Inner Product for a real vector Space V

Chapter 6

Euclidean Inner Product

DEFINITION 1 An ***inner product*** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

- The axioms for a real inner product space are based on properties of the dot product.
- These inner product space axioms will be satisfied automatically if we define the inner product of two vectors **u** and **v** in R^n to be:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- We call R^n with the Euclidean inner product ***Euclidean n-space***.

Norm and Distance

DEFINITION 2 If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad \text{and} \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

THEOREM 6.1.1 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V, *and if* k *is a scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$.

(*b*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

(*c*) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(*d*) $d(\mathbf{u}, \mathbf{v}) \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{v}$.

Weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

- The weights “w” are positive real numbers.
- u and v are vectors in R^n.

Example 1: Weighted Euclidean inner product

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ Interchanging $\mathbf{u}$ and $\mathbf{v}$ does not change the sum so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$

$$\begin{aligned} \langle k\mathbf{u}, \mathbf{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\ &= k(3u_1v_1 + 2u_2v_2) \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

$\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1v_1) + 2(v_2v_2) = 3v_1^2 + 2v_2^2 \geq 0$ with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$

Arithmetic Mean as a weighted Inner Product

Thus, the ***arithmetic average*** of the observed numerical values (denoted by $\bar{x}$) is

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{m}(f_1x_1 + f_2x_2 + \cdots + f_nx_n) \tag{4}$$

If we let

$$\mathbf{f} = (f_1, f_2, \ldots, f_n)$$
$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$
$$w_1 = w_2 = \cdots = w_n = 1/m$$

then (4) can be expressed as the weighted Euclidean inner product

$$\bar{x} = \langle \mathbf{f}, \mathbf{x} \rangle = w_1 f_1 x_1 + w_2 f_2 x_2 + \cdots + w_n f_n x_n$$

Exercise set

1. Let R^2 have the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

and let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$. Compute the stated quantities.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$ (b) $\langle k\mathbf{v}, \mathbf{w} \rangle$ (c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$

(d) $\|\mathbf{v}\|$ (e) $d(\mathbf{u}, \mathbf{v})$ (f) $\|\mathbf{u} - k\mathbf{v}\|$

2. Follow the directions of Exercise 1 using the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \tfrac{1}{2}u_1v_1 + 5u_2v_2$$

Matrix Inner Products

- The Euclidean inner product and the weighted Euclidean inner products are special cases of a general class of inner products on R^n called ***matrix inner products***.

Matrix Inner Product

- Let **u** and **v** be vectors in $\boldsymbol{R^n}$ that are expressed in *column form*, and let $\boldsymbol{A}$ be an ***invertible n × n*** matrix.
- if $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula,

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

- is called the ***inner product on $\boldsymbol{R^n}$ generated by $\boldsymbol{A}$.***

- f **u** and **v** are in column form, then **u** · **v** can be expressed as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A\mathbf{u}$$

Standard Euclidean Inner Product

- The standard Euclidean inner product on R^n is generated by the $n \times n$ identity matrix,

- *Set* $A = \mathrm{I}$ in $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$

- Then, $\langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$

Weighted Euclidean Inner Product

- and the weighted Euclidean inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

- is generated by the matrix:

$$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{w_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix}$$

- **$A^T A$** is **n x n** diagonal matrix whose diagonal entries are weights **w**s.
- Every diagonal matrix with positive diagonal entries generates a weighted inner product.

- The weighted Euclidean Inner product: $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$

- is the inner product on R^2 generated by

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

The Standard Inner Product on M_{nn}

- If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the inner standard Inner product on M_{nn} is:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{tr}(U^T V)$$

Consider the following 2 x 2 matrices:

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

- Calculate:

  - $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) =$
  - $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}\langle U^T U \rangle} =$

  - $u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$
  - $\sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$

Consider the following 2 x 2 matrices:

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

- Calculate (a) u.v (b) norm of u (c) norm of v

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)} = \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2} = \sqrt{14}$$

The Standard Inner Product on Pn

If

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad \mathbf{q} = b_0 + b_1 x + \cdots + b_n x^n$$

are polynomials in P_n, then the following formula defines an inner product on P_n (verify) that we will call the ***standard inner product*** on this space:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n$$

The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

The Evaluation Inner Product on Pn

If

$$\mathbf{p} = p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad \mathbf{q} = q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

are polynomials in P_n, and if $x_0, x_1, \ldots, x_n$ are distinct real numbers (called ***sample points***), then the formula

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n) \tag{10}$$

defines an inner product on P_n called the ***evaluation inner product*** at $x_0, x_1, \ldots, x_n$. Algebraically, this can be viewed as the dot product in R^n of the n-tuples

$$\big(p(x_0), p(x_1), \ldots, p(x_n)\big) \quad \text{and} \quad \big(q(x_0), q(x_1), \ldots, q(x_n)\big)$$

Computing the following quantities using the inner product on R^2 generated by A

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$ (b) $\langle k\mathbf{v}, \mathbf{w} \rangle$ (c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$

(d) $\|\mathbf{v}\|$ (e) $d(\mathbf{u}, \mathbf{v})$ (f) $\|\mathbf{u} - k\mathbf{v}\|$

- $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad \text{and} \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} = 34$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 24 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -39$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -18$

(d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} \right)^{1/2} = \sqrt{89}$

(e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -3 \end{bmatrix} \right)^{1/2} = \sqrt{34}$

(f) $\|\mathbf{u} - k\mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -21 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} -21 \\ -13 \end{bmatrix} \right)^{1/2} = \sqrt{610}$

In Exercises 5–6, find a matrix that generates the stated weighted inner product on R^2.

5. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$

6. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

In Exercises 7–8, use the inner product on R^2 generated by the matrix A to find $\langle \mathbf{u}, \mathbf{v} \rangle$ for the vectors $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

7. $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

▶ In Exercises 9–10, compute the standard inner product on M_{22} of the given matrices. ◀

9. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

10. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

▶ In Exercises 11–12, find the standard inner product on P_2 of the given polynomials. ◀

11. $\mathbf{p} = -2 + x + 3x^2$, $\mathbf{q} = 4 - 7x^2$

12. $\mathbf{p} = -5 + 2x + x^2$, $\mathbf{q} = 3 + 2x - 4x^2$