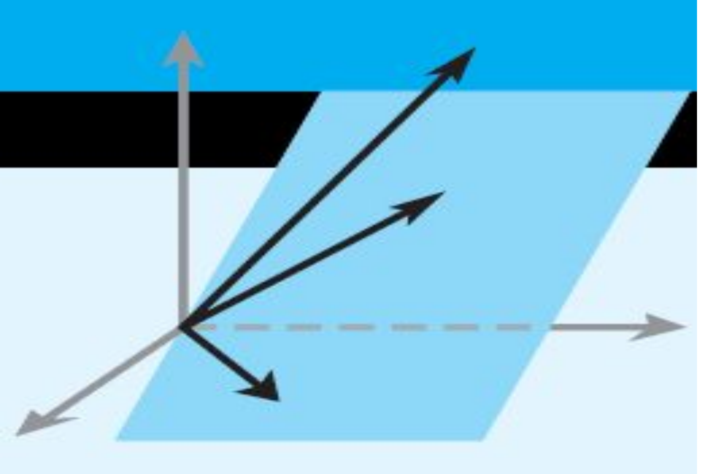

VECTOR SPACES

Chapter # 03 & 04: “Elementary Linear Algebra” by Howard Anton & Chris Rorres

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EUCLIDEAN VECTOR SPACES

3.1: Vector in 2-Space, 3-Space & n-Space

- ***Scalars*** *are quantities that are described by a numerical value alone. For example: temperature, length, and speed etc.*
- ***Vectors*** are quantities that require both a number and a direction. For example: velocity, force etc.
- **Geometric Vectors**:
- **Equivalent Vectors**: Same length & direction even though they may be in different positions.

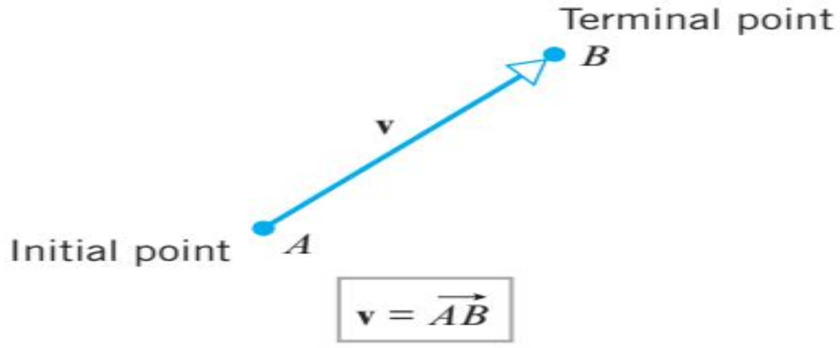


Figure 3.1.2

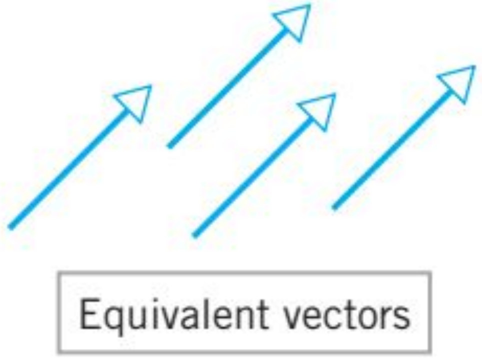


Figure 3.1.3

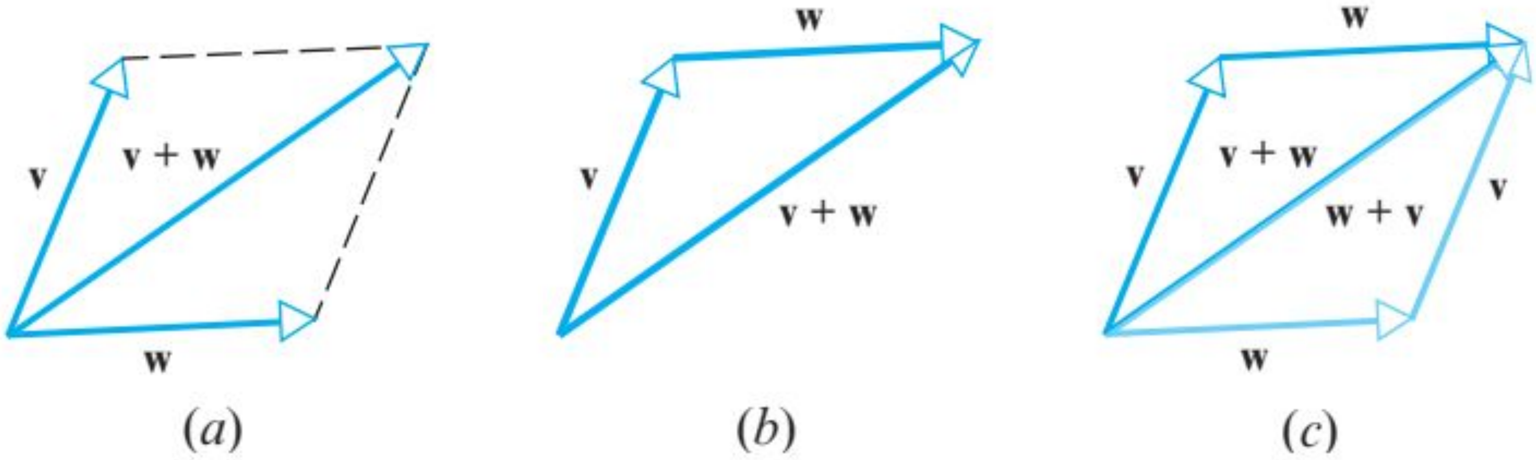
Vector Addition

Parallelogram Rule for Vector Addition If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the ***sum*** $\mathbf{v} + \mathbf{w}$ is the vector represented by the arrow from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram (Figure 3.1.4*a*).

Triangle Rule for Vector Addition If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$, then the ***sum*** $\mathbf{v} + \mathbf{w}$ is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ (Figure 3.1.4*b*).

Vector Addition (Geometrical View)

Parallelogram Rule for Vector Addition | **Triangle Rule for Vector Addition** | **v + w = w + v**

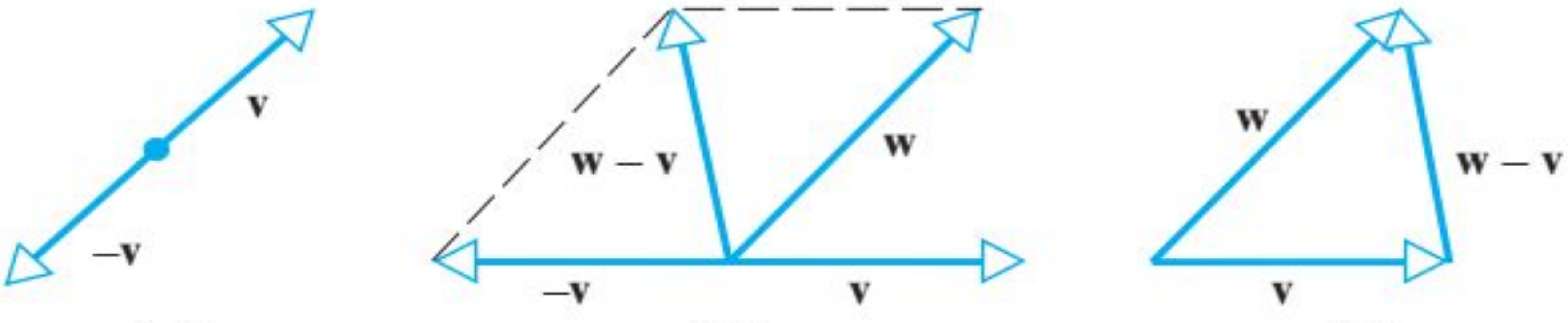


- Figure 3.14:

Vector Subtraction

Vector Subtraction The ***negative*** of a vector $\mathbf{v}$, denoted by $-\mathbf{v}$, is the vector that has the same length as $\mathbf{v}$ but is oppositely directed (Figure 3.1.6a), and the ***difference*** of $\mathbf{v}$ from $\mathbf{w}$, denoted by $\mathbf{w} - \mathbf{v}$, is taken to be the sum

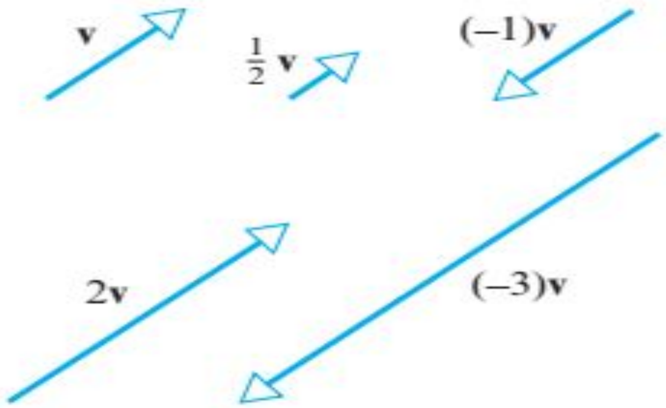
$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) \tag{2}$$



▶ Figure 3.1.6 (a) (b) (c)

Scalar Multiplication

Scalar Multiplication If $\mathbf{v}$ is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the ***scalar product of* v *by* k** to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if k is positive and opposite to that of $\mathbf{v}$ if k is negative. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then we define $k\mathbf{v}$ to be $\mathbf{0}$.



▲ Figure 3.1.7

Parallel and Collinear Vectors

- Suppose that **v** and **w** are vectors in 2-space or 3-space with a common initial point. If one of the vectors is a scalar multiple of the other, then the vectors lie on a common line, so it is reasonable to say that they are *collinear.*
- *The terms* ***parallel and collinear mean the same things*** *when applied to vectors.*

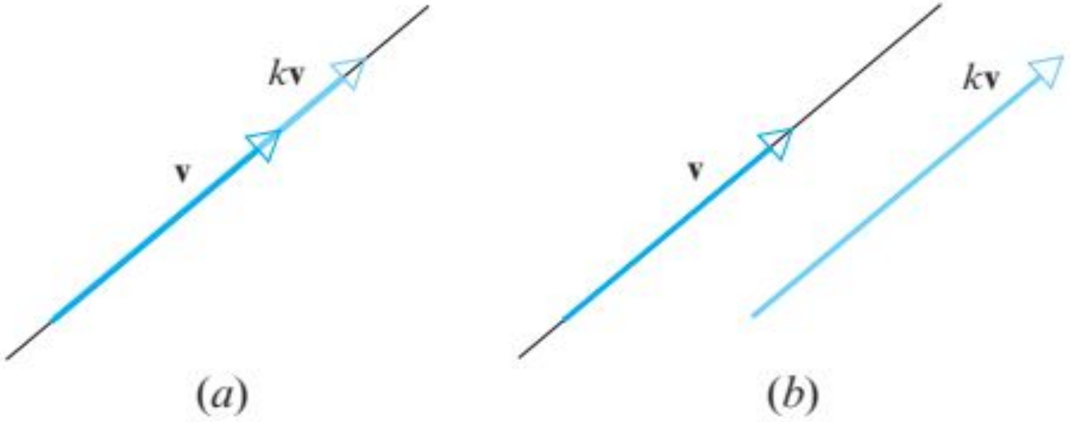
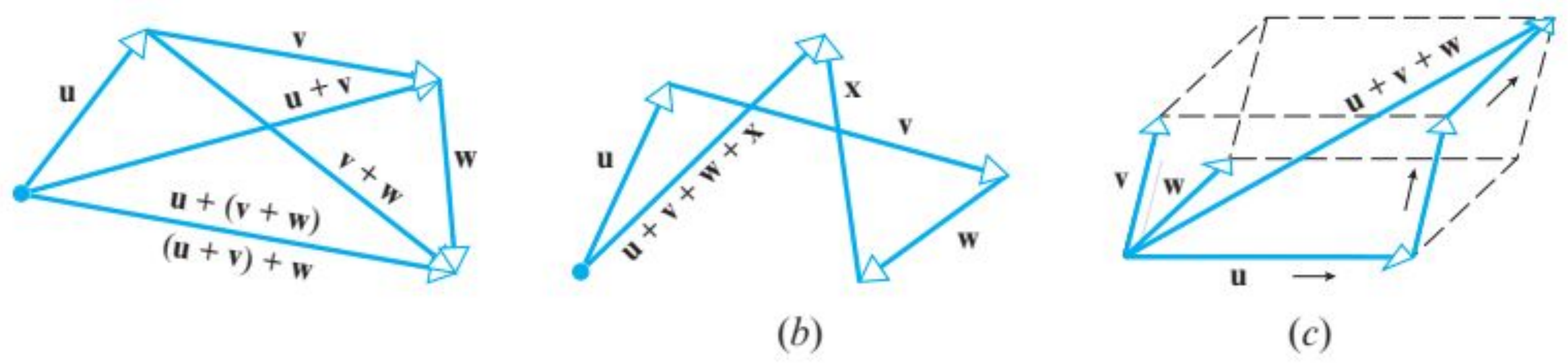


Figure 3.1.8

Sum of three or more vectors

- Vector addition satisfies the *associative law for addition*: **u + (v + w) = (u + v) + w**

- A simple way to construct **u** + **v** + **w** is to place the vectors "tip to tail" in succession and then draw the vector from the initial point of **u** to the terminal point of **w.**



▶ Figure 3.1.9 (a)

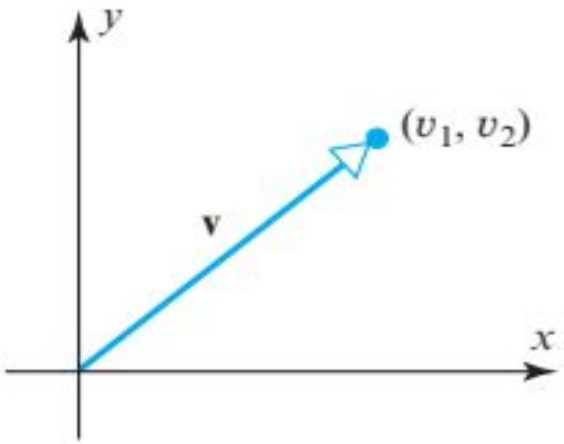
Vectors in Coordinate Systems

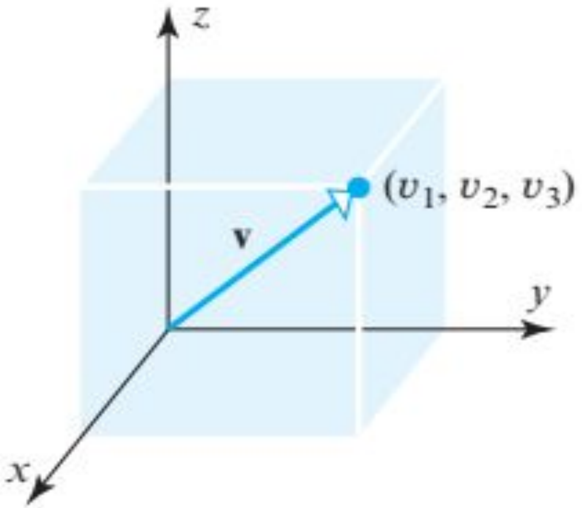
Components of vector v

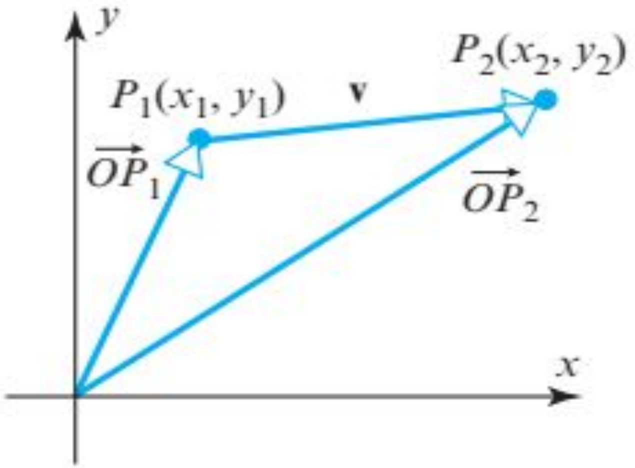
$\mathbf{v} = (v_1, v_2)$

$\mathbf{v} = (v_1, v_2, v_3)$

$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$







- The ordered pair (v_1, v_2) can represent a point or a vector.

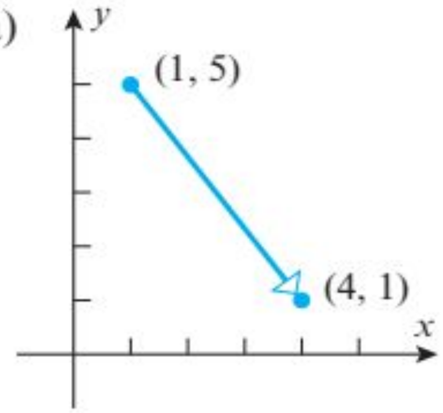
▶ EXAMPLE 1 Finding the Components of a Vector

The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

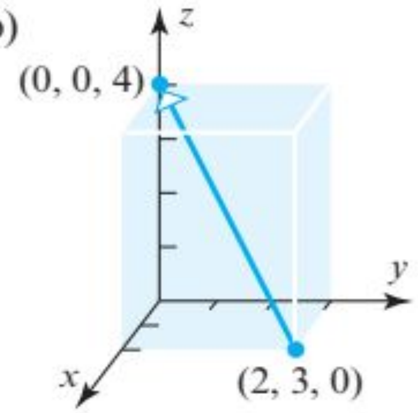
$$\mathbf{v} = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12)$$

In Exercises 1–2, find the components of the vector.

1. (a)



(b)



(a) (3, -4) & (b) (-2,3,4)

In Exercises 3–4, find the components of the vector $\overrightarrow{P_1P_2}$.

3. (a) $P_1(3, 5)$, $P_2(2, 8)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

4. (a) $P_1(-6, 2)$, $P_2(-4, -1)$ (b) $P_1(0, 0, 0)$, $P_2(-1, 6, 1)$

3a. (-1, 3)
3b. (-3, 6, 1)
4a. (2, -3)
4b. (-1, 6, 1)

n-Space

- Set of all real numbers can be represented as a line → R^1
- Set of all ordered pairs of real numbers (2-tuples) → R^2
- Set of all ordered triples of real numbers (3-tuples) → R^3
- Set of all ordered n-tuples: → R^n

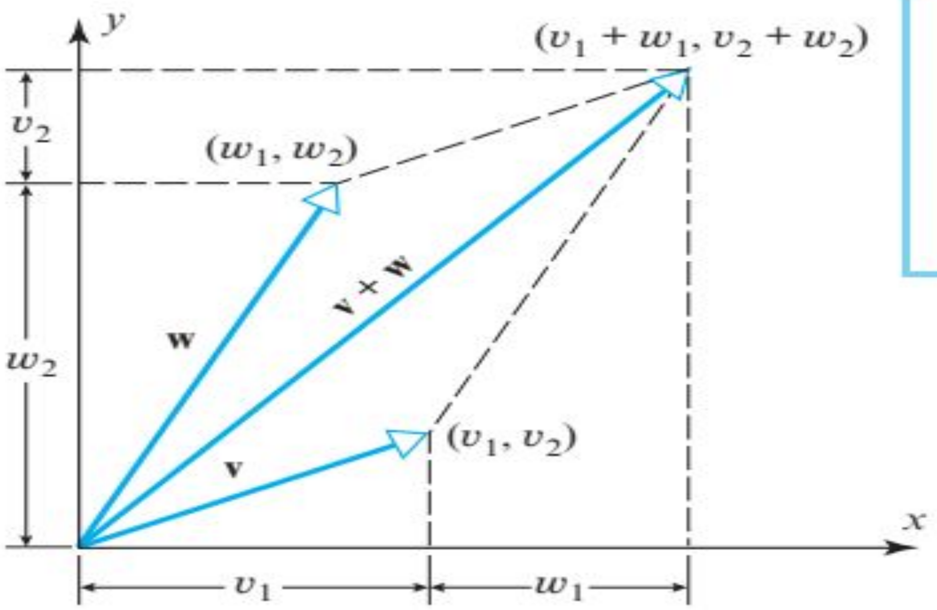
DEFINITION 1 If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n.

DEFINITION 2 Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \ldots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

Operations of Addition, Subtraction, and Scalar multiplication in R^n

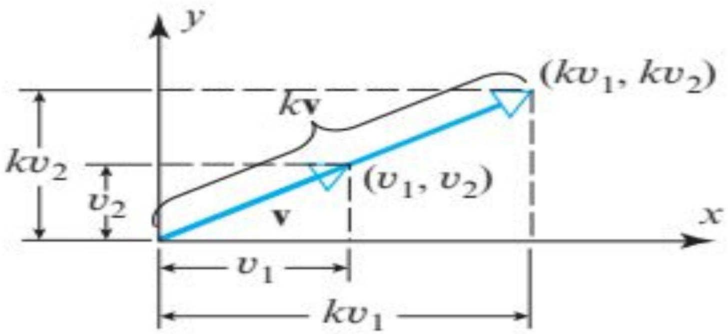


$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2) \tag{6}$$

$$k\mathbf{v} = (kv_1, kv_2) \tag{7}$$

$$-\mathbf{v} = (-1)\mathbf{v} = (-v_1, -v_2) \tag{8}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2) \tag{9}$$



DEFINITION 3 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n) \tag{13}$$

Algebraic Operations Using Components

EXAMPLE 3 Algebraic Operations Using Components

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$
$$-\mathbf{w} = (-4, -2, -1), \qquad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$

Algebraic Operations Using Components (Contd.)

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

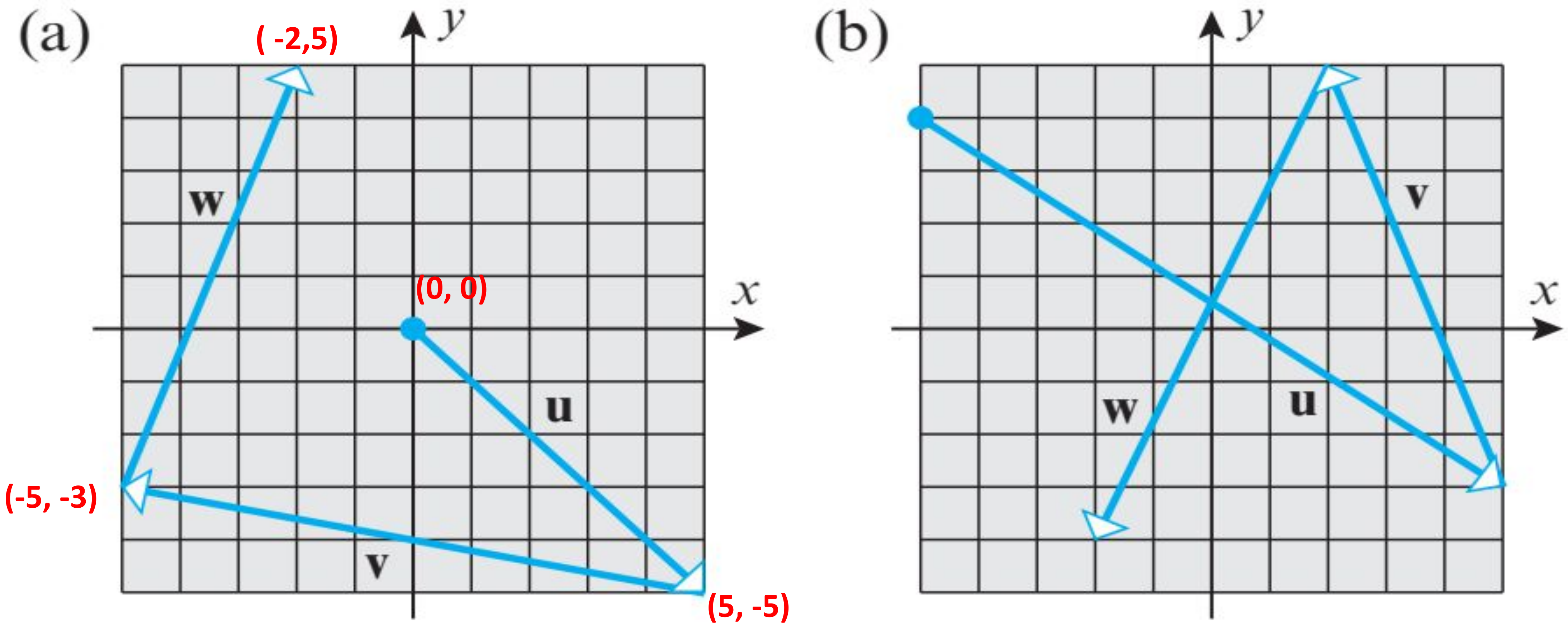
THEOREM 3.1.2 *If* $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then:*

(*a*) $0\mathbf{v} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{v} = -\mathbf{v}$

In each part, find the components of the vector $\mathbf{u} + \mathbf{v} + \mathbf{w}$.



(a) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (5, -5) + (-10, 2) + (3, 8) = (-2, 5)$

(b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (10, -7) + (-3, 8) + (-4, -9) = (3, -8)$

Find an initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3, 0, -5)$ and such that

(a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (4, -2, -1)$.

(b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (4, -2, -1)$.

(a) $(3-4, 0-(-2), -5-(-1))$, i.e., $(-1, 2, -4)$

(b) $(3-(-4), 0-2, -5-1)$, i.e., $(7, -2, -6)$

Linear Combinations

DEFINITION 4 If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

True-False Exercises

(a) Two equivalent vectors must have the same initial point. FALSE

(b) The vectors (a, b) and $(a, b, 0)$ are equivalent. FALSE

(c) If k is a scalar and $\mathbf{v}$ is a vector, then $\mathbf{v}$ and $k\mathbf{v}$ are parallel if and only if $k \geq 0$. FALSE

(d) The vectors $\mathbf{v} + (\mathbf{u} + \mathbf{w})$ and $(\mathbf{w} + \mathbf{v}) + \mathbf{u}$ are the same. TRUE

(e) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. TRUE

(f) If a and b are scalars such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors. FALSE

(g) Collinear vectors with the same length are equal. FALSE

(h) If $(a, b, c) + (x, y, z) = (x, y, z)$, then (a, b, c) must be the zero vector. TRUE

(i) If k and m are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors, then FALSE

$$(k + m)(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + m\mathbf{v}$$

(j) If the vectors $\mathbf{v}$ and $\mathbf{w}$ are given, then the vector equation TRUE

$$3(2\mathbf{v} - \mathbf{x}) = 5\mathbf{x} - 4\mathbf{w} + \mathbf{v}$$

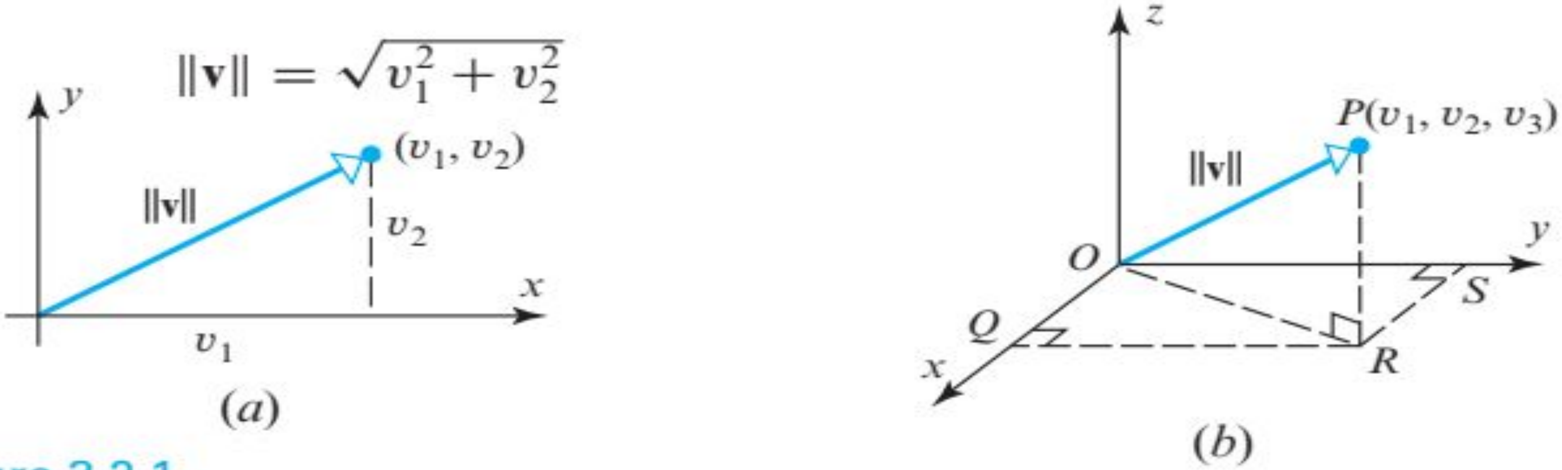
can be solved for $\mathbf{x}$.

(k) The linear combinations $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ and $b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ can only be equal if $a_1 = b_1$ and $a_2 = b_2$. FALSE

3.2: Norm, Dot Product, and Distance in n-space

Norm of a Vector

- The term “norm” is a mathematical synonym for length.



▲ Figure 3.2.1

From the theorem of Pythagoras the norm of v in 2-space (a) and in 3-space (b):

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Norm of a vector in n-space

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

EXAMPLE 1 Calculating Norms

It follows from Formula (2) that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula (3) that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

THEOREM 3.2.1 *If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:*

(a) $\|\mathbf{v}\| \geq 0$

(b) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(c) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

- **Distances are non-negative**
- **The zero vector is the only vector of length zero.**
- **Multiplying a vector by a scalar multiplies its length by the absolute value of that scalar.**

Unit Vector

- A vector of norm 1 is called a ***unit vector***. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand.
- More generally, if **v** is any nonzero vector in R^n, then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

- The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called ***normalizing* v**.

EXAMPLE 2 Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

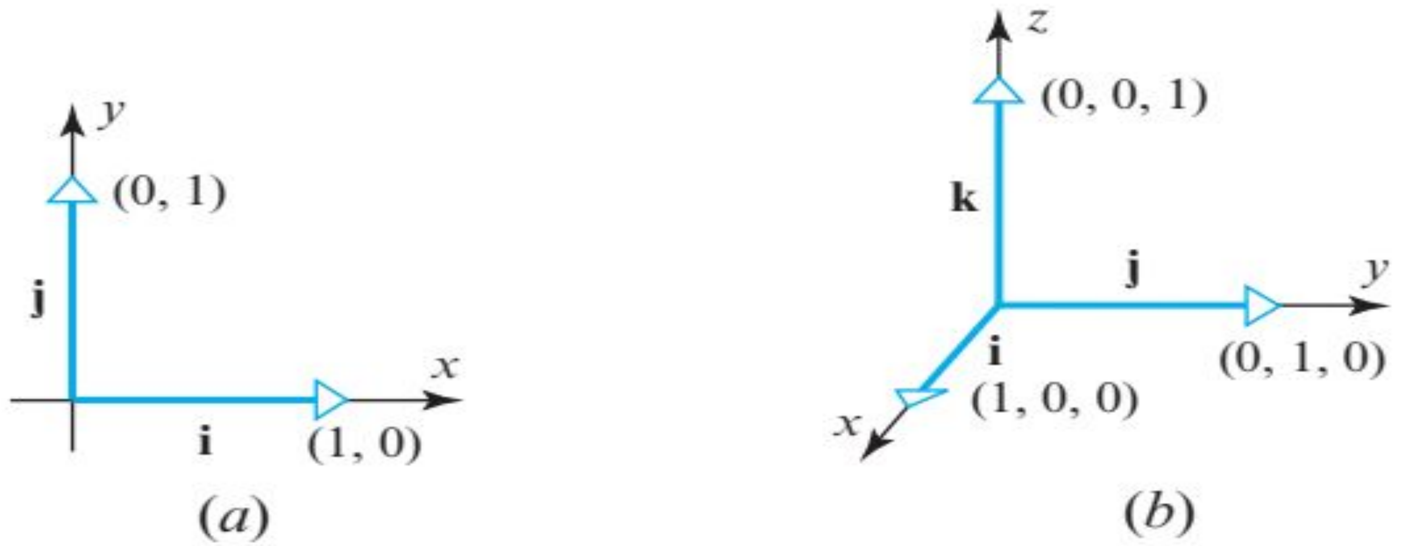
Thus, from (4)

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$. ◀

The standard Unit Vector

When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors***. In R^2 these



Moreover, we can generalize these formulas to R^n by defining the ***standard unit vectors in R^n*** to be

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

in which case every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

EXAMPLE 3 Linear Combinations of Standard Unit Vectors

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4$$

Distance in R^n

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance in R²

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance in R³

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

► EXAMPLE 4 **Calculating Distance in R^n**

If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$ ◄

In Exercises 1–2, find the norm of $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$.

1. (a) $\mathbf{v} = (2, 2, 2)$ (b) $\mathbf{v} = (1, 0, 2, 1, 3)$

2. (a) $\mathbf{v} = (1, -1, 2)$ (b) $\mathbf{v} = (-2, 3, 3, -1)$

In Exercises 3–4, evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

3. (a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$

   (c) $\|-2\mathbf{u} + 2\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

4. (a) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$ (b) $\|\mathbf{u} - \mathbf{v}\|$

   (c) $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$ (d) $\|\mathbf{u}\| - \|\mathbf{v}\|$

1. **(a)** $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$;

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{2\sqrt{3}}(2,2,2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; $-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{2\sqrt{3}}(2,2,2) = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

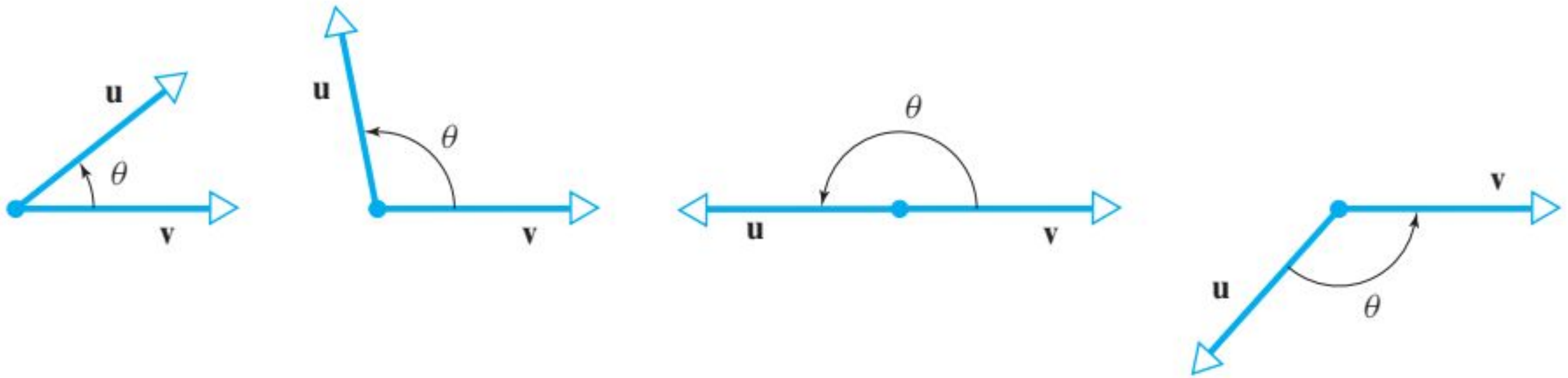
3. **(a)** $\mathbf{u} + \mathbf{v} = (3,-5,7)$; $\|\mathbf{u} + \mathbf{v}\| = \sqrt{3^2 + (-5)^2 + 7^2} = \sqrt{83}$

(b) $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 3^2} + \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{17} + \sqrt{26}$

(c) $-2\mathbf{u} + 2\mathbf{v} = (-4,4,-6) + (2,-6,8) = (-2,-2,2)$;
$\|-2\mathbf{u} + 2\mathbf{v}\| = \sqrt{(-2)^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

(d) $3\mathbf{u} - 5\mathbf{v} + \mathbf{w} = (6,-6,9) - (5,-15,20) + (3,6,-4) = (4,15,-15)$;
$\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\| = \sqrt{4^2 + 15^2 + (-15)^2} = \sqrt{466}$

Angle between two non zero vector



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

Dot Product

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \tag{13}$$

Since $0 \leq \theta \leq \pi$, it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.

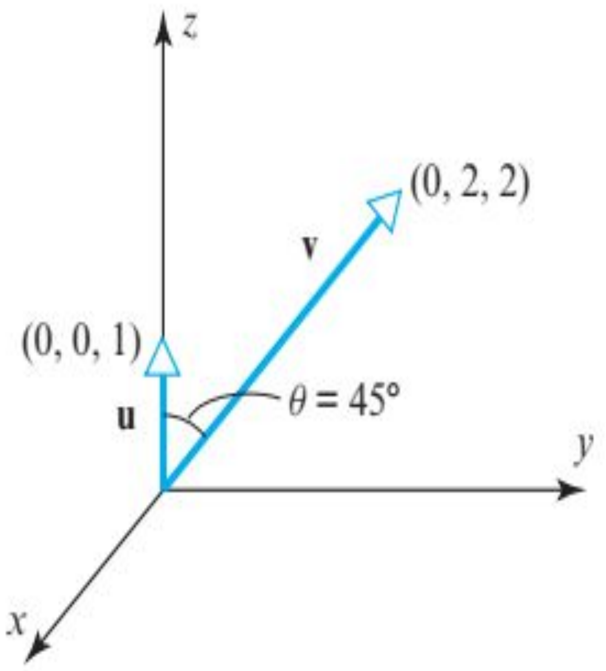


Figure 3.2.5

EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

Solution The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle θ between them is

$$\cos(45^\circ) = 1/\sqrt{2}$$

Thus, it follows from Formula (12) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$$

Components form of dot product

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{17}$$

EXAMPLE 6 Calculating Dot Products Using Components

(a) Use Formula (15) to compute the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 5.

(b) Calculate $\mathbf{u} \cdot \mathbf{v}$ for the following vectors in R^4:

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

Solution (a) The component forms of the vectors are $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)(0) + (0)(2) + (1)(2) = 2$$

which agrees with the result obtained geometrically in Example 5.

Solution (b)

$$\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4$$

Algebraic Properties of Dot Product

In the special case where $\mathbf{u} = \mathbf{v}$ in Definition 4, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \tag{18}$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{19}$$

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then*:

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then*:

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Algebraic Properties of Dot Product (Contd.)

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then*:

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

EXAMPLE 8 Calculating with Dot Products

$$
\begin{aligned}
(\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\
&= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v}) \\
&= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2
\end{aligned}
$$

Some other Theorems

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

THEOREM 3.2.5 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *then:*

(*a*) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**

THEOREM 3.2.6 Parallelogram Equation for Vectors

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n, *then*

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \tag{24}$$

In Exercises 5–6, evaluate the given expression with $\mathbf{u} = (-2, -1, 4, 5)$, $\mathbf{v} = (3, 1, -5, 7)$, and $\mathbf{w} = (-6, 2, 1, 1)$.

5. (a) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$ (b) $\|3\mathbf{u}\| - 5\|\mathbf{v}\| + \|\mathbf{w}\|$

(c) $\|-\|\mathbf{u}\|\mathbf{v}\|$

6. (a) $\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$ (b) $\|\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\|$

In Exercises 9–10, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$.

9. (a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$

(b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

10. (a) $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$

(b) $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

In Exercises 11–12, find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°.

11. (a) $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$

(b) $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$

12. (a) $\mathbf{u} = (1, 2, -3, 0)$, $\mathbf{v} = (5, 1, 2, -2)$

(b) $\mathbf{u} = (0, 1, 1, 1, 2)$, $\mathbf{v} = (2, 1, 0, -1, 3)$

5. (a) $\sqrt{2570}$

(b) $3\sqrt{46} - 10\sqrt{21} + \sqrt{42}$

(c) $\|-\|\mathbf{u}\|\mathbf{v}\| = \sqrt{46}\sqrt{84} = 2\sqrt{966}$

6. (a) $\sqrt{46} + 4\sqrt{21} + 3\sqrt{42}$

(b) $\|\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\| = \sqrt{4788} = 6\sqrt{133}$

9. (a) $\mathbf{u} \cdot \mathbf{v} = -8$

$\mathbf{u} \cdot \mathbf{u} = 26$

$\mathbf{v} \cdot \mathbf{v} = 24$

11. (a) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(3-1)^2 + (3-0)^2 + (3-4)^2} = \sqrt{14}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5}{\sqrt{51}}$; the angle is acute since $\mathbf{u} \cdot \mathbf{v} > 0$

(b) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{59}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-4}{\sqrt{6}\sqrt{45}}$; the angle is obtuse since $\mathbf{u} \cdot \mathbf{v} < 0$

Orthogonality

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

EXAMPLE 1 Orthogonal Vectors

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = \mathbf{0}$$

$$\mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = \mathbf{0}$$

$$\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1, 0, 0) \cdot (0, 0, 1) = 0$$

$$\mathbf{j} \cdot \mathbf{k} = (0, 1, 0) \cdot (0, 0, 1) = 0$$

In Exercises 1–2, determine whether **u** and **v** are orthogonal vectors.

1. (a) $\mathbf{u} = (6, 1, 4)$, $\mathbf{v} = (2, 0, -3)$ Orthogonal

   (b) $\mathbf{u} = (0, 0, -1)$, $\mathbf{v} = (1, 1, 1)$ Non-Orthogonal

   (c) $\mathbf{u} = (3, -2, 1, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$ Non-Orthogonal

   (d) $\mathbf{u} = (5, -4, 0, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$ Non-Orthogonal

In Exercises 11–12, determine whether the given planes are perpendicular.

11. $3x - y + z - 4 = 0$, $x + 2z = -1$ **Not perpendicular**

12. $x - 2y + 3z = 4$, $-2x + 5y + 4z = -1$ Not perpendicular

In Exercises 7–10, determine whether the given planes are parallel.

7. $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$

8. $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$

9. $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$

10. $(-4, 1, 2) \cdot (x, y, z) = 0$ and $(8, -2, -4) \cdot (x, y, z) = 0$

Not parallel

Parallel

Not parallel

Parallel

GENERAL VECTOR SPACES

(Algebraic properties of vectors in $\mathbf{R}^n$ as axioms)

Vector Space

Nonempty set of objects with two operations: Addition & Scalar Multiplication

If the following axioms are satisfied by all objects **u**, **v**, **w** in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If **u** and **v** are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V. Closure under addition
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all **u** in V.
5. For each **u** in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of **u**, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and **u** is any object in V, then $k\mathbf{u}$ is in V. Closure under Scalar Multiplication
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

EXAMPLE 1 The Zero Vector Space

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this the ***zero vector space***.

EXAMPLE 2 R^n Is a Vector Space

Let $V = R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples; that is,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n) \end{aligned}$$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce n-tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1.

▶ EXAMPLE 3 The Vector Space of Infinite Sequences of Real Numbers R^∞

Let V consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \ldots, u_n, \ldots)$$

in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication componentwise by

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) \\
&= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots) \\
k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n, \ldots)
\end{aligned}$$

EXAMPLE 4 The Vector Space of 2 × 2 Matrices

Let V be the set of 2×2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} \tag{1}$$

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

- The set ***V*** is closed under **addition** and scalar **multiplication**.
- Axioms 2, 3, 4, 5, 8, 9, 10 satisfied by **Theorem 1.4.1**.

THEOREM 1.4.1 Properties of Matrix Arithmetic

(a) $A+B=B+A$

(b) $A+(B+C)=(A+B)+C$

(c) $A(BC)=(AB)C$

(d) $A(B+C)=AB+AC$

(e) $(B+C)A=BA+CA$

(f) $A(B-C)=AB-AC$

(g) $(B-C)A=BA-CA$

(h) $a(B+C)=aB+aC$

(i) $a(B-C)=aB-aC$

(j) $(a+b)C=aC+bC$

(k) $(a-b)C=aC-bC$

(l) $a(bC)=(ab)C$

(m) $a(BC)=(aB)C=B(aC)$

EXAMPLE 5 The Vector Space of $m \times n$ Matrices

- **Example 4** is a special case of a more general class of vector spaces.
- The set V of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication is a vector space.
- We will denote this vector space by the symbol M_{mn}.
- The vector space in Example 4 is denoted as M_{22}.

EXAMPLE 6 The Vector Space of Real-Valued Functions

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \tag{2}$$

$$(k\mathbf{f})(x) = kf(x) \tag{3}$$

- Think about these operations is to view the numbers $f(x)$ and $g(x)$ as "components" of $\mathbf{f}$ and $\mathbf{g}$ at the point x

- The set V with these operations is denoted by the symbol $F(-\infty, \infty)$.

▶ EXAMPLE 7 A Set That Is Not a Vector Space

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

- The first nine vector space axioms are satisfied.
- However, Axiom 10 fails to hold for certain vectors. For example,

  if $u = (u_1, u_2)$ is such that $u_2 \neq 0$, then

  $1u = 1(u_1, u_2) = (1.u_1, 0) = (u_1, 0) \neq u$
- Thus, V is not a vector space with the stated operations.

EXAMPLE 8 An Unusual Vector Space

Let V be the set of positive real numbers, let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors (i.e., positive real numbers) in V, and let k be any scalar. Define the operations on V to be

$$u + v = uv \quad \textbf{[Vector addition is numerical multiplication.]}$$
$$ku = u^k \quad \textbf{[Scalar multiplication is numerical exponentiation.]}$$

Thus, for example, $1 + 1 = 1$ and $(2)(1) = 1^2 = 1$—strange indeed, but nevertheless the set V with these operations satisfies the ten vector space axioms and hence is a vector space. We will confirm Axioms 4, 5, and 7, and leave the others as exercises.

- Axiom 4—The zero vector in this space is the number 1 (i.e., $\mathbf{0} = 1$) since

$$u + 1 = u \cdot 1 = u$$

- Axiom 5—The negative of a vector u is its reciprocal (i.e., $-u = 1/u$) since

$$u + \frac{1}{u} = u\left(\frac{1}{u}\right) = 1\ (= \mathbf{0})$$

- Axiom 7—$k(u + v) = (uv)^k = u^k v^k = (ku) + (kv)$. ◀

THEOREM 4.1.1 *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:*

(*a*) $0\mathbf{u} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{u} = -\mathbf{u}$

(*d*) *If* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$.

TRUE-FALSE

(a) A vector is any element of a vector space. TRUE

(b) A vector space must contain at least two vectors. FALSE

(c) If $\mathbf{u}$ is a vector and k is a scalar such that $k\mathbf{u} = \mathbf{0}$, then it must be true that $k = 0$. FALSE

(d) The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers. FALSE

(e) In every vector space the vectors $(-1)\mathbf{u}$ and $-\mathbf{u}$ are the same. TRUE

(f) In the vector space $F(-\infty, \infty)$ any function whose graph passes through the origin is a zero vector. FALSE

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.

$\mathbf{u} + \mathbf{v} = (2, 6)$; $\quad k\mathbf{u} = (0, 6)$

(b) In words, explain why V is closed under addition and scalar multiplication.

- For any $\mathbf{u}$ and $\mathbf{v}$ in V, $\mathbf{u} + \mathbf{v}$ is an ordered pair of real numbers, therefore $\mathbf{u} + \mathbf{v}$ is in V.
- For any $\mathbf{u}$ in V and for any scalar k, $k\mathbf{u} = (0, ku_2)$ is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V.

(c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Show that Axioms 7, 8, and 9 hold.

(d) Axiom 7: $k((u_1, u_2) + (v_1, v_2)) = k(u_1 + v_1, u_2 + v_2) = (0, k(u_2 + v_2)) = (0, ku_2) + (0, kv_2)$

$= k(u_1, u_2) + k(v_1, v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k + m)(u_1, u_2) = (0, (k + m)u_2) = (0, ku_2 + mu_2) = (0, ku_2) + (0, mu_2)$

$= k(u_1, u_2) + m(u_1, u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1, u_2)) = k(0, mu_2) = (0, kmu_2) = (km)(u_1, u_2)$ for all real k, m, u_1, and u_2

(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

$1(u_1, u_2) = (0, u_2)$ does not generally equal (u_1, u_2).

Determine whether each set equipped with the given operations is a vector space.

3. The set of all real numbers with the standard operations of addition and multiplication. Vector Space

4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2. Vector Space

5. The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2. Axioms 6 failed

6. The set of all n-tuples of real numbers that have the form $(x, x, \ldots, x)$ with the standard operations on R^n. Vector Space

7. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$ Axioms 8 failed

8. The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication. Axioms 6 failed

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$ Vector Space

with the standard matrix addition and scalar multiplication.

10. The set of all real-valued functions f defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6. Vector Space

11. The set of all pairs of real numbers of the form $(1, x)$ with the operations

Vector Space

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

12. The set of polynomials of the form $a_0 + a_1x$ with the operations

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x$$

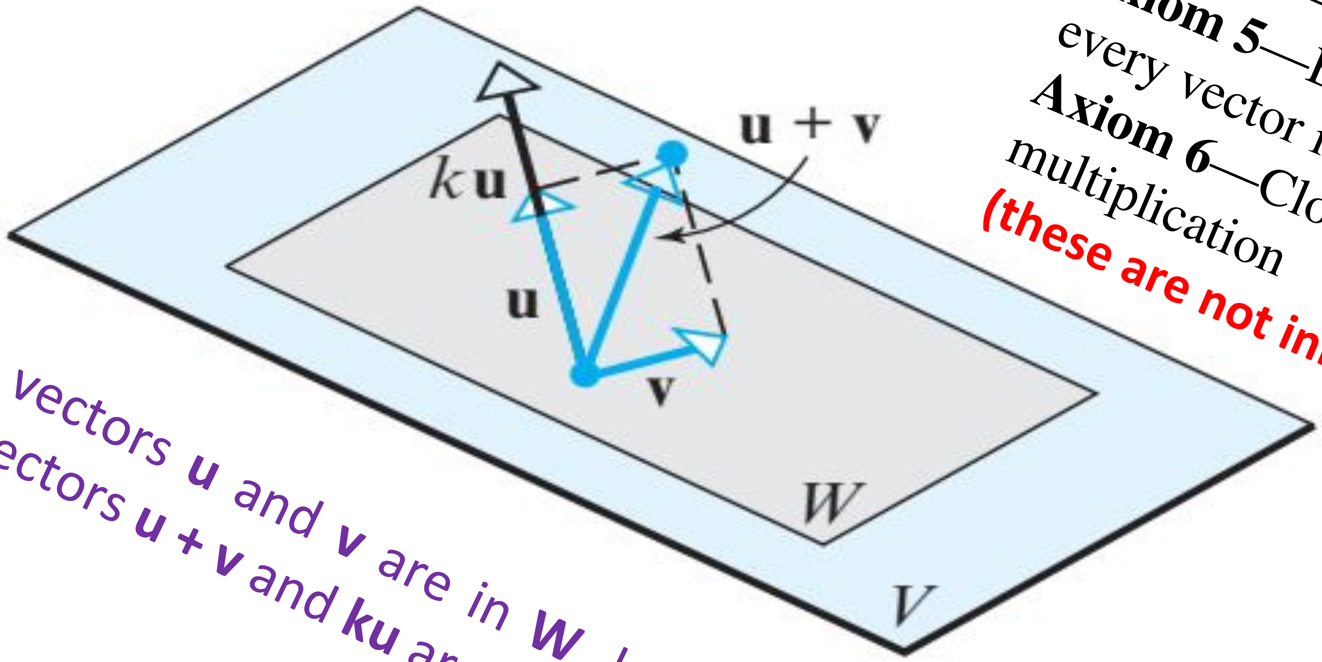
and Vector Space

$$k(a_0 + a_1x) = (ka_0) + (ka_1)x$$

Subspaces

DEFINITION 1 A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Axiom 1—Closure of W under addition
Axiom 4—Existence of a zero vector in W
Axiom 5—Existence of a negative in W for every vector in W
Axiom 6—Closure of W under scalar multiplication
(these are not inherited from V)



The vectors **u** and **v** are in **W**, but the vectors **u + v** and **ku** are not.

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(*a*) *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.*

(*b*) *If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.*

▶ EXAMPLE 1 The Zero Subspace

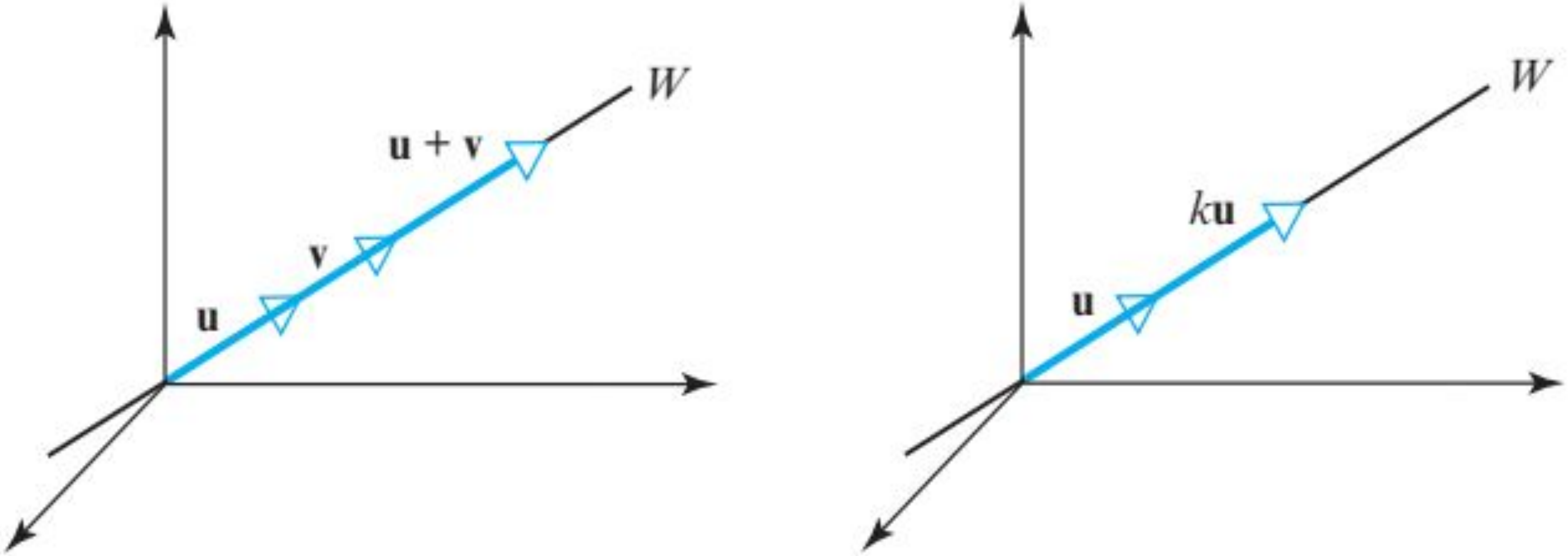
If V is any vector space, and if $W=\{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0}+\mathbf{0}=\mathbf{0} \quad \text{and} \quad k\mathbf{0}=\mathbf{0}$$

for any scalar k. We call W the ***zero subspace*** of V.

EXAMPLE 2 Lines Through the Origin Are Subspaces of R^2 and of R^3

If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication (see Figure 4.2.2 for an illustration in R^3)



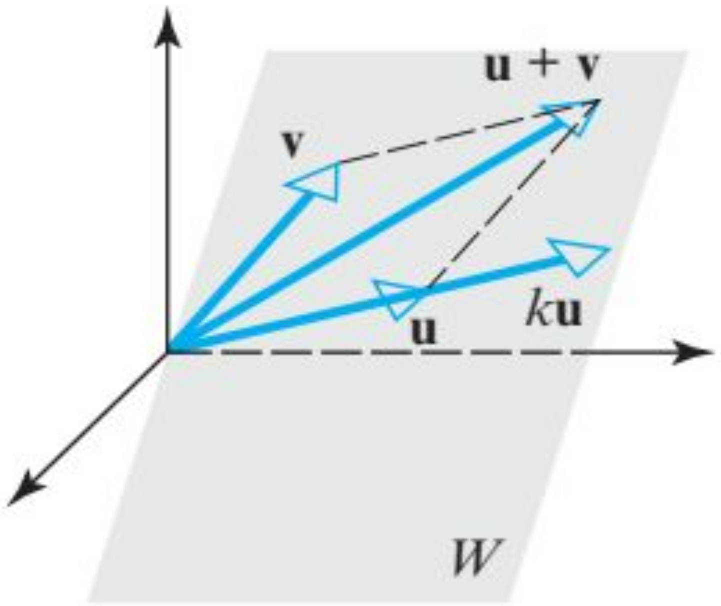
(*a*) W is closed under addition.

(*b*) W is closed under scalar multiplication.

Figure 4.2.2

EXAMPLE 3 Planes Through the Origin Are Subspaces of R^3

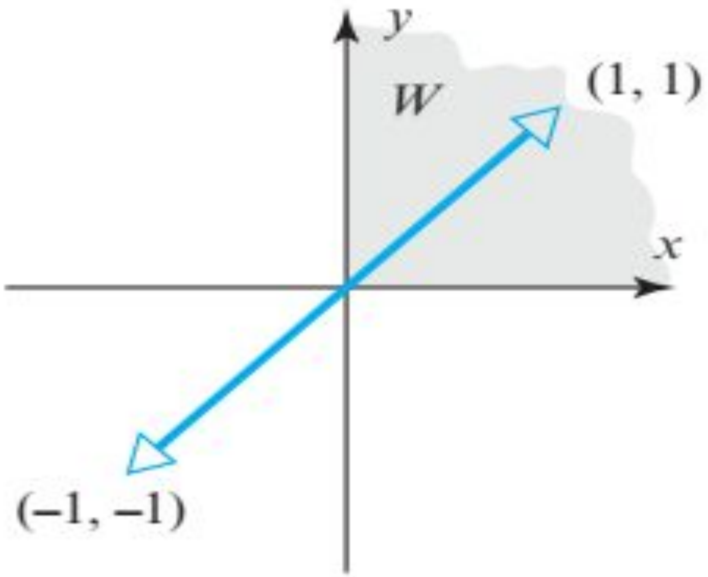
If **u** and **v** are vectors in a plane W through the origin of R^3, then it is evident geometrically that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ also lie in the same plane W for any scalar k (Figure 4.2.3). Thus W is closed under addition and scalar multiplication. ◀



▲ Figure 4.2.3 The vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as **u** and **v**.

EXAMPLE 4 A Subset of R^2 That Is Not a Subspace

- Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$.



- This set is not a subspace of R^2 because it is not closed under scalar multiplication.

- For example, $\mathbf{v} = (1, 1)$ is a vector in W, but $(-1)\mathbf{v} = (-1, -1)$ is not.

EXAMPLE 5 Subspaces of M_{nn}

- Let W be the set of symmetric matrices.

- Then **W** is a subspace of **V** because the sum of two symmetric matrices and the scalar multiplication of symmetric matrices are in W.

- Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

EXAMPLE 6 A Subset of M_{nn} That Is Not a Subspace

- Let **W** be the set of invertible **n × n** matrices. **W** is not a subspace because the zero matrix is not in **W** .

- Consider the matrices: $U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$

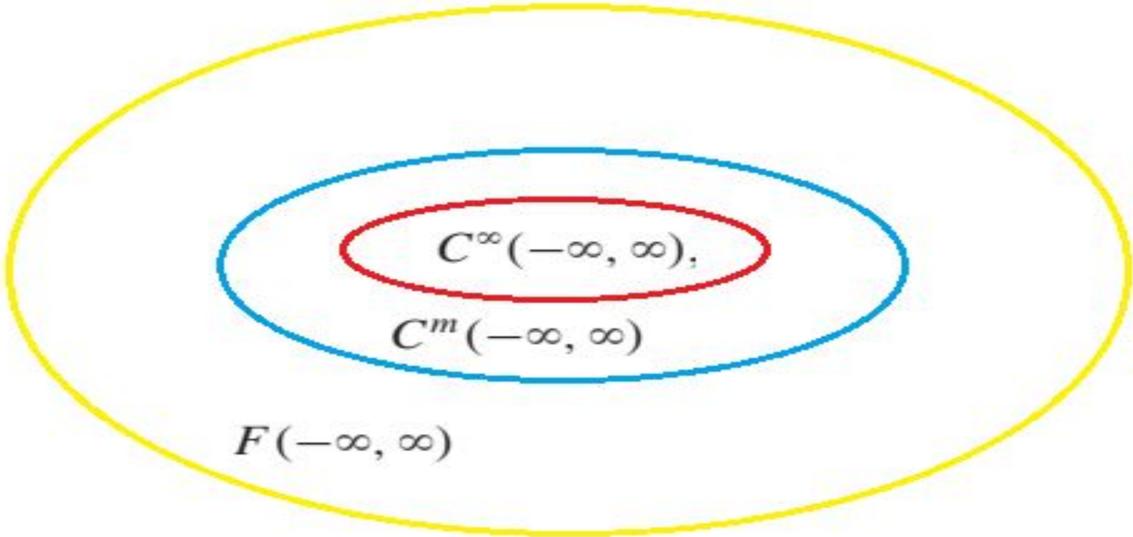
- The matrix $\mathbf{0}\boldsymbol{U}$ is the **2 × 2** zero matrix and hence is not invertible, and the matrix $\boldsymbol{U} + \boldsymbol{V}$ has a column of zeros so it also is not invertible.

EXAMPLE 7 The Subspace $C(-\infty, \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous. Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C(-\infty, \infty)$.

EXAMPLE 8 Functions with Continuous Derivatives

- The functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$.
- The sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable.



▶ EXAMPLE 9 The Subspace of All Polynomials

Recall that a ***polynomial*** is a function that can be expressed in the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants. It is evident that the sum of two polynomials is a

polynomial and that a constant times a polynomial is a polynomial. Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞.

▶ EXAMPLE 10 The Subspace of Polynomials of Degree ≤ *n*

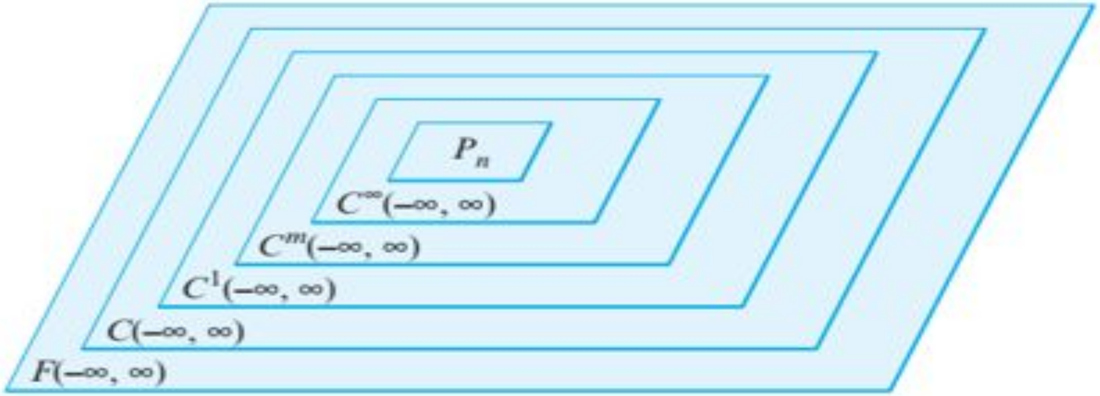
Recall that the ***degree*** of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n. It is *not* true that the set W of polynomials with positive degree n is a subspace of $F(-\infty, \infty)$ because that set is not closed under addition. For example, the polynomials

$$1 + 2x + 3x^2 \quad \text{and} \quad 5 + 7x - 3x^2$$

both have degree 2, but their sum has degree 1. What *is* true, however, is that for each nonnegative integer n the polynomials of degree n *or less* form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n. ◀

The Hierarchy of Function Spaces

- It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$.
- Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$.
- Last 4 examples are "nested " one inside other as illustrated in the figure:



THEOREM 4.2.2 *If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.*

Spanning Set

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

DEFINITION 3 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V ***generated*** by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ ***span*** W. We denote this subspace as

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

- **vectors (v1, v2, …, vr) span a space means: The space consists of all combinations of those vectors.**
- **Span (v1, v2, ... , vr) → set of all possible linear combination of v1, v2, …, vr.**

EXAMPLE 11 The Standard Unit Vectors Span R^n

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Thus, for example, the vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

span R^3 since every vector $\mathbf{v} = (a, b, c)$ in this space can be expressed as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3.

   (a) All vectors of the form $(a, 0, 0)$. W is a subspace of R^3.

   (b) All vectors of the form $(a, 1, 1)$. W is not a subspace of R^3.

   (c) All vectors of the form (a, b, c), where $b = a + c$. W is a subspace of R^3.

   (d) All vectors of the form (a, b, c), where $b = a + c + 1$. W is not a subspace of R^3.

   (e) All vectors of the form $(a, b, 0)$. W is a subspace of R^3.

2. Use Theorem 4.2.1 to determine which of the following are subspaces of M_{nn}.

   (a) The set of all diagonal $n \times n$ matrices. W is a subspace of M_{nn}.

   (b) The set of all $n \times n$ matrices A such that $\det(A) = 0$. W is not a subspace of M_{nn}.

   (c) The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$. W is a subspace of M_{nn}.

   (d) The set of all symmetric $n \times n$ matrices. W is a subspace of M_{nn}.

   (e) The set of all $n \times n$ matrices A such that $A^T = -A$. W is a subspace of M_{nn}.

   (f) The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. W is not a subspace of M_{nn}.

EXAMPLE 14 Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

- $$\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$$
- $(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

- Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$

Similarly, $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equations is inconsistent (verify), so no such scalars k_1 and k_2 exist.

EXAMPLE 15 Testing for Spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

- $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$
- $(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$

- $$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned} \longrightarrow A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$
- since $\det(A) = 0$ (verify), so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) $(2, 2, 2)$ (b) $(0, 4, 5)$ (c) $(0, 0, 0)$

- $a(0,-2,2) + b(1,3,-1) = (2,2,2)$
- $$\begin{array}{rcrcr} 0a & + & 1b & = & 2 \\ -2a & + & 3b & = & 2 \\ 2a & - & 1b & = & 2 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
- The linear system is consistent.

(b) $a(0,-2,2) + b(1,3,-1) = (0,4,5)$

- $$\begin{array}{rcrcr} 0a & + & 1b & = & 0 \\ -2a & + & 3b & = & 4 \\ 2a & - & 1b & = & 5 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- The system is inconsistent, $(0, 4, 5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) The zero vector $(0, 0, 0)$ is a lc of $\mathbf{u}$ and $\mathbf{v}$.

- $0(0,-2,2) + 0(1,3,-1) = (0,0,0)$

In each part, determine whether the vectors span R^3.

(a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$

(b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

(a) • The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination $(b_1, b_2, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$

$$\begin{array}{rcrcrcl} 2k_1 & + & 0k_2 & + & 0k_3 & = & b_1 \\ 2k_1 & + & 0k_2 & + & 1k_3 & = & b_2 \\ 2k_1 & + & 3k_2 & + & 1k_3 & = & b_3 \end{array}$$

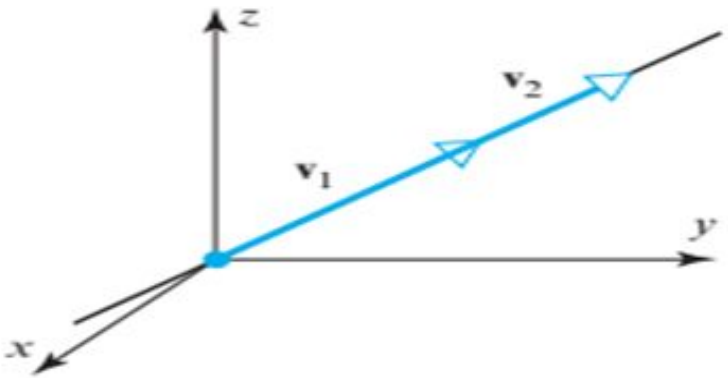
• We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3.

(b) • The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination $(b_1, b_2, b_3) = k_1(2,-1,3) + k_2(4,1,2) + k_3(8,-1,8)$

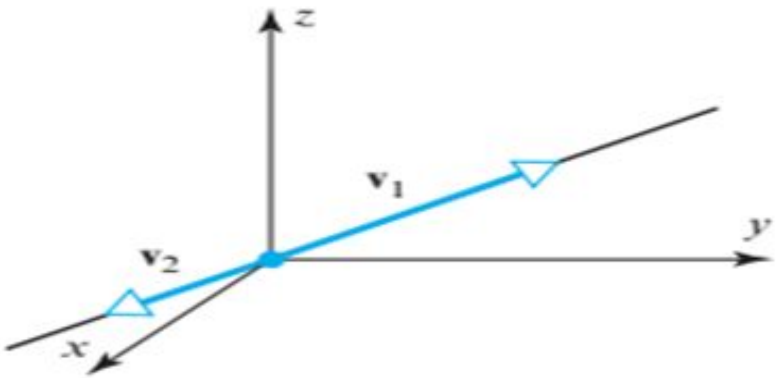
$$\begin{array}{rcrcrcl} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array} \longrightarrow \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0,$$

• Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$. We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

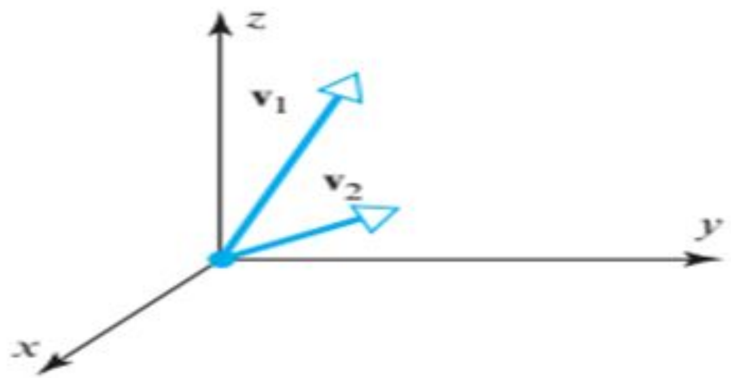
Linear Independence



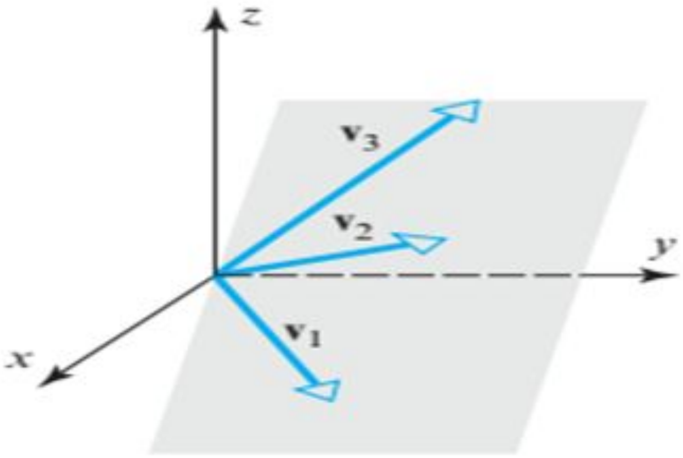
(*a*) Linearly dependent



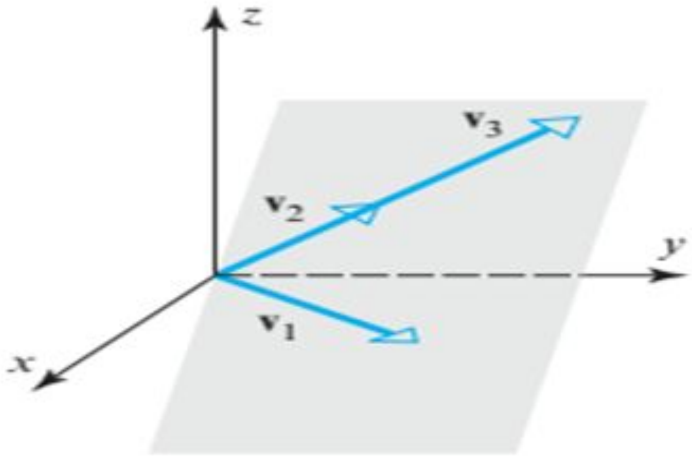
(*b*) Linearly dependent



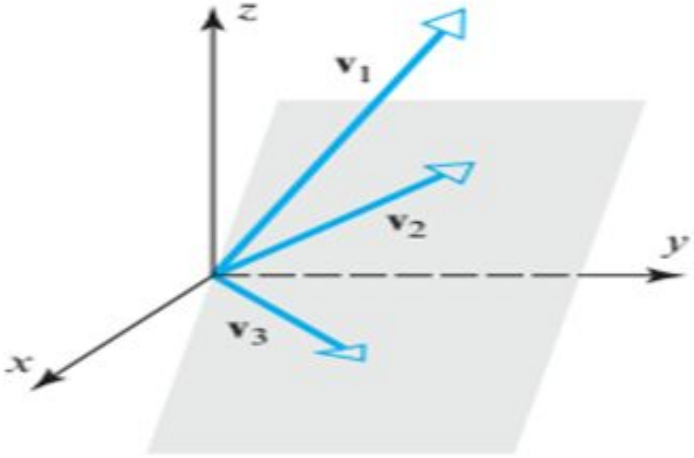
(*c*) Linearly independent



(*a*) Linearly dependent



(*b*) Linearly dependent



(*c*) Linearly independent

Linear Independence (Contd.)

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

THEOREM 4.3.1 *A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \dots, k_r = 0$.

Linear Independence in R^3

Determine whether the vectors: $\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation :

- $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$

- $k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - \ \ k_2 + \ \ k_3 &= 0 \end{aligned}$$

- $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

- This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution

- $$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$
- $$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

- $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$
- Linearly independent vectors.

THEOREM 4.3.2

(*a*) *A finite set that contains* $\mathbf{0}$ *is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not* $\mathbf{0}$.

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

THEOREM 4.3.3 *Let* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *be a set of vectors in* R^n. *If* $r > n$, *then* S *is linearly dependent.*

Explain why the following form linearly dependent sets of vectors.

(a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

(b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

(c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

TRUE- FALSE

(a) A set containing a single vector is linearly independent. FALSE

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k. TRUE

(c) Every linearly dependent set contains the zero vector. FALSE

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k. TRUE

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. TRUE

(f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}. FALSE

(g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent. TRUE

(h) The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2. FALSE

BASIS FOR A VECTOR SPACE

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

- Recall standard unit vectors:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

- Span R^3& linearly independent,
- Thus, they form a basis for R^n that we call the ***standard basis for R^n***.
- In particular, $\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$ is the standard basis for R^3.

EXAMPLE

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

Solution We must show that these vectors are linearly independent and span R^3.

- $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ **(to prove linear independence)**
- $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ **(vector span $\mathbf{R}^3$)**

where, $\mathbf{b} = (b_1, b_2, b_3)$

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad\quad + 4c_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad\quad + 4c_3 &= b_3 \end{aligned} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

- det (A) = -1, which proves that the vector $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3.
- The homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of $b1$, $b2$, and $b3$.

EXAMPLE The Standard Basis for M_{mn}

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

- $c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = 0$ **(to prove linear independence)**
- $c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B$ **(to prove matrices span M22.**

  where, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$$

the matrices span M_{22}. This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}.

non-square matrix and Basis of R2

v1=(1,6), v2=(2, 4), v3=(-1, 2)

c1=c2=0 (linearly independent)

reduced echelon form: (1,0,0) (0,1,0) & (0,0,0)

it is also spanning R2 thus forming a basis for R2.

let B = { b1, b2, b3} is a base for a vector space V. if another vector v is added to B (belongs to V) then B will be a set of linearly dependent variables.

Coordinate Vector relative to basis

DEFINITION 2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the ***coordinates*** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the ***coordinate vector of*** $\mathbf{v}$ ***relative to*** $\boldsymbol{S}$; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n) \tag{6}$$

EXAMPLE 8 Coordinate Vectors Relative to Standard Bases

(a) Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n.

(b) Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

Solution (a) The given formula for $\mathbf{p}(x)$ expresses this polynomial as a linear combination of the standard basis vectors $S = \{1, x, x^2, \ldots, x^n\}$. Thus, the coordinate vector for $\mathbf{p}$ relative to S is

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n)$$

Solution (b) We showed in Example 4 that the representation of a vector

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a linear combination of the standard basis vectors is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is

$$(B)_S = (a, b, c, d)$$

EXAMPLE 9 Coordinates in R^3

(a) We showed in Example 3 that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution (a) To find $(\mathbf{v})_S$ we must first express $\mathbf{v}$ as a linear combination of the vectors in S; that is, we must find values of c_1, c_2, and c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or, in terms of components,

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad\; + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$ (verify). Therefore,

$$(\mathbf{v})_S = (1, -1, 2)$$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7) \end{aligned}$$

it follows from

1. show that the following set of vectors forms a basis for R^3

$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

2. show that the set of vectors is not a basis for R^3.

$$\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$$

3. Find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3.

(a) $\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (3, 3, 3)$

(b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

TRUE-FALSE

(a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. FALSE

(b) Every linearly independent subset of a vector space V is a basis for V. FALSE

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. TRUE

(d) The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$. TRUE

(e) Every basis of P_4 contains at least one polynomial of degree 3 or less. FALSE

Dimension

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

EXAMPLE 1 Dimensions of Some Familiar Vector Spaces

$\dim(R^n) = n$ **[The standard basis has n vectors.]**

$\dim(P_n) = n + 1$ **[The standard basis has $n + 1$ vectors.]**

$\dim(M_{mn}) = mn$ **[The standard basis has mn vectors.]**

For exercise set 4.5 (q1)

THEOREM 4.3.2

(*a*) *A finite set that contains* **0** *is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Dimension of Span (S)

$$\dim[\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r$$

the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

Dimension of a solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 0\\
2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution

- $x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$
- $(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$
- $\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$
- These vactor span the solution space & are linearly independent.
- Thus solution space has dimension 3.

Bases by inspection

Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.

Solution Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis by Theorem 4.5.4.

- find a basis for the solution space of the ho mogeneous linear system, and find the dimension of that space.

1.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

- We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1 .

- We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

7. In each part, find a basis for the given subspace of R^3, and state its dimension.

(a) The plane $3x - 2y + 5z = 0$.

(b) The plane $x - y = 0$.

(c) The line $x = 2t$, $y = -t$, $z = 4t$.

(d) All vectors of the form (a, b, c), where $b = a + c$.

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

Row Space, Column Space & Null Space

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the ***null space*** of A.

THEOREM 4.7.1 *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

THEOREM 4.7.3 *Elementary row operations do not change the null space of a matrix.*

THEOREM 4.7.4 *Elementary row operations do not change the row space of a matrix.*

- ***An elementary row operation can alter its column space.***

EXAMPLE 2 A Vector b in the Column Space of A

Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

EXAMPLE 4 Finding a Basis for the Null Space of a Matrix

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Solution The null space of A is the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, which has the basis

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

THEOREM 4.7.5 *If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*

► EXAMPLE 5 Bases for the Row and Column Spaces of a Matrix in Row Echelon Form

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution Since the matrix R is in row echelon form, it follows from Theorem 4.7.5 that the vectors

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r}_2 &= [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

THEOREM 4.7.6 *If A and B are row equivalent matrices, then:*

(*a*) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(*b*) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

▶ EXAMPLE 7 Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of A.

Solution

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\operatorname{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\operatorname{nullity}(A)$.

► EXAMPLE 1 Rank and Nullity of a 4 × 6 Matrix

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since this matrix has two leading 1's, its row and column spaces are two dimensional and rank*(A)* = 2.

Example # 01 (Contd.)

- To find the nullity of *A*, we must find the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$.
- This system can be solved by reducing its augmented matrix to reduced row echelon form.

$$\begin{aligned} x_1 &= 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 &= 2x_3 + 12x_4 + 16x_5 - 5x_6 \end{aligned}$$

from which we obtain the general solution

$$\begin{aligned} x_1 &= 4r + 28s + 37t - 13u \\ x_2 &= 2r + 12s + 16t - 5u \\ x_3 &= r \\ x_4 &= s \\ x_5 &= t \\ x_6 &= u \end{aligned}$$

or in column vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3)$$

Because the four vectors on the right side of (3) form a basis for the solution space, $\text{nullity}(A) = 4$.

▶ EXAMPLE 2 Maximum Value for Rank

What is the maximum possible rank of an $m \times n$ matrix A that is not square?

Solution Since the row vectors of A lie in R^n and the column vectors in R^m, the row space of A is at most n-dimensional and the column space is at most m-dimensional. Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n. We denote this by writing

$$\text{rank}(A) \leq \min(m, n)$$

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\mathit{rank}(A) + \mathit{nullity}(A) = n \tag{4}$$

EXAMPLE 3 The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\operatorname{rank}(A) = 2 \quad \text{and} \quad \operatorname{nullity}(A) = 4$$

THEOREM 4.8.3 *If A is an $m \times n$ matrix, then*

(*a*) *rank*$(A) =$ *the number of leading variables in the general solution of* $A\mathbf{x} = \mathbf{0}$.

(*b*) *nullity*$(A) =$ *the number of parameters in the general solution of* $A\mathbf{x} = \mathbf{0}$.

QUIZ

- (1) Use the adjoint method to find the inverse of the given matrix, if it exists:

$$\begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$$

- (2) Let $\mathbf{u} = (-2, 0, 4)$, $\mathbf{v} = (3, -1, 6)$, and $\mathbf{w} = (2, -5, -5)$. Compute

  (a) $3\mathbf{v} - 2\mathbf{u}$

  (b) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

  (c) the distance between $-3\mathbf{u}$ and $\mathbf{v} + 5\mathbf{w}$

- (3) determine which of the following are subspaces of M_{nn}.

  (a) The set of all $n \times n$ matrices A such that $\det(A) = 0$.

  (b) The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$.

  (c) The set of all symmetric $n \times n$ matrices.

1. $$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix}$$

$$=3\left(-0 + (-1)\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}\right) - 6\left(-1\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 4 \\ -9 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ -9 & -2 \end{vmatrix}\right) + 0$$
$$-1\left(-1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} + 0 - (-1)\begin{vmatrix} -2 & 3 \\ -9 & 2 \end{vmatrix}\right)$$
$$=3\big(0 - 1(-2) - 1(-8)\big) - 6\big(-1(10) - 1(32) - 1(13)\big) + 0 - 1\big(-1(-8) + 0 + 1(23)\big)$$
$$= 3(10) - 6(-55) + 0 - 1(31)$$
$$= 329$$

2. **(a)** $3\mathbf{v} - 2\mathbf{u} = (9, -3, 18) - (-4, 0, 8) = (13, -3, 10)$

(b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (3, -6, 5);\ \ \|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{3^2 + (-6)^2 + 5^2} = \sqrt{70}$

(c) $-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w}) = (6, 0, -12) - \big((3, -1, 6) + (10, -25, -25)\big) = (-7, 26, 7)$

$d(-3\mathbf{u}, \mathbf{v} + 5\mathbf{w}) = \|-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w})\| = \sqrt{(-7)^2 + 26^2 + 7^2} = \sqrt{774} = 3\sqrt{86}$