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# LA Class Activity

Q1) Express quadratic form in matrix notation $x^T A x$, where A is symmetric matrix.

c) $9x_1^2 - x_2^2 + 4x_2^3 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

Expressing in quadratic form:-

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ans

Q2) b) $-7x_1x_2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -7/2 \\ -7/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ans

2c) $x_1^2 + x_2^2 - 3x_2^3 - 5x_1x_2 + 9x_1x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -5/2 & 9/2 \\ -5/2 & 1 & 0 \\ 9/2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q4) Find formula for quadratic form that does not use matrices.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & 7/2 & 1 \\ 7/2 & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Diagonals are squared coefficients.
Off diagonal inputs are half coefficients of crossed products, therefore:-

$$-2x_1^2 + 0x_2^2 + 3x_3^3 + 7x_1x_2 + 2x_1x_3 + 12x_2x_3$$

$$Q = -2x_1^2 + 3x_3^3 + 7x_1x_2 + 2x_1x_3 + 12x_2x_3$$

Ans

Q6) Find orthogonal change of variables that eliminates cross product terms in the quadratic form Q, and express Q in terms of new variables.

6) $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$

$$[y_1\ y_2\ y_3]\begin{bmatrix}5 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$\det(\lambda I - A)$

$$\det\left(\begin{vmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{vmatrix} - \begin{vmatrix}5 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 4\end{vmatrix}\right) = \det\begin{vmatrix}\lambda-5 & -2 & 0\\ -2 & \lambda-2 & 0\\ 0 & 0 & \lambda-4\end{vmatrix}$$

$(\lambda-5)(\lambda-2)(\lambda-4) - 4(\lambda-4)$

$= (\lambda-5)(\lambda^2-6\lambda+8) + 4\lambda + 16$

$= \lambda^3 - 6\lambda^2 + 8\lambda - 5\lambda^2 + 30\lambda - 40 - 4\lambda + 16$

$= \lambda^3 - 11\lambda^2 + 34\lambda - 24$ {Characteristic Equation}

Roots of equation are:-

$= (\lambda-5)(\lambda-2)(\lambda-4) - 4(\lambda-4)$

$= (\lambda-4)\{(\lambda-5)(\lambda-2) - 4(1)\}$

$= (\lambda-4)(\lambda^2 - 7\lambda + 10 - 4)$

$= (\lambda-4)(\lambda^2 - 7\lambda + 6)$

$\boxed{\lambda_1 = 4}$ $\boxed{\lambda_2 = 6}$ $\boxed{\lambda_3 = 1}$

$$(\lambda I - A)x = 0$$

$$\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce matrix to row echelon form

$R_2 = R_2 - \frac{1}{2} R_1$

$R_3 = -\frac{1}{3} R_3$

$$\begin{bmatrix} -4 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \longleftrightarrow R_2$, $R_1 = -\frac{1}{4} R_1$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + \frac{1}{2} x_2 = 0$

$x_1 = -\frac{1}{2} x_2$

$x_2 = x_2$

$x_3 = 0$

$$x_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}(x_2)$$

$$\left\{ x_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

For $\lambda_2 = 4$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduce to row echelon form

$R_1 \leftrightarrow R_2$ $\quad R_2 - \frac{1}{2} R_1 = R_2$

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = -\frac{1}{3} R_2$, $R_1 - 2R_2 = R_1$, $R_1 = -\frac{1}{2} R$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$, $x_3 = x_3$

$$\left\{ x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda_3 = 6$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Reduce to row echelon form

$R_1 \leftrightarrow R_2$

$R_2 - \frac{1}{2} R_1 = R_2$

$$\begin{bmatrix} -2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\frac{1}{2} R_3 = R_3$ $\quad R_1 = -\frac{1}{2} R_1$

$R_2 \longleftrightarrow R_3$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$, $x_2 = x_2$, $x_3 = 0$

$$\left\{ x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $x_1 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ and $x_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and correspond to $\lambda_1 = 1$ and $\lambda_3 = 6$.

Apply Gram-Schmidt process :-

$$u_1 = x_1 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$$

$$u_2 = x_2 - \frac{(x_2, u_1)}{(u_1, u_1)} u_1$$

$$(x_2, u_1) = \begin{bmatrix}2\\1\\0\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix} = 0$$

$$(u_1, u_1) = \begin{bmatrix}-1\\2\\0\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix} = 5$$

$$u_2 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \frac{0}{5}\begin{bmatrix}-1\\2\\0\end{bmatrix} \qquad u_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

NORMALIZE vectors $u_1, u_2, x_2$. by dividing with their norm :-

$$v_1 = \frac{u_1}{|u_1|} = \frac{1}{\sqrt{1^2+2^2+0^2}}\begin{bmatrix}+1\\-2\\0\end{bmatrix} \Rightarrow v_1 = \begin{bmatrix}-1/\sqrt5\\2/\sqrt5\\0\end{bmatrix}$$

$$v_3 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2^2+1^2+0^2}}\begin{bmatrix}2\\1\\0\end{bmatrix} \Rightarrow v_3 = \begin{bmatrix}2/\sqrt5\\1/\sqrt5\\0\end{bmatrix}$$

$$v_2 = \frac{x_2}{|x_2|} = \frac{1}{\sqrt{0^2+0^2+1^2}}\begin{bmatrix}0\\0\\2\end{bmatrix} \qquad v_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Then :-

$$P = \begin{bmatrix}1/\sqrt5 & 0 & 2/\sqrt5\\ -2/\sqrt5 & 0 & 1/\sqrt5\\ 0 & 1 & 0\end{bmatrix}$$

Find $P^{-1}$ then find $PP^{-1}$.

$$P^{-1} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $P^{-1}AP$ :-

$$P^{-1}AP = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 4 \\ 12/\sqrt{5} & 6/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$Q = y^T (P^T A P) y = y^T D y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \cdot 1 + y_2(0) + y_3(0) & y_1(0) + y_2(4) + y_3(0) & y_1(0) + y_2(0) + y_3(6) \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & 4y_2 & 6y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1 \cdot y_1 + 4y_2 \cdot y_2 + 6y_3 \cdot y_3$$

$$= y_1^2 + 4y_2^2 + 6y_3^2$$

New quadratic form is:-

$$\boxed{Q = y_1^2 + 4y_2^2 + 6y_3^3}$$ Ans

x —— x

Q8) $Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

$$Q_A(x) = x^T A x$$

$$[x_1 \; x_2 \; x_3] \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\det\left( \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{vmatrix} \right) = \det \begin{vmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-5 & 4 \\ 2 & 4 & \lambda-5 \end{vmatrix}$$

$= (\lambda-2)(\lambda^2 - 10\lambda + 9) - 8\lambda + 8$

$= \lambda^3 - 10\lambda^2 + 9\lambda - 2\lambda^2 + 20\lambda - 18 - 8\lambda + 8$

$= \lambda^3 - 12\lambda^2 + 21\lambda - 10$

Eigen values:-

$(\lambda - 1)(\lambda^2 - 11\lambda + 10) = 0$

$\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 1}$ $\boxed{\lambda_3 = 10}$

$$(\lambda I - A)x = 0$$

$$\left( \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{vmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_{1,2} = 1$
$$\begin{vmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, $R_2 = R_2 - \frac{1}{2}R_1$

$$\begin{bmatrix} -2 & -4 & 4 \\ 0 & 0 & 0 \\ 2 & 4 & -4 \end{bmatrix} \xrightarrow[R_1 = -\frac{1}{2}R_1]{R_1 = R_2 + R_3} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 - 2x_3 = 0 \Rightarrow x_1 = -2x_2 + 2x_3$

$x_2 = x_2$ $\qquad$ $x_3 = x_3$

$x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\qquad$ $x_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda_3 = 10$ $\quad \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$ $\quad R_2 = \frac{R_1}{4} + R_2$

$R_3 = R_3 - \frac{1}{4} R_1$

$\begin{bmatrix} 8 & -2 & 2 \\ 0 & 9/2 & 9/2 \\ 0 & 9/2 & 9/2 \end{bmatrix}$ $\quad R_3 - R_2 = R_3$, $R_2 = \frac{9}{2} R_2$ $\quad \begin{bmatrix} 8 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R_1 + 2R_2 = R_1$, $R_1 = \frac{1}{8} R_1$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + \frac{1}{2} x_3 = 0 \Rightarrow x_1 = -\frac{1}{2} x_3$ $\qquad \boxed{x_3 = x_3}$

$x_2 + x_3 = 0 \qquad x_2 = -x_3$

$$x_3 = \left\{ \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$V_1 = x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\quad V_2 = x_2 - \frac{x_2 \cdot V_1}{x_1 \cdot x_1} V_1$

$x_2 \cdot V_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 2(-2) + 0 \cdot 1 + 1 \cdot 0 = -4$

$x_1 \cdot x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = (-2)(-2) + 1 + 0 = 5$

$V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\qquad V_2 = \begin{bmatrix} 2/5 \\ 4/5 \\ 1 \end{bmatrix}$

Normalize vectors :-

$$u_1 = \frac{y_1}{|y_1|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{y_2}{|y_2|} = \frac{1}{\sqrt{9/4}} \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \frac{2}{3} \cdot\cdot \qquad u_2 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$P = \begin{bmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$$

$P^T$ and $PP^T$ :-

$$P^T = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{45} & 4/\sqrt{45} & 5/\sqrt{45} \\ -1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix} \cdot \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{45} & 4/\sqrt{45} & 5/\sqrt{45} \\ -1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $P^{-1}AP$

$$\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{45} & 4/\sqrt{45} & 5/\sqrt{45} \\ -1/3 & -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$x = Py$

$$Q = y^T (P^T AP) y$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & 10y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + y_2^2 + 10y_3^2$$

New quadratic form is

$$\boxed{Q = y_1^2 + y_2^2 + 10y_3^2}$$ Ans