# MAT 330: Differential Equations

## Project One Template

**Complete this template by replacing the bracketed text with the relevant information.**

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[5/27/25]

## **Problem 1:** Consider a time-varying population that follows the logistic population model. The population has a limiting population of 800, and at time $t = 0$, its population of 200 is growing at a rate of 60 per year.

### a) Write a differential equation for the population $P(t)$ that models this scenario.

### Solution:

[We use $\frac{\mathrm{dp}}{\mathrm{dt}} = \mathrm{kP}(M - P)$ to find k, and to find k we subsitute the limiting population. While solving for k, we will get

60 = k*200(800-200). With that we conclude that k = 0.0005. The differential equation will be $\frac{\mathrm{dp}}{\mathrm{dt}} = 0.0005P(800 - P)$, we can simplify this to and we can get $\frac{\mathrm{dp}}{\mathrm{dt}} = 0.4P - 0.0005P^2$.]

### b) Identify an appropriate solution technique and solve this differential equation.

### Solution:

We can simplify this by multiplying numerator and denominator by 2, which gives:

$$P(t) = \frac{800\mathrm{Ce}^{\frac{t}{2.5}}}{1 + \mathrm{Ce}^{\frac{t}{2.5}}}$$

Use initial condition P(0)=200

At t=0, we plug in:

$$P(0) = \frac{800C}{1 + C} = 200$$

Multiply both sides by 1+C:

$$800C = 200(1+C) \Rightarrow 800C = 200 + 200C \Rightarrow 600C = 200 \Rightarrow C = \frac{1}{3}$$

Final particular solution:

Substitute C=1/3 into the formula:

$$P(t) = \frac{800 * e^{\frac{t}{2.5}}}{1 + \frac{1}{3}e^{\frac{t}{2.5}}} = \frac{800e^{\frac{t}{2.5}}}{3 + e^{\frac{t}{2.5}}}$$

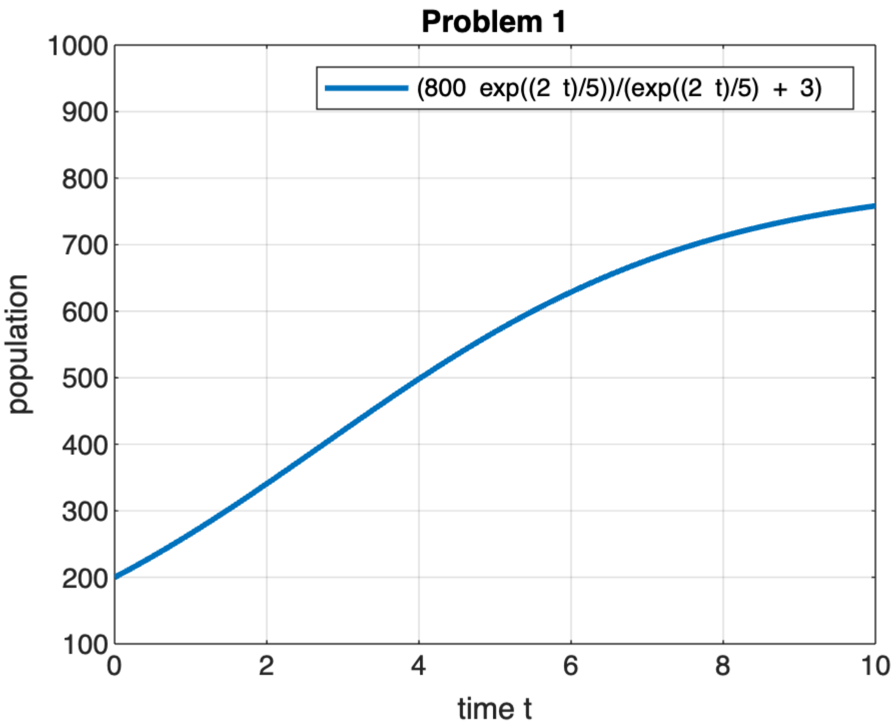
Final Answer:

$$P(t) = \frac{800e^{\frac{t}{2.5}}}{3 + e^{\frac{t}{2.5}}}$$

### c) Use MATLAB to plot your solution for $t = 0$ to $t = 10$.

### Solution:

```
%[Insert your MATLAB code and plot here.]
%define the function/variable of interest
syms P(t) t;

%solution
P(t) = ((800*exp(0.4*t))/(3+exp(0.4*t)));

%plot
%figure
figure;
fplot(P(t), [0, 10], '-', 'LineWidth', 2);
%grid on
grid on;
%label axes and title
xlabel('time t');
ylabel('population');
title('Problem 1');
legend;
%y lims
ylim([100, 1000]);
```




**d) How long will it take for the population to achieve 95% of its maximum population?**

**Solution:**

To find 95% of the max population we start with 0.95*800=760

Then we set our solution = to 760.

$$760 = \frac{800e^{\frac{t}{2.5}}}{3 + e^{\frac{t}{2.5}}}$$

$$760\left(3 + e^{\frac{t}{2.5}}\right) = 800e^{\frac{t}{2.5}}$$

then distribute 760: $2280 + 760e^{\frac{t}{2.5}} = 800e^{\frac{t}{2.5}}$

we rearrange and then simplify the equation: $2280 = (800 - 760)e^{\frac{t}{2.5}}$

$$2280 = 40e^{\frac{t}{2.5}}$$

$$57 = e^{\frac{t}{2.5}}$$

apply ln: $\ln(57) = \frac{t}{2.5}$

t ~ 10.107

the time it will take to reach 95% of the max pop is approximately 10.11 yrs.

**e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.**

**Solution:**

```
%[Insert your MATLAB code and plot here.]
%define the function/ variables of interest
syms P(t) t;

%compute the derivatives of p(t) with respect to t using diff() funct
Dy = diff(P(t));

%define the diff equation
ode = Dy == 0.4*P-0.0005*P^2
```

ode(t) =

$$\frac{\partial}{\partial t} P(t) = \frac{2\,P(t)}{5} - \frac{P(t)^2}{2000}$$

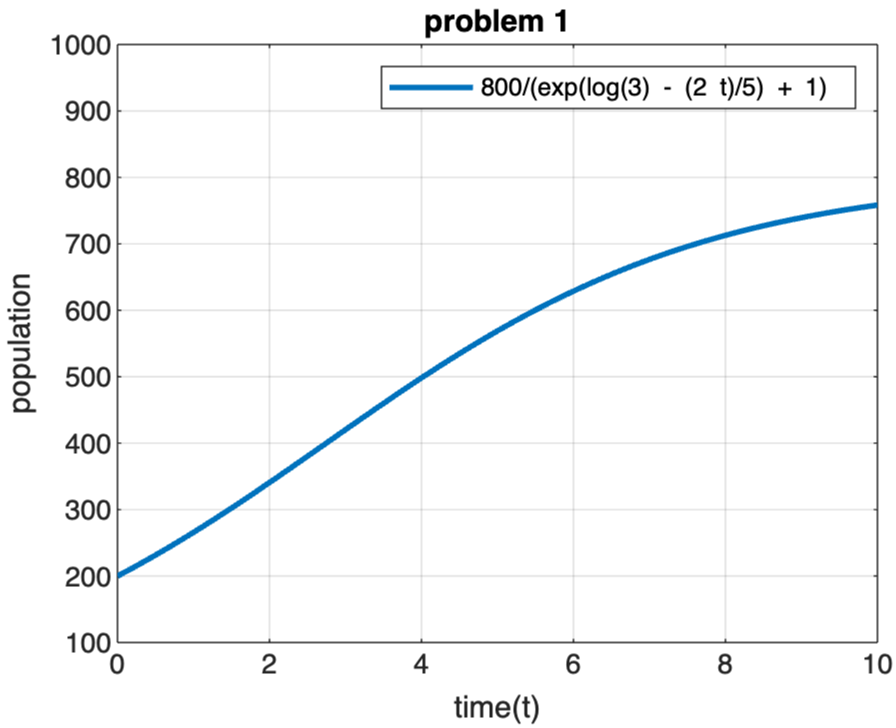
```
%define the conditions
cond1 = P(0) == 200;

%solve the the diff equation using dsolve()
Soln(t) = dsolve(ode, cond1)
```

Soln(t) =

$$\frac{800}{e^{\log(3)-\frac{2t}{5}} + 1}$$

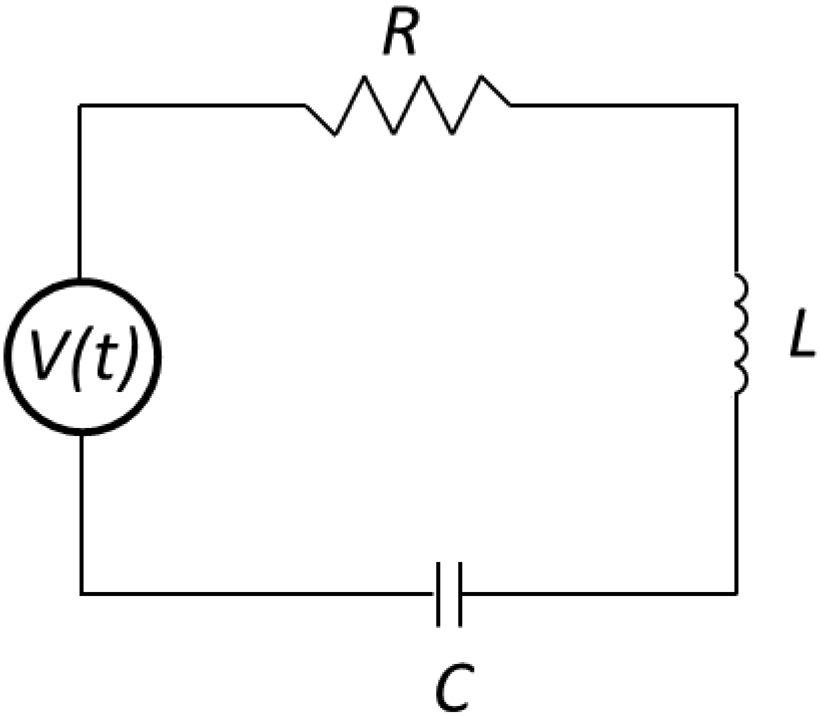
```
%plot the solution
%figure
figure;
fplot(Soln, [0,10], '-', 'LineWidth', 2);
%grid on
grid on;
%label axes & title
xlabel('time(t)');
ylabel('population');
title('problem 1');
%legend
legend;
%y limits
```

```
ylim([100, 1000]);
```




**Problem 2: Consider the RLC circuit shown below, where the inital current flowing through the circuit at time $t = 0$ is $I_0 = 5$, and the initial charge on the capacitor at time $t = 0$ is $Q_0 = 2$. The components have values of**

$$R = 100 \ \ \Omega, L = 5 \ \ \mathrm{H}, \ \text{and } C = \frac{1}{450,500} \ \ \mathrm{F}.$$




**a) Write a differential equation for $Q(t)$, the charge across the capacitor, assuming the voltage source $V(t) = 0$.**

**Solution: We will use Kirchhoffs voltage law using V(t)=0:**

$$L\frac{d^2Q}{\mathrm{dt}^2}+R\frac{\mathrm{dQ}}{\mathrm{dt}}+\frac{1}{C}Q=0$$

Then we plug in the given component values:

$$5\frac{d^2Q}{\mathrm{dt}^2}+100\frac{\mathrm{dQ}}{\mathrm{dt}}+450500Q=0$$

we can simplfy the equation by dividing each term by 5;

$$\frac{d^2Q}{\mathrm{dt}^2}+20\frac{\mathrm{dQ}}{\mathrm{dt}}+90100Q=0$$

**b) Classify the differential equation**

**Solution:**

The differential equation is a second-order, linear, homogenous differential equation with constant coefficients.

### c) Identify an appropriate solution technique, and solve the differential equation for $Q(t)$.

### Solution:

solve using the characteristic equation method:

Replace: $\frac{d^2Q}{\mathrm{dt}^2} \longrightarrow r^2, \ \frac{\mathrm{dQ}}{\mathrm{dt}} \longrightarrow r, \ Q \longrightarrow 1:$

$r^2 + 20r + 90100 = 0$

use the quadratic form:

$r = \frac{-20 + - \sqrt{20^2 - 4(1)(90100)}}{2}$ = -10 +- 300i

Since the roots are complex the general solution is:

$Q(t) = e^{-10t}(\mathrm{A}\cos(300t) + \mathrm{B}\sin(300t))$

apply the initial conditions and use the product rule:

$Q(0) = e^0(\mathrm{A}\cos(0) + \mathrm{B}\sin(0)) = A \Rightarrow A = 2$

$Q'(t) = \frac{d}{\mathrm{dt}}[e^{-10t}(\mathrm{A}\cos(300t) + \mathrm{B}\sin(300t))]$

$Q'(t) = e^{-10}[(-10A + 300B)\cos(300t) + (-10B - 300A)\sin(300t)]$

at t= 0:

$Q'(0) = (-10A + 300B)(1) + (-10B - 300A)(0) = -10A + 300B$

set Q'(0) = 5

$-10(2) + 300B = 5 \Rightarrow -20 + 300B = 5 \Rightarrow 300B = 25 \Rightarrow B = \frac{1}{12}$

final answer:

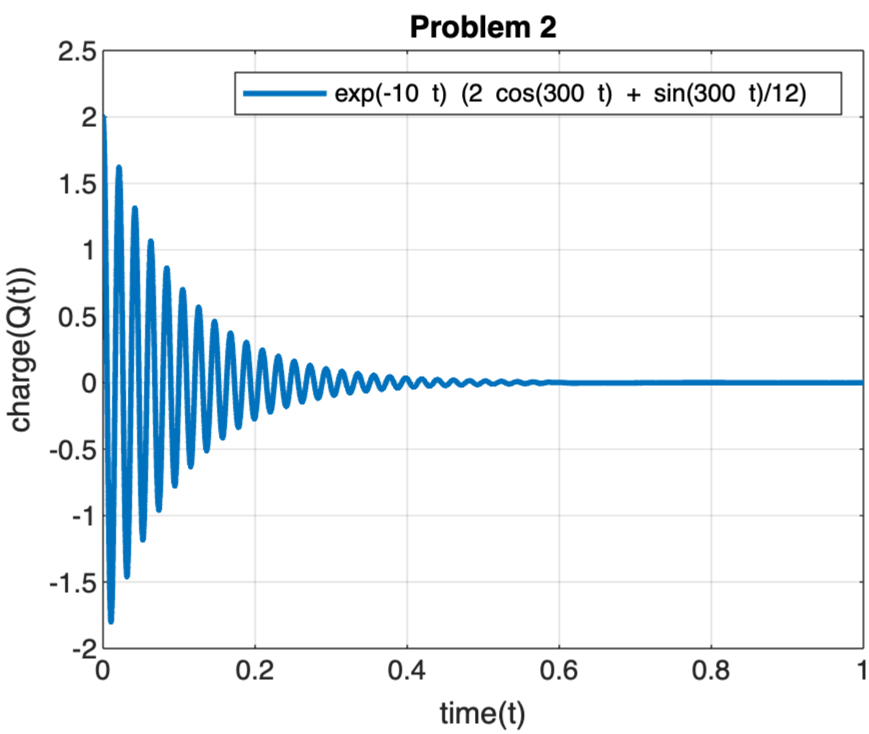
$Q(t) = e^{-10}\left(2\cos(300t) + \frac{1}{12}\sin(300t)\right)$

### d) Use MATLAB to plot your solution for time t = 0 to 1.

### Solution:

```
%[Insert your MATLAB code and plot here.]
%define the function/variables
```

```
syms Q(t) t;

%equation
Q(t) = exp(-10*t)*[2*cos(300*t)+(1/12)*sin(300*t)];

%plot the equation
%figure
figure;
fplot(Q(t), [0,1], '-', 'LineWidth', 2);
%grid on;
grid on;
%label axes & title
xlabel('time(t)');
ylabel('charge(Q(t))');
title('Problem 2');
%legend
legend;
%y limits
ylim([-2, 2.5]);
```




**e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.**

**Solution:**

```
%[Insert your MATLAB code and plot here.]
%define the function/variables
```

```
syms Q(t) t;

%derivatives of Q(t) w/ respect to t using the diff() funct
Dy(t) = diff(Q(t));
Dy2(t) = diff(Q(t),2);

%define the diff equation
ode = Dy2(t)+20*Dy(t)+90100*Q(t) == 0
```

ode =

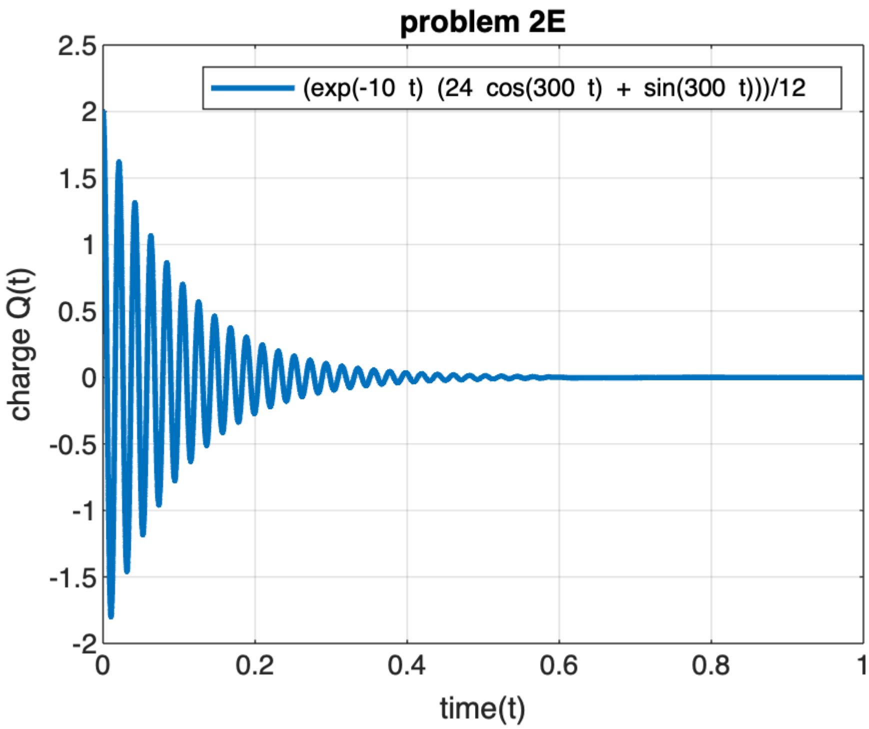
$$\frac{\partial^2}{\partial t^2} Q(t) + 20 \frac{\partial}{\partial t} Q(t) + 90100\, Q(t) = 0$$

```
%define the set of initial conditions
cond1 = Q(0) == 2;
cond2 = Dy(0) == 5;
%put the conditions toether
cond = [cond1; cond2];

%solve the diff equation using dsolve()
Soln = dsolve(ode, cond)
```

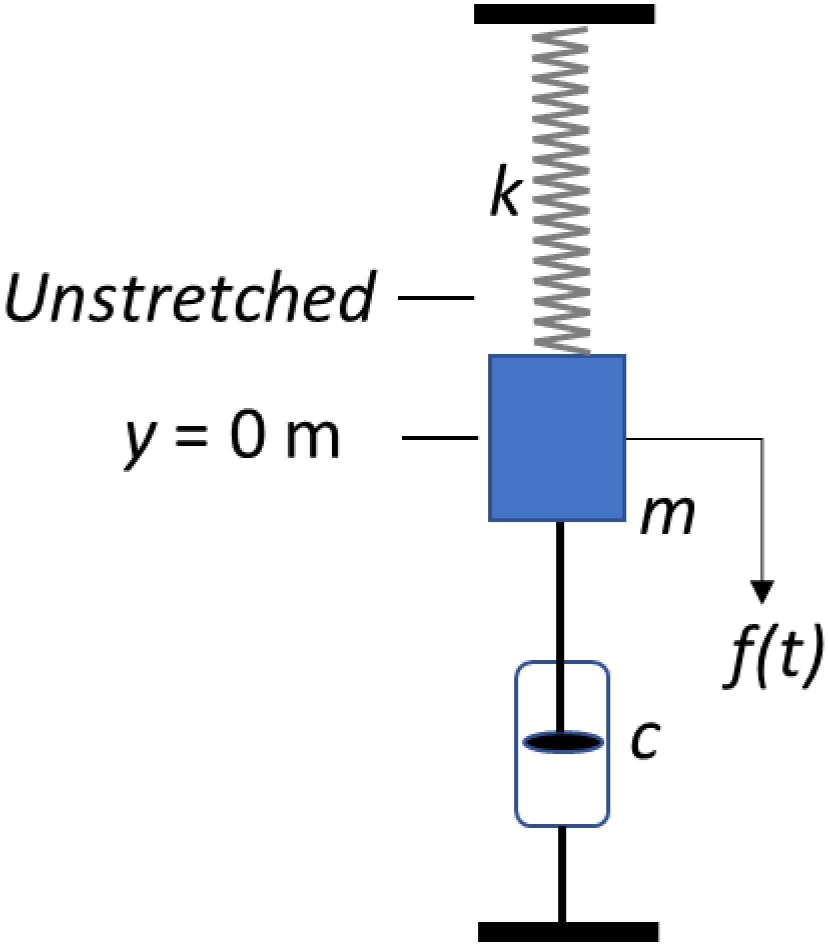
Soln =

$$\frac{e^{-10\,t}\,(24\cos(300\,t) + \sin(300\,t))}{12}$$

```
%plot the solution
%figure
figure;
fplot(Soln, [0, 1], '-', 'LineWidth', 2);
%grid on
grid on;
%label axes &title
xlabel('time(t)');
ylabel('charge Q(t)');
title('problem 2E');
%legend
legend;
%y limits
ylim([-2, 2.5]);
```




**Problem 3: Consider the spring-mass system shown below. A mass of $m = 5$ kg has stretched a spring by $s_0 = 1.5$ meters and is at rest. A damping system is connected that provides a damping force equal to 30 times the instantaneous velocity of the mass. At time $t = 0$, the mass at rest has an external force of $f(t) = 25\sin(10t)$ applied.**




**a) Write a differential equation that models the mass location $y(t)$ as a function of time.**

**Solution:**

the spring constant k: 32.67

Use newtons 2nd law, The standard order ODE for a damped driven spring mass system is:

my''(t) + cy'(t) + ky(t) = f(t)

substitue in the vales:

5y''(t) +30y'(t) +32.67y(t) = 25sin(10t)

**b) Identify an appropriate solution technique, and solve the differential equation for $y(t)$.**

**Solution:**

solve using undetermined coefficients:

5y'' +30y'+32.67y=0

characteristic equation:

$5r^2 + 30r + 32.67 = 0$

$$r = \frac{\left(-30 + - \sqrt{30^2 - 4(5)(32.67)}\right)}{10} = -3 + -1.568$$

which gives us: $\mathrm{C1}e^{-1.43t} + \mathrm{C2}e^{-4.57}$

finding the particular solution:

$5(-100\mathrm{A}\cos(10t) - 100\mathrm{B}\sin(10t)) + 30(-10\mathrm{A}\sin(10t) + 10\cos(10t)) + 32.7(\mathrm{A}\cos(10t) + \mathrm{B}\sin(10t))$

simplifies:

$(-467.3A + 300B)\cos(10t) + (-467.3B - 300A)\sin(10t)$

set up equations:

-467.3A+300B=0

A=0.083

-467.3B-300A=0

B=0.129

general solution:

$y(t) = \mathrm{C1e}^{-1.43t} + \mathrm{C2e}^{-4.57t} + 0.083\sin(10t) - 0.129\cos(10t)$

solve foor C1 and C2:

$y(0) = \mathrm{C1e}^{-1.43(0)} + \mathrm{C2e}^{-4.57(0)} + 0.083\sin(10(0)) - 0.129\cos(10(0))$

$\mathrm{C1} + \mathrm{C2} = 0.129$

use y'(0) =0:

-1.43C1 -4.57C2 = 0.083

-1.43C1 = 0.083 + 4.57C2

$$\mathrm{C1} = \frac{(0.083 + 4.57\mathrm{C2})}{-\frac{1}{43}}$$

$$\frac{(0.083 + 4.47\mathrm{C2})}{-\frac{1}{43}} + \mathrm{C2} = 0.129$$

so: C1= 0.214 and C2 = -0.085

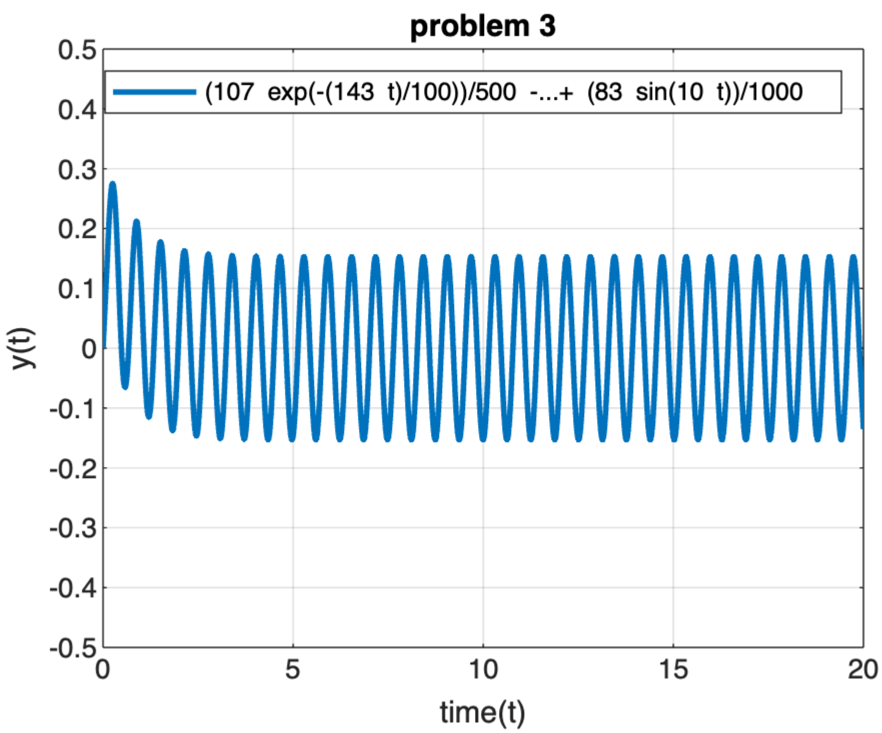
final answer :

$y(t) = 0.214e^{-1.43t} - 0.085e^{-4.57t} + 0.083\sin(10t) - 0.129\cos(10t)$

### c) Use MATLAB to plot your solution for $t = 0$ to $t = 20$.

**Solution:**

```
%[Insert your MATLAB code and plot here.]
%define the function/variables
syms y(t) t;

%equation
y(t) = 0.214*exp(-1.43*t)-0.085*exp(-4.57*t) +
0.083*sin(10*t)-0.129*cos(10*t);

%plot the solution
%figure
figure;
fplot(y(t), [0, 20], '-', 'LineWidth', 2);
%grid on
grid on;
%label axes and title
xlabel('time(t)');
ylabel('y(t)');
title('problem 3');
%legend
legend;
%y limits
ylim([-0.5, 0.5]);
```



### d) What is the steady-state solution of $y(t)$?

### Solution:

The steady state is y(t) = 0.083sin(10t) -0.129cos(10t). The steady state shows what the spring does as time 't' moves in the positive infinity direction. As this exponential function grows it will trend toward 0 showing that to be the steady state.

### e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.

### Solution:

```
%[Insert your MATLAB code and plot here.]
%define the function/ variables
syms y(t) t;

%derivatives of y(t0 w/ respect to t using the diff() function
Dy(t) = diff(y(t));
Dy2(t) = diff(y(t),2);

%define the diff equation
ode = 5*Dy2(t)+30*Dy(t)+32.7*y(t) == 25*sin(10*t)
```

ode =

$$5\frac{\partial^2}{\partial t^2}\ y(t)+30\frac{\partial}{\partial t}\ y(t)+\frac{327\,y(t)}{10}=25\sin(10\,t)$$

```

%define the set of conditions
cond1 = y(0) == 1.5;
cond2 = Dy(0) == 0;
%put the conditions together
cond = [cond1, cond2];

%solve the diff equation using dsolve()
Soln = dsolve(ode, cond)
```

Soln =

$$\frac{25\sqrt{246}\,e^{-t\sigma_2}e^{3t+\sigma_4}\,(\sigma_3-\sin(10t)\,\sigma_2)}{246\,(\sigma_2{}^2+100)}-\frac{e^{-t\sigma_2}\,(8856797268295\sqrt{246}-75711582882873)}{492\,\sigma_5{}^2\,(30\sqrt{246}+5573)}-\frac{25\sqrt{246}}{}$$

where

$$\sigma_1=\frac{\sqrt{246}}{10}-3$$

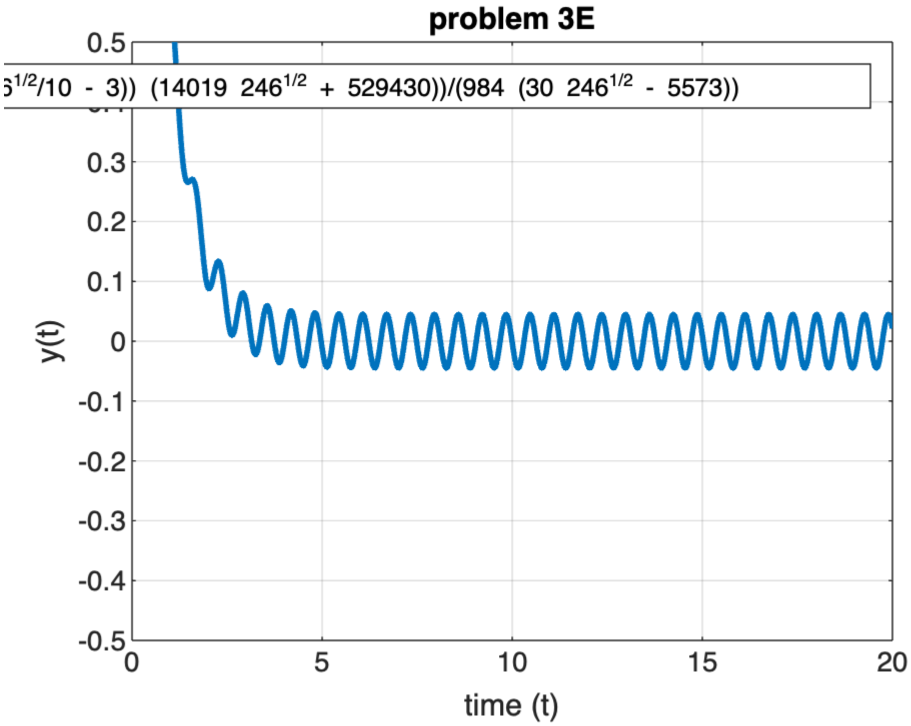
$$\sigma_2=\frac{\sqrt{246}}{10}+3$$

$$\sigma_3=10\cos(10t)$$

$$\sigma_4=\frac{\sqrt{246}\,t}{10}$$

$$\sigma_5=30\sqrt{246}-5573$$

```
%plot the solution
%figure
figure;
fplot(Soln, [0, 20], '-', 'LineWidth', 2);
%grid on
grid on;
%label axes and title
xlabel('time (t)');
ylabel('y(t)');
title('problem 3E');
%legend
legend;
%y limits
ylim([-0.5, 0.5]);
```


problem 3E
ɔ1/2/10 - 3)) (14019 246^1/2 + 529430))/(984 (30 246^1/2 - 5573))
0.5
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
y(t)
0
5
10
15
20
time (t)