MAT 330: Differential Equations

Project Two Template

Complete this template by replacing the bracketed text with the relevant information.

[JoAnn Olney]

[6/16/25]

Problem 1: Consider the system of differential equations $\frac{dx_1}{dt} = x_1 + 3x_2$, $\frac{dx_2}{dt} = 4x_1 + 5x_2$ with initial conditions $x_1(0) = 2$ and $x_2(0) = -6$.

a) Compute the eigenvalues and eigenvectors of the system.

Solution:

[Insert your math/explanation here.]

$$\frac{\mathrm{dx}_1}{\mathrm{dt}} = x_1 + 3x_2$$

$\frac{\mathrm{dx}_2}{\mathrm{dt}} = 4x_1 + 5x_2,$

Where we can put the system in matrix form; $\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{AX}$, where X=$[x_1, x_2]^T$

A ia the coefficient matrix: A = $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$

We will need to use the the characteristic equation to find the eigenvectors and eigenvalues; $\det(A - \lambda I) = 0$, Then sub into the equation;

$\det\left(\begin{bmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix}\right) = 0$
$(1-\lambda)(5-\lambda) - 3(4) = 0$
$\lambda^2 - 6\lambda - 7 = 0$
$(\lambda - 7)(\lambda + 1) = 0$

Now that it has been factored the eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = -1$

To fine the eigenvectors we will solve the syestem $(A-\lambda I)v=0.$ v will be the eigenvector.

I started with $\lambda_1 = 7$;

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} v = 0$$

Then we solve the equations for this and simplify,

$$-6v_1 + 3v_2 = 0$$

$$4v_1 - 2v_2 = 0$$

$$v_1 = \frac{1}{2}v_2$$

This will give us the eigenvector for $\lambda_1 = 7$ as; $v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

If we normalize this eigenvector we will get; $u_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{\sqrt{5}}{\sqrt{2}} \end{bmatrix}$

For $\lambda_2 = -1$ we will have; (A+I)v=0

We will sub A and λ_2 into the equation and then solve the equations;

$$\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} v = 0$$

$2v_1 + 3v_2 = 0$
$4v_1 + 4v_2 = 0$ then we will simplif;

$$v_1 = -\frac{3}{2}v_2$$

Which will give us the eigenvector for $\lambda_2 = -1$;

$$v_2 = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

If we normalize this eigenvector we will get; $u_2 = \begin{bmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{bmatrix}$

b) Compute the solution to the system of differential equations.

Solution:

[Insert your math/explanation here.]

The general solution to the system of diff equations is; $x(t) = \mathrm{c1} * e^{7t}v_1 + \mathrm{c2e}^{-t} * v_2$

We use the initial condition of c1 and c2 as;

$X(0) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, then we will solce for c1 and c2.

$$\mathrm{c1} = \frac{6}{7} * v_2[0] + \frac{2}{7} * v_1[0] = -\frac{10}{7}$$
$$\mathrm{c2} = \frac{6}{7} * v_2[1] + \frac{2}{7} * v_1[1] = -\frac{26}{7}$$

Using the initial conditions x1(0)=2, x2(0)=-6 the solution to the system will be;

$$x(t) = \left(-\frac{10}{7}\right)e^{-7t}[1,2] + \left(-\frac{26}{7}\right)e^{-t}[-3,4]$$

c) Use MATLAB to plot the direction field and eigenvectors of the system.

Solution:

```
%[Insert your MATLAB code and plot here.]
%define the matrix
A = [1 3;4 5];
% calculate the eigenvalues and eigenvectors
[V, D] = eig(A);

%normalize the eigenvectors by the min components
v1 = V(:, 1)./min(V(:,1))
```

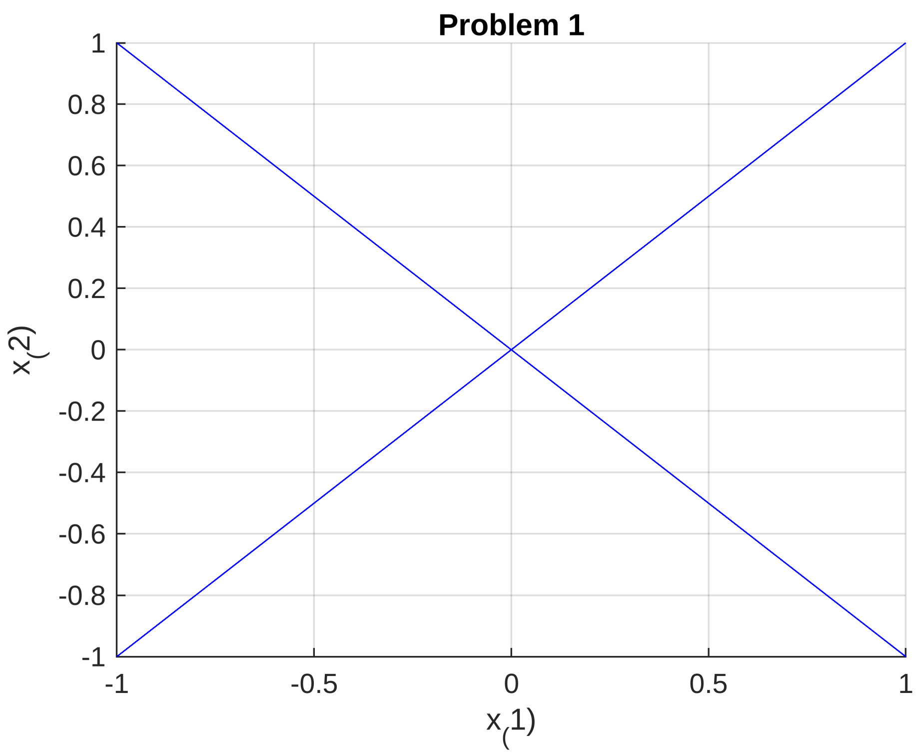
```
v1 = 2×1
    1.0000
   -0.6667
```

```
v2 = V(:, 2)./min(V(:,2))
```

```
v2 = 2×1
    0.5000
    1.0000
```

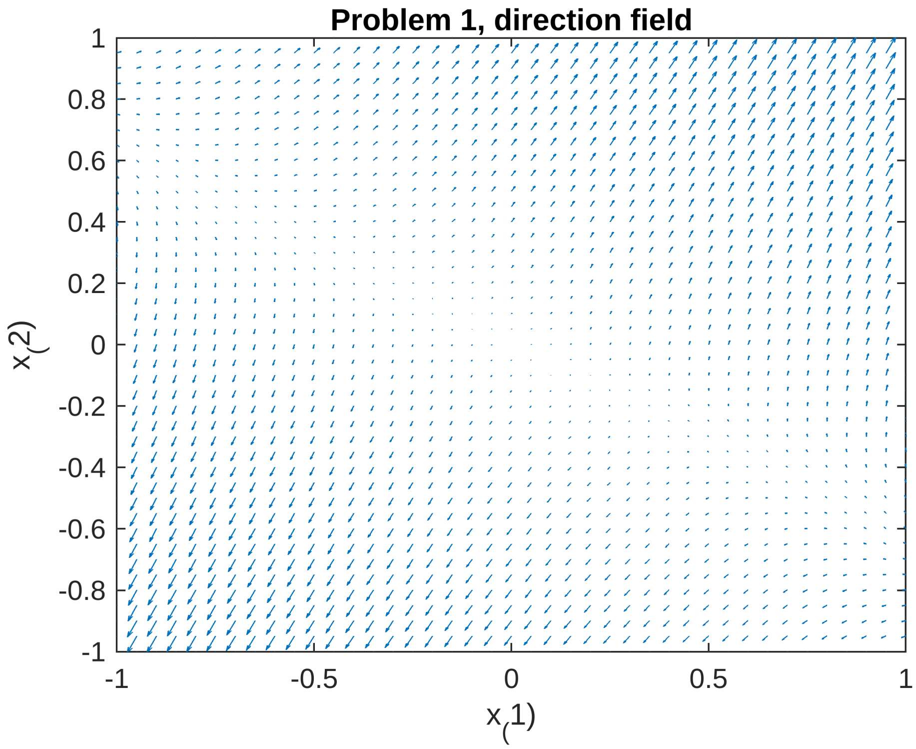
```

%define the range for the vector field
x1Vec = -1:0.05:1;
x2Vec = -1:0.05:1;
%define corresponding lines
vec1 = (1/-1)*x1Vec;
vec2 = (1)*x1Vec;
```

```
%plot the solution
figure;
%hold
hold on;
plot(x1Vec, vec1, 'b');
plot(x1Vec, vec2, 'b');
%label axes and title
xlabel('x_(1)');
ylabel('x_(2)');
title('Problem 1');
%grid
grid on;
%limits
xlim([-1, 1]);
ylim([-1, 1]);
%hold
hold off;
```




```
[x1, x2] = meshgrid(x1Vec, x2Vec);
x1dot = x1 + 3*x2;
x2dot = 4*x1 + 5*x2;

%plot the solution
figure;
quiver(x1,x2,x1dot,x2dot);
%label axes and title
xlabel('x_(1)');
ylabel('x_(2)');
```

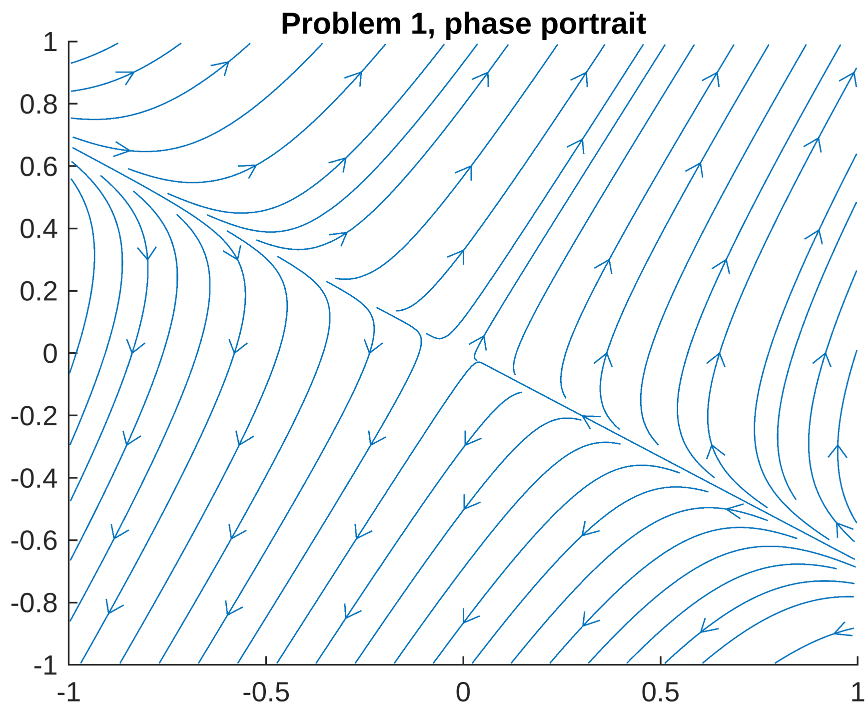
```
title('Problem 1, direction field')
%limits
xlim([-1, 1]);
ylim([-1, 1]);
```



d) Use MATLAB to plot the phase portrait of the system.

Solution:

```
%[Insert your MATLAB code and plot here.]
%plot
figure;
streamslice(x1Vec,x2Vec,x1dot,x2dot);
%title
title('Problem 1, phase portrait');
```



d) Use your phase portrait or numerical analysis to identify and classify any critical points.

Solution:

[Insert your math/explanation here.]

The flow of $x1$ and $x2$ towards negative infinity shows that theres a critical point at (0,0) but it is not stable. Since the trajectory deviates to large negative values, this typically aligns with a saddle point, becausee there is one eigenvalue with a positive real part and one with a negative real part, indicating stability in one direction and instability in another.

Problem 2: Consider the system of differential equations

$$\frac{dx_1}{dt} = -3x_1 - 200x_2, \frac{dx_2}{dt} = 200x_1 - 3x_2.$$

a) Find the linearly independent solutions $x_1(t)$ and $x_2(t)$.

Solution:

[Insert your math/explanation here.]

The matrix for the equation willl be;

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 200 \\ 200 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

x'=Ax where $x' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} -3 & 200 \\ 200 & -3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Finding the eigenvalues of matrix A we will;

$|A - \lambda I| = \left| \begin{bmatrix} -3-\lambda & -200 \\ 200 & -3-\lambda \end{bmatrix} \right| = 0$ in turn giving us;

$$\lambda^2 + 6\lambda + 40009 = 0$$
$$\lambda = -3 \pm 200i$$

This will give us eigenvector of;

$\lambda_1 = -3 + 200i$ $\quad\quad\quad\quad$ $\lambda_1 = -3 - 200i$

$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The two linearly independent solutions x1(t) and x2(t) will be;

$$x_1(t) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(200t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(200t) \right) e^{-3t} = \begin{bmatrix} -\sin(200t) \\ \cos(200t) \end{bmatrix} e^{-3t}$$

$$x_2(t) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(200t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(200t) \right) e^{-3t} = \begin{bmatrix} \cos(200t) \\ \sin(200t) \end{bmatrix} e^{-3t}$$

b) Construct the fundamental matrix $\Phi(t)$.

Solution:

[Insert your math/explanation here.]

The matrix is;

$$\Phi(t) = \begin{bmatrix} -\sin(200t) & \cos(200t) \\ \cos(200t) & \sin(200t) \end{bmatrix} e^{-3t}$$

c) Compute $\Phi(0)$ and $\Phi(0)^{-1}$.

Solution:

[Insert your math/explanation here.]

$$\Phi(0) = \begin{bmatrix} -\sin(200*0) & \cos(200*0) \\ \cos(200*0) & \sin(200*0) \end{bmatrix} e^{-3*0}$$
$$\Phi(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then we have;

$$\Phi^{-1}(0) = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\Phi^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

d) Use your results (b) and (c) to find the solution to the initial value problem $x_1(0) = 2$ and $x_2(0) = -10$.

Solution:

[Insert your math/explanation here.]

Using the initial values of x1(0)=2 and x2(0)=-10;

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -10 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This gives us;

$$\begin{bmatrix} 2 \\ -10 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$$

Then substituting the values into the solution;

$$x(t) = e^{-3t}\left(-10\begin{bmatrix} -\sin(200t) \\ \cos(200t) \end{bmatrix} + 2\begin{bmatrix} \cos(200t) \\ \sin(200t) \end{bmatrix}\right)$$

Then simplify to;

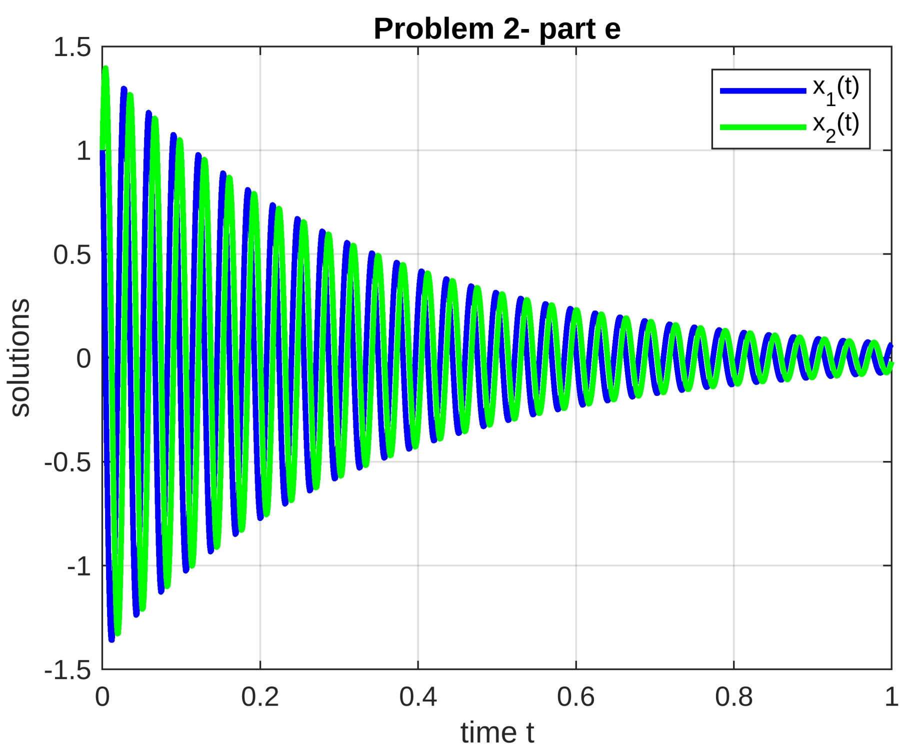
$$x(t) = e^{-3t}\begin{bmatrix} 10\sin(200t) + 2\cos(200t) \\ -10\cos(200t) + 2\sin(200t) \end{bmatrix}$$

e) Use MATLAB to plot your solutions for $x_1(t)$ and $x_2(t)$ on a single figure for time $t = 0$ to $t = 1$.

Solution:

```
%[Insert your MATLAB code and plot here.]
function dXdt=problem2ODE(~,X)
%define the system of equations
x1= X(1);
x2 = X(2);
dx1dt = -3*x1 - 200*x2;
dx2dt = 200*x1 -3*x2;
dXdt = [dx1dt; dx2dt];
```

```
end

%define initial condition
X0 = [1; 1];

%define time span
tspan = [0 1];

%solve the system using ode45
[t, X] = ode45(@problem2ODE, tspan, X0);

%plot the solution
figure;
plot(t, X(:,1), 'b', 'LineWidth', 2);
%hold
hold on;
plot(t, X(:,2), 'g', 'LineWidth', 2);
%label axes and title
xlabel('time t');
ylabel('solutions');
title('Problem 2- part e');
%legend
legend('x_1(t)', 'x_2(t)');
grid on;
```



f) Use the MATLAB ode45() function to solve the original system of equations and plot your solutions on a new figure. The results provided by ode45() and your solution from above should match.

Solution:

```
%[Insert your MATLAB code and plot here.]
function dXdt=problemf2ODE(~,X)
x1 = X(1);
x2 = X(2);
dx1dt = -3*x1 -200*x2;
dx2dt = 200*x1 -3*x2;
dXdt = [dx1dt; dx2dt];
end

%time span
tspan = [0 1];

%initial cond
X0 = [0;1];

%solve using ode45
[t, X] = ode45(@problemf2ODE, tspan, X0);

%plot the equation
figure;
plot(t, X(:,1), 'b', 'LineWidth', 2);
hold on;
plot(t, X(:,2), 'g', 'LineWidth', 2);
%labels
xlabel('time');
ylabel('solution');
title('solution of org system');
legend('x_(t)', 'x_2(t)');
grid on;
```

Problem 3: Consider the differential equation

$$\frac{d^3y}{dt^3} + \frac{19}{12}\frac{d^2y}{dt^2} + \frac{19}{24}\frac{dy}{dt} + \frac{1}{8}y(t) = f(t)$$

that describes a system completely at rest, where the input applied is $f(t) = 1$ for $1 \leq t < 10$ and zero elsewhere.

a) Write the input function $f(t)$ as a difference of time-shifted unit step functions.

Solution:

[Insert your math/explanation here.]

This can be shown as the difference of 2 time shifted unit step functions, where u(t) is the unit step funtion ;

$$f(t) = u(t-1) - u(t-10)$$

b) Take the Laplace transform of the differential equation and solve for $Y(s)$. Use partial fraction expansion (PFE) to simplify your answer.

Solution:

[Insert your math/explanation here].

Using the linearity property of the Laplace transform, we will get;

$$L\left\{\frac{d^3y}{\mathrm{dt}^3}\right\} + \frac{19}{12} * L\left\{\frac{d^2t}{\mathrm{dt}^2}\right\} + \frac{19}{24} * L\left\{\frac{\mathrm{dy}}{\mathrm{dt}}\right\} + \frac{1}{8} * L\{y(t)\} = L\{f(t)\}$$

The Laplace tranforms of the derivatives will be;

$$L\left\{\frac{\mathrm{dy}}{\mathrm{dt}}\right\} = \mathrm{sY}(s) - y(0)$$
$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - \mathrm{sy}(0) - y(0)$$
$$L\left\{\frac{d^3y}{\mathrm{dt}^3}\right\} = s^3Y(s) - s^2y(0) - \mathrm{sy}(0) - y(0)$$

This is where y'''(0), y''(0), y'(0) are the initial conditions of y(t) which then can be assumed to be 0. Then we sub them back into the equation;

$$s^3Y(s) - s^2y(0) - \mathrm{sy}(0) - y(0) + \frac{19}{12}\bigg) * (s^2Y(s) - \mathrm{sy}(0) - y(0)) + \left(\frac{19}{24}\right) * (\mathrm{sY}(s) - y(0)) + \left(\frac{1}{8}\right) * Y(s) = L\{[f(t)]\}$$

Solve for Y(s);

$$Y(s)\left[s^3 + \left(\frac{19}{12}\right)s^2 + \left(\frac{19}{24}\right)s + \frac{1}{8}\right] = L\{f(t)\}$$
$$Y(s) = \frac{L\{f(t)\}}{s^3 + \left(\frac{19}{12}\right)s^2 + \left(\frac{19}{24}\right)s + \frac{1}{8}}$$

c) Find the solution of the differential equation by computing $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Solution:

[Insert your math/explanation here.]

to find the solution we 1st have to find L{f(t)} using the Laplace transform of the unit step fuction u(t);

$$L\{u(t-1)\} = \frac{e^{-s}}{s}$$

$$L\{u(t-10)\} = \frac{e^{-10s}}{s}$$

now we use the linearity property;

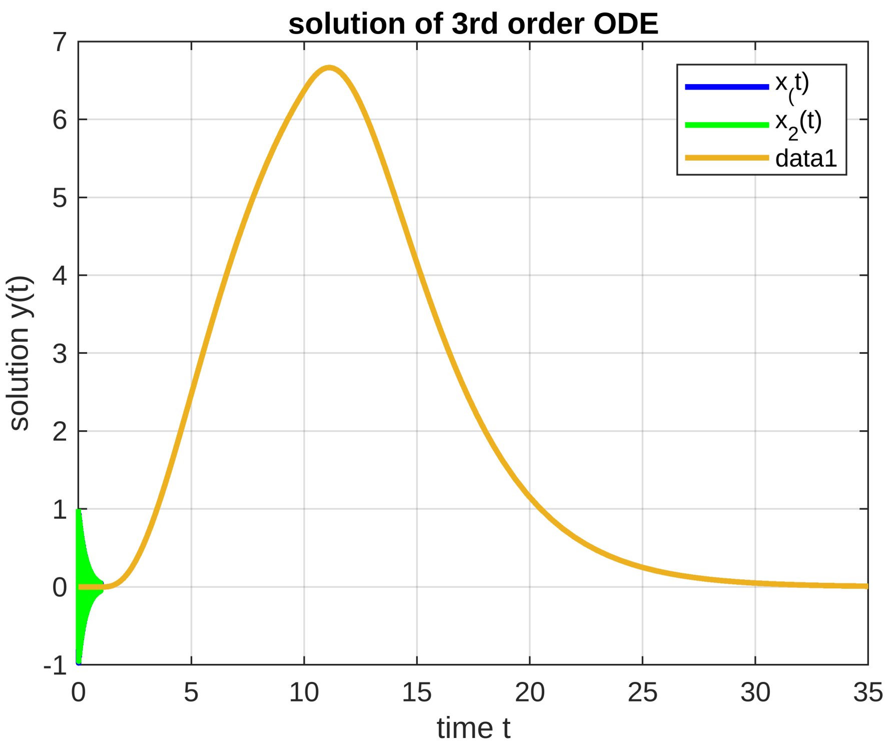
$$L\{f(t)\} = L\{u(t-1) - u(t-10)\} = \left(\frac{e^{s}}{s}\right) - \left(\frac{e^{-10s}}{s}\right)$$

d) Use MATLAB to plot your solution for time $t = 0$ to $t = 35$.

Solution:

```
%[Insert your MATLAB code and plot here.]
%define the system
function dydt = odeSystem(t, y)
    %def the ode system
    if t > 1 && t < 10
        f = 1;
    else
        f = 0;
    end

    dydt = zeros(3,1);
    %y1'
    dydt(1) = y(2);
    %y2'
    dydt(2) = y(3);
    %y3'
    dydt(3) = f - (19/12)*y(3) - (19/24)*y(2) - (1/8)*y(1);
end

%time span
tspan = [0, 35];

%initial cond
y0 = [0; 0; 0];

%solve the equation using ode45
[t, y] = ode45(@odeSystem, tspan, y0);

%plot the solution
plot(t, y(:,1), 'LineWidth', 2);
%labels
xlabel('time t');
ylabel('solution y(t)');
title('solution of 3rd order ODE');
```

```
grid on;
```



e) Re-write the 3rd order differential equation as a system of three first-order differential equations.

Solution:

[Insert your math/explanation here.]

We can start by defining the new variables as;

$$\mathrm{x1} = y, \mathrm{x2} = \frac{\mathrm{dy}}{\mathrm{dt}}, \text{and } \mathrm{x3} = \frac{d^2y}{\mathrm{dt}^2}$$

Then we will want to subsitute these into the original equation;

$$\frac{\mathrm{dx}^3}{\mathrm{dt}} + \left(\frac{19}{12}\right)\frac{\mathrm{dx}^2}{\mathrm{dt}} + \left(\frac{19}{24}\right)\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{1}{8}\mathrm{x1} = f(t)$$

$$\frac{\mathrm{dx}^2}{\mathrm{dt}} = \mathrm{x3}$$

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{x2}$$

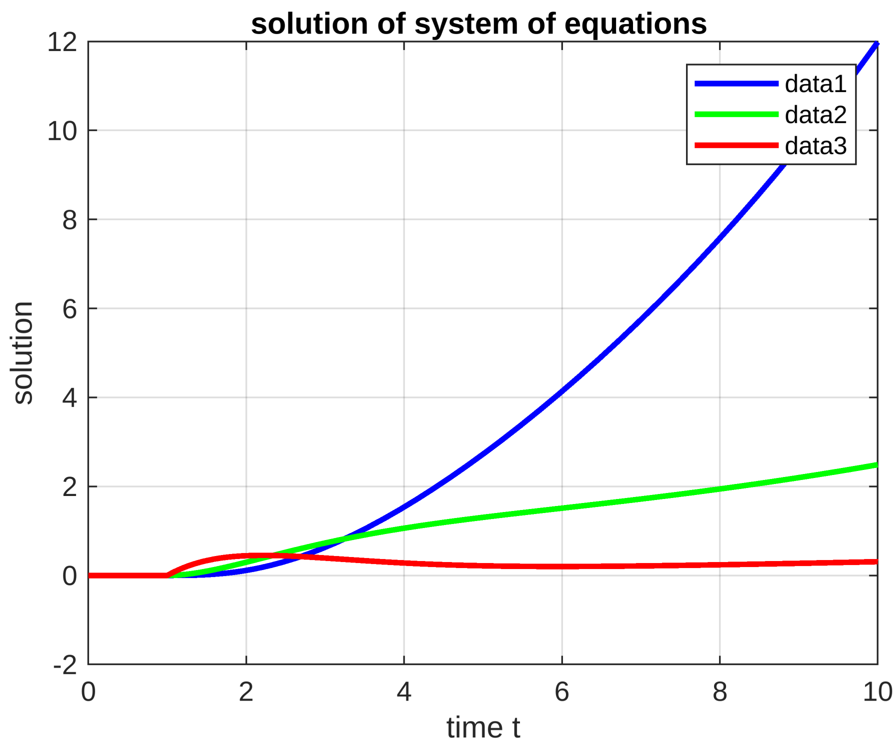
So the equation in first order form will be;

$$\frac{\mathrm{dx}^1}{\mathrm{dt}} = \mathrm{x2}, \frac{\mathrm{dx}^2}{\mathrm{dt}} = \mathrm{x3}, \text{and } \frac{\mathrm{dx}^3}{\mathrm{dt}} = -\frac{19}{12}\mathrm{x3} - \frac{19}{24}\mathrm{x2} - \frac{1}{8}\mathrm{x1} + f(t)$$

e) Use the MATLAB ode45() function to solve the system of equations and plot your solution on a new figure. The result provided by ode45() and your solution from above should match.

Solution:

```
%[Insert your MATLAB code and plot here.]
function dxdt = systemEquation(t, x)
    %define f(t)
    if t > 1 && t < 10
        f = 1;
    else
        f = 0;
    end

    %initialize the output & define system
    dxdt = zeros(3,1);
    dxdt(1) = x(2);
    dxdt(2) = x(3);
    dxdt(3) = -(19/12)*x(3) - (19/24)*x(2) + (1/8)*x(1) + f;
end

%initial conditions
x0 = [0; 0; 0;];

%time span
tspan = [0, 10];

%solve using ODE45
[t, x] = ode45(@systemEquation, tspan, x0);

%plot the solution
figure;
plot(t, x(:,1), 'b', 'LineWidth', 2);
hold on;
plot(t, x(:,2), 'g', 'LineWidth', 2);
plot(t, x(:,3), 'r', 'LineWidth', 2);
%labels
xlabel('time t');
ylabel('solution');
title('solution of system of equations ');
%legend
legend;
grid on;
```



Problem 4: Consider the differential equation $y'' + 6y' + 5y = 3\delta(t-4)$ with initial conditions $y(0) = 5$ and $y'(0) = 2$.

a) Take the Laplace transform of the differential equation and solve for $Y(s)$. Use partial fraction expansion (PFE) to simplify your answer.

Solution:

[Insert your math/explanation here.]

I started by; $L(y'') + 6L(y') + 5L(y) = 3L(\delta(t-4))$

$(s^2L(y) - sy(0) - y(0)) + 6(sL(y) - y(0)) + 5L(y) = 3e^{-4s}$
$(s^2 + 6s + 5)L(y) - 5s - 2 - 5 = 3e^{-4s}$

Then;

$$L(y) = \frac{5s + 7 + 3e^{-4s}}{s^2 + 5s + 6}$$

$$= \frac{5}{s+3} + \frac{3(e^{-4s} - 1)}{s+2} - \frac{3(e^{-4s} - 1)}{s+3}$$

Therefore;

$$L(y) = \frac{3e^{-4s}}{s+2} - \frac{3e^{-4s}}{s+3} + \frac{2}{s+3} - \frac{3}{s+2}$$

b) Find the solution of the differential equation by computing $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Solution:

[Insert your math/explanation here.]

We already know; $L(y) = \frac{3e^{-4s}}{s+2} - \frac{3e^{-4s}}{s+3} + \frac{2}{s+3} - \frac{3}{s+2}$

moving on to solve;

$$y = L^{-1}\left(\frac{3e^{-4s}}{s+2} - \frac{3e^{-4s}}{s+3} + \frac{2}{s+3} - \frac{3}{s+2}\right)$$

$$= L^{-1}\left(\frac{3e^{-4s}}{s+2}\right) + L^{-1}\left(-\frac{3e^{-4s}}{s+3}\right) + L^{-1}\left(\frac{2}{s+3}\right) + L^{-1}\left(-\frac{3}{s+2}\right)$$

$$= 3e^{-2(t-4)}u(t-4) - 3e^{-3(t-4)}u(t-4) + 2e^{-3t} - 3e^{-2t}$$

So the solution of thw differential equation will be;

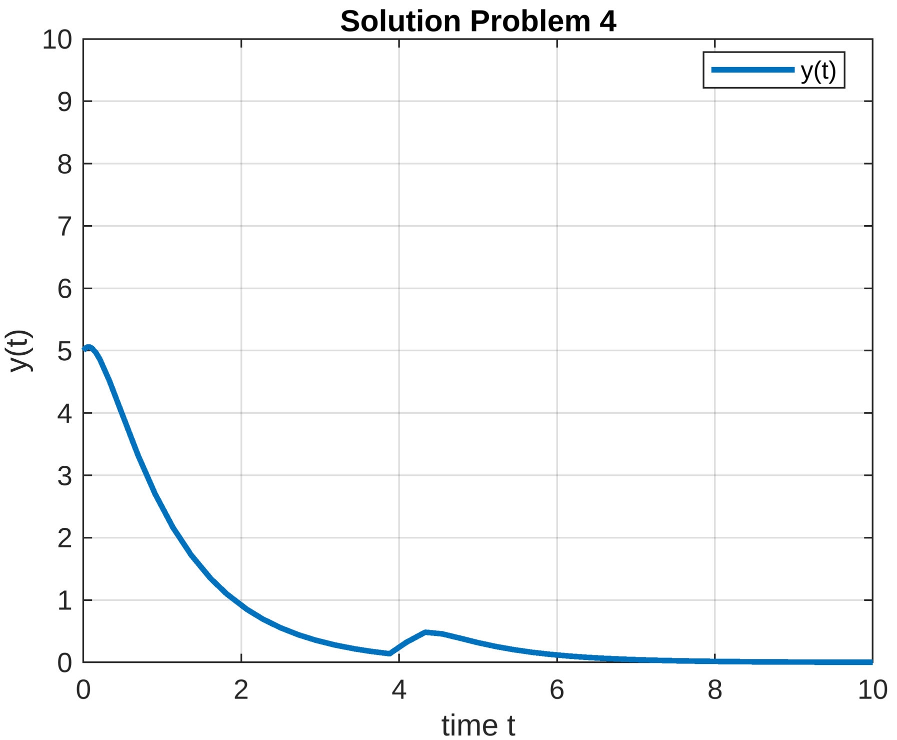
$$y = 3e^{-2(t-4)}u(t-4) - 3e^{-3(t-4)}u(t-4) + 2e^{-3t} - 3e^{-2t}$$

c) Use MATLAB to plot your solution for time $t = 0$ to $t = 10$.

Solution:

```
%[Insert your MATLAB code and plot here.]
%define variables
syms y(t) t;

y(t) = 3*heaviside(t-4)*((exp(-(t-4))-exp(-5*(t-4)))/4)+(27/4)*(exp(-
t))-(7/4)*(exp(-5*t));

%plot solution
figure;
fplot(y(t), [0, 10], '-', 'LineWidth', 2);
%labels
xlabel('time t');
ylabel('y(t)');
title('Solution Problem 4');
grid on;
%legend
```

```
legend('y(t)', 'Location', 'best');
%y limits
ylim([0, 10]);
```

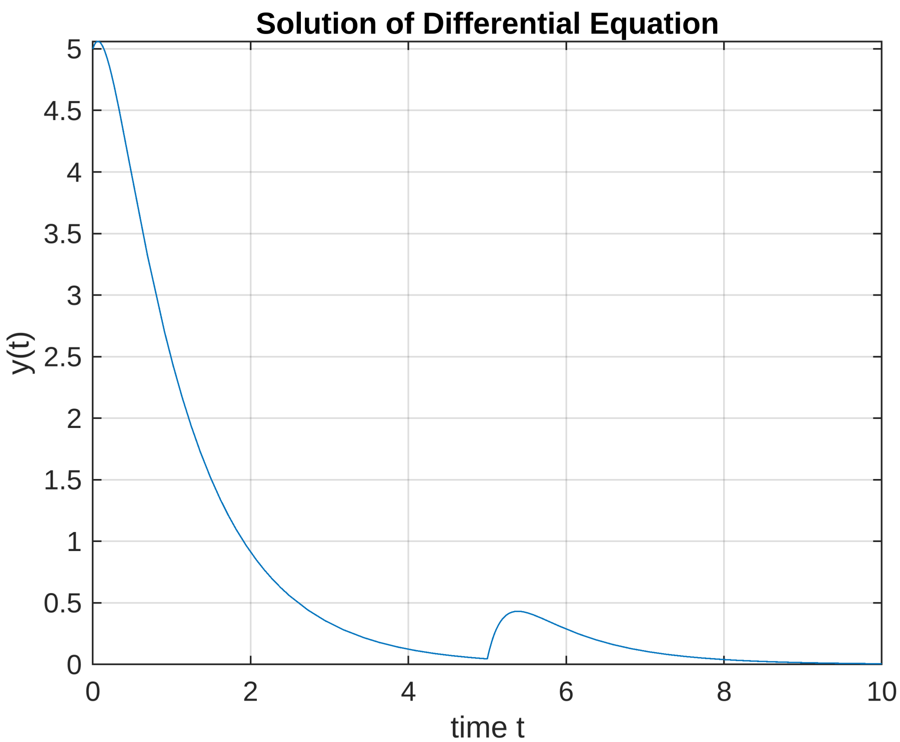


d) Use the MATLAB dsolve() function to solve the system of equations and plot your solution on a new figure. The result provided by dsolve() and your solution from above should match.

Solution:

```
%[Insert your MATLAB code and plot here.]
%define variables
syms y(t)

eqn = diff(y,t,2)+6*diff(y,t)+5*y==3*dirac(t-5);
Dy = diff(y,t);

%conditions
cond = [y(0)==5, Dy(0)==2];

%use dsolve with the condition and eqn
ySol(t) = dsolve(eqn, cond);

%plot solution
fplot(ySol, [0, 10])
```

```
%labels
xlabel('time t');
ylabel('y(t)');
title('Solution of Differential Equation');
grid on;
```




```
function dxdt = differentialEquation(~, x)
    dxdt = zeros(2,1);
    dxdt(1)=-3*x(1)-200*x(2);
    dxdt(2)=200*x(1)-3*x(2);
end
function dydt = systemEquations(~, y)
    dydt = zeros(3,1);
    dydt(1)=y(1);
    dydt(2)=y(2);
    dydt(3)=y(3);
end
```