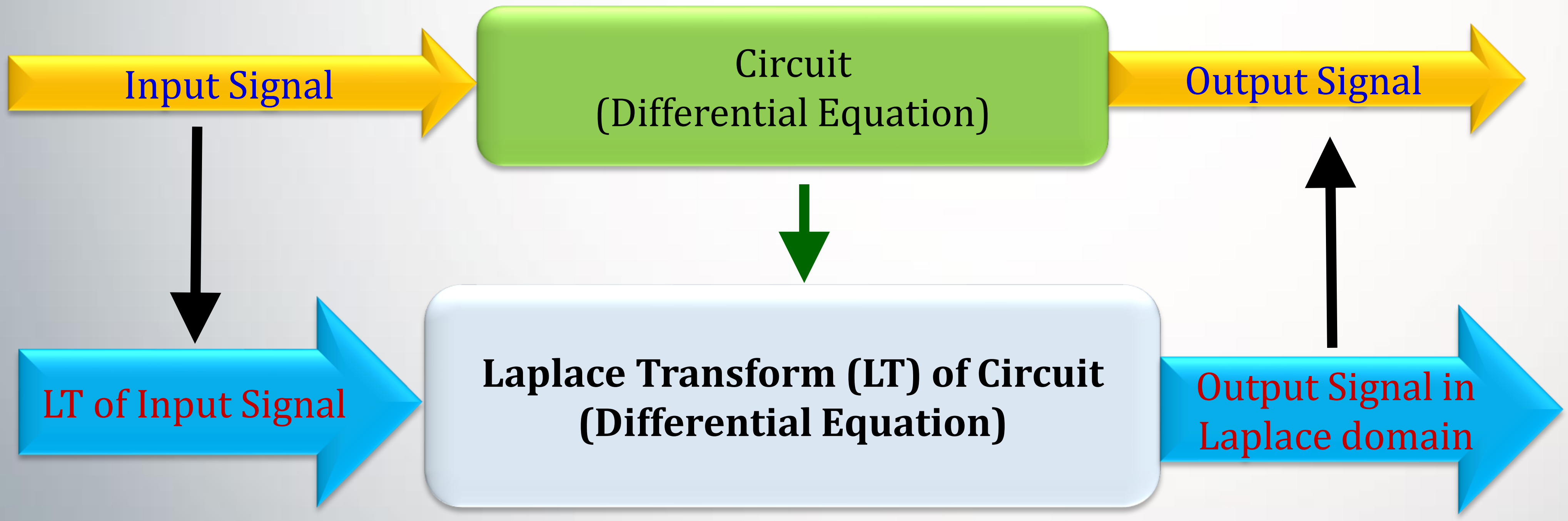
Laplace Transform Analysis

Linear Circuit Analysis II
EECE 202

Laplace Transform Analysis

Laplace transform analysis technique transforms the time domain analysis of circuit, system, or differential equation to the frequency domain thus making it easier to solve.



Laplace Transform Analysis

The one-side *Laplace Transform* of a Signal, a Function, or an Excitation is given by:

$$F(S) = L[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

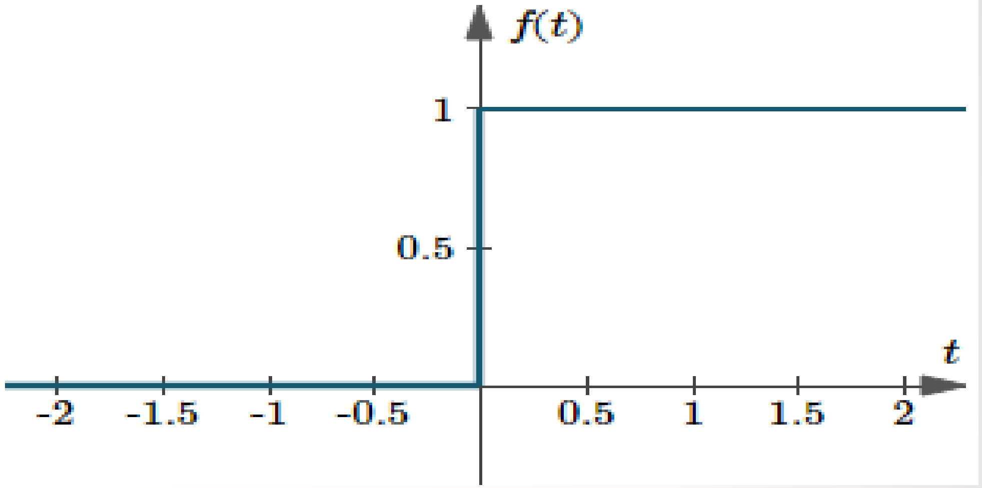
s =jω is a complex variable (*a complex frequency*) and j=$\sqrt{-1}$

F(s) is the frequency domain counterpart of f(t).

Analysis using Laplase transforms is often called <u>frequency domain analysis</u>.

Example 12.4 (p. 556)

Find the Laplace tranform of the unit step U(s)



Unit step function: $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$U(s) = L[u(t\,)] = \int_{0^-}^{\infty} U(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-st}dt = \frac{e^{-st}}{-s}\Big]_{0^-}^{\infty}$$

$$= \frac{e^{-st}}{-s}\Big]_{0^-}^{\infty} = \frac{e^{-st}}{-s}\Big]_{t\ldots\infty} - \frac{e^{-st}}{-s}\Big]_{0^-} = \frac{1}{s}$$

Example 12.5 (p. 557)

Find F(s) for $f(t)=Ke^{-at}u(t)$

$$\mathbf{F(s)} = \mathrm{L}[K\,e^{-at}\,u(t)] = \int_{0-}^{\infty} K\,e^{-at}u(t)e^{-st}dt = \int_{0-}^{\infty} K\,e^{-at}e^{-st}dt$$

$$= K\int_{0-}^{\infty} e^{-(s+a)t}dt = \frac{\boldsymbol{K}}{\boldsymbol{s+a}}$$

Example 1

Recall Delta function (or Impulse function)

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Find Delta (s)

Delta(s) = L[δ(t)]

$$\textbf{Delta(s)} = \mathrm{L}[\delta\,(t\,)] = \int_{0-}^{\infty} \delta(t)e^{-st}dt$$

$$= e^{-st}\ \Big]_0 = 1$$

Properties of Laplace Transform

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$r(t) = tu(t)$	$\frac{1}{s^2}$
$t^2u(t)$	$\frac{2}{s^3}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
$cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$

$f(t)$	$F(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)u(t-T)$	$e^{-sT}F(s)$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dot{f}(0^-)$
$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(t)dt}{s}$

Laplace transform properties and examples

1. Linearity property

The Laplace Transform operation is Linear

For $f(t) = a_1 f_1(t) + a_2 f_2(t)$

$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

Example 12.6 (p. 557)

Find F(s) when $f(t) = K_1 u(t) + K_2 e^{-at} u(t)$

Recall LT pairs

$$Ku(t) = \frac{K}{s}$$

$$e^{-at}u(t) = \frac{1}{s+a}$$

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+a}$$

Laplace transform properties and examples

2. Time shift property

If $L[f(t)u(t)] = F(s)$, then, for $T>0$

$$L[f(t-T)u(t-T)] = e^{-sT}F(s)$$

Proof

$$\mathbf{F(s)} = L[f(t\text{-}T)\,u(t\text{-}T)] = \int_{0^-}^{\infty} f(t\text{-}T)\,u(t\text{-}T)e^{-st}dt = \int_{T^-}^{\infty} f(t-T)\,e^{-st}dt$$

Now let q= (t-T), then, t= q+T, and as t approaches T^- then q approacehs 0^-

As T is constant, then $\mathbf{dq = dt}$

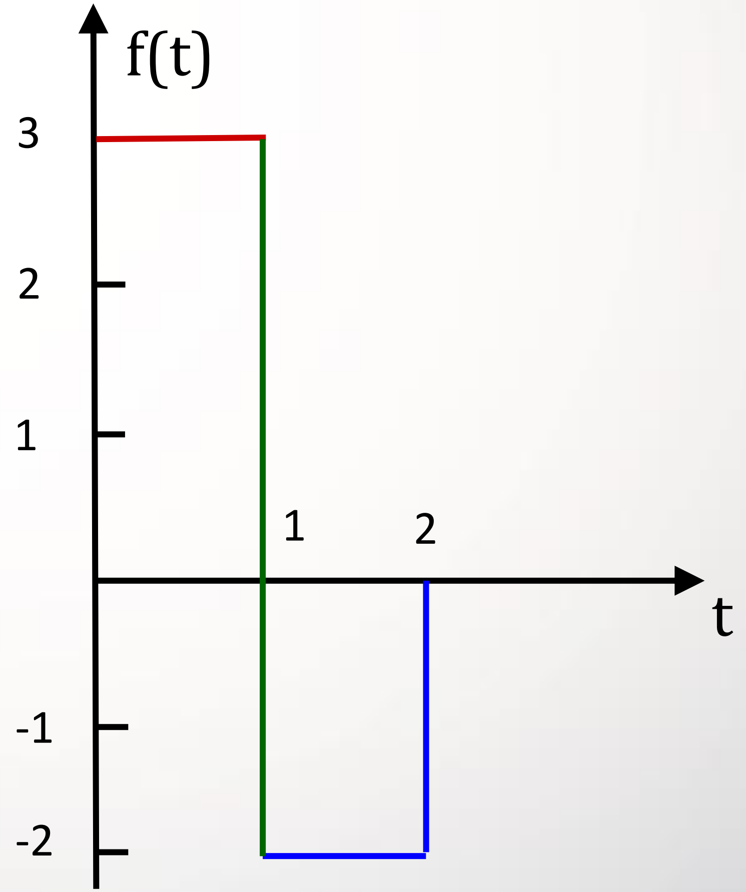
$$\mathbf{F(s)} = \int_{0^-}^{\infty} f(t\text{-}T)\,u(t\text{-}T)e^{-st}dt = \int_{T^-}^{\infty} f(t-T)\,e^{-st}dt = \int_{0^-}^{\infty} f(q)\,e^{-s(q+T)}dq$$

$$= e^{-sT}\int_{0^-}^{\infty} f(q)\,e^{-sq}dq = \mathbf{e^{-sT}F(s)}$$

Example 12.7 (p. 559)

Find F(s) for the signal f(t) given in the following figure

$$f(t) = 3u(t) - 5u(t-1) + 2u(t-2)$$



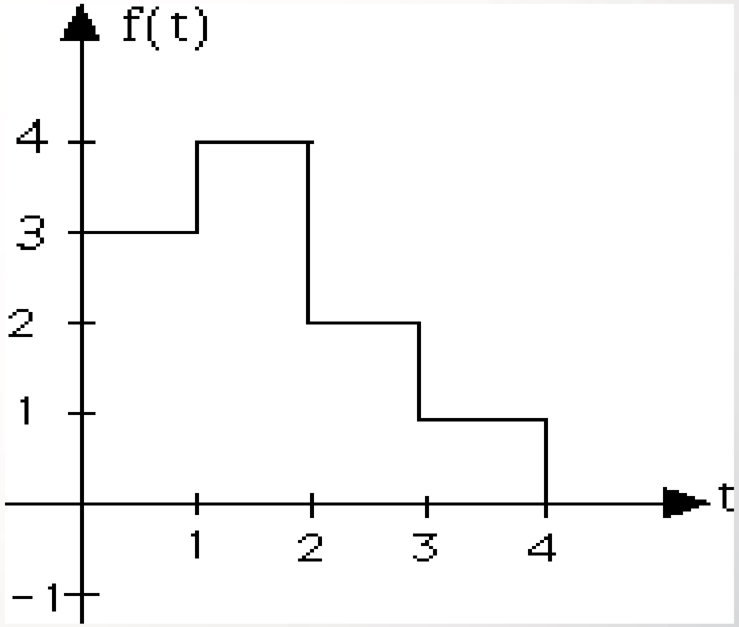
Recall LT pairs

$$F(s) = \frac{3}{s} - \frac{5e^{-s}}{s} + \frac{2e^{-2s}}{s}$$

Example 2

Find F(s) for the signal f(t) given in the following figure

$$f(t) = 3u(t) + u(t-1) - 2u(t-2) - u(t-3) - u(t-4)$$



Recall LT pairs

$$F(s) = \frac{3}{s} + \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}$$

Laplace transform properties and examples

3. Multiplication-by-t property

$$\boxed{\text{If } F(s) = L[f(t)], \text{ then } L[t \times f(t)] = -\frac{d}{ds}F(s)}$$

Example 12.8 (p. 560)

Find the Laplace transform of the ramp function R(s)=L[r(t)]

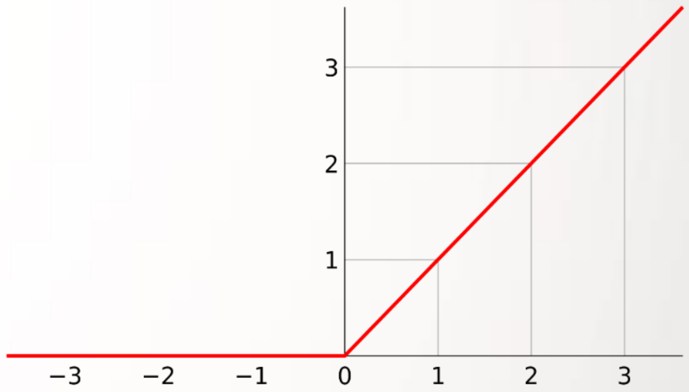
Recall Ramp function

$$R(s) = L[r(t)] = \int_{0^-}^{\infty} tu(t)e^{-st}dt$$

$$= \int_{0^-}^{\infty} te^{-st}dt = \int_{0^-}^{\infty} -\frac{d}{ds}(e^{-st})dt$$

$$= -\frac{d}{ds}\int_{0^-}^{\infty} e^{-st}dt = -\frac{d}{ds}\left(\left[\frac{e^{-st}}{-s}\right]_{0^-}^{\infty}\right)$$

$$= \frac{-d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$



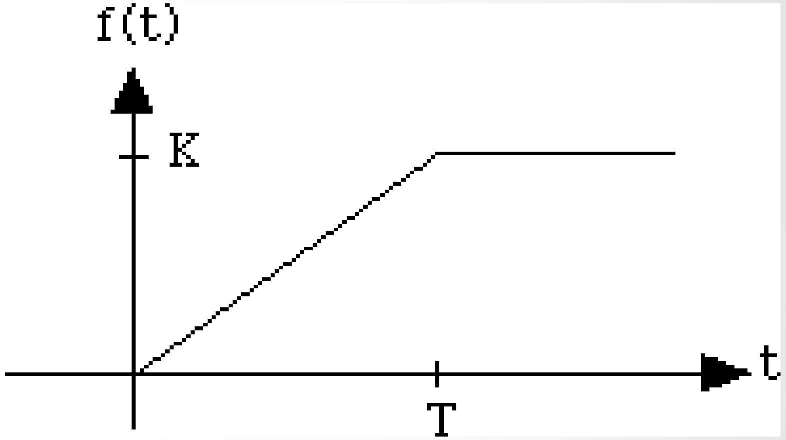
$$r(t) = tu(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

Example 3

Find F(s) for the signal f(t) given in the following figures

A)

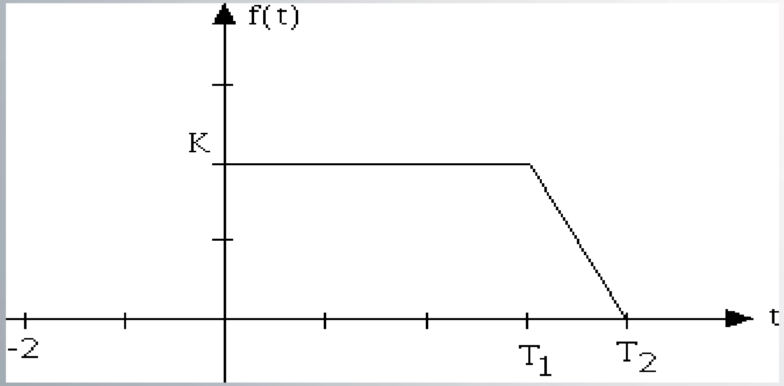
$$f(t) = \frac{K}{T} r(t) - \frac{K}{T} r(t - T)$$



Recall LT pairs

$$F(s) = \frac{K}{T} \times \frac{1}{s^2} - \frac{K}{T} \times \frac{e^{-Ts}}{s^2} = \frac{K}{T \times s^2}\left(1 - e^{-Ts}\right)$$

B)

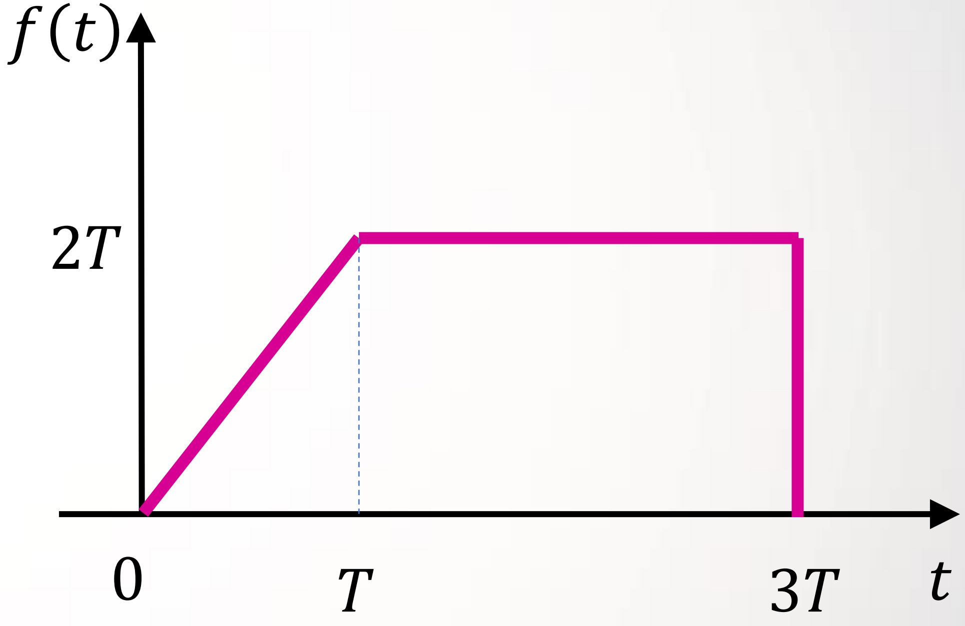


$$f(t) = Ku(t) - \frac{K}{T_2 - T_1} r(t - T_1) + \frac{K}{T_2 - T_1} r(t - T_2)$$

$$F(s) = \frac{K}{s} - \frac{K}{T_2 - T_1} \times \frac{e^{-T_1 s}}{s^2} + \frac{K}{T_2 - T_1} \times \frac{e^{-T_2 s}}{s^2}$$

Example 12.10 (p. 561)

Find F(s) for the signal f(t) depicted in the following figure

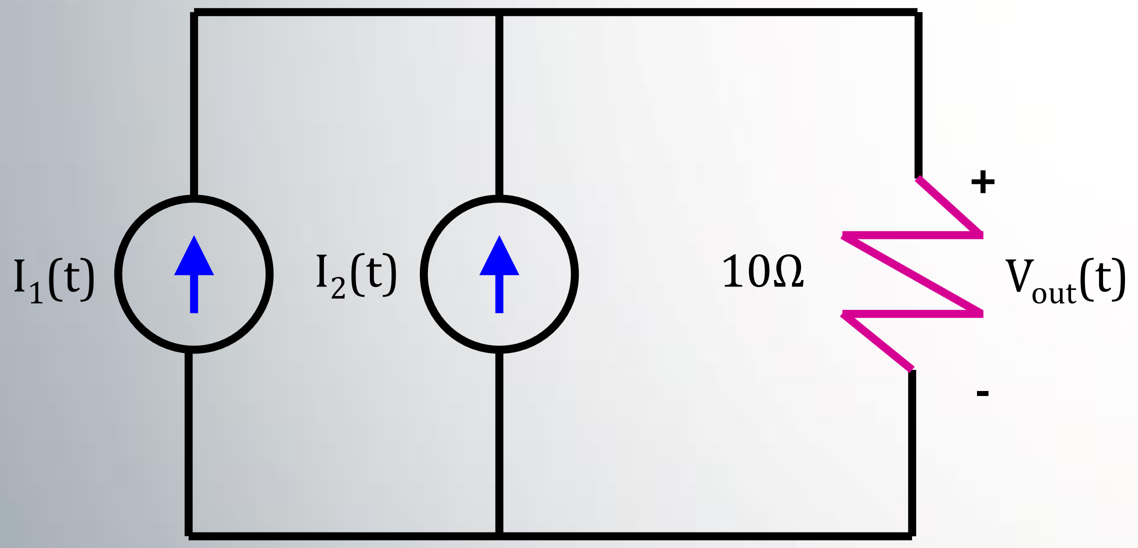


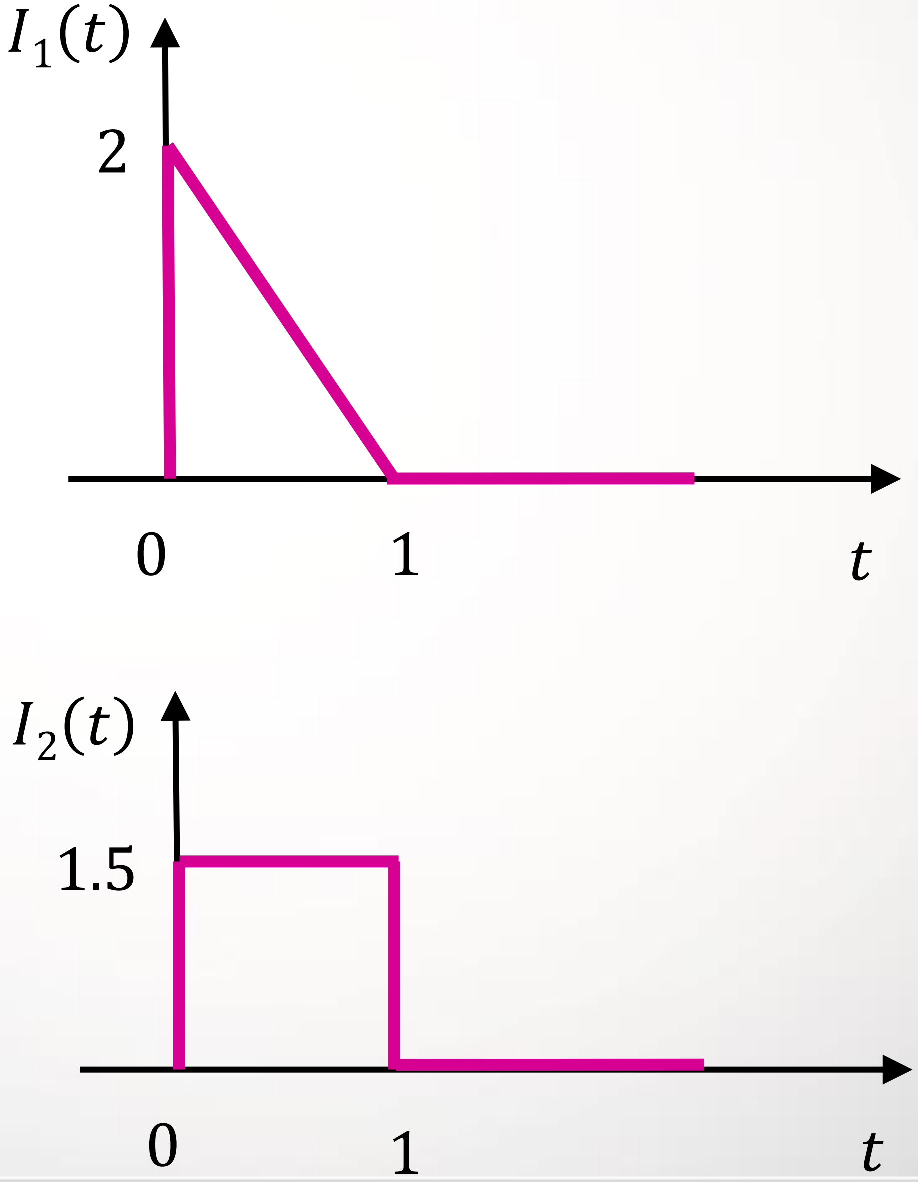
$$f(t) = \frac{2T}{T} r(t) - \frac{2T}{T} r(t-T) - 2Tu(t-3T)$$

$$F(s) = \frac{2}{s^2} - \frac{2e^{-sT}}{s^2} - \frac{2Te^{-3sT}}{s}$$

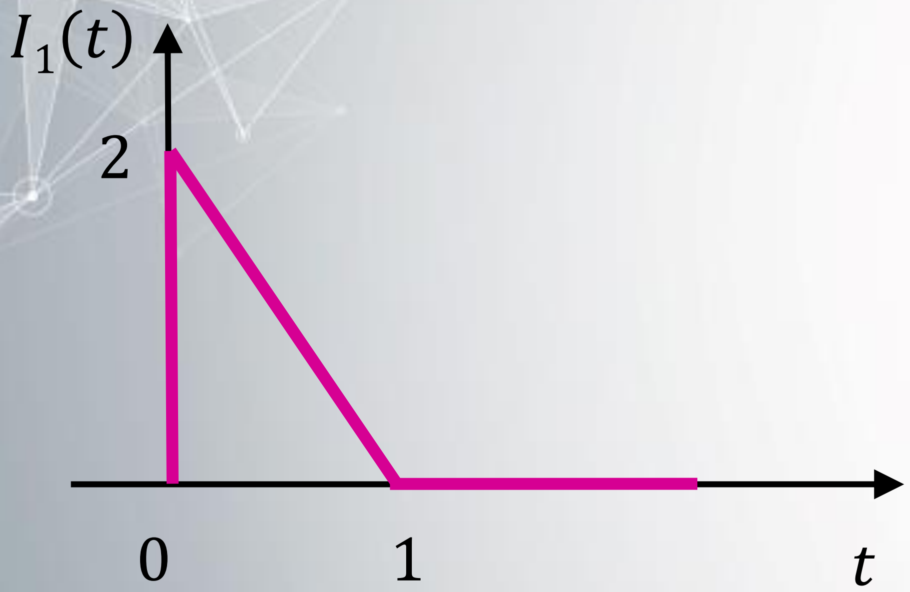
Example 12.11 (p. 562)

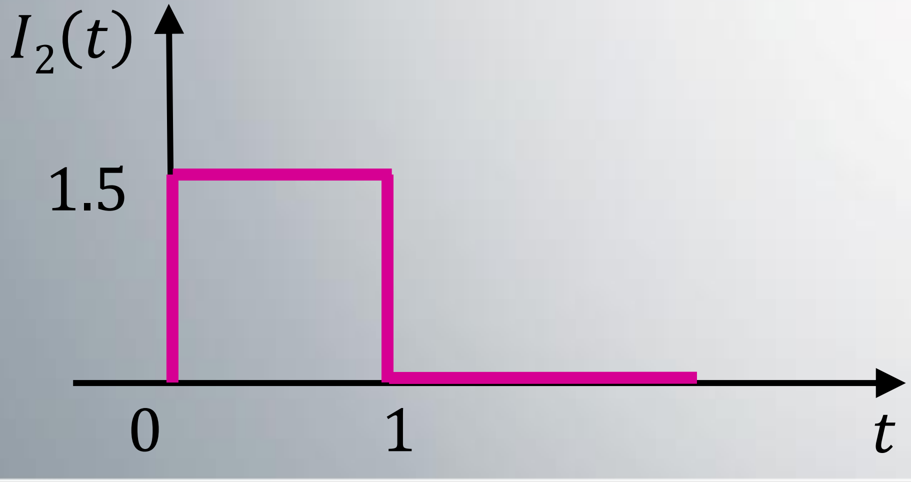
Find $V_{out}(s)$ for the following Circuit





Example 12.11 (p. 562) cont.





$$\mathrm{V_{out}(t)} = 10\times(\mathrm{I_1(t)} + \mathrm{I_2(t)})$$

From linearity: $\mathrm{V_{out}(s)} = 10\times(\mathrm{I_1(s)} + \mathrm{I_2(s)})$

$\mathrm{I_1(t)=2u(t)\text{-}2r(t)+2r(t\text{-}1)}$

$$\mathrm{I_1(s)} = \frac{2}{s} - \frac{2}{s^2} + \frac{2e^{-s}}{s^2} = \frac{2}{s} - \frac{2-2e^{-s}}{s^2} = \frac{2s+2e^{-s}-2}{s^2}$$

$\mathrm{I_2(t)=1.5u(t)\text{-}1.5u(t\text{-}1)}$

$$\mathrm{I_2(s)} = \frac{1.5}{s} - \frac{1.5\times e^{-s}}{s} = \frac{1.5-1.5e^{-s}}{s} = \frac{1.5s(1-e^{-s})}{s^2}$$

$$\mathrm{V_{out}(s)} = 10\times\left(\frac{2s+2e^{-s}-2}{s^2} + \frac{1.5s(1-e^{-s})}{s^2}\right)$$

$$\mathrm{V_{out}(s)} = \frac{35}{s} - \frac{20}{s^2} + e^{-s}\left(\frac{20}{s^2} - \frac{15}{s}\right)$$

Laplace transform properties (Table 12.2. p.584)

Property	Transform Pair
Linearity	$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$
Time shift	$\mathcal{L}[f(t-T)u(t-T)] = e^{-sT}F(s),\ T > 0$
Multiplication by t	$\mathcal{L}[tf(t)u(t)] = -\frac{d}{ds}F(s)$
Multiplication by t^n	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$
Frequency shift	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
Time differentiation	$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0^-)$
Second-order differentiation	$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2F(s) - sf(0^-) - f^{(1)}(0^-)$
nth-order differentiation	$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f^{(1)}(0^-) - \ldots - f^{(n-1)}(0^-)$

Property	Transform Pair
Time integration	(i) $\mathcal{L}\left[\int_{-\infty}^{t} f(q)dq\right] = \frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(q)dq}{s}$ (ii) $\mathcal{L}\left[\int_{0^-}^{t} f(q)dq\right] = \frac{F(s)}{s}$
Time/Frequency scaling	$\mathcal{L}[f(at)] = 1/a\, F(s/a)$

Example 4

Determine Laplace Transform for the following function:

$f(t) = 12\, t\, e^{-3(t-4)}\, u(t-4)$

$f(t) = t \times 12\, e^{-3(t-4)}\, u(t-4) = t \times f_0(t)$

$f_0(t) = 12\, e^{-3(t-4)}\, u(t-4)$, then $F_0(s) = \frac{12e^{-4s}}{s+3}$

Time Shift Property

$$F(s) = -\frac{d(F_0(s))}{ds} = -\frac{d\left(\frac{12e^{-4s}}{s+3}\right)}{ds} = \frac{48e^{-4s}}{(s+3)} + \frac{12e^{-4s}}{(s+3)^2}$$

Time Multiplication Property

Example 5

Determine Laplace Transform for the following function:

$f(t) = 10\ t^3\ e^{-2t}\ u(t)$

$$f(t) = t^3 \times 10\ e^{-2t}\ u(t) = t^3 \times f_0(t)$$

$$F_0(s) = \frac{10}{s+2}$$

$$F(s) = -\frac{d^3\left(\frac{10}{s+2}\right)}{ds^3} = \frac{60}{(s+2)^4}$$

Suggested Additional Problems for Ch. 12:

Example 12.25 (p. 585), 12.26 (p. 587) ?