UNIT - 7

PARTIAL ORDERING

7.1 Partial Order Relation.

7.2 Partial Ordered Set (POSET)

7.3 Representation of POSET, Construction of Hasse Diagram

7.4 Chains, Anti-Chains, Maximal and Minimal Elements.

7.5 Upper Bound, Lower Bound, Least Upper Bound (LUB), Greatest Lower Bound (GLB)

7.6 Lattice, Lattice Operators, Sub lattice, Properties of Lattices

7.7 Types of Lattices: Bounded Lattice, Distributive Lattice, Complemented Lattice.

7.8 Boolean Algebra.

* Partially Ordered Relation.

* Definitions:

⇒ Partially ordered relation: A relation R on a set A is called partial order if R is Reflexive, anti-symmetric and transitive poset.

⇒ The set A together with the partial order R is called a partially ordered set or poset. It is denoted by (A, R)

⇒ A relation R on a set A is called a partial order relation iff R is

i) Reflexive relation

$a R a \ , \ \forall a \in A$

i.e. $(a, a) \in R \ , \ \forall a \in A$

ii) Antisymmetric relation

if aRb and bRa then $a = b$

i.e. $(a, b) \in R, (b, a) \in R \Rightarrow a = b$, where $a, b \in A$

iii) Transitive relation

aRb and bRc then aRc

i.e. $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$, where $a, b, c \in A$

Ex1 N is a set of natural numbers. R is a relation defined on set N.

$R = \{(a,b) ; a \text{ divides } b \text{ or } a | b\}$.

aRb iff $a|b$; a is related with b iff a divides b.

Solution :-

(i) Reflexive Relation : aRa, $\forall a \in R$

$\therefore aRa$

$\therefore a|a$, $\forall a \in N$.

$(a,a) \in R$

$\therefore$ R is reflexive relation.

(ii) Antisymmetric Relation : aRb and bRa then $a=b$.

$a, b \in N$ and

if $a|b$ and $b|a$ then $a=b$.

$\therefore$ R is antisymmetric relation

(iii) Transitive Relation: aRb and bRc then aRc

$a, b, c \in N$

if $a|b$ and $b|c$ then $a|c$

i.e. $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$.

$\therefore$ R is transitive relation.

Hence, Being reflexive, anti-symmetric and transitive, then R is a partial order relation

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Ex2 Show that the relation "$\geq$" is a partial
Q 3.18 ordering on the set of integers. or Show that $(Z, \geq)$ is a poset

Solution (1) Reflexive Relation: aRa ; $\forall a \in R$.

$\therefore a \geq a$ $\forall$ integer a

$\therefore \geq$ is a reflexive.

(ii) Anti-symmetric relation: $a R b$ and $b R a$ then $a = b$

If $a \geq b$ and $b \geq a$; then $a = b$.

Hence, $\geq$ is antisymmetric.

(iii) Transitive relation: $a R b$ and $b R c$ then $a R c$

If $a \geq b$ and $b \geq c$ then $a \geq c$.

Hence, $\geq$ is transitive.

$\therefore$ $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

Ex 3 Define the relation R on the set (Z) by
QB 821
aRb if $a-b$ is non-negative even integer.
Verify that R defines a partial order for Z.
Is this partial order a total order?

Solution.

(i) R is Reflexive :

$\therefore a - a = 0$

$\therefore a R a$

(ii) R is Antisymmetric :-

if $a \neq b$ then $a \not R b$ or $b \not R a$.

Let $a - b = 2n$ where n = integer no. if bRa, then

Since $a \neq b$ $a-b=2n$ but $b-a=-2n \notin R$

$a \neq b$

$b - a = -2n$

$b \not R a$

(iii) R is Transitive :-

Let $a R b$ and $b R c$

ie. $a - b = 2n_1$ and $b - c = 2n_2$.

$\Rightarrow a - \cancel{b} + \cancel{b} - c = 2n_1 + 2n_2$

$\Rightarrow a - c = 2(n_1 + n_2)$

$=$ even positive integer

Hence, R is partial order.

Ex4 Draw the digraph for the following relation
QB
Q20 and determine whether the relation is reflexive, symmetric, transitive and antisymmetric
$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $x R y$ whenever
y is divisible by x.

Solution $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8)$
$(2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4)$
$(4,8), (5,5), (6,6), (7,7), (8,8)\}$.

(i) R is Reflexive: $a R a$, $\forall a \in A$.

$\therefore (a,a) \in R$, $\forall a \in A$.

$\therefore$ R is reflexive

(ii) R is Symmetric : $a R b$ then $b R a$, $\forall a, b \in A$.

$\therefore (1,2) \in R$ but $(2,1) \notin R$
(as 2 is divisible by 1) (as 1 is not divisible by 2)

$\therefore$ R is not symmetric.

(iii) R is transitive : $a R b$ and $b R c$ then $a R c$.
$\forall a, b, c \in A$.

Let. $(x, y), (y, z) \in R$

$\Rightarrow$ y is divisible by x and z is divisible by y

$\Rightarrow$ $\frac{y}{x} = a$ and $\frac{z}{y} = b$

$\Rightarrow$ $\frac{y}{x} \cdot \frac{z}{y} = a \cdot b$

$\Rightarrow$ $\frac{z}{x} = a \cdot b$

$\Rightarrow$ z is divisible by x.

$\Rightarrow$ $(x, z) \in R$.

$\therefore$ R is transitive.

(iv) R is antisymmetric : a R b and b R a then a = b.

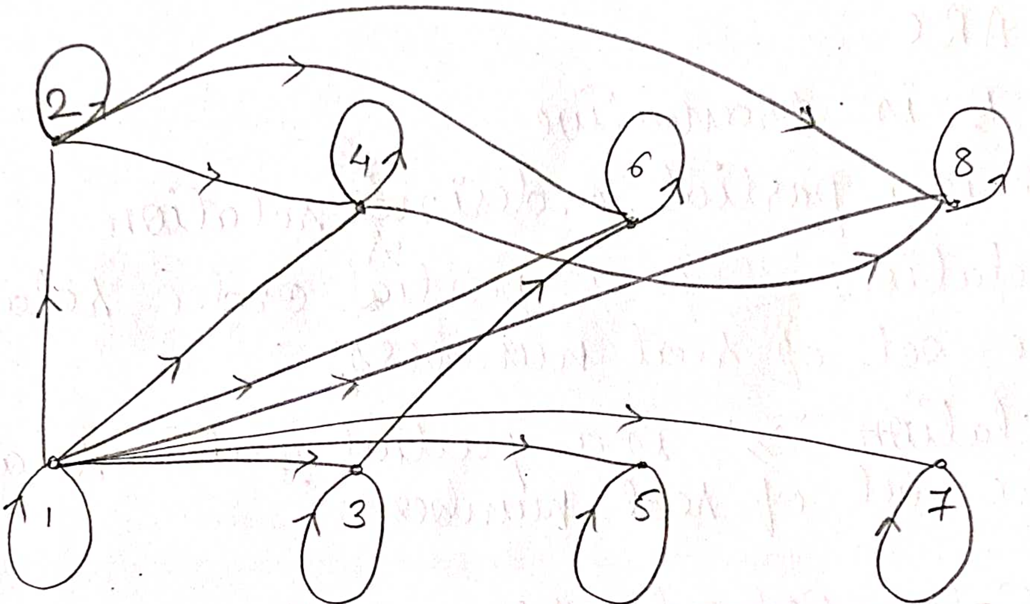
Let x R y and y R x

⇒ y is divisible by x and x is divisible by y.

⇒ Since x and y are integers. if x = y.

∴ R is antisymmetric.

Hence, R is partial order relation and A is partial order set.



Ex 5. Let $S = \{1, 2, 3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

R is a relation defined on set P(S).

A R B iff $A \subseteq B$.

then R is partial ordering relation on P(S)

Solution: (1) R is Reflexive

Every set is subset of itself

$A \subseteq A$.

⇒ A R A ; ∀ A ∈ P(S)

Hence, R is reflexive relation.

(ii) R is antisymmetric

Let ARB and BRA

$\Rightarrow A \subseteq B$ and $B \subseteq A$

$\Rightarrow A = B$.

Hence, R is antisymmetric relation

(iii) R is transitive

Let ARB and BRC

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow ARC$

Hence, R is transitive

$\therefore$ R is a partial ordering relation

* The relation '$\leq$' is a partial order relation on the set of real numbers.
* The relation '$\geq$' is a partial order relation on the set of real numbers.

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* <u>Partially Ordered Set or POSET.</u>

If A is any non-empty set and R is a partial ordered relation on set A, then the ordered pair (A, R) is called partially ordered set or poset.

Examples

1. S is a non-empty set and $P(S)$ is a power set of S the relation $\subseteq$ is partial order relation on set $P(S)$.

   Hence, $(P(S), \subseteq)$ is known as Poset.

2. aRb iff $a|b$ (a divides b) is a partial order relation on set of natural numbers set.

   Hence, (N, R) is a poset.

* Hasse Diagram

Poset can be represented by digraphs.
A simpler way of representating poset is Hasse diagram.

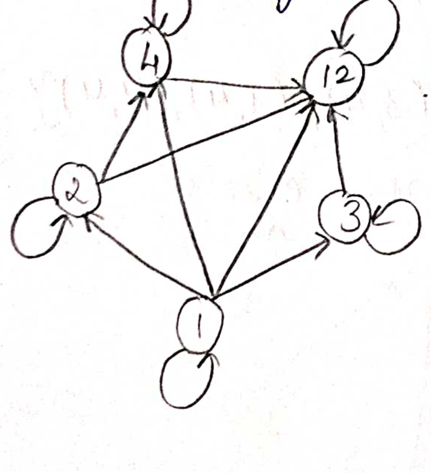
Method to find Hasse diagram

1. Omitt loops as relation is reflexive on poset.
2. All arrows that appear on the edges are omitted.
3. Eliminate all edges that are implied by transitive relation.
   eg if aRb, bRc then aRc, so (a,c) eliminate.
4. An arc pointing upward is drawn from a to b if $a \neq b$ and aRb.

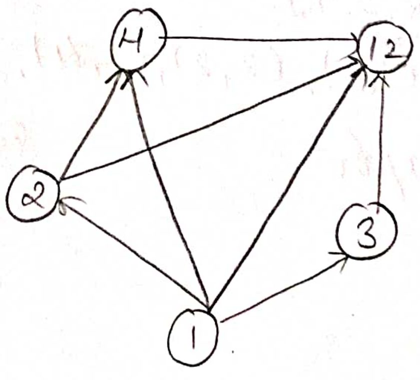
Ex1 QB 327 Draw Hasse digram for the following relations on set $A = \{1,2,3,4,12\}$.

$R = \{(1,1),(2,2),(3,3),(4,4),(12,12),(1,2),(4,12),(1,3),(1,4),(1,12),(2,4),(2,12),(3,12)\}$.

Solution Digraph



Step-1 Remove reflexive pairs



Step-2 Remove Transitive pairs

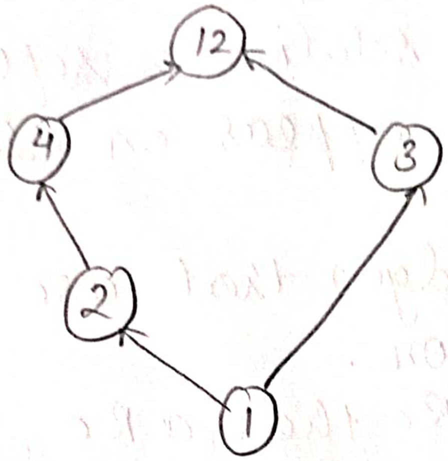
1R2 , 2R4 ⇒ 1R4

2R4 , 4R12 → 2R12

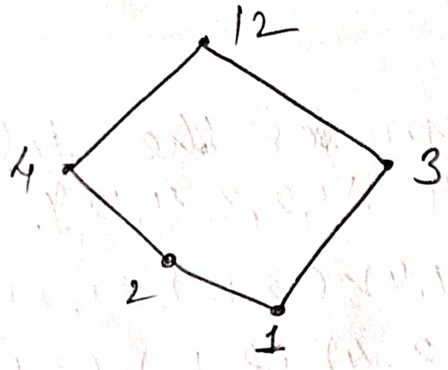
1R4 , 4R12 ⇒ 1R12

Eliminated edges (1, 4), (2, 12), (1, 12).

all arrows are pointing upwards.



Step-3: Make sure that all edges are pointing upwards, then remove arrows from edges, replace circles by dots.

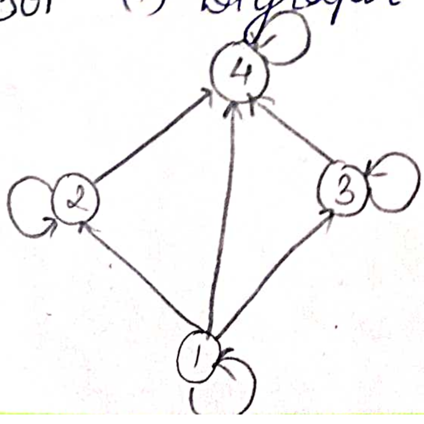


Ex 2 Determine the Hasse Diagram of the relation R

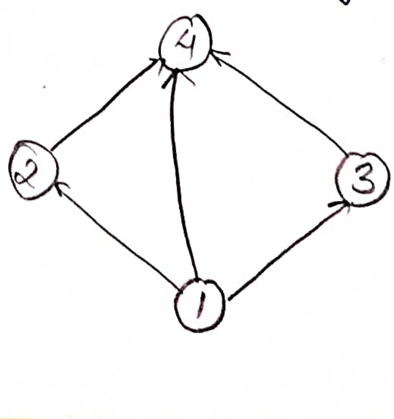
(i) $A = \{1, 2, 3, 4\}$.

$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$.

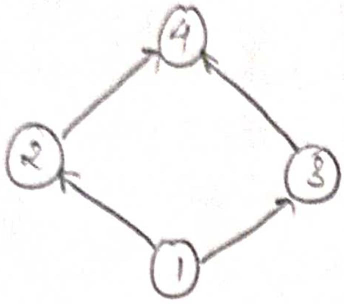
Soln (i) Digraph.



Step-1 · Remove cycles

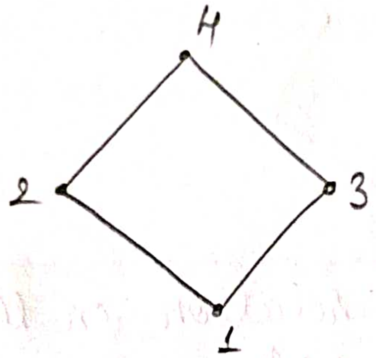


Step-2 Remove transitive edge
(1, 4)



Step-3 Make sure that all edges are pointing upwards, then remove arrows from edges, replace circles by dots.

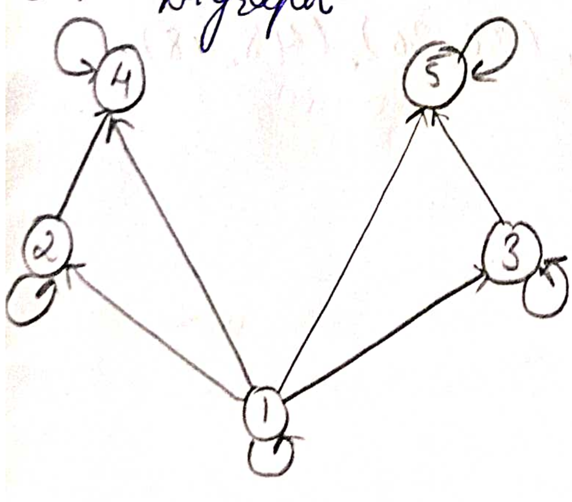
Hasse diagram



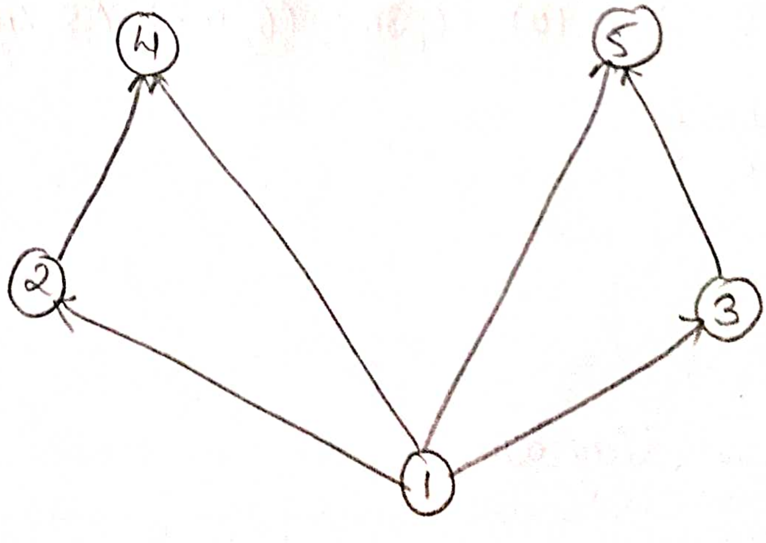
(2) $A = \{1, 2, 3, 4, 5\}$

QB 330 $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,4), (3,5), (2,2), (3,3), (4,4), (5,5)\}$.

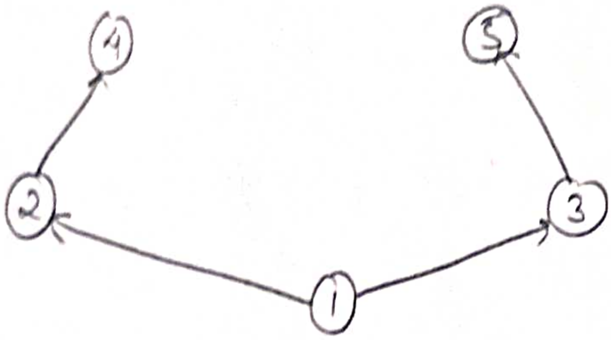
Sol^n Digraph



Step-1 Remove cycles.

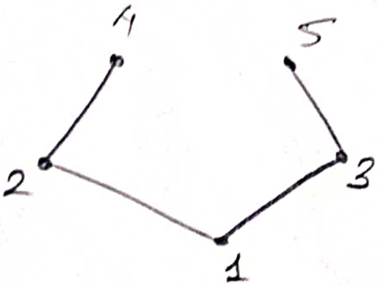


Step-2 Remove transitive edge (1, 4), (1, 5)



Step-3 All edges are pointing upwards. remove arrows from edges replace circles by dots.

Hasse diagram:



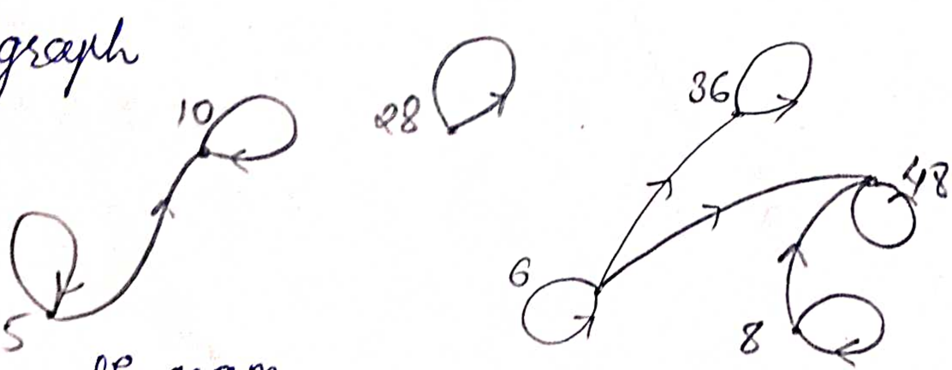
Ex 3 Let R be the relation on the set A.
QB 322 $A = \{5, 6, 8, 10, 28, 36, 48\}$

Let $R = \{(a, b) \mid a \text{ is divisor of } b\}$. Draw Hasse diagram. Compare with digraph. Determine whether R is equivalence relation.

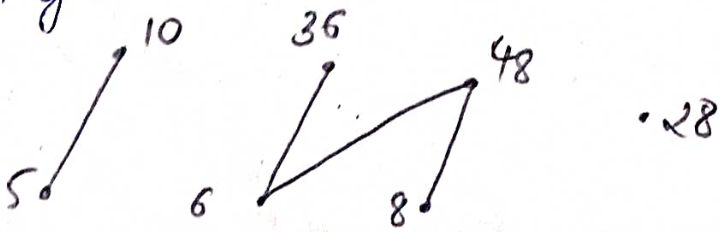
<u>Solution</u> $A = \{5, 6, 8, 10, 28, 36, 48\}$

$R = \{(5,5), (6,6), (8,8), (10,10), (28,28), (36,36), (48,48),$
$(5,10), (6,36), (6,48), (8,48)\}$

Digraph



Hasse diagram



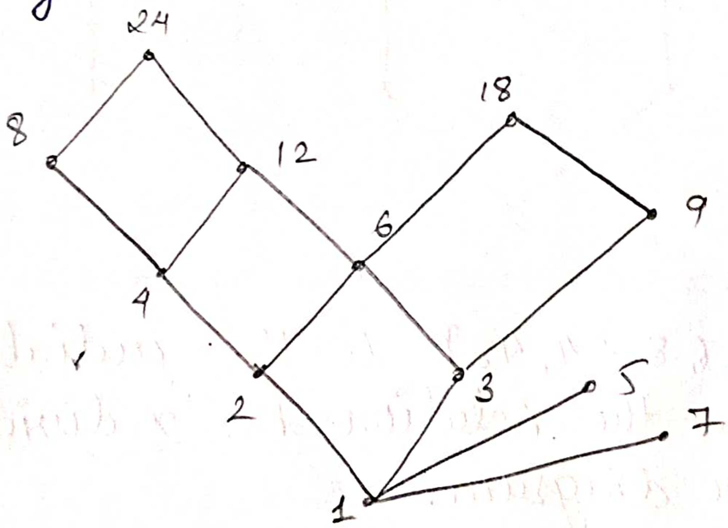
Here R is reflexive relation but not symmetric relation.
Hence, R is not equivalence relation.

Ex 4 WB 323 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$ be ordered by the relation x divides y. Show that the relation is partial ordering and draw the Hasse diagram.

Solution:- Here $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,12), (1,18), (1,24), (2,2), (2,4), (2,8), (2,12), (2,18), (2,24), (3,3), (3,6), (3,9), (3,12), (3,18), (3,24), (4,4), (4,8), (4,12), (4,24), (5,5), (6,6), (6,12), (6,18), (6,24), (7,7), (8,8), (8,24), (9,9), (9,18), (12,12), (12,24), (24,24)\}$.

Hasse diagram:-



(i) R is reflective relation:- aRa, $\forall a \in A$.
Every natural number is divisible by itself.

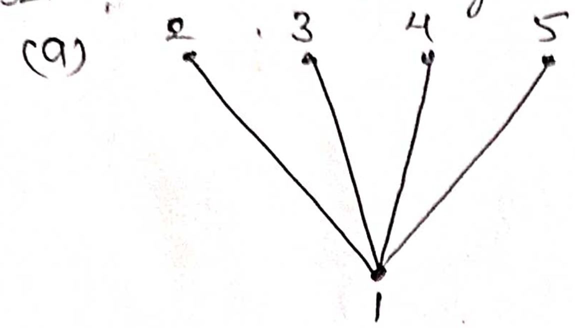
(2) R is antisymmetric relation; aRb and bRa then $a=b$
if $a|b$ and $b|a$ then $a = b$.

(3) R is transitive relation: aRb and bRc then aRc
if $a|b$ and $b|c$ then $a|c$

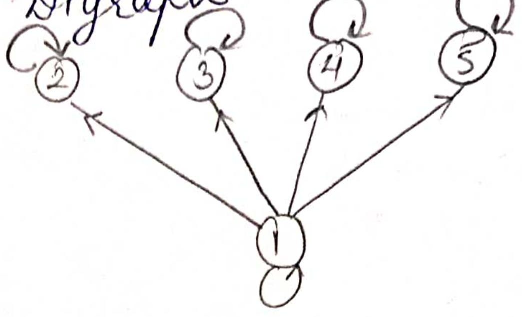
Hence, R is a partial ordering relation

Ex 11 Determine the matrix of the partial order whose Hasse diagram is given.
QB 332

(a)



Digraph

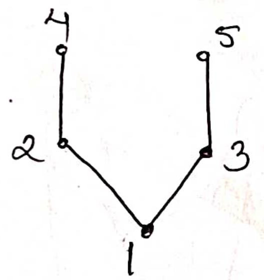


Relation set for above digraph is

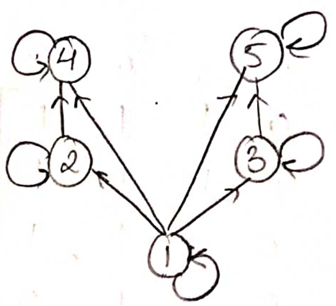
$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4), (1,5)\}$

Matrix $M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$

(b)



Digraph



Relation set for above digraph is

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,4), (1,4), (1,3)$
$(3,5), (1,5)\}$

Matrix

$M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$

Ex12 Draw Hasse Diagram for the partial ordering

QB 325. $\{(A,B) \mid A \subseteq B\}$ on the power set $P(S)$, where $S=\{a,b,c\}$.

Solution. $R=\{(A,B) / A \subseteq B\}$.

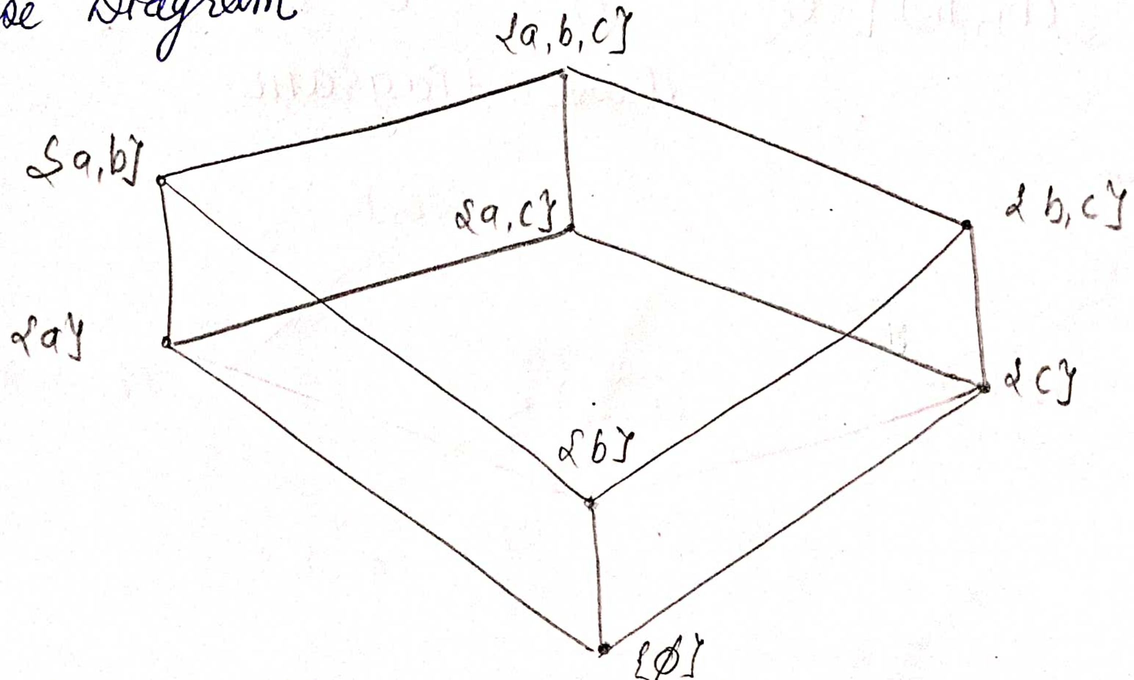
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$R = \{\{\{\phi\},\{\phi\}\}, \{\{\phi\},\{a\}\}, \{\{\phi\},\{b\}\}, \{\{\phi\},\{c\}\}, \{\{\phi\},\{a,b\}\},$
$\{\{\phi\},\{a,c\}\}, \{\{\phi\},\{b,c\}\}, \{\{\phi\},\{a,b,c\}\}, \{\{a\},\{a,b\}\},$
$\{\{a\},\{a,c\}\}, \{\{b\},\{a,b\}\}, \{\{b\},\{b,c\}\}, \{\{c\},\{a,c\}\},$
$\{\{c\},\{b,c\}\}, \{\{a,b\},\{a,b,c\}\}, \{\{a,c\},\{a,b,c\}\},$
$\{\{b,c\},\{a,b,c\}\}, \{\{a,b,c\},\{a,b,c\}\}.$

Matrix M_R

$$M_R = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{c\} \\ \{a,b\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \phi & \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Hasse Diagram



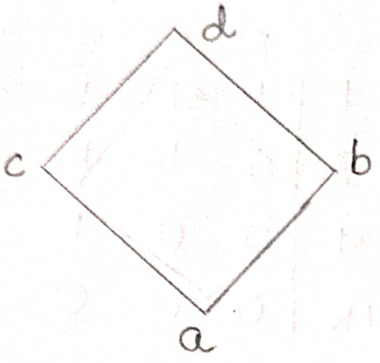
* Chain and Antichain

Let $(A, \leq)$ be a partially ordered set.

⇒ A subset of A is called a chain if every two elements in the subset are related.

⇒ A subset of A is called an antichain, if no two distinct elements in the subset are related.

Eg



$A = \{a, b, c, d\}$

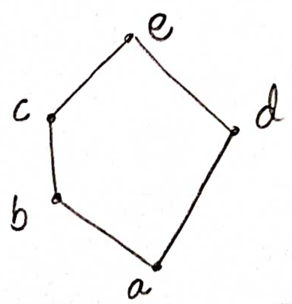
Chain: $\{a, b\}, \{b, d\}, \{a, d\}$.

Antichain: $\{a, c, b\}, \{c, b\}$

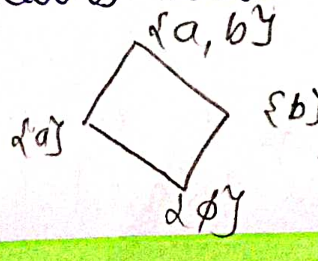
* Totally Ordered Set or Linearly Ordered Set

A partially ordered set $(A, \leq)$ is called a "totally ordered set" or linear ordered set or simple ordered set, if A is a chain.

Ex1 For partially ordered set mention $A = \{a, b, c, d, e\}$ we have $\{a, b, c, e\}, \{a, b, c\}, \{a, b\}, \{a, d, e\}, \{a, d\}, \{a\}$ are chains and $\{b, d\}, \{c, d\}$ are antichains.



Ex 2 Let $A = \{a, b\}$ and consider its poset $(P(A), \subseteq)$ then $\{\{\emptyset\}, \{a\}, \{a, b\}\}, \{\{\emptyset\}, \{b\}, \{a, b\}\}, \{\{\emptyset\}, \{a\}\}, \{\{\emptyset\}, \{b\}\}, \{\{a\}, \{a, b\}\} \{\{b\}, \{a, b\}\}$ are chains and $\{\{a\}, \{b\}\}$, is antichain



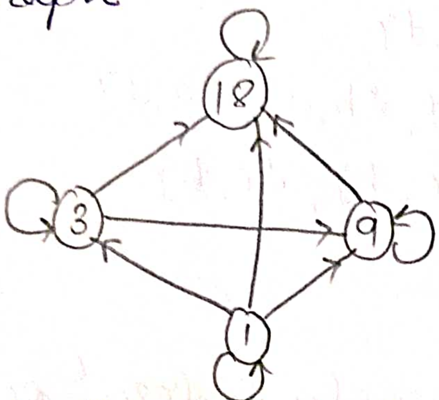
Ex1 Draw the Hasse diagram of the following sets under partial ordering relation "divides" and indicate those which are chains.

QB 333

a) $\{1, 3, 9, 18\}$

$R = \{(1,1), (1,3), (1,9), (1,18), (3,3), (3,9), (3,18), (9,9), (9,18), (18,18)\}.$

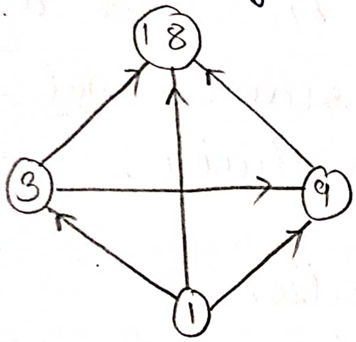
Digraph



Matrix

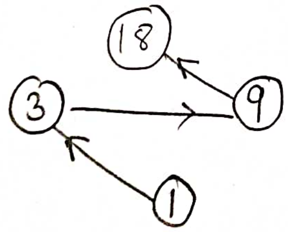
$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Step-1 Remove cycles

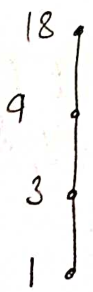


Step-2 Remove transitive

(1,18), (3,18), (1,9)



Step-3

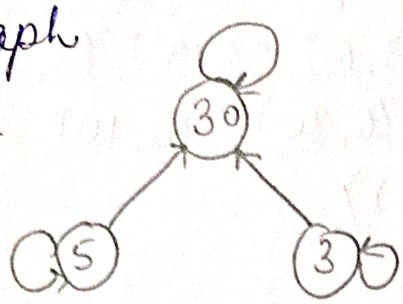


This poset is a chain because every two elements are related.

(b) $\{3, 5, 30\}$

$R = \{(3,3), (3,30), (5,5), (5,30), (30,30)\}$

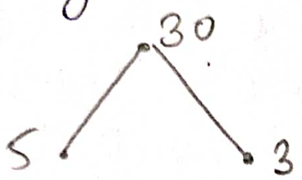
Digraph



Matrix

$$M_R = \begin{array}{c} \\ 3 \\ 5 \\ 30 \end{array}\begin{array}{c} \begin{array}{ccc} 3 & 5 & 30 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Hasse Diagram



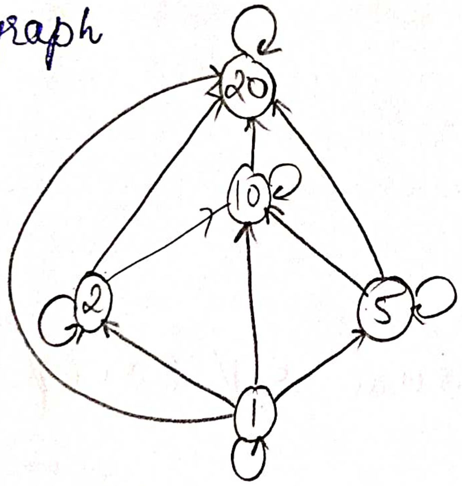
Given poset is not a chain because,

$3 \not R 5$ or $5 \not R 3$

(c) $\{1, 2, 5, 10, 20\}$

$R = \{(1,1), (1,2), (1,5), (1,10), (1,20), (2,2), (2,10), (2,20),$
$(5,5), (5,10), (5,20), (10,10), (10,20), (20,20)\}$

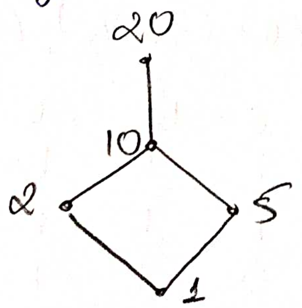
Digraph



Matrix

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 10 \\ 20 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 5 & 10 & 20 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Hasse Diagram



Given poset is not a chain because

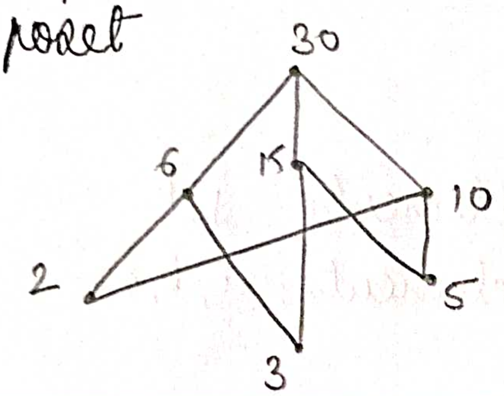
$2 \not R 5$ or $5 \not R 2$.

* Maximal and Minimal Elements

An element $a \in A$ is called a maximal element of A, if there is no element c in A such that $a < c$

An element $b \in A$ is called a minimal element of A, if there is no element c in A such that $c < b$

Example: $A = \{2, 3, 6, 5, 10, 15, 30\}$ Hasse diagram of poset

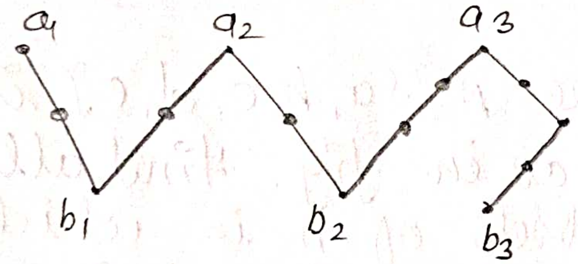


Maximal element = 30

Minimal element = 2, 3, 5

Examples

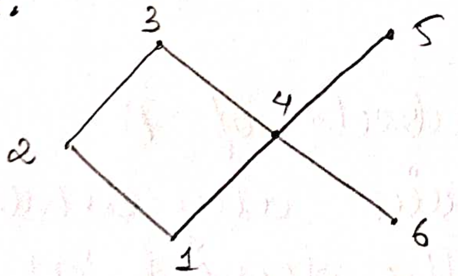
1.



Maximal element: a_1, a_2, a_3

Minimal elements: b_1, b_2, b_3

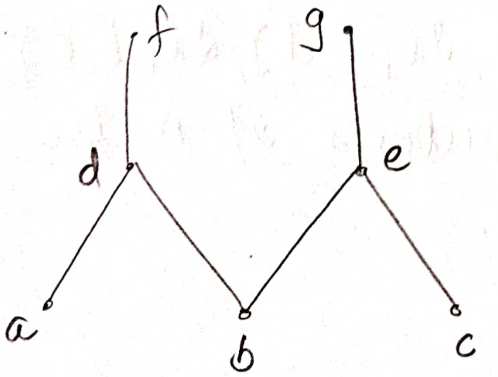
2.



Maximal elements : 3, 5

Minimal elements : 1, 6

3.



Maximal elements : f, g

Minimal elements : a, b, c

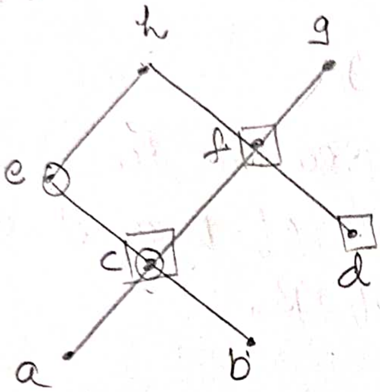
* Upper Bounds and Lower Bounds.

Let $(A, \leq)$ be a poset, and a subset B of A.

⇒ An element $a \in A$ is called an upper bound of B if $b \leq a \ \forall \ b \in B$.

⇒ An element $a \in A$ is called an lower bound of B if $a \leq b$, $\forall \ b \in B$.

Example



Let $B = \{a, b, c, d, e, f, g, h\}$

(i) $B_1 = \{e, c\}$

Upper bound $= \{h, e\}$

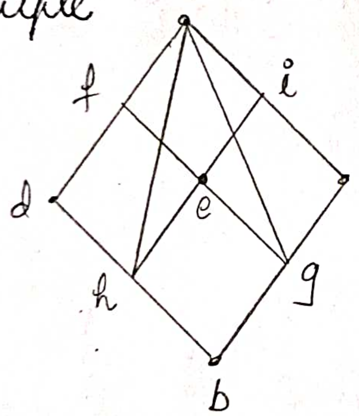
Lower bound $= \{a, b, c\}$.

(ii) $B_2 = \{c, f, d\}$

Upper bound $= \{h, g, f\}$

Lower bound $= \{\phi\}$

Example



Let $B = \{a, b, c, d, e, f, g, h, i\}$

(i) $B_1 = \{e, f\}$

Upper bound $= \{f, a, i, e\}$

Lower bound $= \{b, g\}$

(ii) $B_2 = \{h, e, g, c\}$

Upper bound $= \{a, i\}$

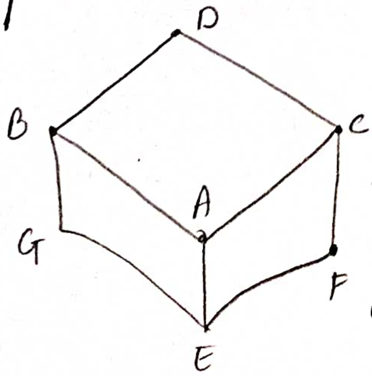
Lower bound $= \{b\}$

(iii) $B_3 = \{a\}$

Upper bound $= \{a\}$

Lower bound $= \{f, d, i, e, h, b, g, c\}$

Example



Let $B = \{A, B, C, D, E, F, G\}$.

(i) $B_1 = \{B, D\}$ Upper $= \{D\}$
Lower $= \{E, G, B, A\}$

(ii) $B_2 = \{A, C\}$ Upper $= \{D, C\}$
Lower $= \{E, A\}$

(iii) $B_3 = \{B, C\}$ Upper $= \{D\}$
Lower $= \{A, E\}$

Supremum or Join or $\vee$ join

* Least Upper Bound (LUB) $\vee$

- Let A be a poset and B be a subset of A.

An element $a \in A$ is called a least upper bound (LUB) of B if a is an upperbound of B and $a \leq a'$, whenever a is an upper bound of B.

Thus, $a = LUB$

(B) if $b \leq a$ for all $b \in B$ and if whenever $a' \in A$ is also an upper bound of B, then $a' \leq a$.

Infimum or Meet or $\wedge$

* Greatest Lower Bound (GLB) meets $\wedge$

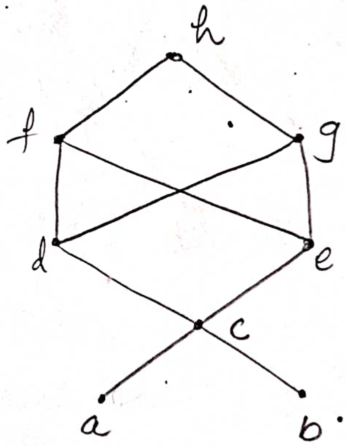
An element $a \in A$ is called a greatest lower bound (GLB) of B if a is a lower bound of B and $a' \leq a$ whenever, a' is a lower bound of B.

Thus, $a = GLB$

(B) if $a \leq b$ for all $b \in B$ and if whenever $a' \in A$ is also a lower bound of B, then $a' \leq a$

Examples: 1 $\quad A = \{a, b, c, d, e, f, g, h\}$.

$B_1 \subseteq A$, $B_2 \subseteq A$.



(i) $B_1 = \{a, b\}$

Upper bound $= \{c, d, e, f, g, h\}$

Lower bound $=$ none or $\{\phi\}$

Least upper bound $= \{c\}$

Greatest lower bound $=$ none or $\{\phi\}$

(ii) $B_2 = \{c, d, e\}$.

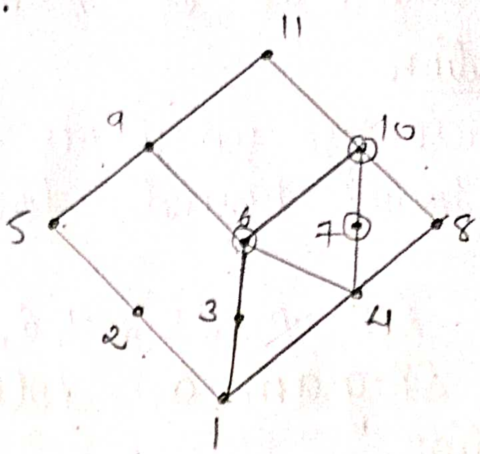
Upper bound $= \{f, g, h\}$

Lower bound $= \{c, a, b\}$

Least upper bound $=$ None or $\{\phi\}$

Greatest Lower bound $= \{c\}$

Ex:2 Let $A = \{1, 2, 3, 4, 5, \dots, 11\}$ be the poset whose Hasse diagram is shown. Find LUB & GLB of $B = \{6, 7, 10\}$, if they exist.



Solution

Upper bound = 10, 11

Least upper bound = 10

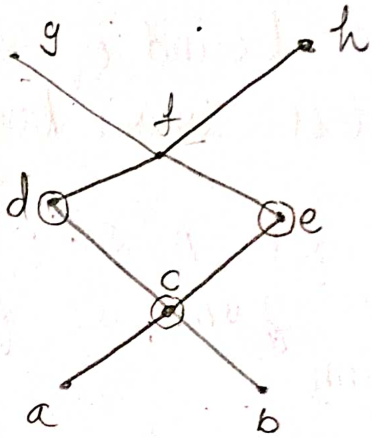
Lower bound = 4, 1

Greatest lower bound = 4

Ex 3 Let $A = \{a, b, c, d, e, f, g, h\}$ be the poset whose Hasse diagram is below. Find GLB & LUB of

QB 335

$B = \{c, d, e\}$



Solution

Upper bound = f, g, h

Least upper bound = f

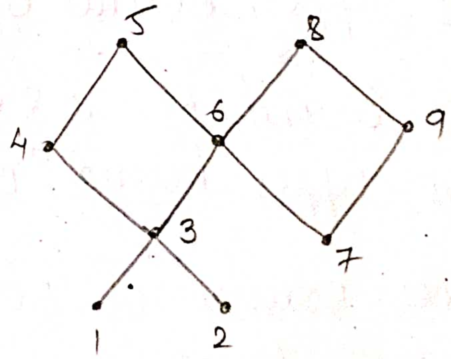
Lower bound = c, a, b.

Greatest lower bound = c

Ex4. Let A be poset whose Hasse diagram is

QB 336 shown in fig $A = \{1, 2, \dots, 9\}$. Find GLB & LUB.

of set $B = \{3, 4, 6\}$



Solution

Upper bound = 5

Least upper bound = 5

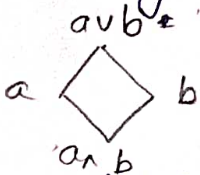
Lower bound = 3, 1, 2

Greatest lower bound = 3

$a \wedge b = a$
$a \vee b = b$

* Lattice

A lattice is a poset $(A, \leq)$ in which every subset $\{a, b\} \in A$ has LUB and GLB.



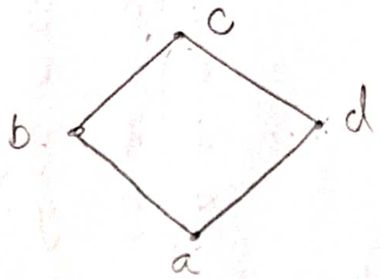
we denote :

LCM LUB $(\{a,b\})$ by $a \vee b$ (call join of a and b)

GCD GLB $(\{a,b\})$ by $a \wedge b$ (call meet of a and b)

Ex1 Determine whether the following Hasse diagram represent lattice or not.

(a)



LUB

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

GLB

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Hence, it is lattice

(b)

LUB

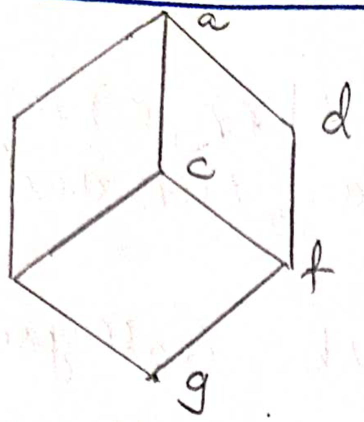
$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

GLB

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Hence, this is lattice. because every pair of elements has LUB and GLB

(c) QB
8H)



LUB

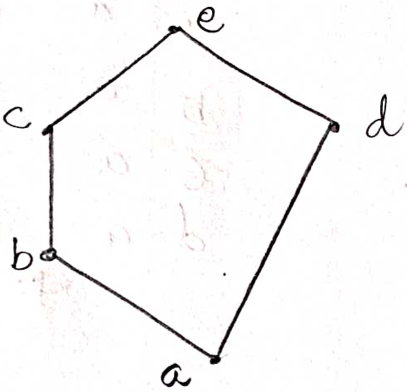
v	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

GLB

∧	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

Hence, it is lattice, because each subset of two elements has LUB and GLB.

(d)



LUB

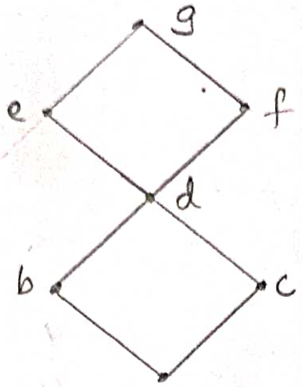
v	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	c	c	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

GLB

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	a	b
c	a	b	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

This is a lattice because every pair of elements has a LUB and GLB

(e)
QB
342



LUB

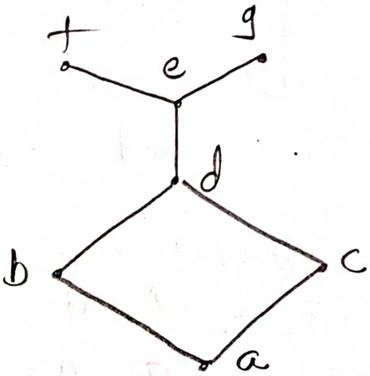
v	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

GLB

∧	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	d	e
f	a	b	c	d	d	f	f
g	a	b	c	d	e	f	g

Hence, it is lattice.

(f)
QB
343



LUB

v	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	–
g	g	g	g	g	g	–	g

GLB

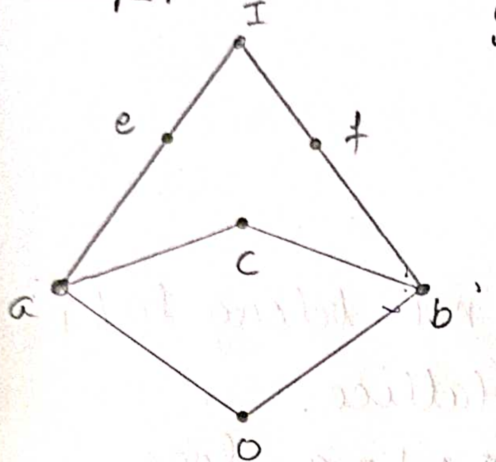
∧	a	b	c	d	e	f	g
a	a	d	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

Hence, it is not lattice, because $f \vee g$ does not exist

* Sublattice

Let $(L, \leq)$ be a lattice. A nonempty subset S of L' is called a sublattice of L, if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$

Example. Consider the lattice, determine whether or not is a sublattice of L.



(a) $B = \{a, b, e, f, I\}$

(b) $C = \{a, b, e, f, I, O\}$

(a) $B = \{a, b, e, f, I\}$.

LUB

$\vee$	a	b	e	f	I
a	a	$\boxed{c}$	e	I	I
b	$\boxed{c}$	b	I	f	I
e	e	I	e	I	I
f	I	f	I	f	I
I	I	I	I	I	I

GLB

$\wedge$	a	b	e	f	I
a	a	$\boxed{O}$	a	$\boxed{O}$	a
b	$\boxed{O}$	b	$\boxed{O}$	b	b
e	a	$\boxed{O}$	e	$\boxed{O}$	e
f	$\boxed{O}$	b	$\boxed{O}$	f	f
I	a	b	e	f	I

Hence, it is not sublattice.

(b) $C = \{a, b, e, f, I, O\}$

LUB

$\vee$	a	b	e	f	I	O
a	a	$\boxed{c}$	e	I	I	a
b	$\boxed{c}$	b	I	f	I	b
e	e	I	e	I	I	e
f	I	f	I	f	I	f
I	I	I	I	I	I	I
O	a	b	e	f	I	O

GLB

$\wedge$	a	b	e	f	I	O
a	a	O	a	O	a	O
b	O	b	O	b	b	O
e	a	O	e	O	e	O
f	O	b	O	f	f	O
I	a	b	e	f	I	O
O	O	O	O	O	O	O

Hence, it is not sublattice.

* Properties of Lattice

1. Idempotent Properties : $a \vee a = a$
   $a \wedge a = a$

2. Commutative Properties : $a \vee b = b \vee a$
   $a \wedge b = b \wedge a$

3. Associative Properties : $a \vee (b \vee c) = (a \vee b) \vee c$
   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties : $a \vee (a \wedge b) = a$
   $a \wedge (a \vee b) = a$

---

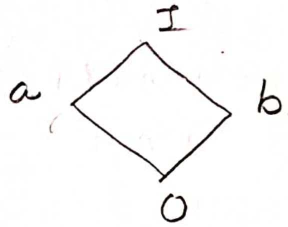
* Types of Lattices

1. **Bounded lattice**

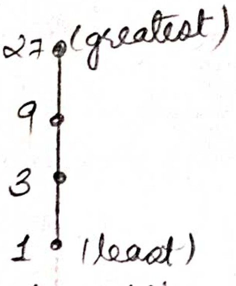
⇒ A lattice L is said to be bounded if it has a greatest element (I) and a least element (O).
If L is bounded lattices, then for all $a \in A$.

$$O \leq a \leq I.$$

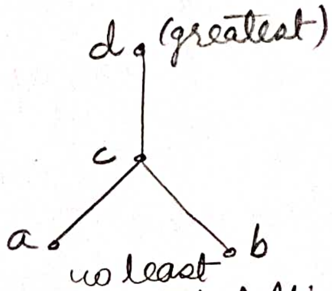
$a \vee O = a$, $a \wedge O = O$
$a \vee I = I$, $a \wedge I = a$



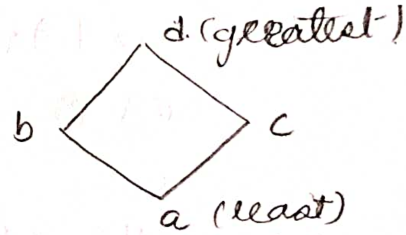
Eg



Bounded lattice



Not Bounded lattice



Bounded lattice

Eg $(P(S), \subseteq)$ ⇒ Least $= \emptyset$, Greatest $= S$.

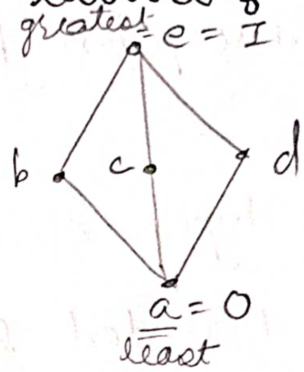
2. Distributive Lattice

⇒ A lattice L is said to be distributive lattice, if for every element $a, b, c \in L$.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Ex 1 Consider the following Hasse Diagram. Is the given lattice a distributive lattice?



(i) $LHS = b \vee (c \wedge d)$

$= b \vee a$

$= b$

(ii) $RHS = (b \vee c) \wedge (b \vee d)$

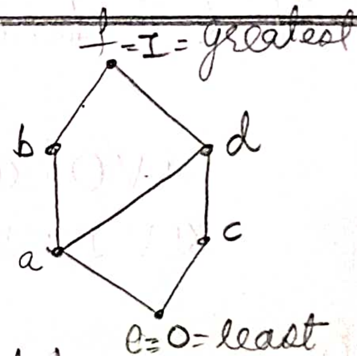
$= e \wedge e$

$= e$

$\therefore LHS \neq RHS$

∴ It is not distributive lattice

Ex 2



(i) $LHS = a \wedge (b \vee c)$

$= a \wedge I$

$= a$

(ii) $RHS = (a \wedge b) \vee (a \wedge c)$

$= a \vee O$

$= a$

$\therefore LHS = RHS$.

∴ It is distributive lattice.

Complement of b.

① d is not complement of b

$LUB(b,d) = f$, $GLB(b,d) = a \neq e$

② c is complement of b

$LUB(c,b) = f$ $GLB(c,b) = e$

Complement of d

① = b is not complement of d

$LUB(b,d) = f$, $GLB = a \neq e$

② a is not complement of d

$LUB(a,d) = d \neq f$, $GLB(a,d) = a \neq e$

⇒ Complement of d does not exist.

"A lattice L is said to be a distributive lattice if every element in L has "atmost one complement" (0 or 1)

Complement of b

⇒ c is the complement of b

$LUB(c,b) = e$

$GLB(c,b) = a$

⇒ d is the complement of b

$LUB(d,b) = e$

$GLB(d,b) = a$

⇒ b has two complements

⇒ It is not a distributive

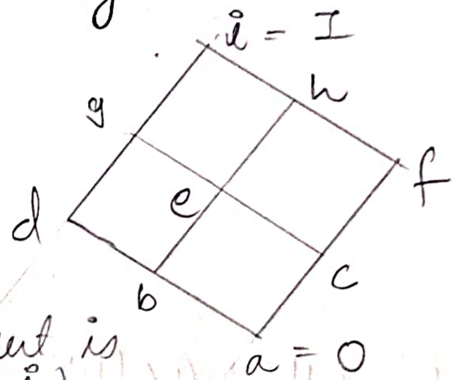
3. Complemented Lattice

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$. An element $a' \in L$ is called a complement of a if

$$a \vee a' = I \qquad a \wedge a' = 0$$

A Lattice L is called complemented if it is bounded and if every element in L has a complement. (at least one complement) (1 or more than one)

Ex1 Consider the following Hasse Diagram. Is the given lattice a complemented lattice?



Sol^n given lattice is a bounded lattice.

$\Rightarrow$ Greatest element $= i$ (every element is related to i)
Least element $= a$ (" " to a)

Now, lets check whether the given lattice is a complemented lattice or not.

$\Rightarrow$ Complement of d

① e is not the complement of d because

$LUB(e,d) = g \neq i$

$GLB(e,d) = b \neq a$

② f is the complement of d because

$LUB(f,d) = i$

$GLB(f,d) = a$

Therefore, f is the complement of d and d is the complement of f.

⇒ Complement of e

① d is not the complement of e because

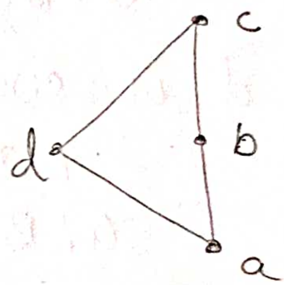
$$LUB(d,e) = g \neq i$$
$$GLB(d,e) = b \neq a$$

② f is not the complement of e because

$$LUB(f,e) = h \neq i$$
$$GLB(f,e) = c \neq a$$

Therefore, the complement of e does not exist
Hence, the given lattice is not a complemented lattice.

Ex2. Consider the following Hasse Diagram:



Is the above lattice a complemented lattice?

Solution

⇒ b is complement of d and
d is complement of b

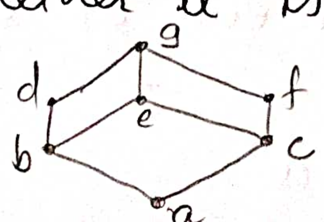
⇒ $LUB(b,d) = LUB(d,b) = c$
$GLB(b,d) = GLB(d,b) = a$

⇒ c is complement of a and
a is complement of c

⇒ $LUB(c,a) = LUB(a,c) = c$
$GLB(c,a) = GLB(a,c) = a$

Hence, the given lattice is a complemented lattice

Ex3 Determine whether the following Hasse diagram is a bounded lattice or not. If yes, then check whether it is complemented lattice.



Solution: the given lattice is a bounded lattice because it has the greatest and least element

Greatest element = g

least element = a

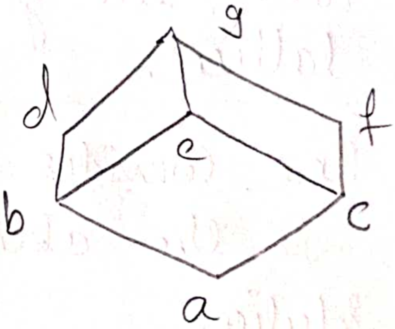
Let's check whether the given lattice is a complemented lattice or not.

⇒ Complement of d

① e is not the complement of d because

LUB(e,d) = g

GLB(e,d) = b ≠ a



② f is complement of d because

LUB(f,d) = g

GLB(f,d) = a

③ d is complement of c because

LUB(d,c) = g

GLB(d,c) = a

④ f is complement of b because

LUB(f,b) = g

GLB(f,b) = a

⑤ a is complement of g because

LUB(a,g) = g

GLB(a,g) = a

⑥ d is not the complement of e because

LUB(d,e) = g

GLB(d,e) = b ≠ a

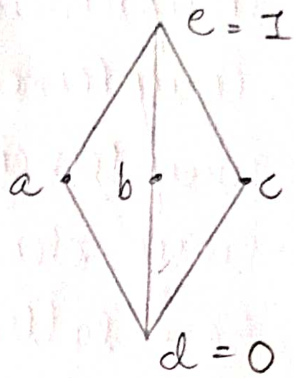
⑦ f is not the complement of e because

LUB(f,e) = g

GLB(f,e) = c ≠ a

Here, the complement of e does not exists.
Therefore, the given lattice is not a complemented lattice.

Ex 3 Show that lattice shown is complemented.



Solution. Greatest = e = I, Least = d = 0

$a \vee b = e = I$, $a \wedge b = d = 0$

∴ complement of a is b.
∴ Complement of b is a.

$a \vee c = e = I$. $a \wedge c = d = 0$

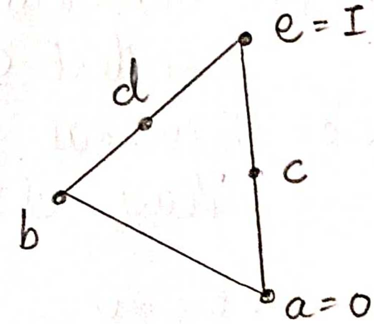
∴ Complement of a is c
∴ Complement of c is a

$b \vee c = e = I$, $b \wedge c = d = 0$

∴ Complement of b is c
∴ Complement of c is b.

∴ Every element has a complement.
So, it is complemented lattice.

Ex 4 Consider the complemented lattice in fig
Give the complements of each elements.



Solution Greatest element = e = I
Least element = a = 0

⇒ $b \vee c = e$, $b \wedge c = a$
∴ Complement of b is c.
∴ Complement of c is b.

⇒ $d \vee c = e$, $d \wedge c = a$
∴ Complement of d is c
∴ Complement of c is d.

* Boolean Algebra

A Lattice 'L' is said to be Boolean Algebra, if it is complemented and distributive lattice

one or more comp.   0 or 1 pair comp.

NOTE :-

(1) Lattice contains 2^n elements, for $n \geq 0$.

(2) Lattice contains least element and greatest element

Eg

Ex1 Determine whether the following posets is
QB Boolean algebras. Justify your answer.
352 $A = \{1, 2, 3, 6\}$ with divisibility.

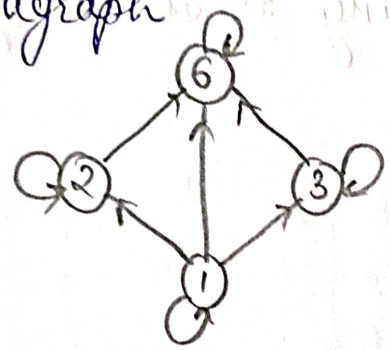
Solution given set $A = \{1, 2, 3, 6\}$.

Partial order relation of divisibility on set A is

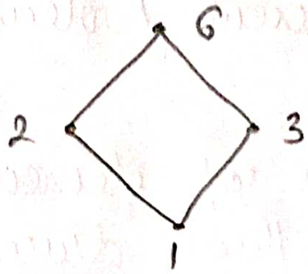
$R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph



Hasse diagram.



LUB

$\vee$	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB

$\wedge$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Every pair of elements in A has GLB & LUB
Therefore, A is a lattice.

$\therefore$ A has least element $= 1 = 0$
& a greatest element $= 6 = I$.

$\therefore$ Number of elements in A is $4 = 2^2$

$\Rightarrow$ Complement of 2 is 3 and 3 is 2.

$2 \vee 3 = 6 = I$, $2 \wedge 3 = 1 = 0$

∴ A is a complement lattice
⇒ Distributive lattice.

LHS $= 2 \vee (2 \wedge 3)$ RHS $= (2 \vee 2) \wedge (2 \vee 3)$

$= 2 \vee 1$ $= 2 \wedge 6$

$= 2$ $= 2$

∴ LHS = RHS

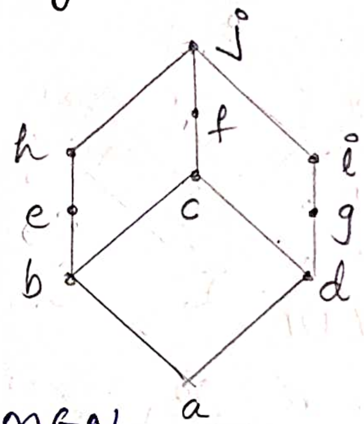
∴ A is distributive lattice.

∴ A satisfies all requirements of Boolean Algebra

∴ A under divisibility is a Boolean Algebra.

Ex2 Determine whether the following Hasse Diagram
QB 353 represent Boolean Algebra.

Solution



Number of elements in the above Hasse diagram is 10.

But every Boolean algebra having 2^n elements for some $n \in N$.

As $10 \neq 2^n$.

Hence, the Hasse diagram above does not represent Boolean algebra.

Ex3 Is D_{70} with the relation "divisibility" a
QB 354 Boolean algebra? Justify your answers.

Solution. Divisor of 70 are $A = \{1, 2, 5, 7, 10, 14, 35, 70\}$.

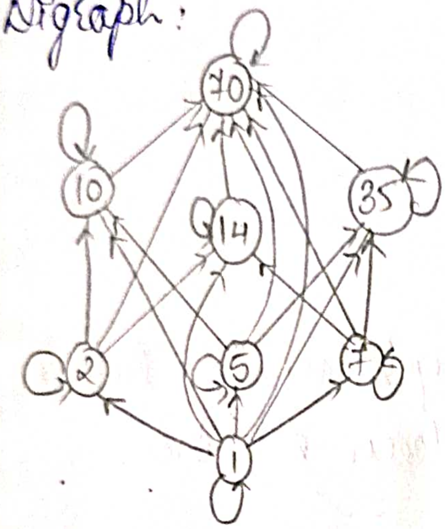
Partial order relation of divisibility on set A is

$R = \{(1,1), (1,2), (1,5), (1,7), (1,10), (1,14), (1,35), (1,70), (2,2), (2,10), (2,14), (2,70), (5,5), (5,10), (5,35), (5,70), (7,7), (7,14), (7,35), (7,70), (10,10), (10,70), (14,14), (14,70), (35,35), (35,70), (70,70)\}$

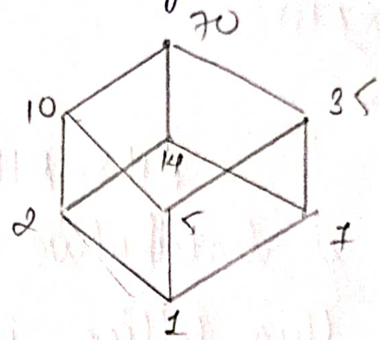
Matrix of the above relation is

$$M_R = \begin{array}{c|cccccccc} & 1 & 2 & 5 & 7 & 10 & 14 & 35 & 70 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 7 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 14 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 35 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Digraph:



Hasse Diagram



LUB

∨	1	2	5	7	10	14	35	70
1	1	2	5	7	10	14	35	70
2	2	2	10	14	10	14	35	70
5	5	10	5	35	10	70	35	70
7	7	14	35	7	70	14	35	70
10	10	10	10	70	10	70	70	70
14	14	14	70	14	70	14	70	70
35	35	35	35	35	70	70	35	70
70	70	70	70	70	70	70	70	70

GLB

∧	1	2	5	7	10	14	35	70
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
5	1	1	5	1	5	1	5	5
7	1	1	1	7	1	7	7	7
10	1	2	5	1	10	2	5	10
14	1	2	1	7	2	14	7	14
35	1	1	5	7	5	7	35	35
70	1	2	5	7	10	14	35	70

Every pair of elements in A has GLB and LUB.
therefore A is a lattice

A has least element (0) = 1
and a greatest element (I) = 70
Number of elements in A is $8 = 2^3$

$2^c = 35$ $7^c = 10$ $5^c = 14$
$35^c = 2$ $10^c = 7$ $14^c = 5$

⇒ Complement of 2.

$2 \vee 35 = 70 = I$, $2 \wedge 35 = 1 = 0$

$2 \vee 7 = 14 \neq I$ $\quad 2 \wedge 7 = 1 = 0$

$2 \vee 5 = 10 \neq I$ $\quad 2 \wedge 5 = 1 = 0$

∴ Complement of 2 is 35 and 35 is 2. $(2^c = 35, 35^c = 2)$

∴ A is complement lattice.

⇒ Distributive lattice.

$LHS = 2 \wedge (5 \vee 10)$ $\qquad RHS = (2 \wedge 5) \vee (2 \wedge 10)$

$= 2 \wedge 10$ $\qquad = 1 \vee 2$

$= 2$ $\qquad = 2$

∴ LHS = RHS.

∴ A is distributive lattice

Hence, this lattice is complemented lattice
Hence, this lattice is Boolean Algebra.

Ex 3 Is D_{75} with the relation "divisibility".
OB 354 a Boolean Algebra? Justify your answers.

Solution Here the set of divisors are

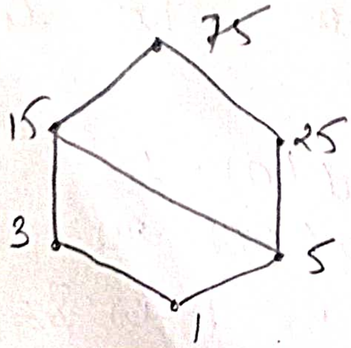
$A = \{1, 3, 5, 15, 25, 75\} = D_{75}$

Here, D_{75} contains 6 elements and

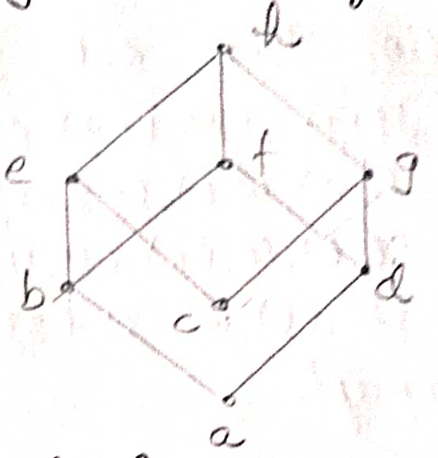
$6 \neq 2^n$. for any integer $n \geq 0$

∴ D_{75} is not a Boolean Algebra.

Hasse Diagram



Ex 4 Determine whether the following Hasse Diagram represent Boolean Algebra.
DB 355



Solution

Here the number of elements in the above Hasse diagram is 8 which is verified.

$2^n = 2^3 = 8$, for some $n \in N$.

LUB

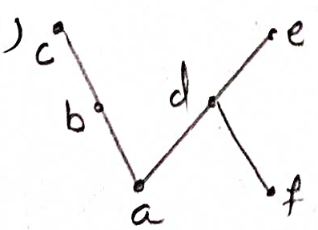
∨	a	b	c	d	e	f	g	h
a	a	b	e	d	e	f	g	h
b	b	b	e	f	e	f	h	h
c	e	e	c	g	e	h	g	h
d	d	f	g	d	h	f	g	h
e	e	e	e	h	e	h	h	h
f	f	f	h	f	h	f	h	h
g	g	h	g	g	h	h	g	h
h	h	h	h	h	h	h	h	h

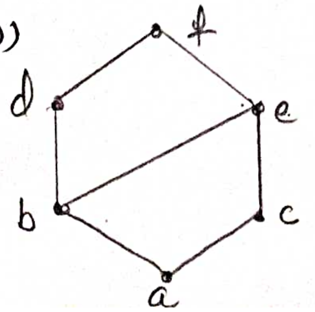
GLB

∧	a	b	c	d	e	f	g	h
a	a	a	–	a	a	a	a	a
b	a	b	–	a	b	b	a	b
c	–	–	c	–	c	–	c	c
d	a	a	–	d	a	d	d	d
e	a	b	c	a	e	b	a	e
f	a	b	–	d	b	f	d	f
g	a	a	c	d	a	d	g	g
h	a	b	c	d	e	f	g	h

There is no GLB for pairs (a,c), (b,c), (c,d), (c,f)
So, the Hasse Diagram doesn't represent a lattice.
∴ Hence, it is not Boolean Algebra.

Ex 5 Determine whether the following posets represents Boolean algebra.
DB 356 357

(a)


(b)


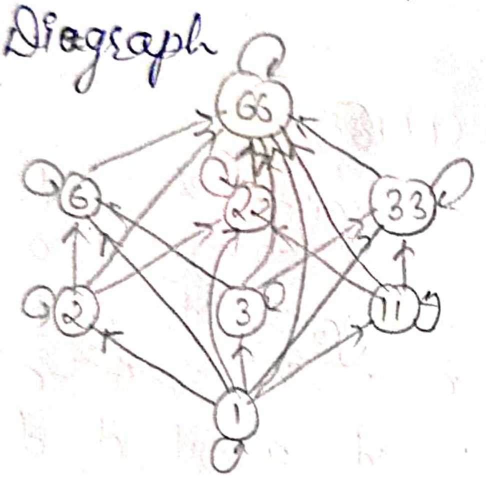
In both the Hasse diagram, number of elements is 6, which is not equal to 2^n, for any integer $n \geq 0$, so we conclude that given posets do not represent Boolean algebra.

Ex 6 QB 359 Prove that $\langle S_{66}, D \rangle$ is a Boolean Algebra.

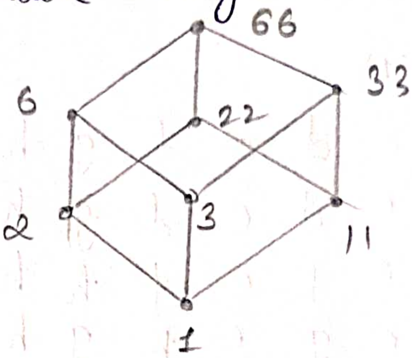
Solution Here set $A = \{1, 2, 3, 6, 11, 22, 33, 66\}$

$R = \{(1,1), (1,2), (1,3), (1,6), (1,11), (1,22), (1,33), (1,66),$
$(2,2), (2,6), (2,22), (2,66), (3,3), (3,6), (3,33), (3,66),$
$(6,6), (6,66), (11,11), (11,22), (11,33), (11,66), (22,22),$
$(22,66), (33,33), (33,66) (66,66)\}$.

Diagraph



Hasse diagram



LUB

∨	1	2	3	6	11	22	33	66
1	1	2	3	6	11	22	33	66
2	2	2	6	6	22	22	66	66
3	3	6	3	6	33	66	33	66
6	6	6	6	6	66	66	66	66
11	11	22	33	66	11	22	33	66
22	22	22	66	66	22	22	66	66
33	33	66	33	66	33	66	33	66
66	66	66	66	66	66	66	66	66

GLB

∧	1	2	3	6	11	22	33	66
1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2
3	1	1	3	3	1	1	3	3
6	1	2	3	6	1	2	3	6
11	1	1	1	1	11	11	11	11
22	1	2	1	2	11	22	11	22
33	1	1	3	3	11	11	33	33
66	1	2	3	6	11	22	33	66

Every pair of elements in A has a GLB and LUB.
Therefore, A is a lattice.

A has a least element $(O) = 1$.
and a greatest element $(I) = 66$
Number of elements $= 8 = 2^3$.

$\Rightarrow$ Complement lattice.

* Complement of 2

$2 \vee 33 = 66 = I$ $\quad$ $2 \wedge 33 = 1 = O$ $\quad$ $2^c = 33$, $33^c = 2$

$2 \vee 11 = 22 \neq I$ $\quad 2 \wedge 11 = 1 = 0$

$2 \vee 3 = 6 \neq I$ $\quad 2 \wedge 3 = 1 = 0$

* complement of 6

$6 \vee 33 = 66 = I$, $\quad 6 \wedge 33 = 3 \neq 0$ $\quad 6^c = 11$

$6 \vee 22 = 66 = I$ $\quad 6 \wedge 22 = 2 \neq 0$ $\quad 11^c = 6$

$6 \vee 11 = 66 = I$ $\quad 6 \wedge 11 = 1 = 0$

* complement of 3.

$3 \vee 2 = 6 \neq I$, $\quad 3 \wedge 3 = 1 = 0$

$3 \vee 11 = 33 \neq I$ $\quad 3 \wedge 11 = 1 = 0$ $\quad 3^c = 22$

$3 \vee 22 = 66 = I$ $\quad 3 \wedge 22 = 1 = 0$ $\quad 22^c = 3$

* Complement of 11

$11 \vee 2 = 22 \neq I$ $\quad 11 \wedge 2 = 1 = 0$

$11 \vee 3 = 33 \neq I$ $\quad 11 \wedge 3 = 1 = 0$ $\quad 11^c = 6$

$11 \vee 6 = 66 = I$ $\quad 11 \wedge 6 = 1 = 0$ $\quad 6^c = 11$

* complement of 33.

$33 \vee 2 = 66 = I$ $\quad 33 \wedge 2 = 1 = 0$ $\quad 33^c = 2$

$33 \vee 6 = 66 = I$ $\quad 33 \wedge 6 = 3 \neq 0$ $\quad 2^c = 33$

$33 \vee 22 = 66 = I$ $\quad 33 \wedge 22 = 11 \neq 0$

$\therefore$ A is complemented lattice.

$\Rightarrow$ Distributive Lattice.

$LHS = 2 \vee (3 \wedge 11)$ $\quad RHS = (2 \vee 3) \wedge (2 \vee 11)$

$= 2 \vee 1$ $\quad = 6 \wedge 22$

$= 2$ $\quad = 2$

$\therefore LHS = RHS$

$\therefore$ A is distributive lattice

$\therefore$ A satisfies all requirements of Boolean Algebra

$\therefore$ A is a Boolean Algebra.