Class Activity 19K-1084

Q1c

Quadratic form

$$Q_A(u) = u^T A u.$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad A = \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix}$$

$$Q_A(u) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Q2c

Quadratic form is

$$Q_A(u) = u^T A u$$

$$u_1^2 + u_2^2 - 3u_3^2 - 5u_1u_2 + 9u_1u_3 = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{5}{2} & \frac{9}{2} \\ -\frac{5}{2} & 1 & 0 \\ \frac{9}{2} & 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Q2b

$$Q_A(x) = x^T A x$$

$$-7x_1x_2 = [x_1 \ x_2] \begin{bmatrix} 0 & -7/2 \\ -7/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q4

$$Q_A(x) = x^T A x$$

$$Q_A(x) = [x_1 \ x_2 \ x_3] \begin{bmatrix} -2 & 7/2 & 1 \\ 7/2 & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\boxed{Q_A(x) = -2x_1^2 + 3x_3^2 + 7x_1x_2 + 2x_1x_3 + 12x_2x_3}$$

Q6

$$Q_A(x) = x^T A x$$

$$Q_A(x) = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$$

$$Q_A(x) = [x_1 \ x_2 \ x_3] \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = 0.$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) = \det \begin{vmatrix} \lambda-5 & -2 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix}$$

$= (\lambda-5)(\lambda-2)(\lambda-4) - 4(\lambda-4)$

$= \lambda^3 - 11\lambda^2 + 34\lambda - 24$

$= (\lambda-4)(\lambda^2 - 7\lambda + 6)$

$\lambda = 4$ $\qquad \lambda = 6, \ \lambda = 1$

$(\lambda I - A)u = 0.$

when $\lambda = 4$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Cramsian

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$u_1 = 0, \ u_2 = 0, \ u_3 = t.$

$$\boxed{t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

When $\lambda = 1$

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + \frac{1}{2} x_2 = 0 \Rightarrow x_1 = -\frac{1}{2} z$$

$$x_2 = z$$

$$x_3 = 0$$

$$\boxed{z = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}$$

When $\lambda = 6$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian:-

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2y$$

$$x_2 = y$$

$$x_3 = 0$$

$$\boxed{y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

Apply Gram-Schmit process.

$$v_1 = t = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle x_2 \, v_1 \rangle = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = 2(-1) + 1 \times 2 + 0 = 0$$

$$\langle v_1, v_1 \rangle \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1(-1) + 2 \times 2 + 0 = 5.$$

$$\boxed{U_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{5}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

$$V_1 = \frac{U_1}{\|U_1\|}$$

$$V_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}}$$

$$V_2 = \frac{z}{\|z\|} = \frac{1}{\sqrt{1}}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$V_3 = \frac{U_2}{\|U_2\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{V_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}}$$

$$P = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = PAP^T$$

$$= \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}}$$

$$Q = y^T D y$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boxed{Q = y_1^2 + 4y^2 + 6y^2}$$

$$Q_A(x) = x^T A x$$

$$Q_A(x) = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Scanned with CamScanner

$$\det(\lambda I - A) = 0.$$

$$\det\left(\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{bmatrix} 2 & +2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}\right) = \det\begin{vmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-5 & 4 \\ 2 & 4 & \lambda-5 \end{vmatrix}$$

$$= (\lambda-2)(\lambda^2 - 10\lambda + 25 - 16) + 2(-2\lambda + 2) + 2(-2\lambda + 2)$$
$$= \lambda^3 - 12\lambda^2 + 21\lambda - 10.$$

$$\lambda = 1,\ \lambda_3 = 10.,$$

$$\Rightarrow (\lambda I - A)x = 0.$$

When $\lambda = 1$

$$\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 2x_3 = 0.$$
$$x_2 = t$$
$$x_3 = z$$
$$x_1 = -2t + 2z.$$

$$\boxed{t = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}, \quad \boxed{z = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$$

Scanned with CamScanner

When $\lambda = 10$

$$\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1 + \tfrac{1}{2} u_3 = 0 \Rightarrow u_1 = -\tfrac{1}{2} k$$

$$u_2 + u_3 = 0 \Rightarrow u_2 = -k$$

$$u_- = k \qquad \Rightarrow u_3 = k$$

$$\boxed{k = \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}}$$

$$v_1 = u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle u_2, v_1 \rangle = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 2(-2) + 0 \cdot 1 + 1 \cdot 0 = -4$$

$$\langle v_1, v_1 \rangle = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2(-2) + 1 \cdot 1 + 0 = 5$$

$$v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

Normalizer

$$v_1 = \frac{u}{\|u_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix}$$

$$v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{4/9}}\begin{bmatrix}-1/2\\-1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix}$$

$$v_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{\frac{21}{25}}}\begin{bmatrix}2/5\\4/5\\1\end{bmatrix} = \begin{bmatrix}2/\sqrt{45}\\4/\sqrt{45}\\5/\sqrt{45}\end{bmatrix}$$

$$\boxed{P = \begin{bmatrix}-2/\sqrt{5} & 2/\sqrt{45} & -1/3\\1/\sqrt{5} & 4/\sqrt{45} & -2/3\\0 & 5/\sqrt{45} & 2/3\end{bmatrix}}$$

$$D = PAP^T$$

$$D = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 10\end{bmatrix}$$

$$Q_A(y) = y^T D y$$

$$= [y_1\ y_2\ y_3]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$\boxed{Q_A(y) = y_1^2 + y_2^2 + 10y_3^2}$$