Name: Md Salman Rahman Homework Chapter 3 & 4

3.10

Solution: Given Table 1.1 (Hardness Data) - page-5

(a) Linear Regression Model for Hardness Data:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where $i = 1, 2, \ldots n$

Where,
$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\bar{y} = \frac{\sum y_i}{n}$$

$$\bar{x} = \frac{\sum x_i}{n}$$

y = Hardness

x = Temperature

(b) <u>14x2 matrix X:</u>

$$X = \begin{bmatrix} 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 40 \\ 1 & 40 \\ 1 & 40 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \end{bmatrix}_{14\times 2}$$

(c) 14×1 vector of responses y:

$$y = \begin{bmatrix} 55.8 \\ 59.1 \\ 54.8 \\ 54.6 \\ 43.1 \\ 42.2 \\ 45.2 \\ 31.6 \\ 30.9 \\ 30.8 \\ 17.5 \\ 20.5 \\ 17.2 \\ 16.9 \end{bmatrix}_{14\times 1}$$

(d) 2×2 matrix $X'X$:

$$X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 30 & 30 & 30 & 30 & 40 & 40 & 40 & 50 & 50 & 50 & 60 & 60 & 60 & 60 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 30 & 30 & \cdots & 60 \end{bmatrix}_{2\times 14} \begin{bmatrix} 1 & 30 \\ 1 & 30 \\ \vdots & \vdots \\ 1 & 60 \\ 1 & 60 \end{bmatrix}_{14\times 2}$$

$$X'X = \begin{bmatrix} 14 & 630 \\ 630 & 30300 \end{bmatrix}$$

Ans

Note: Alternatively this can be found by

$$X'X = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

where $n = 14$, $\sum x_i = 630$

$\sum x_i^2 = 30300$

(e) $(X'X)^{-1}$:-

we know the formula

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{So, } (X'X)^{-1} = \frac{1}{14\times 30300 - 630\times 630} \begin{bmatrix} 30300 & -630 \\ -630 & 14 \end{bmatrix}$$

$$= \frac{1}{27300} \begin{bmatrix} 30300 & -630 \\ -630 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1098 & -0.023 \\ -0.023 & 0.0005 \end{bmatrix}$$ Ans

(f) Write down the expression for the least squares estimates in $\hat{\beta} = (X'X)^{-1}X'y$:

$$\hat{\beta} = (X'X)^{-1} X'y$$

$$= \begin{bmatrix} 1.1098 & -0.023 \\ -0.023 & 0.00051 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 30 & 30 & \cdots & 60 \end{bmatrix} \begin{bmatrix} 55.8 \\ 59.1 \\ 54.8 \\ \vdots \\ 17.2 \\ 16.9 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1098 & -0.023 \\ -0.023 & 0.00051 \end{bmatrix} \begin{bmatrix} 520.2 \\ 20940.0 \end{bmatrix}$$

$$= \begin{bmatrix} 94.134 \\ -1.266 \end{bmatrix}$$

Ans:

** whole problem done manually & in R. (both).

** - The R code (whole problem done in R also) in share along with Homework.

Problem 3.13:

Solution:

Here X is $n \times p$ matrix

and $A = (X'X)^{-1}X'$

$H = XA$

(a) (i) Show that $HH = H$:

Here, $H = XA$

$= X(X'X)^{-1}X'$

$$HH = X\underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}X'$$

$= X(X'X)^{-1}X'$

$= H.$

So, $HH = H$ (showed)

(ii) Show that $(I-H)(I-H) = (I-H)$:

From (i) we see that $HH = H$

So, $(I-H)(I-H) = I(I-H) - H(I-H)$

$= I - H - H + HH$

$= I - 2H + HH$

H matrix (H is idempotent)

$$H \cdot H = H$$

So, $(I-H)(I-H) = I - 2H + H$

$$= (I-H)$$

So, $(I-H)(I-H) = (I-H)$

(iii) Show that $HX = X$:

$$HX = \underbrace{X(X'X)^{-1}X'X}_{I}$$

$$= X$$

So, $HX = X$ [showed]

(b) (i) $A(I-H) = AI - AH$

$$= (X'X)^{-1}X' - (X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X'$$

$$= (X'X)^{-1}X' - (X'X)^{-1}X'$$

$$= 0$$

(ii) $(I-H)A' = A' - HA'$

$$= \left((X'X)^{-1}X'\right)' - X(X'X)^{-1}X'\left((X'X)^{-1}X'\right)'$$

$$= \left((X'X)^{-1}X'\right)' - X\underbrace{(X'X)^{-1}X'X}_{I}\left((X'X)^{-1}\right)'$$

$$= \left((X'X)^{-1}X'\right)' - X\left((X'X)^{-1}\right)' \quad \left[\because (AB)' = B'A'\right]$$

$$= X\left((X'X)^{-1}\right)' - X\left((X'X)^{-1}\right)'$$

$$= 0$$

(iii) $H(I-H) = H - HH$

Hat matrix H is idempotent (prove in a(i))

So, $HH = H$.

So, $H(I-H) = H - HH = H - H = 0$

(iv) $(I-H')H' = H' - H'H'$

we know that Hat matrix H is symmetric $H' = H$.

$$H' = \left[X(X'X)^{-1}X'\right]' = (X')'\left[X(X'X)^{-1}\right]'$$

$$= X\left((X'X)^{-1}\right)'X'$$

$$= X\left((X'X)'\right)^{-1}X'$$

$$= X(X'X)^{-1}X' = H \quad \left[\because (AB)' = B'A'\right]$$

So, $H' = H$

So, $(I - H')H' = H' - H'H' = H - HH$

Also, we know, H is idempotent
$HH = H$,

So, $(I - H')H' = H - H = 0$ Ans

Problem 4.2:

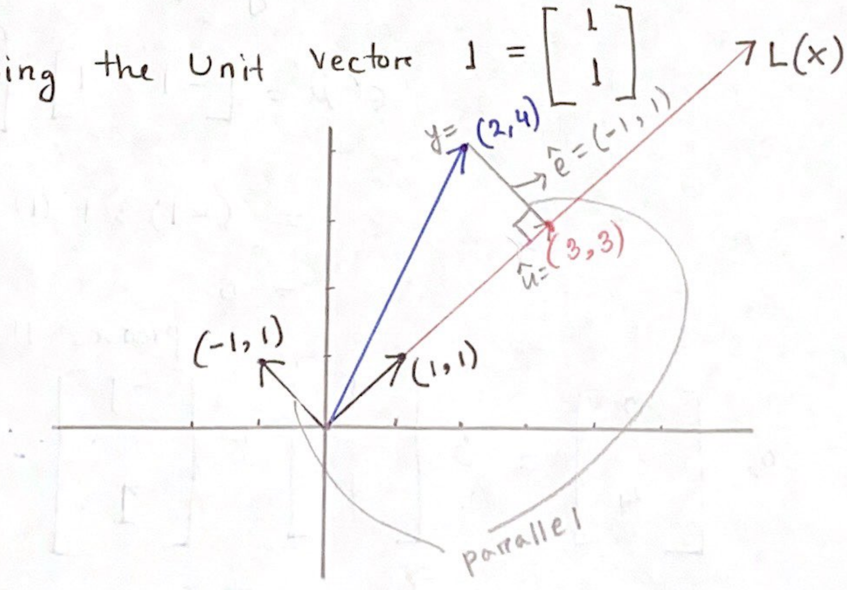
Solution: Here, $y_t = \beta_0 + \varepsilon_t$ with $n = 2$ and $y' = (2, 4)$

$$y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

As here $n = 2$, we are looking into two-dimensional Euclidean space.

Step 1: Drawing the Unit vector $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Step 2: Drawing the response vector $y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Step 3: Here $p = 0$, so $p+1 = 1$, $L(\mathbf{1})$ is given and shown by red line in figure.

Step 4: The projection leads to the vector of fitted values $\hat{\mu} = 3\vec{1} = (3, 3)^T$ and the LSE $\hat{\beta}_0 = 3$.

The estimate is the average of two observations, 2 and 4.

Step 5: The residual vector, $e = y - \hat{\mu}$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and the vector of fitted values $\hat{\mu} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

are orthogonal. means,

$$e'\hat{\mu} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= (-1)\,3 + (1)\,3$$

$$= 0$$

proves the orthogonality.

So, $$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$y = \beta_0 \mathbf{1} \quad + \varepsilon_t$$

Problem 4.4:

Solution: Here $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, 2, 3$ with

$$x = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Step 1: Drawing the unit vector $\vec{1} = (1, 1, 1)'$ and $\vec{x} = (1, 3, 2)'$. These two vectors are graphed in three dimensional space. (Next page: Fig).

Any linear combination of these two vectors results in a vector that lies in the two dimensional space that is spanned by the vectors $\vec{1}$ and $\vec{x}$.

Step 2: We highlight the plane . Here $L(\vec{1}, \vec{x})$ is a Subspace of $\mathbb{R}^3$ and dimension = 2.

Step 3: Here $\vec{1}$ and $\vec{x}$ are not linearly dependent that means one of the two vectors cannot be written as a multiple of others. So, the Matrix X will be

$$X = \begin{bmatrix} \vec{1} & \vec{x} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Full column rank = 2

Step 4: The orthogonal projection leads to the vector of fitted values:

The subspace $L(x) = L(\vec{1}, \vec{x})$

we know that $y - \hat{y} \perp \vec{x}$ (perpendicular)

and $y - \hat{y} \perp \vec{1}$

So, we can write $\Rightarrow$ $\hat{y} = a\vec{1} + b\vec{x}$

because $\langle y - \hat{y}, x \rangle = 0$

and $\langle y - \hat{y}, 1 \rangle = 0$

Here a & b two unknown

For $\langle y - \hat{y}, x \rangle = 0$

we have

$$y - \hat{y} = y - a\vec{1} - b\vec{x}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - b\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - a - b \\ 4 - a - 3b \\ 6 - a - 2b \end{bmatrix}$$

So, $\langle y - \hat{y}, x \rangle = 0$

$$\left\langle \begin{bmatrix} 2 - a - b \\ 4 - a - 3b \\ 6 - a - 2b \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\rangle = 0$$

$$\Rightarrow (2 - a - b) + 3(4 - a - 3b) + 2(6 - a - 2b) = 0$$

$$\Rightarrow 2 - a - b + 12 - 3a - 9b + 12 - 2a - 4b = 0$$

$$\Rightarrow 26 - 6a - 14b = 0 \quad \text{———— (1)}$$

Similarly, $< y - \hat{y}, 1 > = 0$,

$$y - \hat{y} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - b \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - a - b \\ 4 - a - 3b \\ 6 - a - 2b \end{bmatrix}$$

$$< y - \hat{y}, 1 > = 0$$

$$< \begin{bmatrix} 2 - a - b \\ 4 - a - 3b \\ 6 - a - 2b \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} > = 0$$

$$\Rightarrow 2 - a - b + 4 - a - 3b + 6 - a - 2b = 0$$

$$\Rightarrow 12 - 3a - 6b = 0 \quad \text{———— (2)}$$

So For solving (1) & (2) multiplying eq (2) by 2

$$26 = \cancel{6a} + 14b$$

$$\begin{array}{rcl} 24 & = & \cancel{6a} + 12b \\ (-) & & (-) \quad (-) \\ \hline 2 & = & 2b \end{array}$$

$$\boxed{\therefore b = 1}$$

So From (1)

$$26 - 6a - 14 \times 1 = 0$$

$$\Rightarrow \quad 6a = 26 - 14$$

$$\Rightarrow \quad 6a = 12$$

$$\therefore \boxed{a = 2}$$

So the equation, $\hat{y} = a\vec{1} + b\vec{x}$

$$= 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$ This in the estimate of $\hat{y}$.

we say $\hat{u} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$

Step 5: The residual vector,

$$e = y - \hat{u}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Step 6: parameter $\hat{\beta}$ of $Y = X\beta + e$

$$\hat{\beta} = (X'X)^{-1} X'y$$

Here $X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ $X' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

$$X'X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1 + 1\times1 + 1\times1 & 1\times1 + 3\times1 + 2 \\ 1 + 3 + 2 & 1 + 9 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(formula)}$$

$$= \frac{1}{3\times14 - 36} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$= \begin{bmatrix} 7/3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7/3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2+4+6 \\ 2+12+12 \end{bmatrix}$$

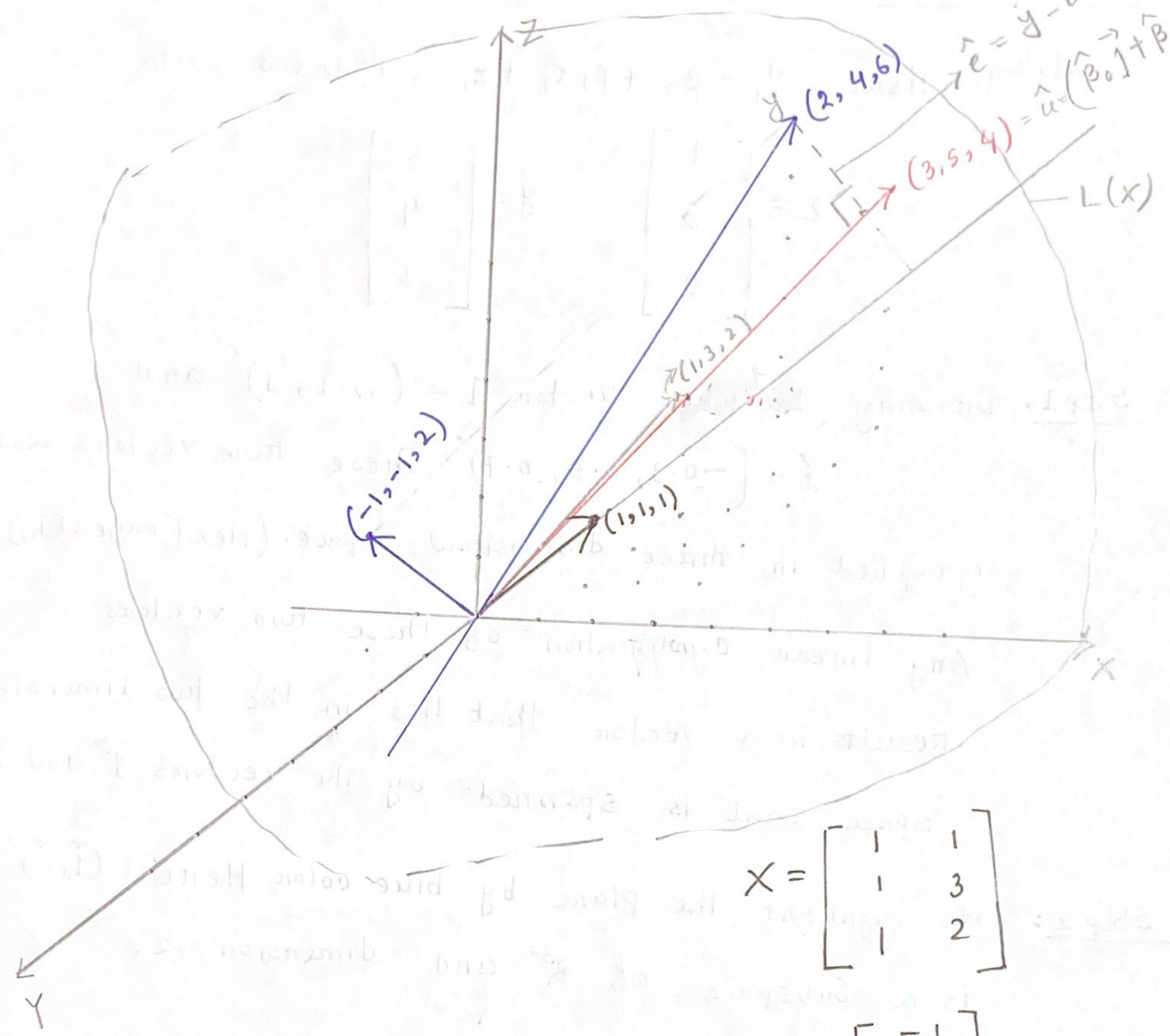
$$= \begin{bmatrix} 7/3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{3}\times 12 - 26 \\ -12 + \frac{1}{2}\times 26 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$$Y = X \quad \beta + \varepsilon$$



$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$e = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\hat{u} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

Problem - 4.5:

Solution: Here $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$

on 15 cases

MSE $s^2 = 3$

$$(X'X)^{-1} = \begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.6 \\ 0.3 & 6.0 & 0.5 & 0.4 \\ 0.2 & 0.5 & 0.2 & 0.7 \\ 0.6 & 0.4 & 0.7 & 3.0 \end{bmatrix}$$

(a) estimate, $V(\hat{\beta}_i) = \sigma^2 V_{ii}$

$$V(\hat{\beta}_1) = s^2 V_{11} = 3 \times 6.0 = 18$$

(b) estimate $\wedge$ $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = s^2 V_{13} = 3 \times 0.4 = 1.2$

(c) estimate $\wedge$ $\text{Corr}(\hat{\beta}_1, \hat{\beta}_3) = \dfrac{V_{13}}{(V_{11} \; V_{33})^{1/2}} = \dfrac{0.4}{(6.0 \times 3.0)^{1/2}} = 0.0943$

(d) estimate $\wedge$ $V(\hat{\beta}_1 - \hat{\beta}_3) = V(\hat{\beta}_1) + V(\hat{\beta}_3) - 2\,\text{Cov}(\hat{\beta}_1, \hat{\beta}_3)$

$$= s^2 V_{11} + s^2 V_{33} - 2 \times 1.2$$

$$= 3 \times 6.0 + 3 \times 3.0 - 2 \times 1.2$$

$$= 24.6$$

4.6

$\because \quad E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \qquad n = 15$ cases

$\hat{\beta}_0 = 10, \ \hat{\beta}_1 = 12, \ \hat{\beta}_2 = 15, \ s^2 = 2$

$$(X'X)^{-1} = \begin{bmatrix} 1 & 0.25 & 0.25 \\ 0.25 & 0.5 & -0.25 \\ 0.25 & -0.25 & 2 \end{bmatrix}$$

Solution:

(a) Estimate $V(\hat{\beta}_2)$:

$$V(\hat{\beta}_2) = \sigma^2 V_{22} = \sigma^2 \times 2 = 2\sigma^2$$

As they said estimate, so, $V(\hat{\beta}_2) = 2 \times 2 = 4$

(b) Test of hypothesis $\beta_2 = 0$:

Here, $\hat{\beta}_2 = 15$

$$s.e.(\hat{\beta}_2) = s\sqrt{V_{22}} = \sqrt{2} \times \sqrt{2} = 2$$

So, the t- statistics for $\hat{\beta}_2$ is

$$t_0(\hat{\beta}_2) = \frac{\hat{\beta}_2 - 0}{s.e(\hat{\beta}_2)}$$

$$= \frac{15 - 0}{2} = 7.5$$

Here the Subscript of t_0 (zero) indicates that we test the hypothesis $\beta_2 = 0$.

Here we have 15 cases (15 observation) and

DF = 12. (As there are 3 parameter so, DF = 15 − 3 = 12)

The probability value of this test statistics for a two-sided alternative ($\beta_2 \neq 0$) is given by:

$$P(|T| > 7.5) = 2P(T > 7.5)$$
$$\simeq 2.417 \times 10^{-8} \text{ [Using R]}$$

Here t distribution with 12 degree of freedom.

The probability is very small - smaller than ~~th~~ any reasonable significance level.

Thus, there is very strong evidence that $\hat{\beta}_2$ differ from 0.

(c) $Cov(\hat{\beta}_1, \hat{\beta}_2) = \sigma^2 V_{ij}$

estimate,

$$Cov(\hat{\beta}_1, \hat{\beta}_2) = s^2 V_{12}$$
$$= 2 \times (-0.25)$$
$$= -0.5$$