}_r}{\mathcal{L}_q})$$

# Model Comparison: Deviance

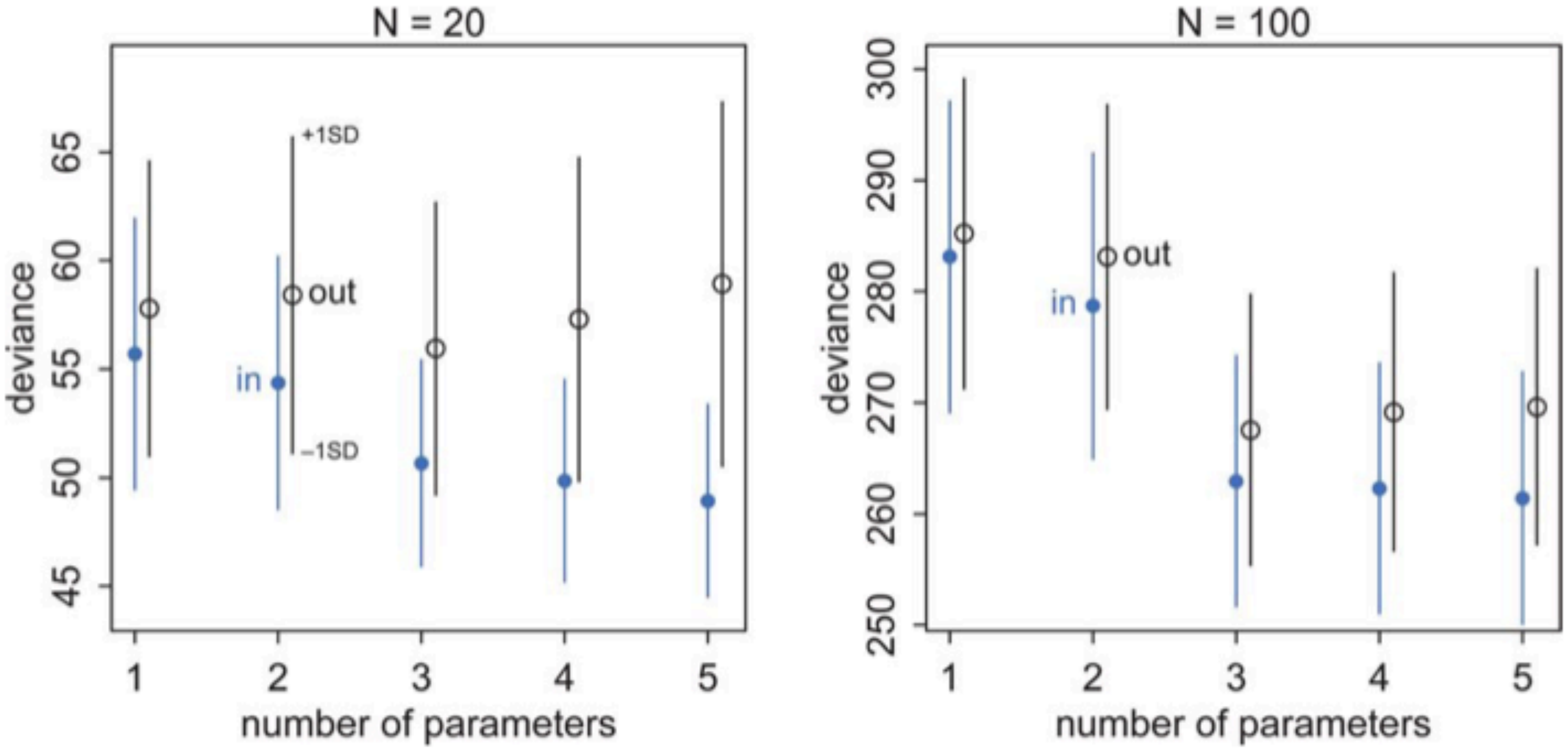
You only need the sample averages of the logarithm of $r$ and $q$:

$$D_{KL}(p,q) - D_{KL}(p,r) = \langle log(r) \rangle - \langle log(q) \rangle$$

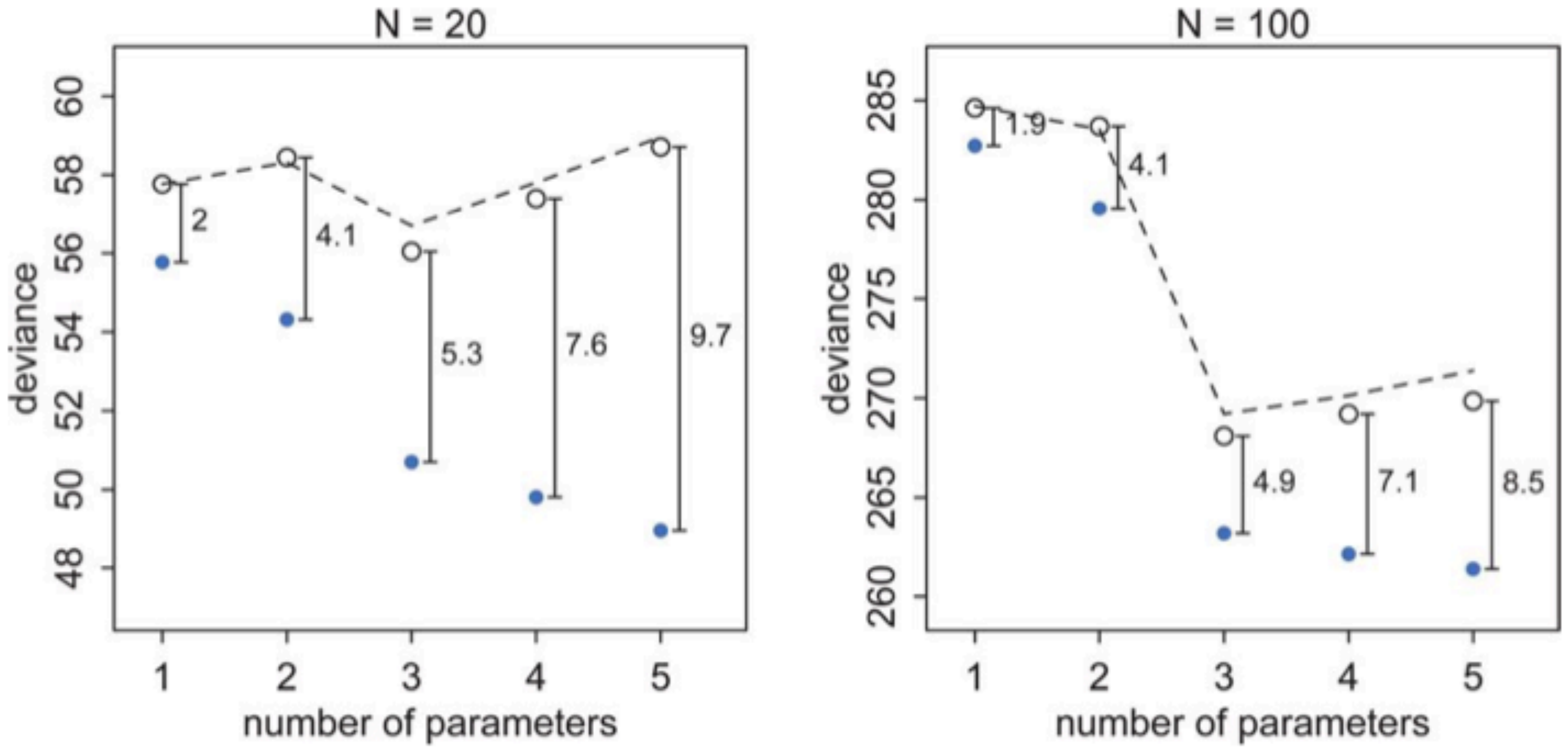
Define the deviance: $D(q) = -2 \sum_i log(q_i)$, a risk (e.g., -$2 \times \ell$, although the distribution need not be a likelihood)...

$$D_{KL}(p,q) - D_{KL}(p,r) = \frac{2}{N}(D(q) - D(r))$$

# Train to Test



# AIC




The test set deviances are $2 * p$ above the training set ones.

# Akake **Information Criterion**:

AIC **estimates out-of-sample deviance**

$$AIC = D_{train} + 2p$$

- Assumption: likelihood is approximately multivariate gaussian.
- penalized log-likelihood or risk if we choose to identify our distribution with the likelihood: REGULARIZATION
- high $p$ increases the out-of-sample deviance, less desirable.