BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI
First Semester, 2020-21
MATH F444, Numerical Solutions to Ordinary Differential Equations

Assignment 2 **Date: 17/11/2020**

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Gas flows through micro/nano scale channels find their applications in broad areas of science, engineering and bio-medical sciences. We consider processes that fall into the class of steady shear flows, mainly steady Poiseuille flows [M. Torrilhon and H. Struchtrup].

Let us consider shear flow which is homogeneous in z −direction and for the velocity we assume that $v_y = v_z = 0$, thus the velocity vector is given by

$$\mathbf{V} = [v_x(y), \quad 0 \quad 0].$$

The flow is driven by a body force (gravity or a pressure gradient) acting only in x −direction,

$$\mathbf{f} = [F, \quad 0 \quad 0].$$

This setting is valid for channel flows as displayed in Fig. 1. The fluid (ideal gas) is confined between two infinite plates at distance L and is moving solely in x −direction.

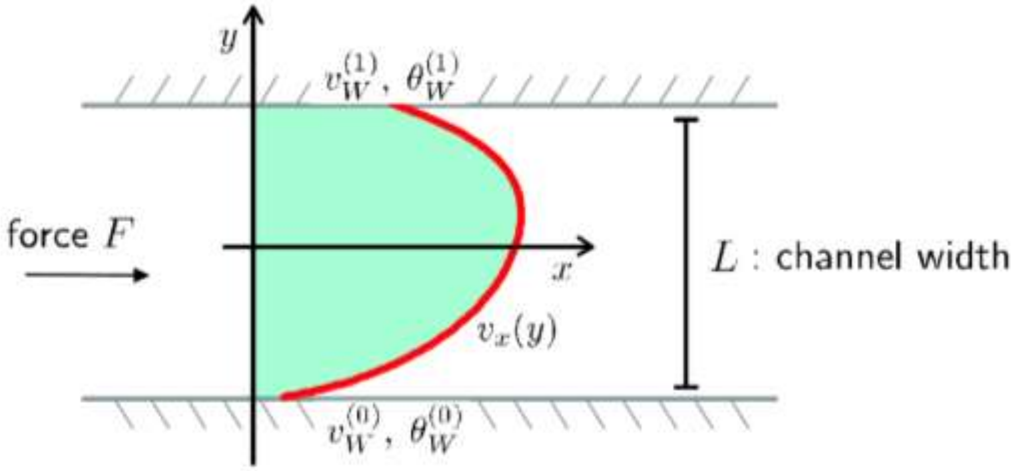


Fig. 1: General shear flow setting. The gas flows between infinite plates with velocities $v_w^{(0,1)} = 0$ and temperature $\theta_w^{(0,1)} = 1$. The force F is given by gravity or a pressure gradient.

The differential equations, describing this process are given by the conservation laws:

$$\begin{aligned} \frac{d\sigma}{dy} &= \rho F, \\ \frac{dq_y}{dy} + \sigma \frac{dv_x}{dy} &= 0. \end{aligned}$$

Here, $v_x(y)$ is the velocity of the gas in x-direction, $\rho(y)$ is gas density, which is given by the ideal gas law $\rho(y) = p_0/\theta(y)$, where $p_0 = 1$ is the dimensionless pressure (a constant) in the gas across the channel and $\theta(y)$ is the dimensional temperature of the gas.

The heat flux in y −direction $q_y(y)$ and the shear stress $\sigma(y)$ are given by the Fourier's law and the Navier-Stokes relations, respectively as

$$q_y = -\frac{15}{4} Kn \frac{d\theta}{dy}, \text{ and } \sigma = -Kn \frac{dv_x}{dy}.$$

Here, Kn is defined as the Knudsen number, a parameter which dictates the degree of rarefaction in the gas.

This system of four ODEs needs to be solved for four unknowns $(v_x(y),\ \sigma(y), \theta(y), q_y(y));\ 0 \leq y \leq 1$. The required boundary conditions for such systems are given by the velocity-slip and temperature-jump boundary conditions, as

$$\sigma = -n_y \sqrt{\frac{2}{\pi\theta}} v_x, \text{ and } q_y = -2n_y \sqrt{\frac{2}{\pi\theta}} (\theta - 1)$$

Where following the setting of Fig.1 these boundary conditions have to hold on both sides of the channel with $n_y = \pm 1$ for lower ($y = 0$) and upper wall ($y = 1$), respectively.

Our task is to use the **mid-point finite difference method** in order to solve the above system of BVPs with $F = 0.23$, and with Knudsen number $Kn = 0.068, 0.1, 0.5$ along with discretized points $N = 200$.

Part 1

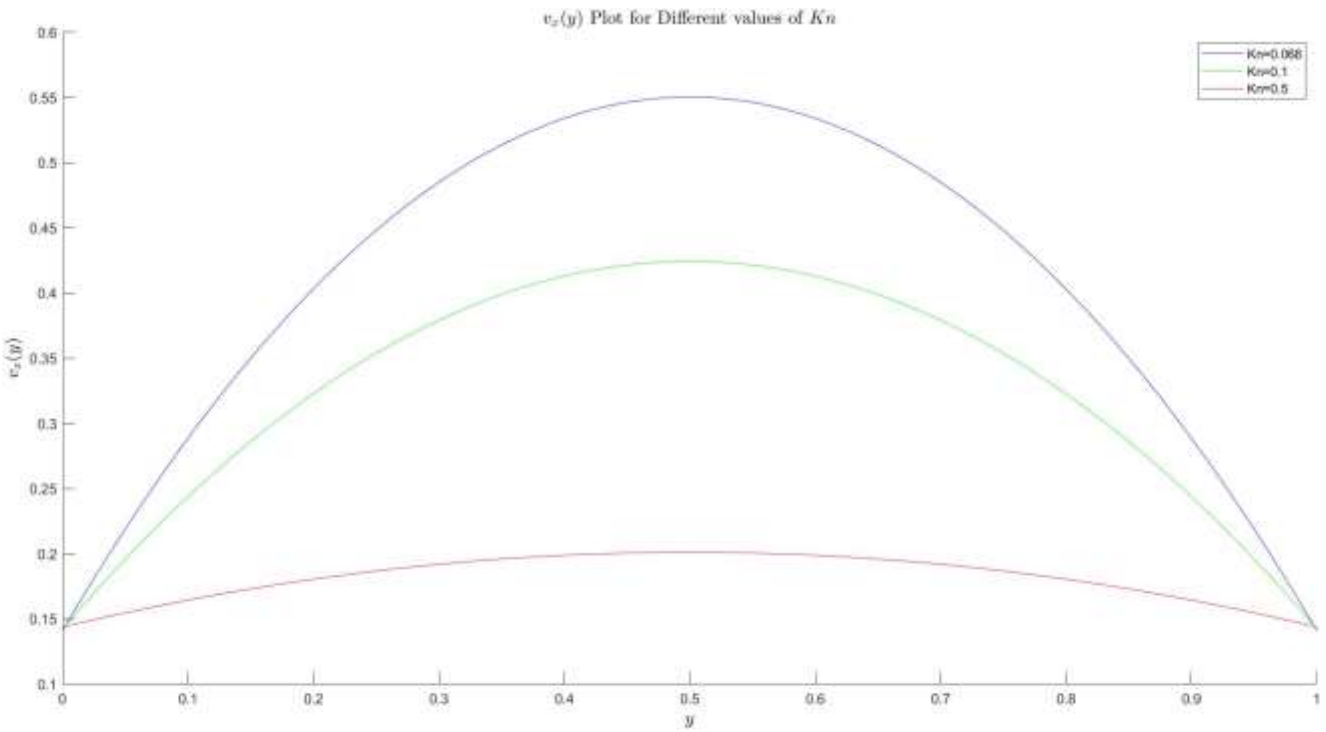
(a) Plot velocity v_x vs y for $Kn = 0.068, 0.1, 0.5$ in the same plot.

(b) Plot velocity σ vs y for $Kn = 0.068, 0.1, 0.5$ in the same plot.

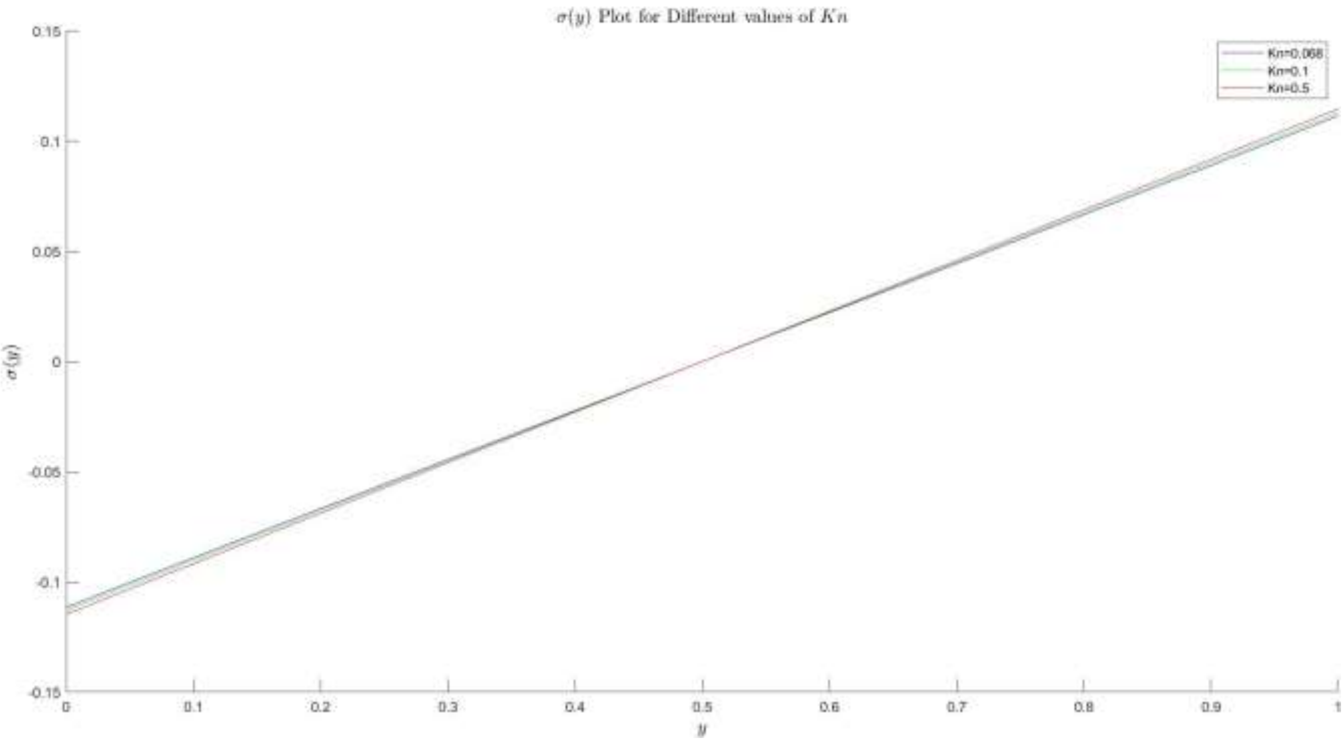
(c) Plot velocity θ vs y for $Kn = 0.068, 0.1, 0.5$ in the same plot.

(d) Plot velocity q_y vs y for $Kn = 0.068, 0.1, 0.5$ in the same plot.

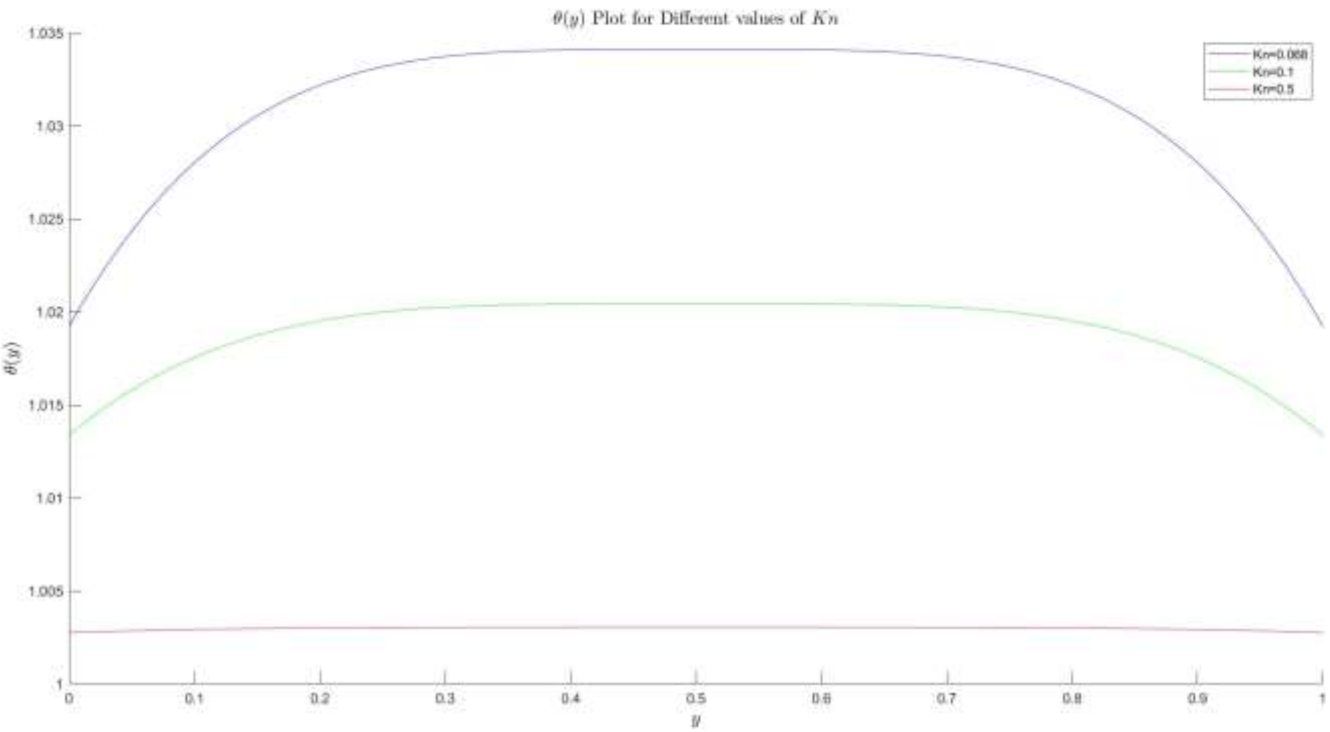
INSERT FIGURE A HERE



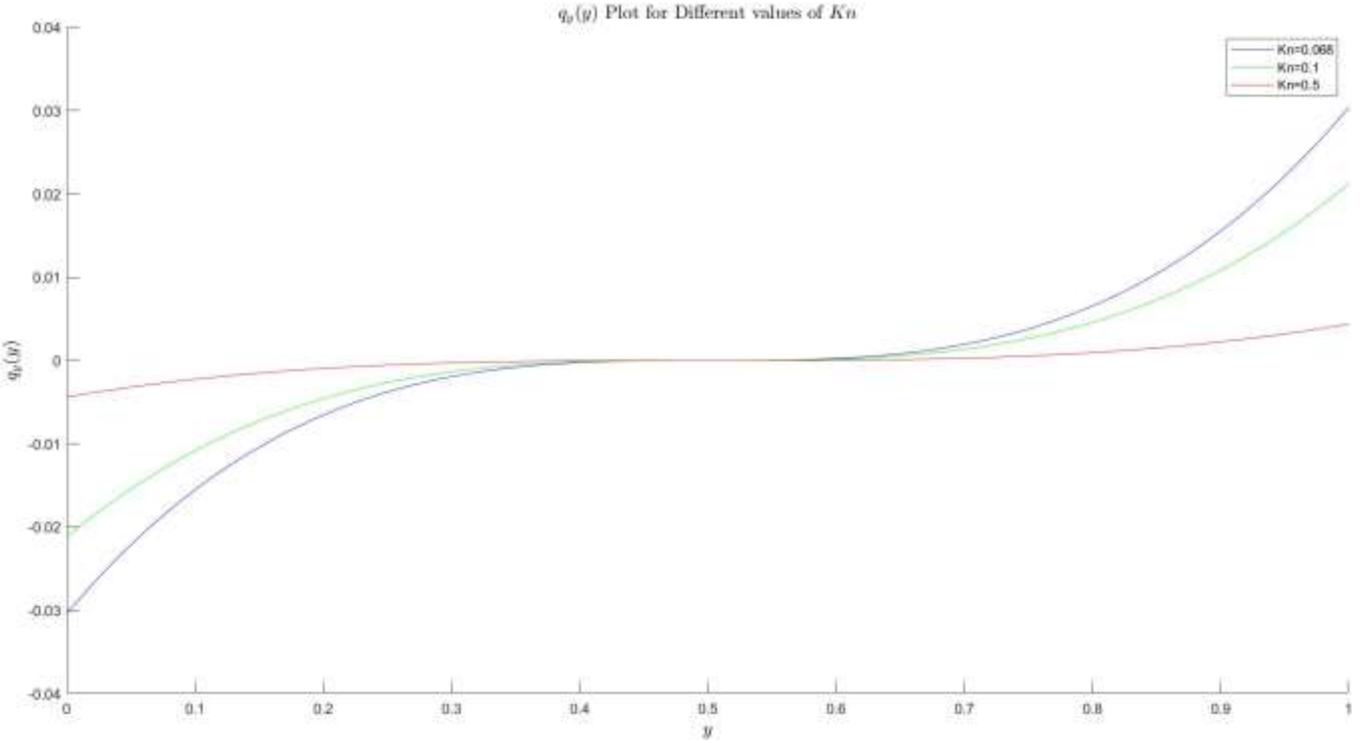
INSERT FIGURE B HERE



INSERT FIGURE C HERE



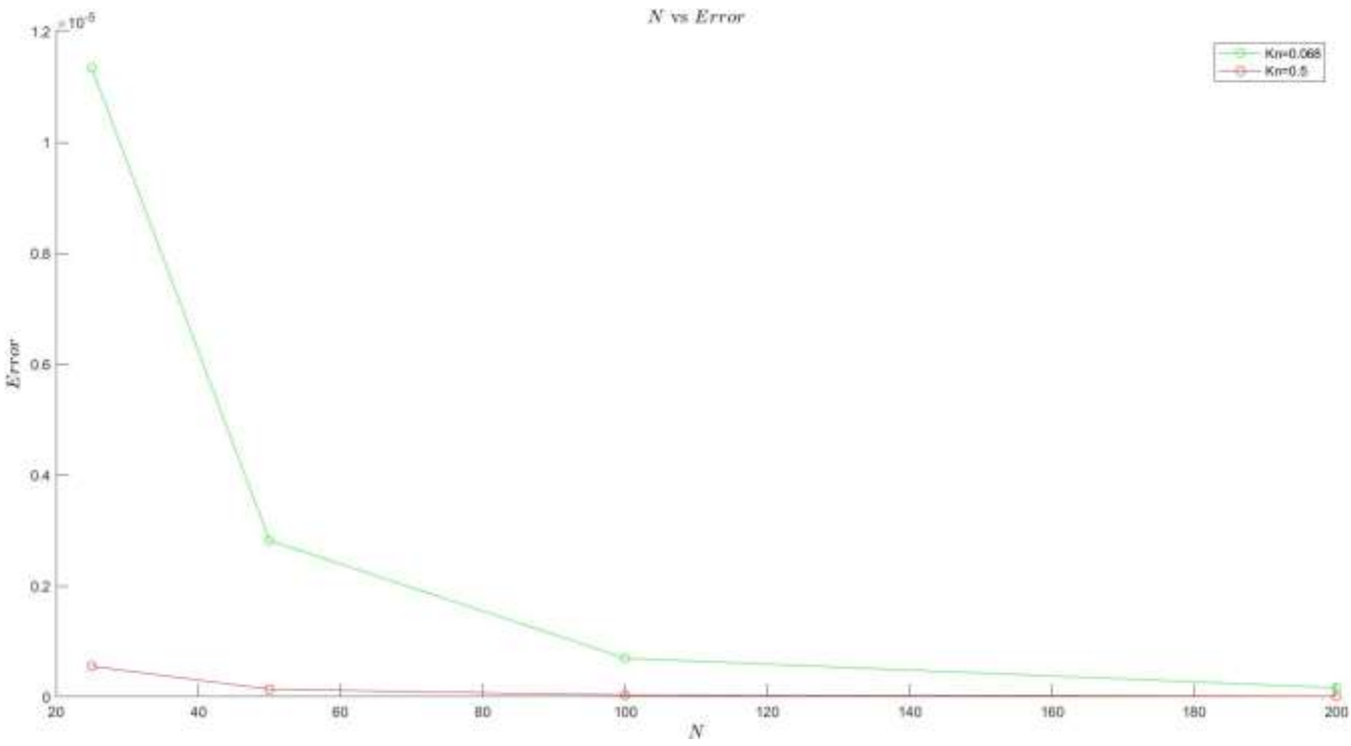
INSERT FIGURE D HERE

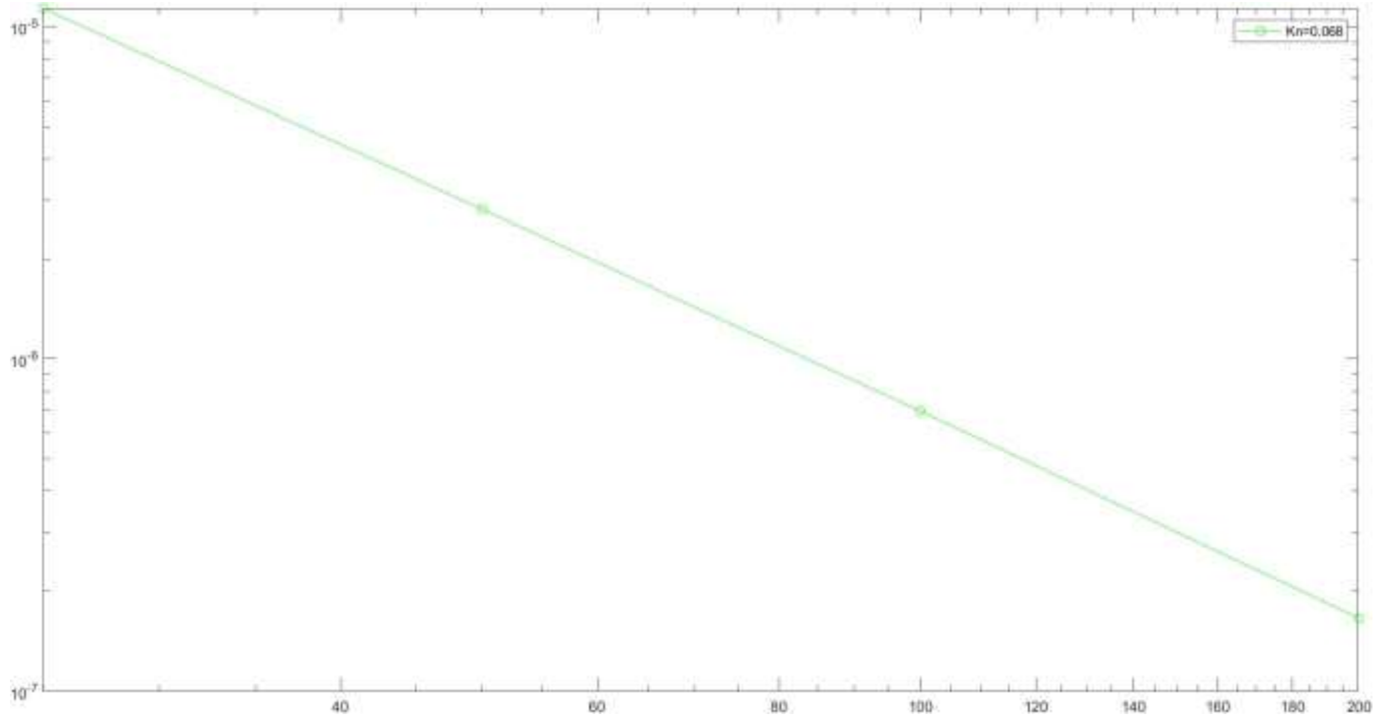


Part 2

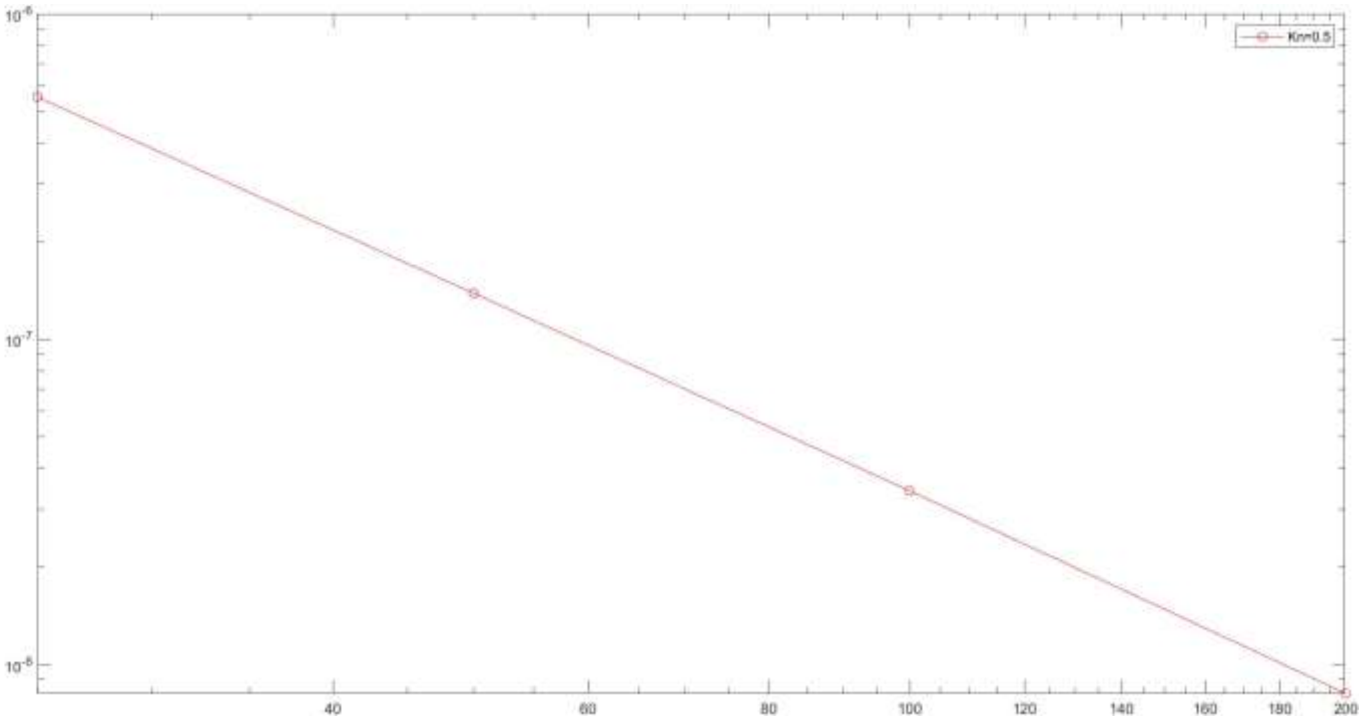
Perform an empirical error of convergence (EOC) analysis of the numerical method in velocity, with $Kn =$ 0.068 and 0.5.

INSERT FIGURE EOC HERE





Kn = 0.068, Slope of best fit line = 2.0304



Kn = 0.5, Slope of best fit line = 2.0292

How the EOC is affected by the Knudsen number?

Ans: Larger Knudsen number (i.e. $Kn = 0.5$) shows smaller error in $v_x(y)$. This might be because $\frac{dv_x(y)}{dy} = -\frac{\sigma}{Kn}$ and as Kn increases, $\frac{dv_x(y)}{dy}$ tends to 0 i.e. $v_x(y)$ tends towards being a constant. This maybe the reason why error decreases. For both $Kn = 0.068$ and $Kn = 0.5$, the method is 2nd order as we can see from the slope of the best fit line which is 2.03 in case of $Kn = 0.068$ and 2.03 in case of $Kn = 0.5$. This can be seen from **Figure A** in part **A.** There is no change in order of the method as Kn changes.