19K-1043 SALMAN AHMED KHAN

Q. Find P that orthogonally diagonalizes A, find $P^{-1}AP$.

8. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(\lambda I - A) = 0$

$\det\left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-3 & \cancel{\lambda}-1 \\ -1 & \lambda-3 \end{bmatrix}\right\} = 0$

$= (\lambda-3)^2 - 1 = 0$

$\lambda^2 - 6\lambda + 9 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda^2 - 4\lambda - 2\lambda + 8 = 0$

$\lambda(\lambda-4) - 2(\lambda-4) = 0$

$(\lambda-2)(\lambda-4) = 0$

So Either,

$\boxed{\lambda = 2}$, $\boxed{\lambda = 4}$

For $\lambda = 2$:-

$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$ => homogenous System

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 = 0$

$x_2 = t$, $x_1 = -t$

$\therefore P_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 2,2$

$\begin{bmatrix} 4-3 & -1 \\ -1 & 4-3 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - x_2 = 0$

$x_1 = x_2$

$\therefore x_2 = t$, $x_1 = t$

$\therefore P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Apply GramSchmidt on P_1

$V_1 = P_1$

~~For~~ Orthonormality

$P_1 = \dfrac{V_1}{\|V_1\|} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

B

Apply Gram Schmidt Process on P_2

$$V_2 = P_2 - \frac{(P_2, V_1)}{\|V_1\|}$$

$$(P_2, V_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = -1 + 1 = 0$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Orthonormality: $P_2 = \dfrac{V_2}{\|V_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Now, $D = P^T A P$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -3/\sqrt{2} + 1/\sqrt{2} & 3/\sqrt{2} + 1/\sqrt{2} \\ -1/\sqrt{2} + 3/\sqrt{2} & 1/\sqrt{2} + 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2/\sqrt{2} & 4/\sqrt{2} \\ 2/\sqrt{2} & 4/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2/2 + 2/2 & 4/2 + 4/2 \\ -2/2 + 2/2 & 4/2 + 4/2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}$$

12). $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\det \begin{bmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\cancel{(\lambda-1)}\ (\lambda((\lambda-1)^2 - 1)) = 0$$

$$\lambda((\lambda^2 - 2\lambda + \cancel{1}) \cancel{-1}) = 0$$

$$\lambda^3 - 2\lambda^2 = 0$$

$$\lambda^2(\lambda-2) = 0$$

Either, $\boxed{\lambda = 2}$, $\boxed{\lambda = 0}$

For $\lambda = 0$:-

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

Parametric: $x_2 = t$ $x_3 = s$, $x_1 = -t$

$P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 2$:-

$$\left[\begin{array}{ccc|c} 2-1 & -1 & 0 & 0 \\ -1 & 2-1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$x_1 - x_2 = 0 \quad , \quad x_1 = x_2$$

$$x_3 = 0$$

$$P_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using Gram Schmidt

$V_1 = P_1$.

$$V_2 = P_2 - \frac{(P_2, V_1)\, V_1}{\|V_1\|}$$

$$(P_2, V_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} = 0.$$

$V_2 = P_2$

$$V_3 = P_3 - \frac{(P_3, V_2)}{\|V_2\|} - \frac{(P_3, V_1)\, V_1}{\|V_1\|}$$

$$(P_3, V_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = 0$$

$V_3 = P_3$

For Orthonormality

$$P_1 = \frac{V_1}{\|V_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$P_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \qquad P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

Using $D = P^T A P$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q. Find Characteristic Eq. & dimensions

Q1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$$\det\left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{bmatrix}\right\} = 0$$

$(\lambda-1)(\lambda-4) - 4 = 0$

$\lambda^2 - 4\lambda - \lambda + \cancel{4} - \cancel{4} = 0$

$\lambda^2 - 5\lambda = 0$

$\boxed{\lambda = 0}$ $\boxed{\lambda = 5}$

$\lambda = 0 \Rightarrow$ One dimension

$\lambda = 5 \Rightarrow$ One dimension

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$$\det \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix} = 0$$

$(\lambda-1)\{(\lambda-1)^2 - 1\} + 1\{-\lambda + \cancel{1} - \cancel{1}\} - 1\{\cancel{1} + \lambda - \cancel{1}\} = 0$

$-\lambda \quad -\lambda$

$(\lambda-1)(\lambda^2 - 2\lambda) \cancel{-\lambda - \lambda} = 0$

$\lambda^3 - 2\lambda^2 - \lambda^2 + \cancel{2\lambda} \; \cancel{-2\lambda} = 0$

$\lambda^3 - 3\lambda^2 = 0$

$\lambda^2(\lambda - 3) = 0$

$\boxed{\lambda = 3} \rightarrow$ one dimension

$\boxed{\lambda = 0} \Rightarrow$ ~~one~~ two dimension

Q. $\begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\det(\lambda I - A) = 0$

$$\det \begin{bmatrix} \lambda - 4 & -4 & 0 & 0 \\ -4 & \lambda - 4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & -4/\lambda - 4 & 0 & 0 \\ 0 & \frac{\lambda^2 - 8\lambda}{\lambda - 4} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0$$

$\frac{\lambda^2 - 8\lambda}{\lambda - 4} \times \lambda^2 = 0$

$\lambda^2(\lambda^2 - 8\lambda) = 0$

$\lambda^3(\lambda - 8) = 0$

$\lambda = 0 \quad \lambda = 8 \rightarrow$ dimension = 1

λ^3 ↳ (λ^n) dimension $\Rightarrow$ 3

QR Decomposition

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $A = QR$

$= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Here rank = 2 that is not linearly indepedent

QR - decomposition not possible.

(b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ $Q \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$

$R = \begin{bmatrix} (u_1, q_1) & (u_2, q_1) & (u_3, q_1) \\ 0 & (u_2, q_2) & (u_3, q_2) \\ 0 & 0 & (u_3, q_3) \end{bmatrix}$ $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $u_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$(u_1, q_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \overset{0}{\cancel{-1/\sqrt{3}}} & 1/\sqrt{2} \end{bmatrix} = 2/\sqrt{2}$

$(u_2, q_1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = 2/\sqrt{2}$

$(u_3, q_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = 2/\sqrt{2}$

$$(v_2, q_2) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} \cancel{0} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = 3/\sqrt{3}$$

$$(v_3, q_2) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = -1/\sqrt{3}$$

$$(v_3, q_3) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} = 4/\sqrt{6}.$$

$$A = QR$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 3/\sqrt{3} & -1/\sqrt{3} \\ 0 & 0 & 4/\sqrt{6} \end{bmatrix}$$