MAT 230 EXAM ONE

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

(a) The domain for all variables in the expressions below is the set of real numbers. **Determine whether each statement is true or false.**

(i) $\forall x \, \exists y \, (x + y \geq 0)$

$\forall x \, \exists y \, (x + y \geq 0)$
The statement is true

(ii) $\exists x \, \forall y \, (x \cdot y > 0)$

This statement asserts that there exists at least one x,
such that for all values of y the product
$x \cdot y$ is grater than zero. But if we take $y = 0$:
$x \cdot 0 = 0$
This doesnt satisfy the inequality, therefore the statement is false.

(b) **Translate each of the following English statements into logical expressions.**

(i) There are two numbers whose ratio is less than 1.

$\exists x \exists y \left(\frac{x}{y} < 1 \right)$

(ii) The reciprocal of every positive number is also positive.

$\forall x \left(x > 0 \to \frac{1}{x} > 0 \right)$

PROBLEM 2

Prove the following using the specified technique:

(a) Let x and y be two real numbers such that $x+y$ is rational. Prove by contrapositive that if x is irrational, then $x-y$ is irrational.

Assume: $x+y$ is rational, this is given.
Assume: $x-y$ is rational, assumption for the contrapositive
If both $x+y$ and $x-y$ are rational numbers, we can add them together:
$(x+y)+(x-y)=2x$
Since the sum of two rational numbers is also rational:
$2x$ is rational.
If $2x$ is rational then dividing by 2, a nonzero
rational number, gives another rational number.
So $x=\frac{2x}{x}$ is rational
Therefore we showed that if $y-x$ is rational,
then x muxt also be rational. By contrapositive this means:
If x is irrational then $x-y$ is irrational.

(b) Prove by contradiction that for any positive two real numbers, x and y, if $x \cdot y \leq 50$, then either $x<8$ or $y<8$.

Assume $x>0$, $y>0$, and $x \cdot y \leq 50$
But assume the conclusion is false, which is:
$x \geq 8$ and $y \geq 8$
Then we have: $x \cdot y \geq 8 \cdot 8 = 64$
So:
$x \cdot y \geq 64 \quad \text{and} \quad x \cdot y \leq 50$
This shows to be a contradicction because
a number cant be both less than or equal to 50 and
greater than or equal to 64
Therefore our assumption is false, so it must be that either:
$x<8$ or $y<8$

Problem 3

Let $n \geq 1$, x be a real number, and $x \geq -1$. **Prove the following statement using mathematical induction.**

$$(1 + x)^n \geq 1 + nx$$

Base case: First we will verify the base case $n = 1$:
For $n = 1$:
$(1 + x)^1 = 1 + x \geq 1 + 1 \cdot x$
This is true for all $x \geq -1$
Inductive step:
Assume the statement is true for $n = m$
$(1 + x)^m \geq 1 + mx \quad \forall x \geq -1$
We have to show that this is true for:
$n = m + 1$
$(1 + x)^{m+1} = (1 + x)(1 + x)^m$
Now to use the inductive hypothesis:
$(1 + x)^{m+1} \geq (1 + x)(1 + mx)$
Now expand the right side out:
$(1 + x)(1 + mx) = 1 + mx + x + mx^2$
Simplify: $1 + (m + 1)x + mx^2$
Now to show:
$1 + (m + 1)x \geq 1 + (m + 1)x$
This is true.
Thus, By induction the statement holds for all $n \geq 1$

Problem 4

Solve the following problems:

(a) How many ways can a store manager arrange a group of 1 team leader and 3 team workers from his 25 employees?

Choose 1 team leader from 25:$\binom{25}{1}$
Choose 3 team workers from the remaining 24:
$\binom{24}{3}$
So the total number of ways is:
$\binom{25}{1} \cdot \binom{24}{3}$
$= 25 \cdot \frac{24 \cdot 23 \cdot 22}{6} = \boxed{50600}$

(b) A states license plate has 7 characters. Each character can be a capital letter ($A - Z$), or a non-zero digit ($1 - 9$). How many license plates start with 3 capital letters and end with 4 digits with no letter or digit repeated?

three distinct capital letter $A - Z$ $26 \cdot 25 \cdot 24$
four distinct digits from $1 - 9$:
$10 \cdot 9 \cdot 8 \cdot 7$
The total license plates:
$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$
$\boxed{78624000}$

(c) How many binary strings of length 5 have at least 2 adjacent bits that are the same ("00" or "11") somewhere in the string?

Total num of binary strings of length five is:
$2^5 = 32$
We subtact the num of binary strings with no adjacent bits equal.
These alternate between zero and one;
Valid alternating strings of length five: 01010 and 10101
So the num of binary strings with at two adjacent equal bits is:
$32 - 2 = \boxed{30}$

Problem 5

A class with n kids lines up for recess. The order in which the kids line up is random with each ordering being equally likely. There are two kids in the class named Betty and Mary. The use of the word "*or*" in the description of the events, should be interpreted as the inclusive or. That is "*A or B*" means that A is true, B is true, or both A and B are true.

What is the probability that Betty is first in line or Mary is last in line as a function of n? Simplify your final expression as much as possible and include an explanation of how you calculated this probability.

We will compute $P(A \cup B)$,
the probability that betty is first OR mary is last
Using the formula for the union of two events:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Given the lineup is random and position is equally likely:
$P(A) = \frac{1}{n}, \quad P(B) = \frac{1}{n}$
To compute:
Fix betty in the first position and mary in last.
The remaining $n-2$ kids can be arranged in $(n-2)!$
The total num of arrangements of n kids is $n!$
Therefore:
$P(A \cap B) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$
Now we will substitube:
$P(A \cup B) = \frac{1}{n} + \frac{1}{n} - \frac{1}{n(n-1)}$
$= \frac{2}{n} - \frac{1}{n(n-1)}$
Simplify:
$P(A \cup B) = \frac{2(n-1)-1}{n(n-1)} = \frac{2n-3}{n(n-1)}$
$P = \frac{2n-3}{n(n-1)}$

PROBLEM 6

The general manager, marketing director, and 3 other employees of Company A are hosting a visit by the vice president and 2 other employees of Company B. The eight people line up in a random order to take a photo. Every way of lining up the people is equally likely.

(a) What is the probability that the general manager is next to the vice president?

There are 8 total arrangements of the 8 people.
There are 7 possible positions for the GM-VP block and for each such
position, GM and VP can be arranged in 2 ways in the block. The other 6 people are arranged in 6! ways.
favorable outcomes $= 7 \times 2 \times 6!$
total outcomes $= 8!$
$(GMnexttoVP) = \frac{7\times2\times6!}{8!}$
$\frac{1}{4}$

(b) What is the probability that the marketing director is in the leftmost position?

There are 8 people in total and each one has
an equal chance of being in the leftmost position
MD in leftmost position $= \frac{1}{8}$

(c) Determine whether the two events are independent. Prove your answer by showing that one of the conditions for independence is either true or false.

$A =$ event that GM next to VP
$B =$ event that MD is leftmost
We know:
$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{8}$
$\rightarrow P(A) \cdot P(B) = 132$
There are now 7 people left and
we treat the GM and VP as a block:
favorable outcomes $= 6 \times 2 \times 5! = 1440$
total arrangements w MD fixed $= 7! = 5040$
$P(A \cap B) = \frac{1440}{5040} = \frac{2}{7}$
since;
$P(A \cap B) = \frac{2}{7} \neq \frac{1}{32}$
$= P(A) \cdot P(B)$
The two events are not independent.