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QR- Decomposition:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

$A = QR$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$R_2 + R_4 = R_4$ || $R_1 - R_3 = R_3$.

$$A = \begin{bmatrix} 1 & 0 & 1. \\ -1 & 1 & \cancel{0}1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ Rank = 2. which is less than 3 (no of varil

So that columns vector are not Linearly Independen.: Therefore QR - decomposition not possible.

3. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$; $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$.

for QR decomposition:

$A = QR$.

where $R = \begin{bmatrix} (u_1, q_1) & (u_2, q_1) & (u_3, q_1) \\ 0 & (u_2, q_2) & (u_3, q_2) \\ 0 & 0 & (u_3, q_3) \end{bmatrix}$ $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$; $u_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

- $(u_1, q_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$= \sqrt{2}$

- $(u_2, q_1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$= \sqrt{2}$

- $(u_3, q_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}, 0, 1/\sqrt{2} \end{bmatrix}$

$(u_3, q_1) = \sqrt{2}$

- $(u_2, q_2) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$= \sqrt{3}$

- $(u_3, q_2) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = -\sqrt{3}/3$

$$(v_3, q_2) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}.$$

$$= 2\sqrt{6}/3$$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3}/3 \\ 0 & 0 & 2\sqrt{6}/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3}/3 \\ 0 & 0 & 2\sqrt{6}/3 \end{bmatrix}$$

→ Find characteristic equation and dimension of eigenspace.

1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\det(\lambda I - A) = 0.$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 0.$$

$$\det\begin{bmatrix} \lambda - 1 & -2 \\ 0 - 2 & \lambda - 4 \end{bmatrix} = 0.$$

$$(\lambda - 1)(\lambda - 4) - 4 = 0.$$

$$\lambda^2 - 4\lambda - \lambda + \cancel{4} - \cancel{4} = 0$$

$$\lambda^2 - 5\lambda = 0.$$

$$\lambda(\lambda - 5) = 0.$$

$\lambda_1 = 0.$ → one dimension

$\lambda_2 = 5$ → one dimension.

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\det(\lambda I - A) = 0.$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = 0.$$

$$\det\begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix} = 0.$$

$$(\lambda-1)\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix} + 1\begin{bmatrix} -1 & -1 \\ -1 & \lambda-1 \end{bmatrix} - 1\begin{bmatrix} -1 & \lambda-1 \\ -1 & -1 \end{bmatrix}$$

$$(\lambda-1)\left(\lambda^2-\lambda-\lambda+\cancel{1}-\cancel{1}\right)+1\left(-\lambda+\cancel{1}-\cancel{1}\right)-1\left(1+\lambda-\cancel{1}\right)=0$$

$$(\lambda-1)(\lambda^2-2\lambda)+\cancel{\lambda}-\lambda-\lambda=0$$

$$(\lambda-1)(\lambda^2-2\lambda)-2\lambda=0$$

$$\lambda^3-2\lambda^2-\lambda^2+2\cancel{\lambda}-2\cancel{\lambda}=0$$

$$\lambda^3-3\lambda^2=0.$$

$$\lambda^2(\lambda-3)=0.$$

$\boxed{\lambda_1 = 0}$
$\boxed{\lambda_2 = 0.}$ → two dimension.

$\boxed{\lambda_3 = 3}$ one dimension.

5) $\begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\det(\lambda I - A) = 0.$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} \lambda-4 & -4 & 0 & 0 \\ -4 & \lambda-4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0.$$

$\frac{1}{(\lambda-4)} R_1 = R_1$ $\qquad 4R_1 + R_2 = R_2.$

$$\det \begin{bmatrix} 1 & -4/(\lambda-4) & 0 & 0 \\ 0 & \frac{\lambda^2-8\lambda}{\lambda-4} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0.$$

lower triangle matrix $\therefore$ determinant.

$$\frac{(\lambda^2-8\lambda)}{\lambda-4} \times \lambda \times \lambda = 0.$$

$$\lambda^2(\lambda^2-8\lambda) = 0.$$

$$\lambda^4 - 8\lambda^3 = 0.$$

$$\lambda^3(\lambda-8) = 0.$$

$\boxed{\lambda = 8}$ $\quad \boxed{\lambda = 0.}$

$\downarrow$ one dimension $\quad \downarrow$ three dimension.

Find P matrix that orthoganally diagonalizes:

8- $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

show that $P^T A P = D$.

$\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{bmatrix} = 0$

det $(\lambda-3)(\lambda-3) - 1 = 0.$

$\lambda^2 - 3\lambda - 3\lambda + 9 - 1 = 0.$

$\lambda^2 - 6\lambda + 8 = 0.$

$\lambda^2 - 4\lambda - 2\lambda + 8 = 0.$

$\lambda(\lambda-4) - 2(\lambda-4) = 0$

$\boxed{\lambda = +2}$.

$\boxed{\lambda = 4}$

Now for $\lambda = 2$.

$\det \begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} = 0.$

$\det \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = 0.$

det. $-1 \,/\, -1 \,/ = 0.$

$\Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

$\Rightarrow. \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$

$R_1 - R_2 = R_2$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-x_1 - x_2 = 0$$

$$x_1 = -x_2 \qquad x_2 = 1$$

$$P_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for $\lambda = 4$

$$\begin{bmatrix} 4-3 & -1 \\ -1 & 4-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$R_1 + R_2 = R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = x_2 \qquad x_2 = 1$$

$$P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

By Gram schmidt processs

$$PP_1 = \frac{P_1}{|P_1|} = \left[-1/\sqrt{2}, 1/\sqrt{2}\right]$$

$$PP_2 = \frac{P_2}{|P_2|} \qquad \left[1/\sqrt{2}, 1/\sqrt{2}\right]$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T = \begin{bmatrix} +1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

$D = P^T A P.$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -2\sqrt{2} & -2\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & +4 \end{bmatrix}$$ proved.

12) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(\lambda I - A) = 0.$

$\det \begin{bmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda-1 & 0. \\ 0 & 0 & \lambda \end{bmatrix} = 0.$

$\Rightarrow \lambda-1 \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{bmatrix} + 1 \begin{bmatrix} -1 & 0 \\ 0 & \lambda \end{bmatrix} + 0$

$(\lambda-1)(\lambda^2-\lambda). \quad +1(-\lambda).$

$\Rightarrow (\lambda-1)$

$\lambda^3 - \lambda^2 - \lambda^2 + \cancel{\lambda} - \cancel{\lambda} = 0$

$\Rightarrow$

$\lambda^3 - 2\lambda^2 = 0.$

$\lambda^2(\lambda - 2) = 0$

$\boxed{\lambda_3 = 2} \quad \boxed{\lambda_1 = 0} \quad \boxed{\lambda_2 = 0}$

Now for $\lambda = 0.$

$\left[\begin{array}{ccc|c} -1 & -1 & 0. & 0 \\ -1 & -1 & 0. & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

$R_1 - R_2 = R_2$

$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

$-x_1 - x_2 = 0. \rightarrow x_1 = -x_2.$

$x_3 = x_3.$

$\therefore x_2 = t.$

$x_3 = s.$

$$P_1 = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$P_2 = s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now for $\lambda = 2$:

$$\left[\begin{array}{ccc|c} 2-1 & -1 & 0 & 0 \\ -1 & 2-1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right].$$

$R_1 + R_2 = R_2$ // $\frac{1}{2}R_3 = R_3$.

$$\left[\begin{array}{ccc|c} +1 & -1 & 0 & 0. \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right].$$

$x_1 - x_2 = 0.$ $\qquad x_2 = t.$

$x_2 = x_2.$ $\qquad \cancel{x_2} = t.$

$x_3 = 0_3.$

$$P_3 = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Apply Gram Schmidt process.

$\rightarrow PP_1 = \frac{P_1}{|P_1|}$.

$\rightarrow PP_2 = \frac{P_2}{|P_2|}$

$\rightarrow PP_3 = \frac{P_3}{|P_3|}$.

- $PP_1 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$
- $PP_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
- $PP_3 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

$$P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

$$P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}.$$

$D = P^T A P$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$