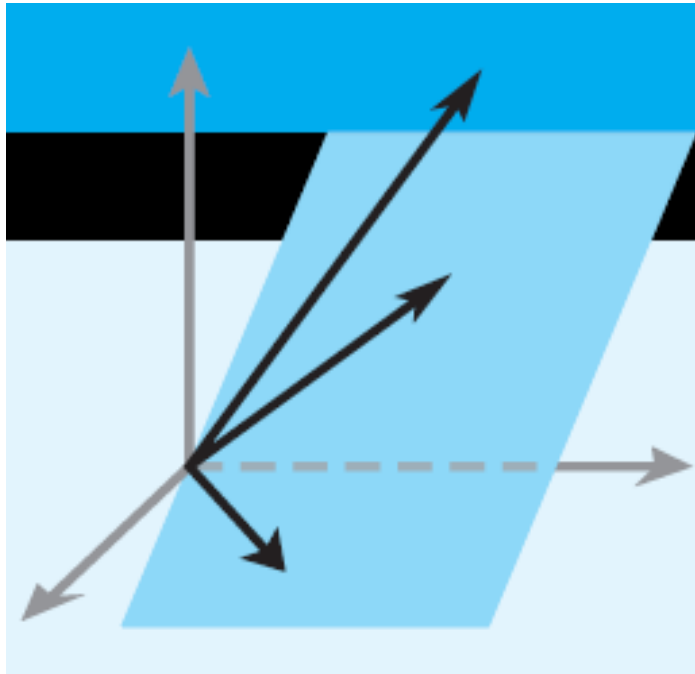


# Elementary Linear Algebra



## Chapter 3

Howard Anton

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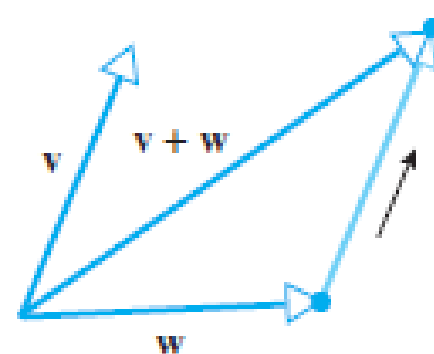
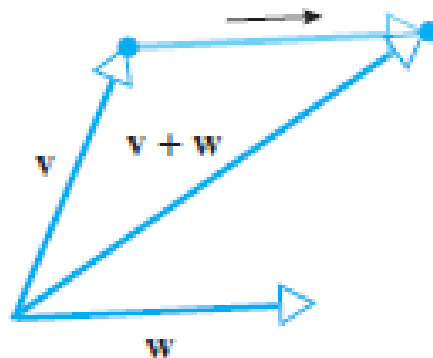
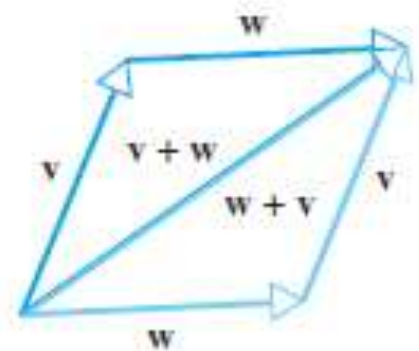
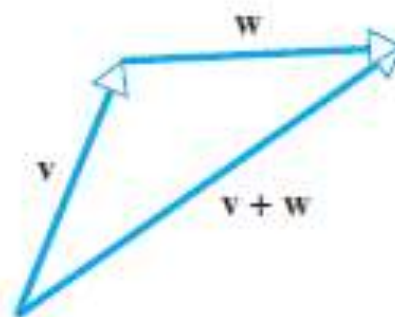
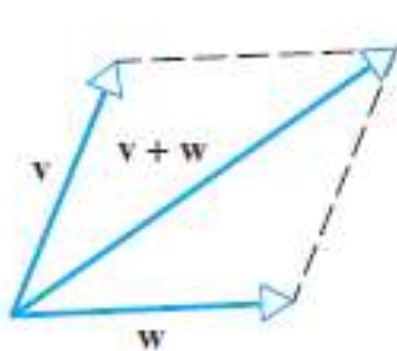
# Chapter 3

## Euclidean Vector Spaces

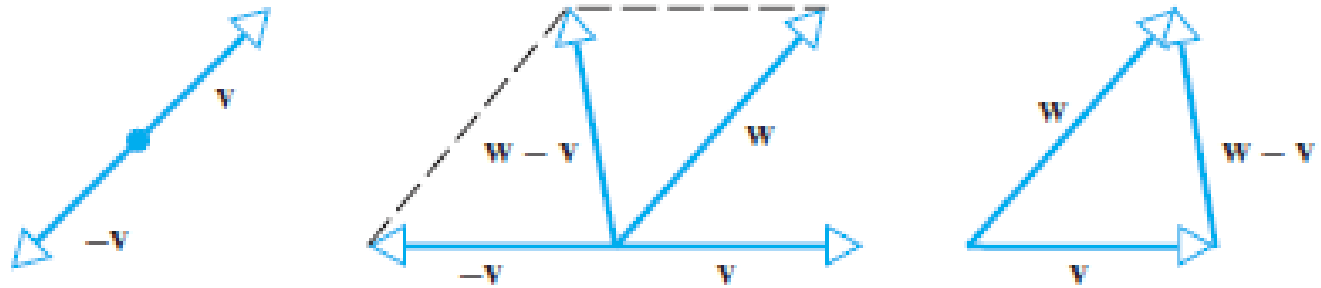
- 3.1 Vectors in 2-Space, 3-Space, and  $n$ -Space
- 3.2 Norm, Dot Product, and Distance in  $\mathbb{R}^n$
- 3.3 Orthogonality
- 3.4 The Geometry of Linear Systems
- 3.5 Cross Product

# Section 3.1 Vectors

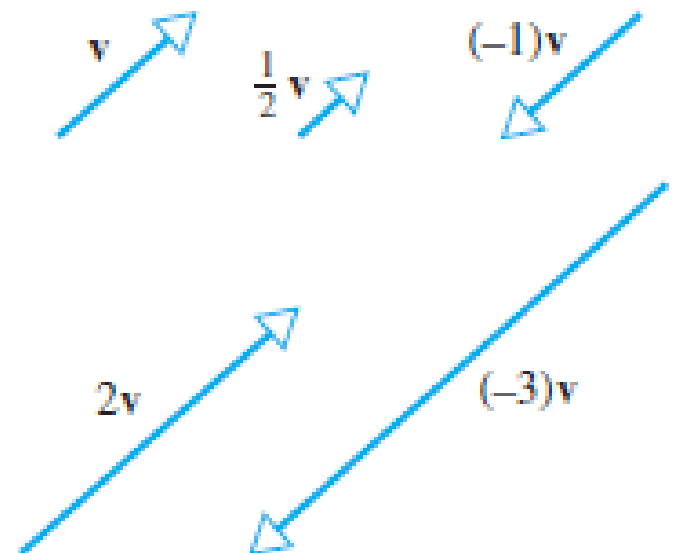
Addition of vectors by the  
parallelogram or triangle rules



# Subtraction:



# Scalar Multiplication:



# Properties of Vectors

**THEOREM 3.1.1** *If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in  $R^n$ , and if  $k$  and  $m$  are scalars, then:*

- (a)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (b)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (c)  $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- (d)  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- (e)  $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (f)  $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
- (g)  $k(m\mathbf{u}) = (km)\mathbf{u}$
- (h)  $1\mathbf{u} = \mathbf{u}$

# Section 3.2 Norm, Dot Product, and Distance in $R^n$

Norm:

**DEFINITION 1** If  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  is a vector in  $R^n$ , then the *norm* of  $\mathbf{v}$  (also called the *length* of  $\mathbf{v}$  or the *magnitude* of  $\mathbf{v}$ ) is denoted by  $\|\mathbf{v}\|$ , and is defined by the formula

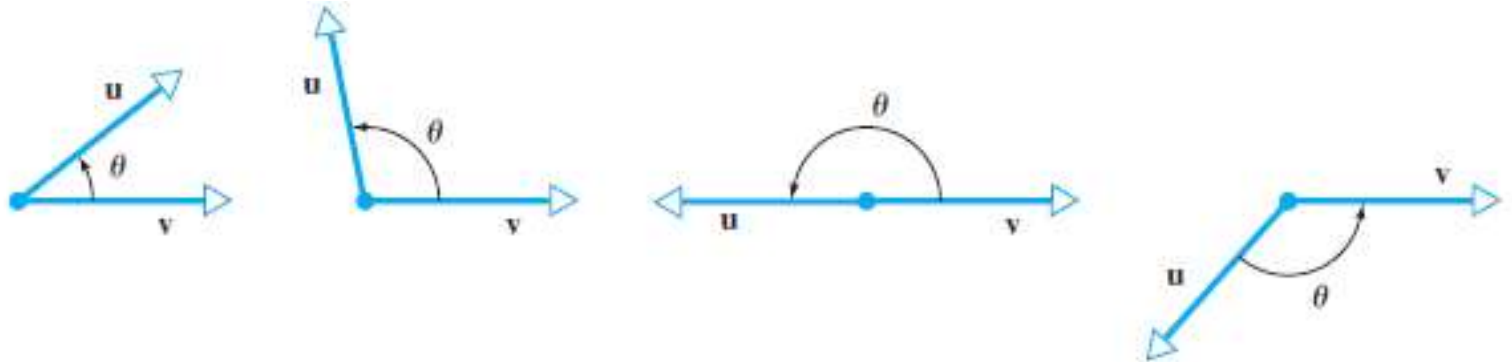
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \quad (3)$$

Unit Vectors:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$



# The Dot Product



The angle  $\theta$  between  $\mathbf{u}$  and  $\mathbf{v}$  satisfies  $0 \leq \theta \leq \pi$ .

**DEFINITION 3** If  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors in  $R^2$  or  $R^3$ , and if  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , then the *dot product* (also called the *Euclidean inner product*) of  $\mathbf{u}$  and  $\mathbf{v}$  is denoted by  $\mathbf{u} \cdot \mathbf{v}$  and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (12)$$

If  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ , then we define  $\mathbf{u} \cdot \mathbf{v}$  to be 0.

The sign of the dot product reveals information about the angle  $\theta$  that we can obtain by rewriting Formula (12) as

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad (13)$$

# The Dot Product

**DEFINITION 4** If  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  are vectors in  $R^n$ , then the *dot product* (also called the *Euclidean inner product*) of  $\mathbf{u}$  and  $\mathbf{v}$  is denoted by  $\mathbf{u} \cdot \mathbf{v}$  and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad (17)$$



# Properties of the Dot Product

**THEOREM 3.2.2** *If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in  $R^n$ , and if  $k$  is a scalar, then:*

- (a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  [Symmetry property]
- (b)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$  [Distributive property]
- (c)  $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$  [Homogeneity property]
- (d)  $\mathbf{v} \cdot \mathbf{v} \geq 0$  and  $\mathbf{v} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{v} = \mathbf{0}$  [Positivity property]

**THEOREM 3.2.3** *If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in  $R^n$ , and if  $k$  is a scalar, then:*

- (a)  $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
- (b)  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- (c)  $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$
- (d)  $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$
- (e)  $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

# Cauchy-Schwarz Inequality

## THEOREM 3.2.4 Cauchy-Schwarz Inequality

If  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  are vectors in  $R^n$ , then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad (22)$$

or in terms of components

$$|u_1 v_1 + u_2 v_2 + \dots + u_n v_n| \leq (u_1^2 + u_2^2 + \dots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2} \quad (23)$$

# Dot Products and Matrices

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}^T \mathbf{v} = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T \mathbf{u} = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a column matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v} = \mathbf{v}^T \mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}\mathbf{v} = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T \mathbf{u}^T = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a row matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}\mathbf{u} = \mathbf{u}^T \mathbf{v}^T$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{v}\mathbf{u} = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ $\mathbf{u}^T \mathbf{v}^T = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a row matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{u}\mathbf{v}^T = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}\mathbf{u}^T = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$

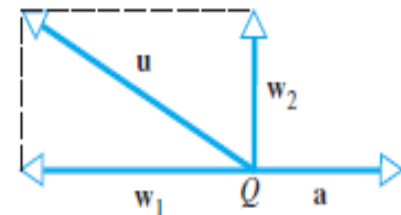
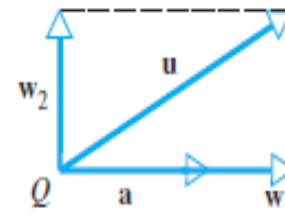
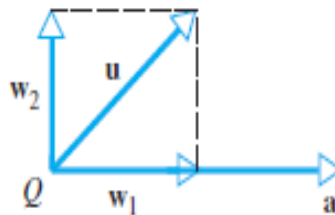
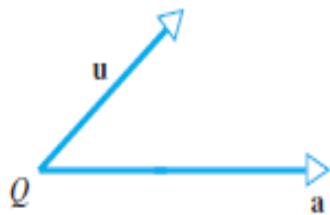
# Section 3.3 Orthogonality

**DEFINITION 1** Two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $R^n$  are said to be *orthogonal* (or *perpendicular*) if  $\mathbf{u} \cdot \mathbf{v} = 0$ . We will also agree that the zero vector in  $R^n$  is orthogonal to *every* vector in  $R^n$ .

# Orthogonal Projections

## THEOREM 3.3.2 Projection Theorem

If  $\mathbf{u}$  and  $\mathbf{a}$  are vectors in  $R^n$ , and if  $\mathbf{a} \neq \mathbf{0}$ , then  $\mathbf{u}$  can be expressed in exactly one way in the form  $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ , where  $\mathbf{w}_1$  is a scalar multiple of  $\mathbf{a}$  and  $\mathbf{w}_2$  is orthogonal to  $\mathbf{a}$ .





# Point-line and Point-plane Distance Formulas

## THEOREM 3.3.4

- (a) In  $R^2$  the distance  $D$  between the point  $P_0(x_0, y_0)$  and the line  $ax + by + c = 0$  is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \quad (15)$$

- (b) In  $R^3$  the distance  $D$  between the point  $P_0(x_0, y_0, z_0)$  and the plane  $ax + by + cz + d = 0$  is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (16)$$

# Section 3.4

## The Geometry of Linear Systems

**THEOREM 3.4.1** Let  $L$  be the line in  $R^2$  or  $R^3$  that contains the point  $\mathbf{x}_0$  and is parallel to the nonzero vector  $\mathbf{v}$ . Then the equation of the line through  $\mathbf{x}_0$  that is parallel to  $\mathbf{v}$  is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (1)$$

If  $\mathbf{x}_0 = \mathbf{0}$ , then the line passes through the origin and the equation has the form

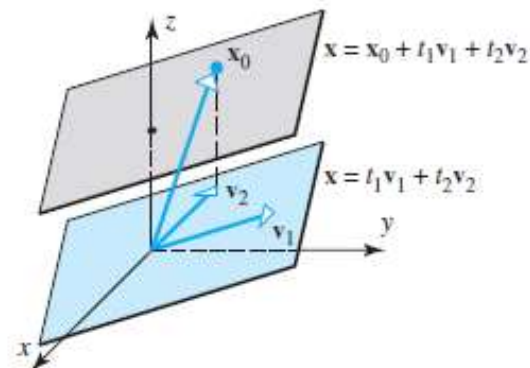
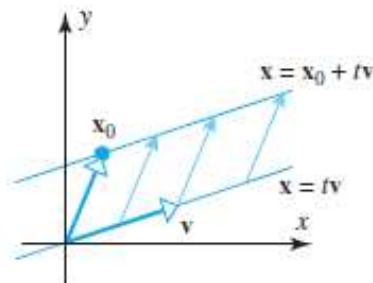
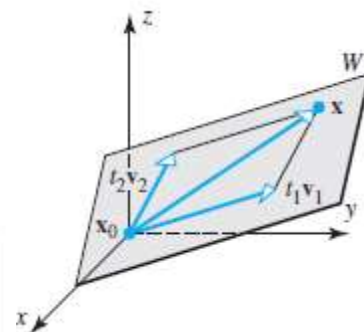
$$\mathbf{x} = t\mathbf{v} \quad (2)$$

**THEOREM 3.4.2** Let  $W$  be the plane in  $R^3$  that contains the point  $\mathbf{x}_0$  and is parallel to the noncollinear vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Then an equation of the plane through  $\mathbf{x}_0$  that is parallel to  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is given by

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (3)$$

If  $\mathbf{x}_0 = \mathbf{0}$ , then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (4)$$





**DEFINITION 1** If  $\mathbf{x}_0$  and  $\mathbf{v}$  are vectors in  $R^n$ , and if  $\mathbf{v}$  is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (5)$$

defines the *line through  $\mathbf{x}_0$  that is parallel to  $\mathbf{v}$* . In the special case where  $\mathbf{x}_0 = \mathbf{0}$ , the line is said to *pass through the origin*.

**DEFINITION 2** If  $\mathbf{x}_0$ ,  $\mathbf{v}_1$ , and  $\mathbf{v}_2$  are vectors in  $R^n$ , and if  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (6)$$

defines the *plane through  $\mathbf{x}_0$  that is parallel to  $\mathbf{v}_1$  and  $\mathbf{v}_2$* . In the special case where  $\mathbf{x}_0 = \mathbf{0}$ , the plane is said to *pass through the origin*.

# Section 3.5 Cross Product

**DEFINITION 1** If  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  are vectors in 3-space, then the *cross product*  $\mathbf{u} \times \mathbf{v}$  is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \quad (1)$$

# Cross Products and Dot Products

## THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in 3-space, then

- (a)  $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$  *( $\mathbf{u} \times \mathbf{v}$  is orthogonal to  $\mathbf{u}$ )*
- (b)  $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$  *( $\mathbf{u} \times \mathbf{v}$  is orthogonal to  $\mathbf{v}$ )*
- (c)  $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$  *(Lagrange's identity)*
- (d)  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$  *(relationship between cross and dot products)*
- (e)  $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$  *(relationship between cross and dot products)*

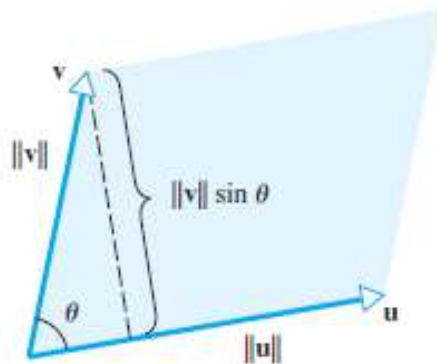
# Properties of Cross Product

## THEOREM 3.5.2 Properties of Cross Product

*If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are any vectors in 3-space and  $k$  is any scalar, then:*

- (a)  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- (b)  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- (c)  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- (d)  $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
- (e)  $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
- (f)  $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

# Geometry of the Cross Product



$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

## THEOREM 3.5.3 Area of a Parallelogram

*If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in 3-space, then  $\|\mathbf{u} \times \mathbf{v}\|$  is equal to the area of the parallelogram determined by  $\mathbf{u}$  and  $\mathbf{v}$ .*



# Geometry of Determinants

## THEOREM 3.5.4

(a) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

*is equal to the area of the parallelogram in 2-space determined by the vectors  $\mathbf{u} = (u_1, u_2)$  and  $\mathbf{v} = (v_1, v_2)$ . (See Figure 3.5.7a.)*

(b) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

*is equal to the volume of the parallelepiped in 3-space determined by the vectors  $\mathbf{u} = (u_1, u_2, u_3)$ ,  $\mathbf{v} = (v_1, v_2, v_3)$ , and  $\mathbf{w} = (w_1, w_2, w_3)$ . (See Figure 3.5.7b.)*