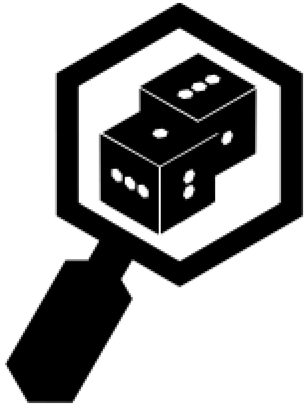
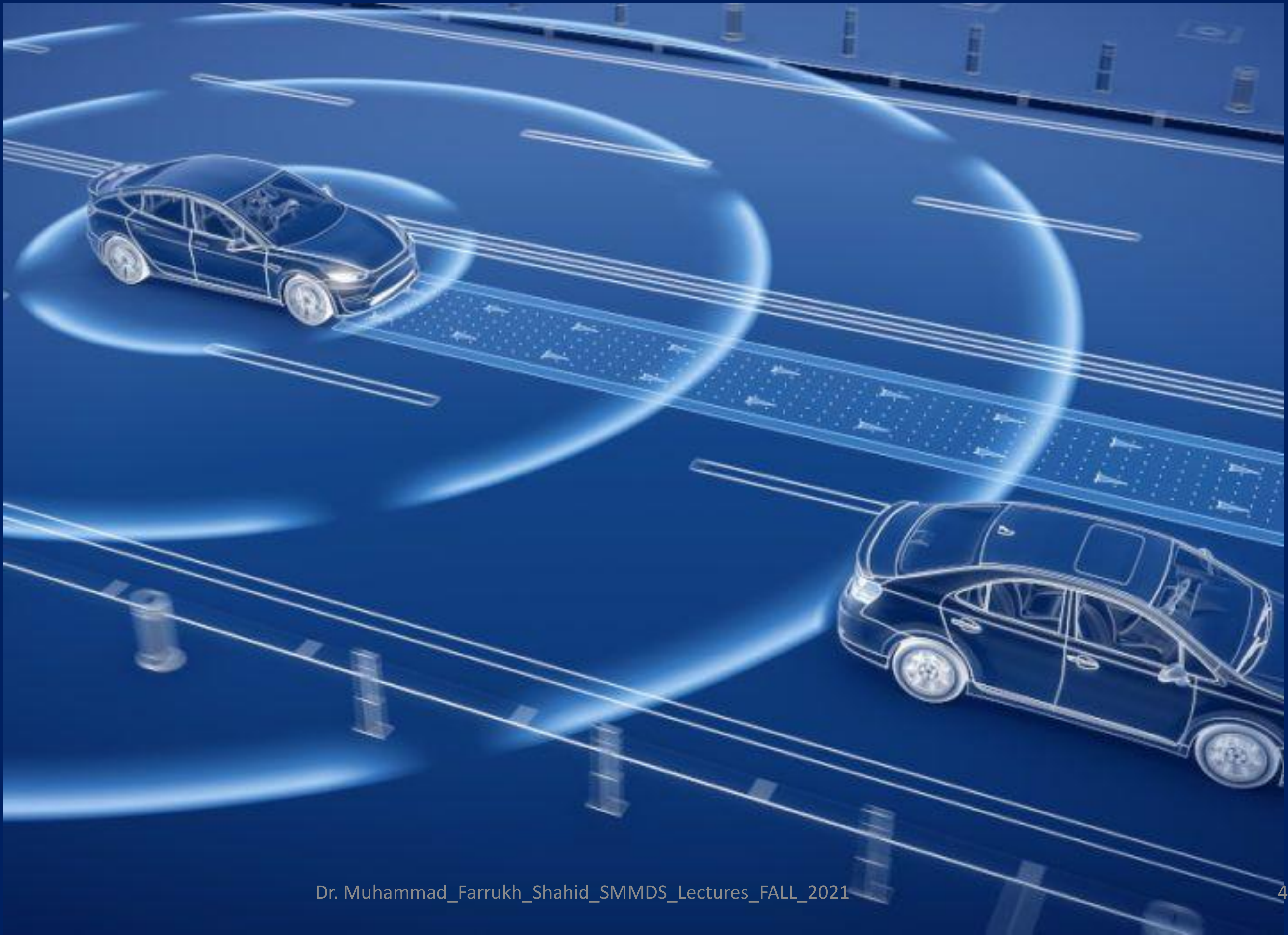
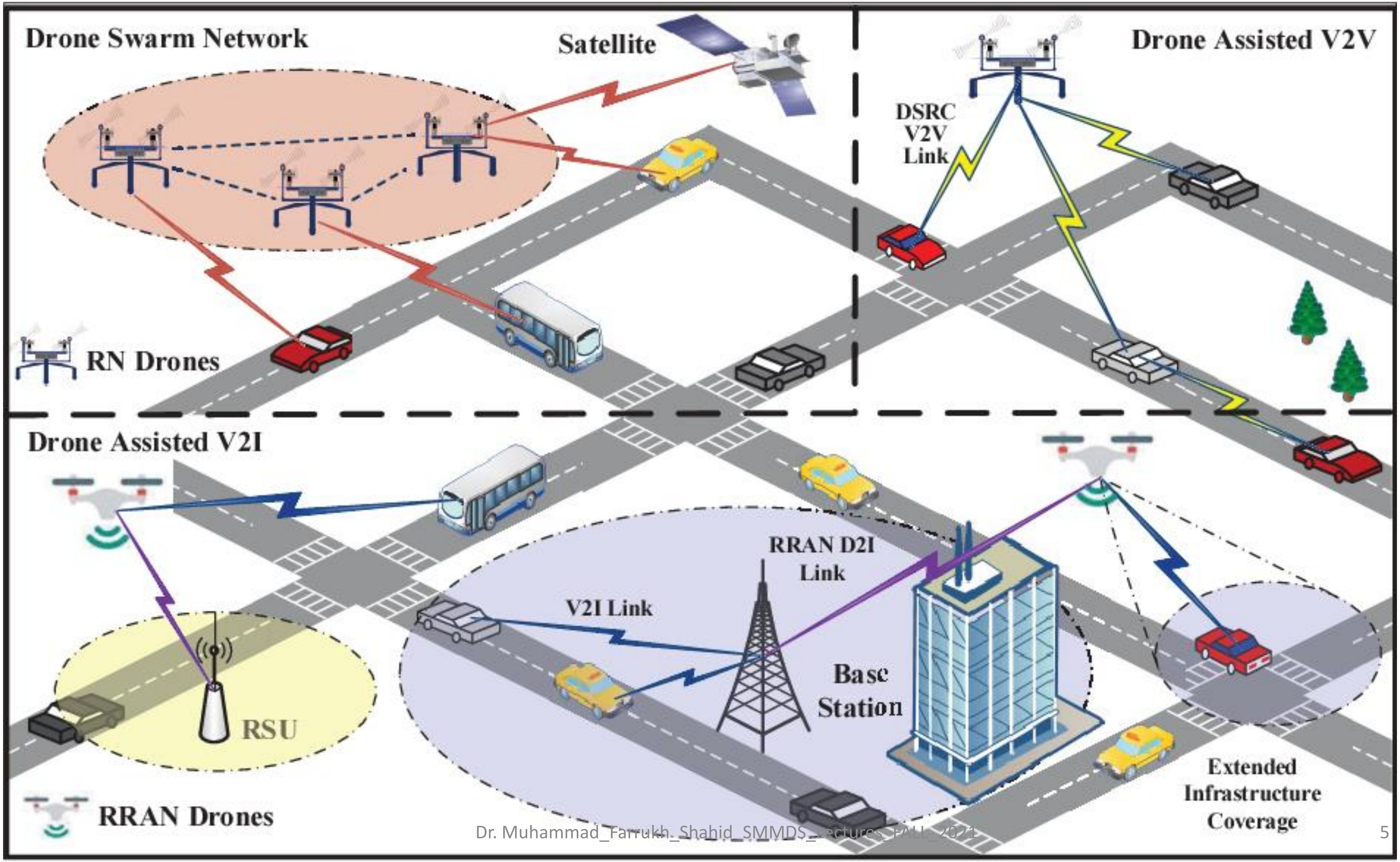
Probability

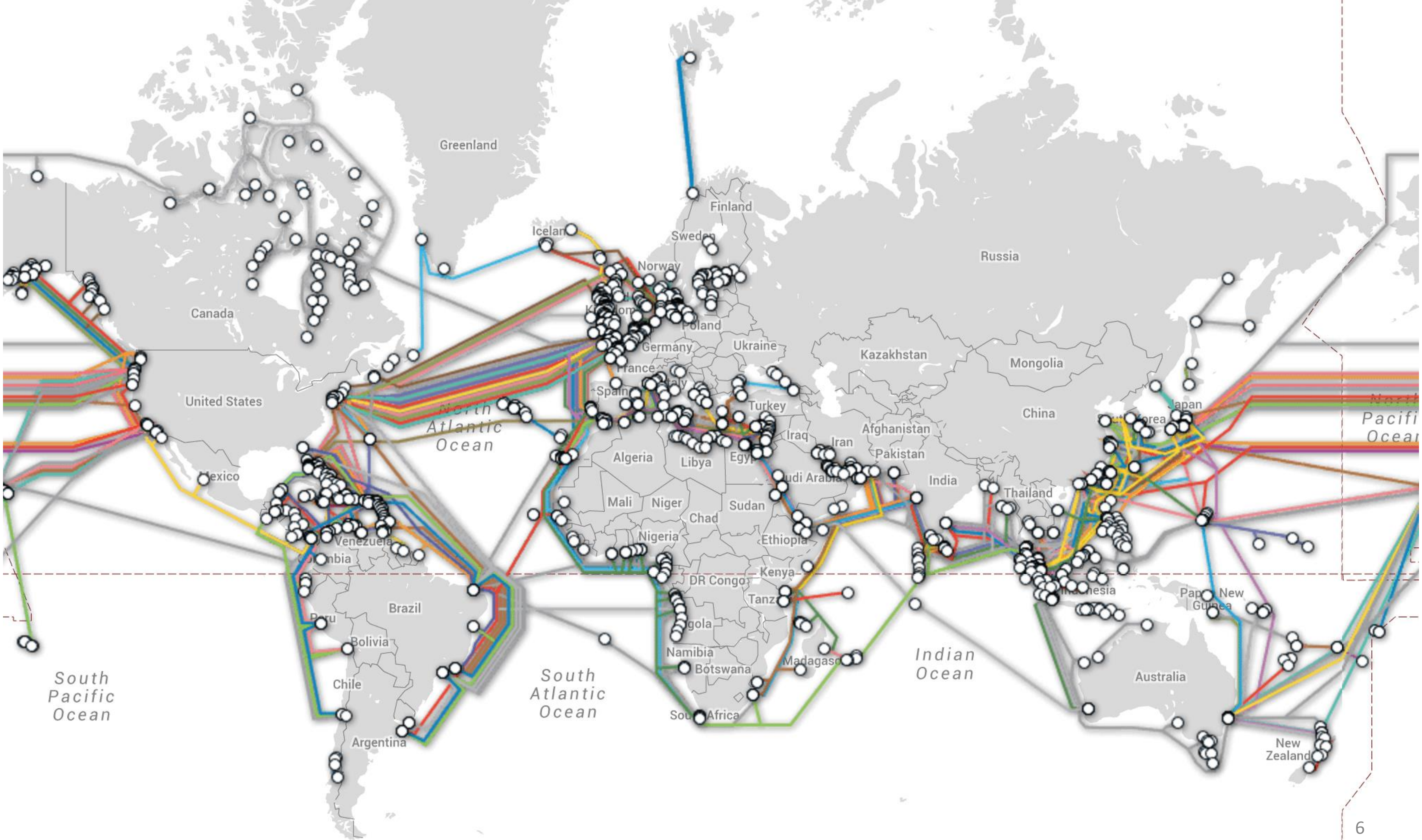


Dealing with uncertainty

Motivation !




Drone Swarm Network
Satellite
Drone Assisted V2V
DSRC
V2V
Link
RN Drones
Drone Assisted V2I
RRAN D2I
Link
V2I Link
Base
Station
RSU
Extended
Infrastructure
Coverage
RRAN Drones


Greenland
Finland
Russia
Norway
Canada
Poland
Germany
Ukraine
Kazakhstan
Mongolia
France
United States
Turkey
China
Japan
North Atlantic Ocean
Afghanistan
Algeria
Libya
Iraq
Iran
Pakistan
India
Thailand
Mali
Niger
Chad
Sudan
Nigeria
Ethiopia
Venezuela
Kenya
DR Congo
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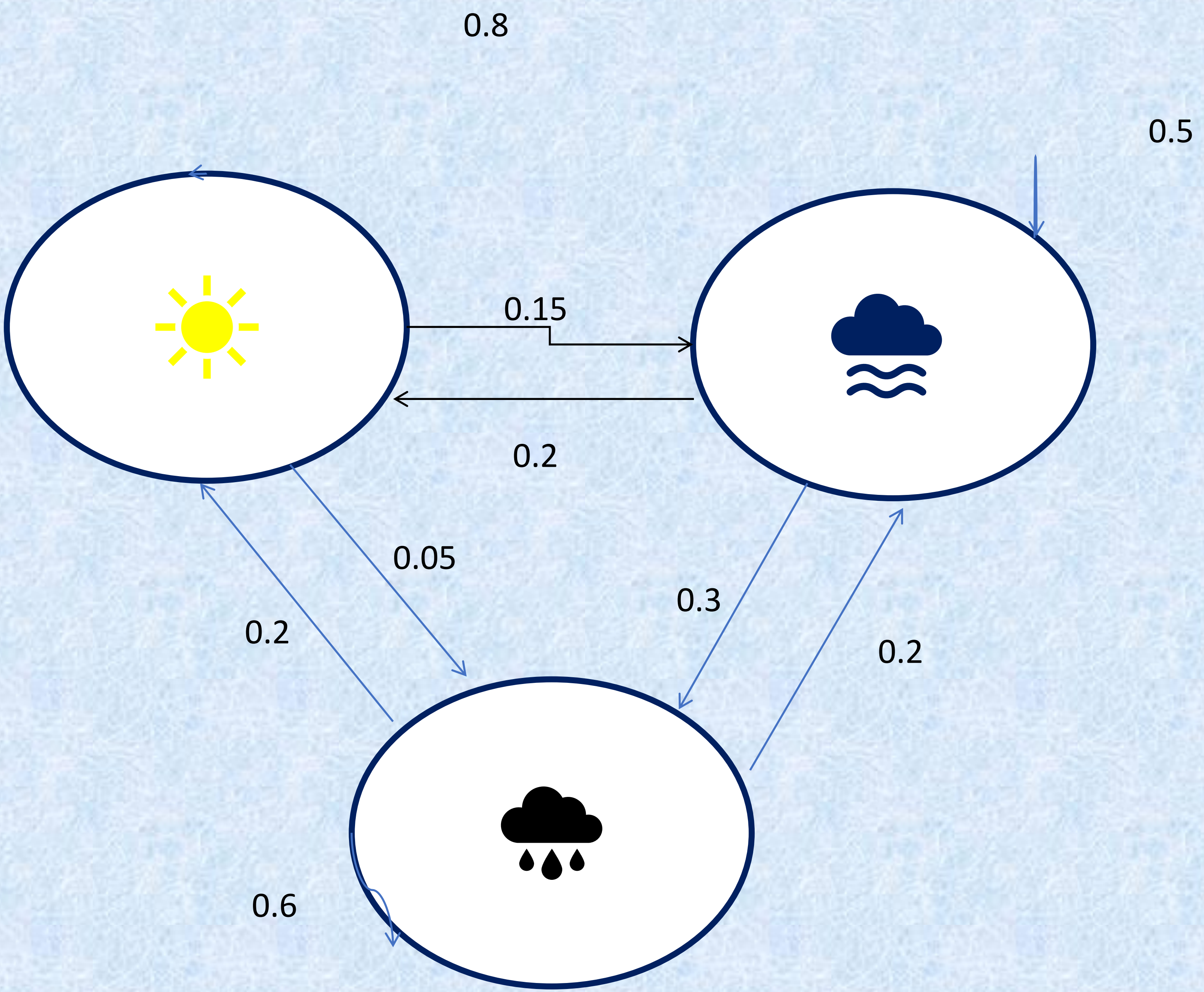
Bolivia
Namibia
Botswana
Madagascar
Indian Ocean
Australia
Chile
South Pacific Ocean
South Atlantic Ocean
Argentina
New Zealand


0.8
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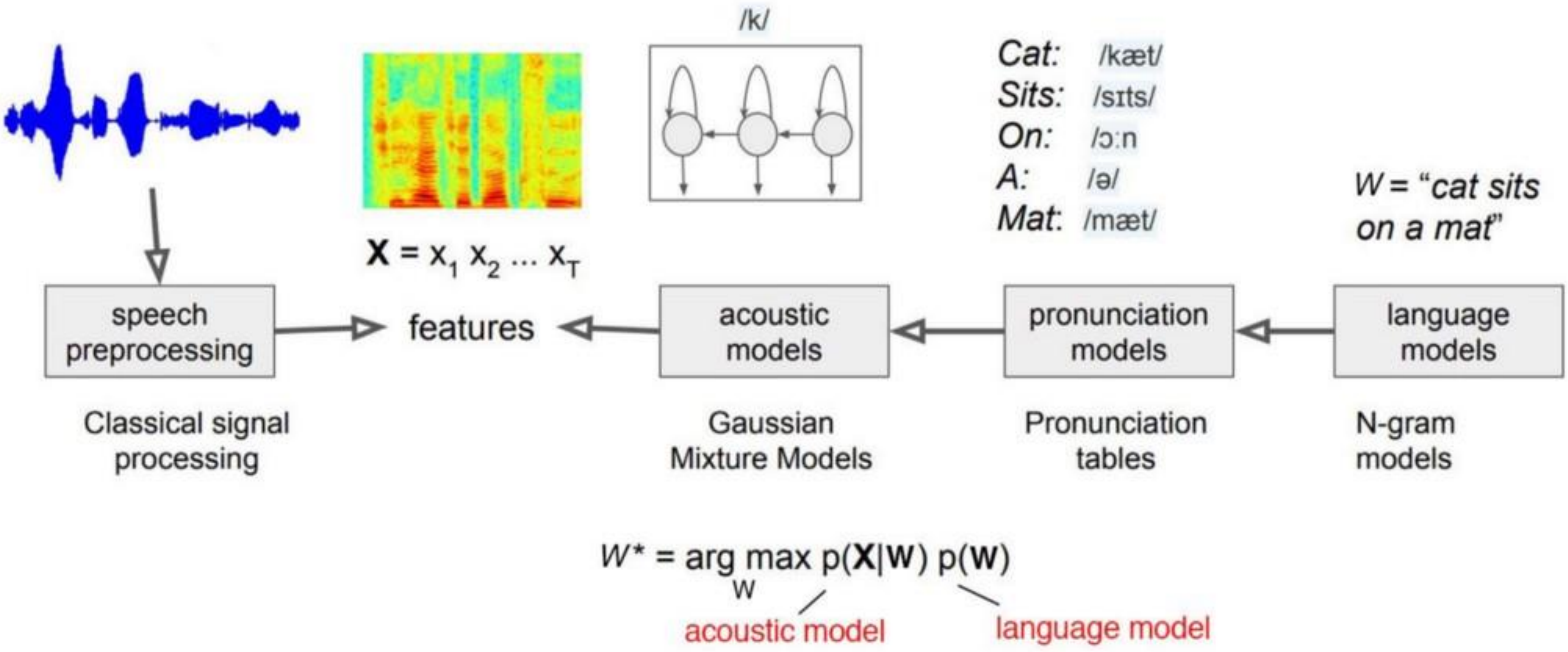
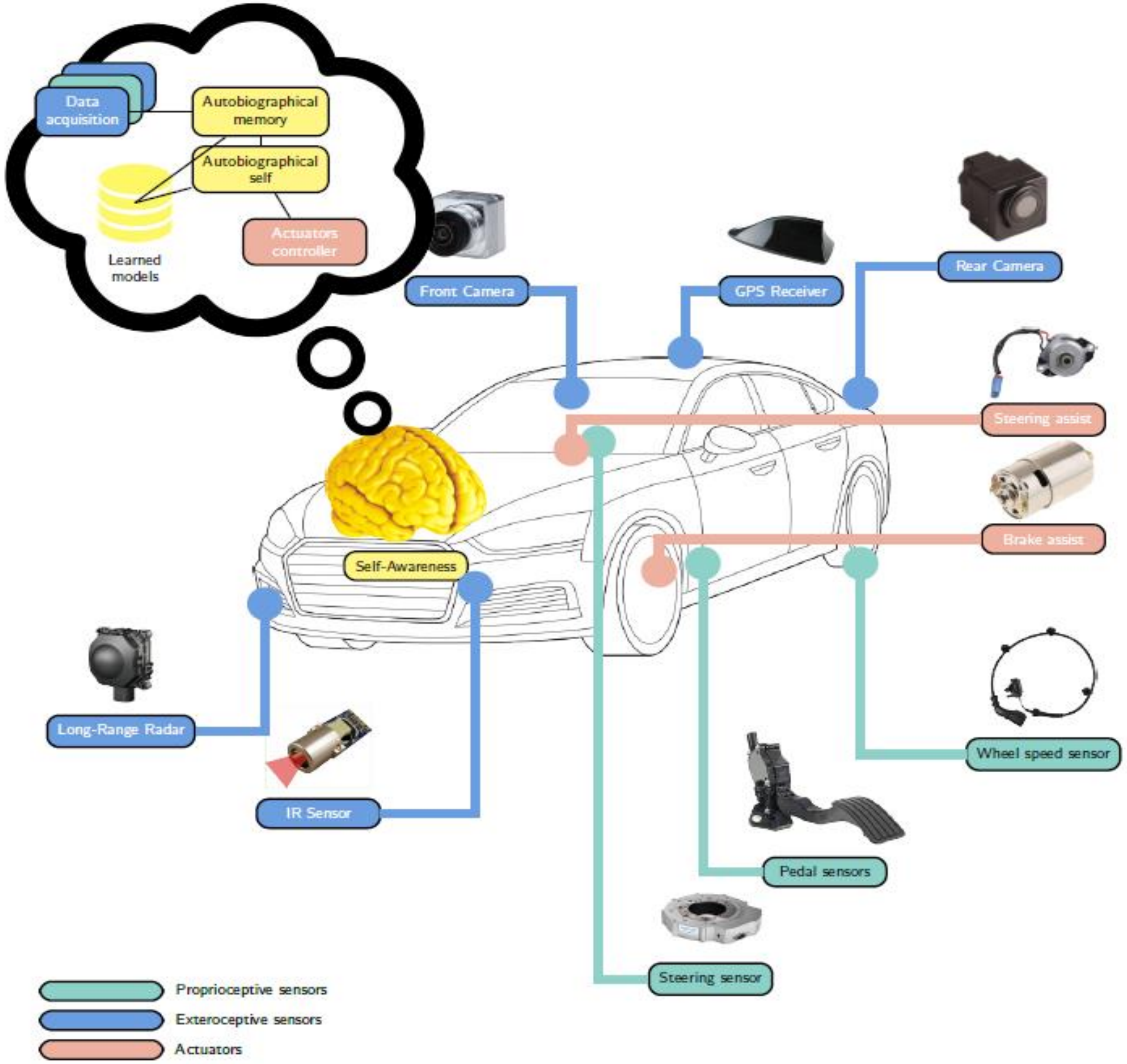
0.6



Source: https://jonathan-hui.medium.com/speech-recognition-gmm-hmm-8bb5eff8b196

An artificial agent, e.g., a car, can be endowed with **multiple sensors** that capture information from the outside world (exteroceptive data) or from its own states (proprioceptive data).



BIO-INSPIRED SELF-AWARENESS THEORIES

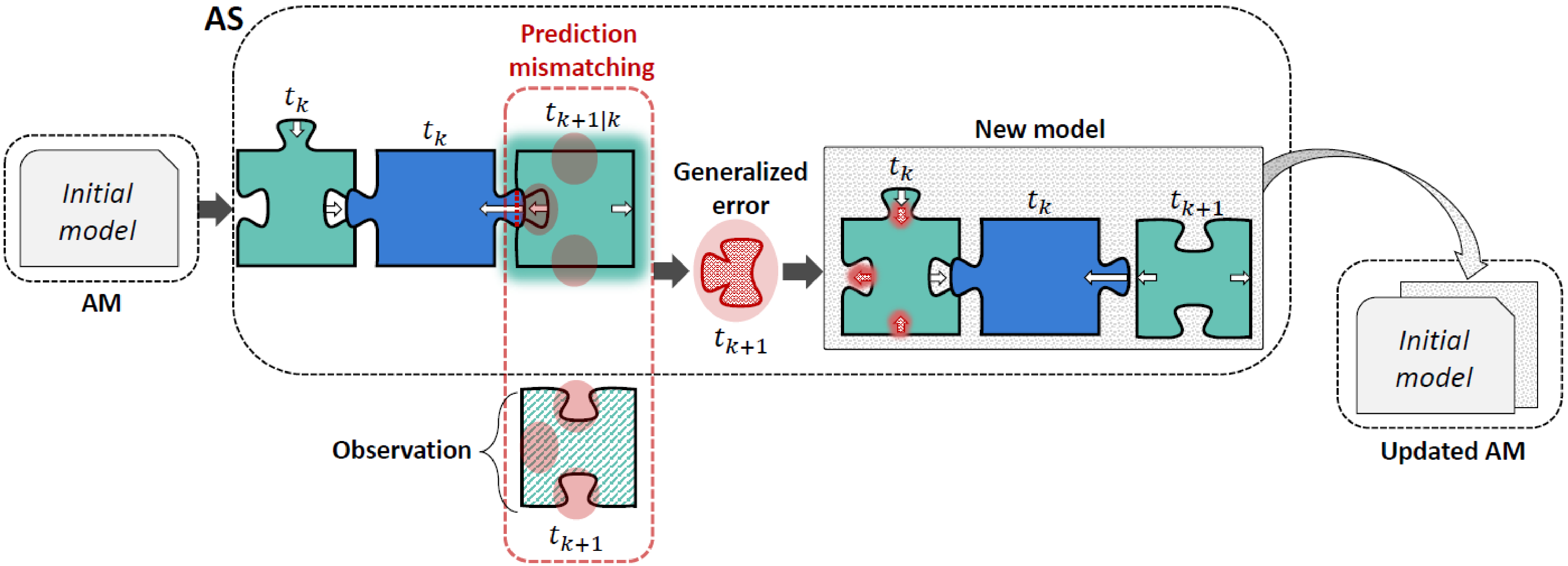
- Three fundamental bio-inspired theories that have studied SA from diverse viewpoints:
  - **Damasio** [1]
  - **Haykin** [2]
  - **Friston** [3]

[1] A. R. Damasio, Looking for Spinoza: Joy, Sorrow, and the Feeling Brain, 1st ed. Orlando: Harcourt, 2003. [Online]. Available: http://lccn.loc.gov/2002011347

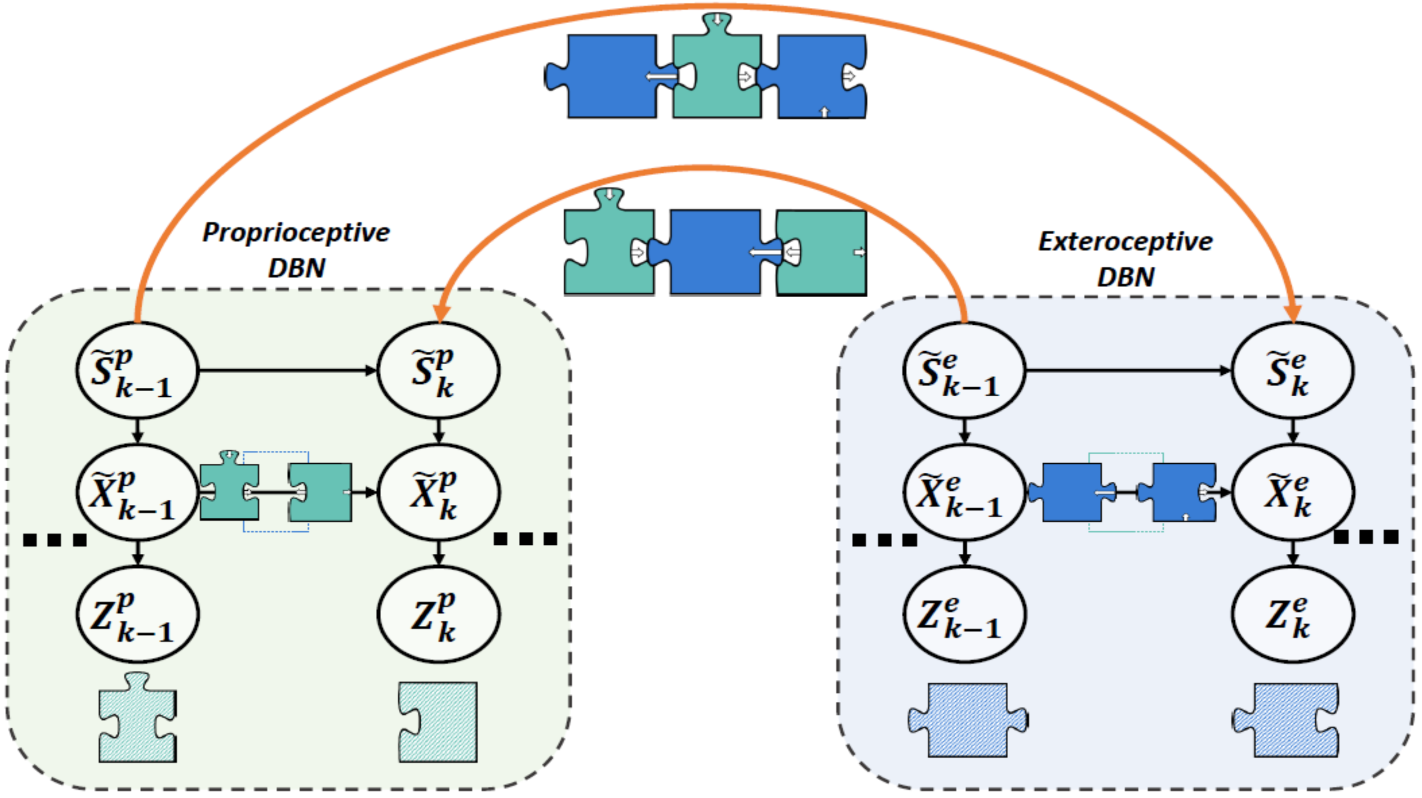
[2] S. Haykin, Cognitive Dynamic Systems: Perception-action Cycle, Radar and Radio, ser. Cognitive Dynamic Systems: Perception–action Cycle, Radar, and Radio. Cambridge University Press, 2012.

[3] K. J. Friston, B. Sengupta, and G. Auletta, "Cognitive dynamics: From attractors to active inference," Proceedings of the IEEE, vol. 102, no. 4, pp. 427–445, 2014. [Online]. Available: https://doi.org/10.1109/JPROC.2014.2306251

Friston’s model (initial model)



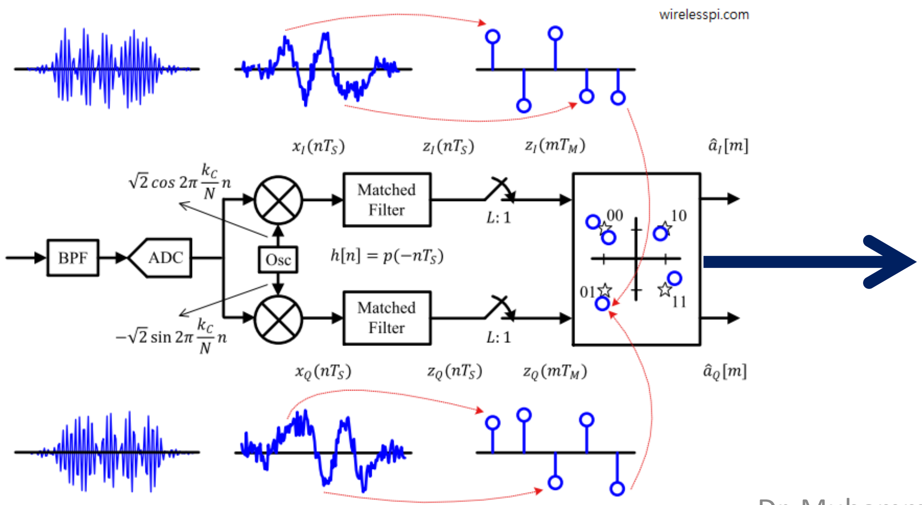
Friston’s model (interactions)



Formal Definitions

1. The set of all possible outcomes of a statistical experiment is called a sample space and is represented by the symbol **S**.

S = { Signal, Jammer }



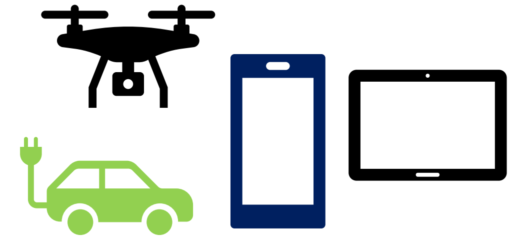
S = { ? }

Event

2. An Event is a subset of a sample space.

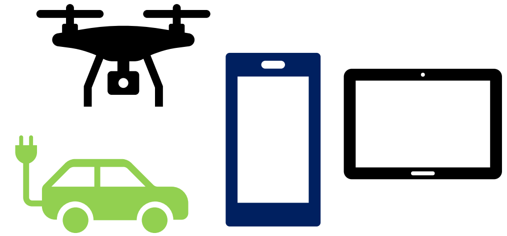
$S = \{ \quad \}$

3. The **complement** of an event **A** with respect to **S** is the subset of all elements of **S** that are not in **A**. We denote the complement of **A** by the symbol **A'**



S = { Self Driving Car, UAV, Satellite, Radio devices, Mobile phones, Laptops }

A = {UAV, Satellite, Radio devices, Mobile phones}

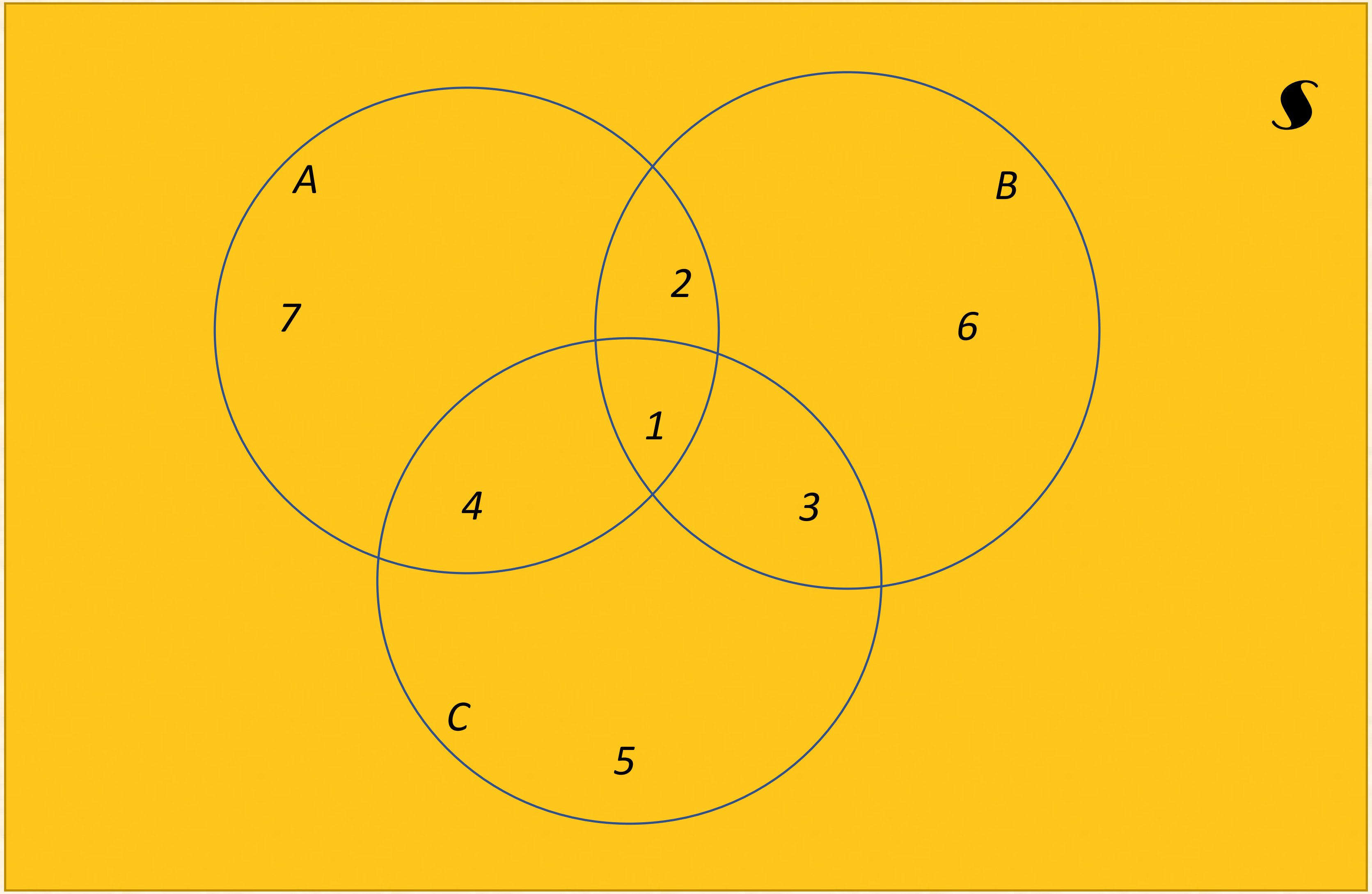


A' = {---}

4. The intersection of two events **A** and **B**, denoted by $\mathbf{A} \cap \mathbf{B}$, is the event containing all elements that are common to **A** and **B**.

5. Two events are mutually exclusive or disjoint if $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$**,** that is, if A and B have no elements in common.

6. The union of the two events A and B, denoted by the symbol $\mathbf{A} \cup \mathbf{B}$, is the event containing all the elements that belong to A or B or both.



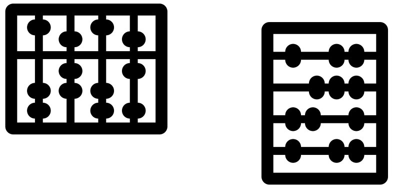
$A \cap B$ *1 and 2*

$A \cup C$ *1,2,3,4,5 and 7*

$B \cap C$ *1 and 3*

$A \cap B \cap C$ *1*

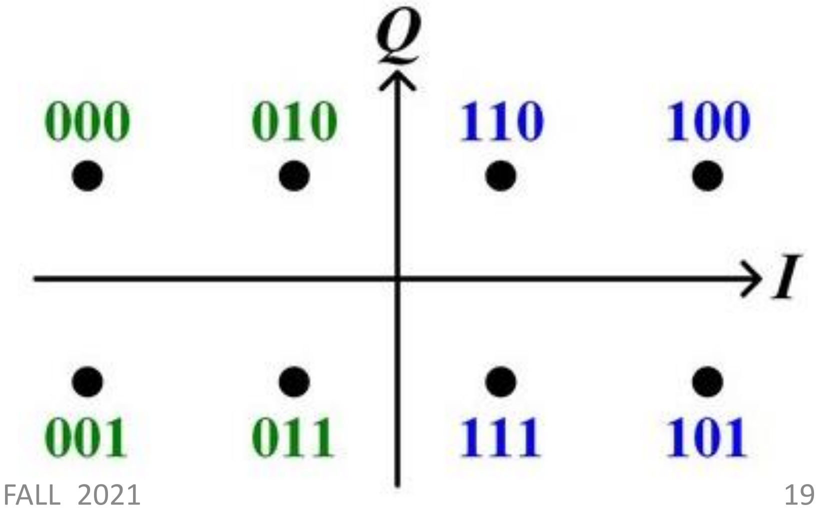
Counting Sample Points



- If an operation can be performed in n_1 ways and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in n_1 and n_2 ways.

How many constellation points can be drawn when two signals having 8-QAM modulation are transmitted ?

$8 \times 8 = 64$



Counting Sample Points

- **Permutation:** It is an arrangement of all or part of a set of objects.

$$\binom{n}{r} = \frac{n!}{(n-r)!}$$

- **Combination:** The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Problem 01: Three awards (**researcher, teaching and service**) will be given one year for a class of 25 graduate students in a CS department. If each student can receive at most one award, how many possible selections are there?

Since the awards are distinguishable, it is a permutation problem. The total number of sample points is:

$$\binom{n}{r} = \frac{n!}{(n-r)!} = \frac{25!}{22!} = 25x24x23 = 13{,}800$$

Probability of an Event

- The probability of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \qquad P(\emptyset) = 0, and\ P(S) = 1$$

- If A_1, A_2, A_3… is sequence of ***mutually exclusive*** events then.

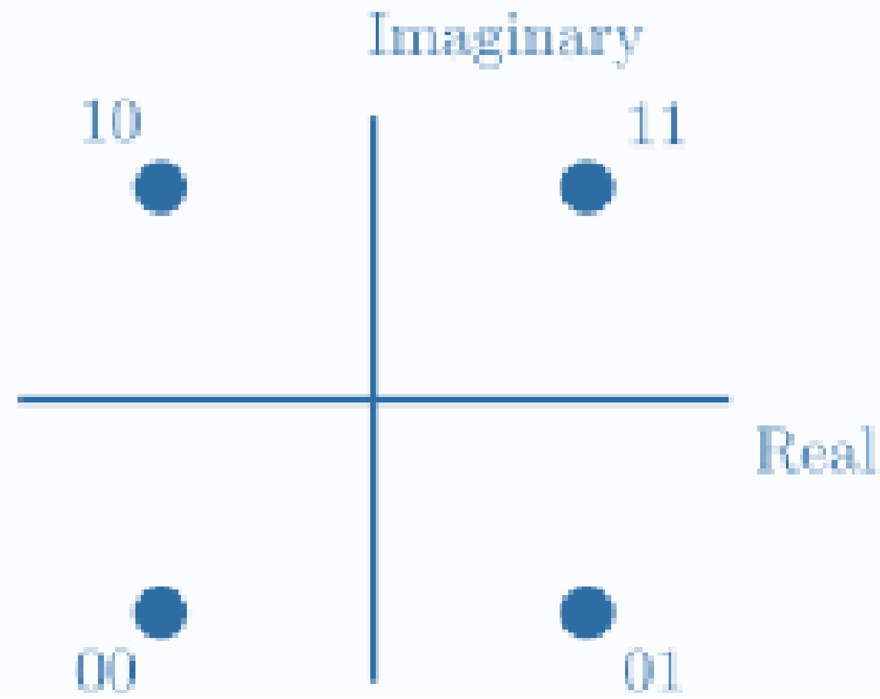
$$P(A_1 \cup A_2 \cup ..) = P(A_1) + P(A_2) + \cdots$$

Problem 02: If a signal is transmitted using ***4-QAM*** modulation. What is the probability that at least 1 occurs in any given symbol ?

S ={00, 01 , 10 , 11 }

A = { 01, 10,11}

P(A) = 1/4 + 1/4 + 1/4 = 3/4



Problem 03: A SA car has 6 sensors built in the car front end. The sensors read the data in such a way that 2D sensors data are twice likely to read as 1D sensors data. The sample space of sensors are given as:

SA = {Sensor1, Sensor2, Sensor3, Sensor4, Sensor5, Sensor6}

Senosr1, Sensor3 and Sensor5 takes 1D data
Senosr2, Sensor4 and Sensor6 takes 2D data

If **E** is the event that contains two 1D sensor data and one 2D sensor data. Find *P(E)*.

E = {Senosr1, Sensor2 and Sensor3 }

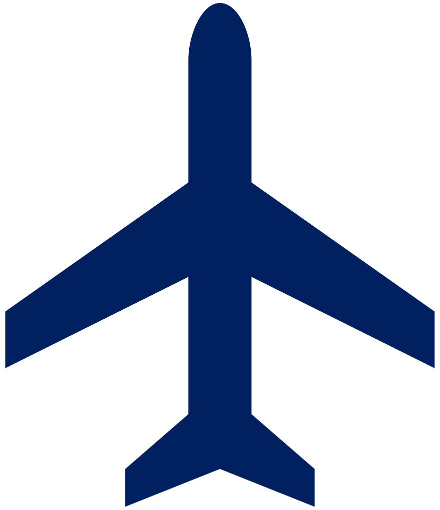
P(E) = 1/9 +2/9+1/9 = 4/9

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}$$

Problem 04:

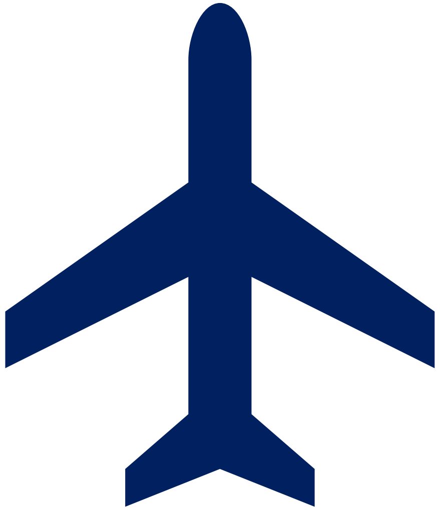
An Airbus A380 assembly plant uses 3 different sectors (Finance, mechanical and electronic) which work together during plane assembling process. There are 26 people in Finance, 12 in mechanical and 13 in electronic departments. An Airbus makes use of AI technology by using Robots which select people randomly to perform special tasks. Find the probability that the person chosen is belong to the electronic department.



Problem 04:

An Airbus A380 assembly plant uses 3 different sectors (Finance, mechanical and electronic) which work together during plane assembling process. There are 26 people in Finance, 12 in mechanical and 13 in electronic departments. An Airbus makes use of AI technology by using Robots which select people randomly to perform special tasks. Find the probability that the person chosen is belong to the electronic department.

$$P(E) = 13 / (26 + 12 + 13)$$



- If A and B are any two events then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

- If A and A' are complementary events, then

$$P(A) + P(A') = 1$$

Conditional Probabilities

The probability of an event B occurring when it is known that some event A has occurred is called as conditional probability and denoted by $P(B|\mathrm{A})$.

Probability of B given A

$$\boldsymbol{P(B|A) = \frac{P(A \cap B)}{P(A)}}$$

If $P(A) > 0$

Example ?

	Data Scientist	Software Engineer	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

A certain organising is conducting interviews for new hiring against different posts. The HR consider the following event for the post of Associate researcher:

M - A man is chosen,
DS- the one chosen is data scientist

We need to find $P(M\backslash DS)$?

$$P(M|\mathrm{DS}) = \frac{P(M \cap DS)}{P(DS)}$$

$$P(\mathrm{DS}) = \frac{600}{900} = \frac{2}{3}$$

$$P(M \cap \mathrm{DS}) = \frac{460}{900} = \frac{23}{45}$$

$$P(M|\mathrm{DS}) = \frac{23/45}{2/3} = \frac{23}{30}$$

The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

Consider the events

L: length defective, T: texture defective.

Given that the strip is length defective, the probability that this strip is texture defective is given by

$$P(T|L) = \frac{P(T \cap L)}{P(L)} = \frac{0.008}{0.1} = 0.08.$$

Thus, knowing the conditional probability provides considerably more information than merely knowing $P(T)$.

- Two events A and B are independent if and only if

$$P(B|A) = P(B) \;\; or \quad P(A|B) = P(A)$$

Otherwise A and B are dependent.

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Thus, the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability that B occurs, given that A occurs. Since the events $A \cap B$ and $B \cap A$ are equivalent, it follows from Theorem 2.10 that we can also write

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B).$$

In other words, it does not matter which event is referred to as A and which event is referred to as B.

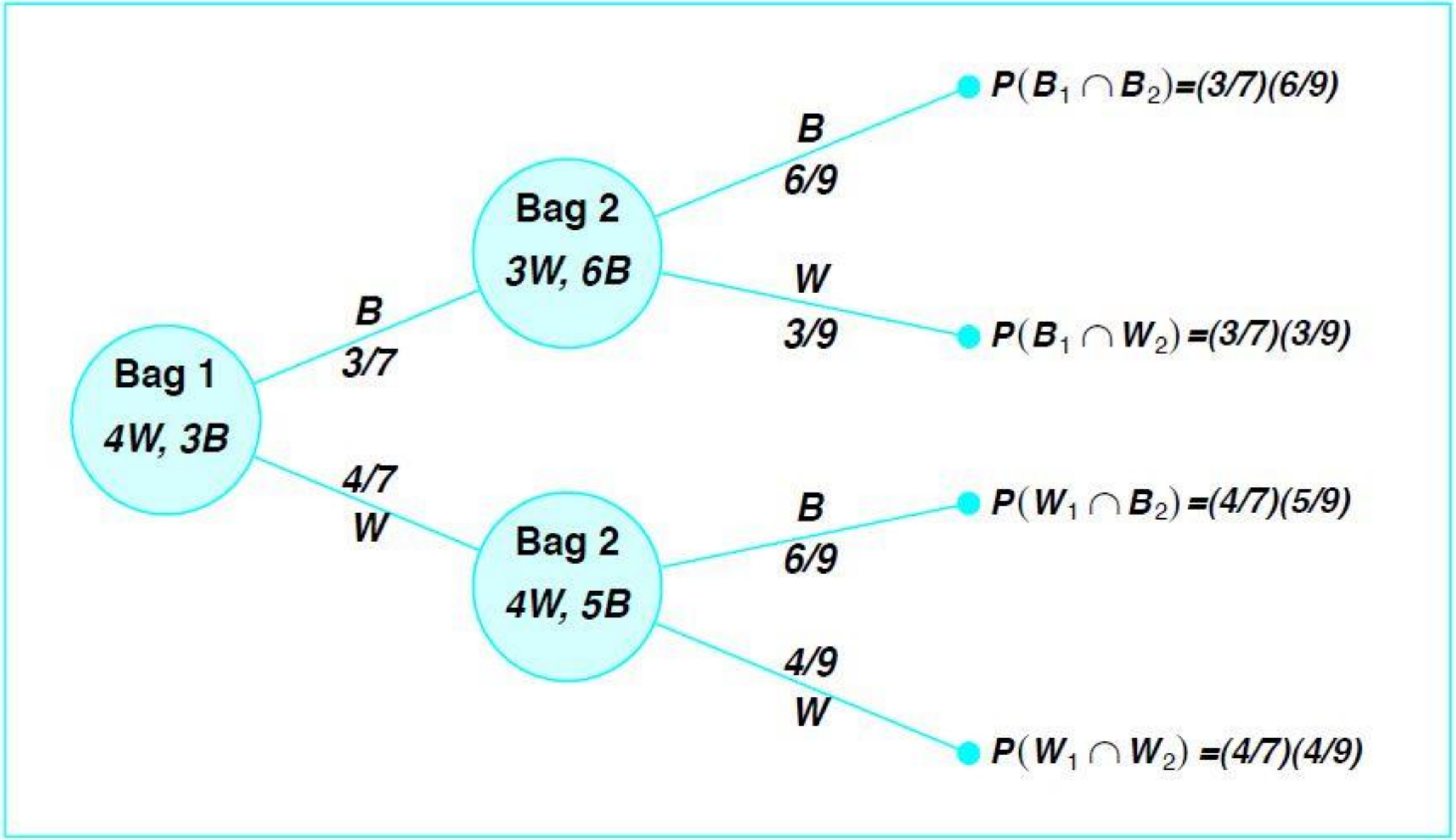
Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?
We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective; then we interpret $A \cap B$ as the event that A occurs and then B occurs after A has occurred. The probability of first removing a defective fuse is $1/4$; then the probability of removing a second defective fuse from the remaining 4 is $4/19$. Hence,

$$P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19}.$$

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in Figure 2.8. Now

$$\begin{aligned} P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}. \end{aligned}$$


Bag 1
4W, 3B
B
3/7
4/7
W
Bag 2
3W, 6B
Bag 2
4W, 5B
B
6/9
W
3/9
B
6/9
4/9
W
$P(B_1 \cap B_2)$=(3/7)(6/9)
$P(B_1 \cap W_2)$=(3/7)(3/9)
$P(W_1 \cap B_2)$=(4/7)(5/9)
$P(W_1 \cap W_2)$=(4/7)(4/9)

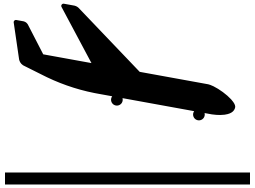
Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Problem 05: The probability that a regularly scheduled flight departs on time is P(D) = 0.83; the probability that it arrives on time is P(A) = 0.82; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$.

Find the probability that a plane
(a) arrives on time given that it departed on time
(b) Departed on time given that it has arrived on time.



a) $P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94$

b) $P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$

Problem 06:

A smart town has one fire **drone** engine and one **UAV** ambulance available for emergencies. The probability that the fire drone is available when needed is 0.98 and the probability that UAV ambulance is available is 0.92. In the case of an injury resulting from the burning building, find the probability that both the UAV ambulance and the fire drone will be available.

$$P(A \cap B) = P(A)P(B) = 0.98\ x\ 0.92 = 0.9016$$

: If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

: First we define the events

A_1: the first card is a red ace,

A_2: the second card is a 10 or a jack,

A_3: the third card is greater than 3 but less than 7.

Now

$$P(A_1) = \frac{2}{52}, \quad P(A_2|A_1) = \frac{8}{51}, \quad P(A_3|A_1 \cap A_2) = \frac{12}{50},$$

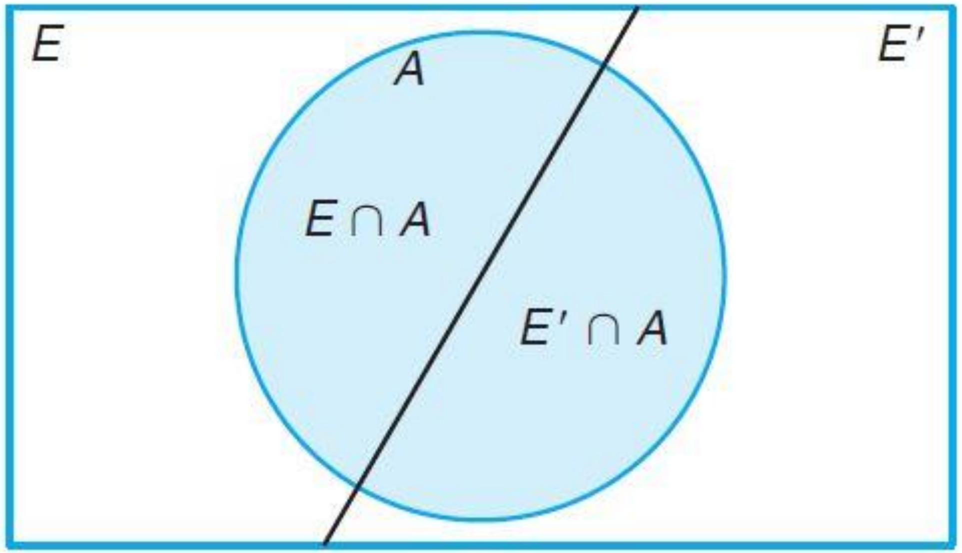
and hence, by Theorem 2.12,

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \\ &= \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{8}{5525}. \end{aligned}$$

The property of independence stated in Theorem 2.11 can be extended to deal with more than two events. Consider, for example, the case of three events A, B, and C. It is not sufficient to only have that $P(A \cap B \cap C) = P(A)P(B)P(C)$ as a definition of independence among the three. Suppose $A = B$ and $C = \phi$, the null set. Although $A \cap B \cap C = \phi$, which results in $P(A \cap B \cap C) = 0 = P(A)P(B)P(C)$, events A and B are not independent. Hence, we have the following definition.

Total Probability

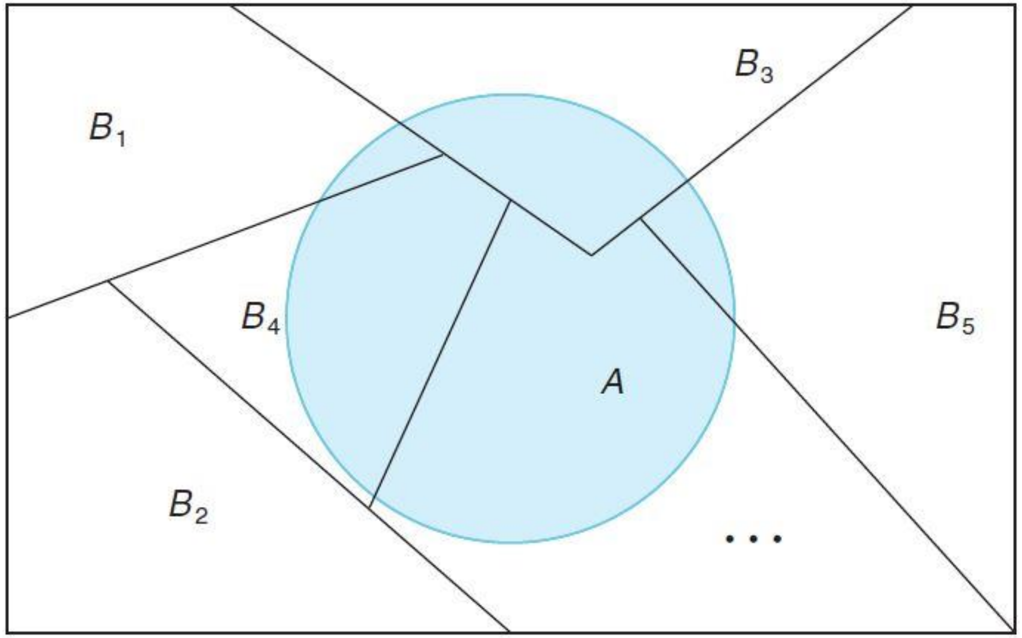
$$
\begin{aligned}
P(A) &= P[(E \cap A) \cup (E' \cap A)] = P(E \cap A) + P(E' \cap A) \\
&= P(E)P(A|E) + P(E')P(A|E').
\end{aligned}
$$



Rule of Elimination

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$



***Proof*:** Consider the Venn diagram of Figure 2.14. The event A is seen to be the union of the mutually exclusive events

$$B_1 \cap A,\ B_2 \cap A,\ \ \ldots,\ B_k \cap A;$$

that is,

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_k \cap A).$$

Using Corollary 2.2 of Theorem 2.7 and Theorem 2.10, we have

$$\begin{aligned} P(A) &= P[(B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_k \cap A)] \\ &= P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_k \cap A) \\ &= \sum_{i=1}^{k} P(B_i \cap A) \\ &= \sum_{i=1}^{k} P(B_i)P(A|B_i). \end{aligned}$$

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

***Solution*:** Consider the following events:

A: the product is defective,

B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3.

Applying the rule of elimination, we can write

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

Referring to the tree diagram of Figure 2.15, we find that the three branches give the probabilities

$$\begin{aligned} P(B_1)P(A|B_1) &= (0.3)(0.02) = 0.006, \\ P(B_2)P(A|B_2) &= (0.45)(0.03) = 0.0135, \\ P(B_3)P(A|B_3) &= (0.25)(0.02) = 0.005, \end{aligned}$$

and hence

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

Baye's Rule

(Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum\limits_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum\limits_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01, \qquad P(D|P_2) = 0.03, \qquad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

From the statement of the problem

$$P(P_1) = 0.30, \quad P(P_2) = 0.20, \quad \text{and} \quad P(P_3) = 0.50,$$

we must find $P(P_j|D)$ for $j = 1, 2, 3$. Bayes' rule (Theorem 2.14) shows

$$\begin{aligned} P(P_1|D) &= \frac{P(P_1)P(D|P_1)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)} \\ &= \frac{(0.30)(0.01)}{(0.3)(0.01) + (0.20)(0.03) + (0.50)(0.02)} = \frac{0.003}{0.019} = 0.158. \end{aligned}$$

Similarly,

$$P(P_2|D) = \frac{(0.03)(0.20)}{0.019} = 0.316 \text{ and } P(P_3|D) = \frac{(0.02)(0.50)}{0.019} = 0.526.$$