American University Of The Middle East

# Laplace Transform Analysis

## Linear Circuit Analysis II
## EECE 202

www.aum.edu.kw

# Announcement

1. Project Groups should be complete
2. Quiz during Week 4
3. GCA1 during Week 5

# Recap

1. Laplace Transform Analysis

# New Material

1. Proper and Improper Rational Functions
2. Inverse Laplace Transforms
3. Differential Property

# Learning Outcomes

2. an ability to compute responses of linear circuits with and without initial conditions via one-sided Laplace transform techniques. SO [1]

# Partial Fraction Expansions

## *1. Distinct Poles*

$$F(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s^1 + a_0}{s^n + b_{n-1} s^{n-1} + \cdots b_1 s^1 + b_0} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s^1 + a_0}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

*$p_1, p_2, \ldots\ldots\ldots\; p_n$ are the zeros of the denominator polynomial*

If F(s) is a proper rational function with distinct (simple) poles, $p_1, \ldots, p_n$. The partial fraction expansion can be represented as:

$$F(s) = K + \frac{A}{(s - p_1)} + \frac{B}{(s - p_2)} + \cdots + \frac{D}{(s - p_n)}$$

**For m≤n**

# Partial Fraction Expansions

**where**

$K = \lim_{s\to\infty}(\mathrm{F(s)})$ (note: $K$=0 when m<n)

$A = \lim_{s\to p_1}((s-p_1)\mathrm{F(s)})$

$B = \lim_{s\to p_2}((s-p_2)\mathrm{F(s)})$

.

.

.

$D = \lim_{s\to p_n}((s-p_n)\mathrm{F(s)})$

# Partial Fraction Expansions

## 2. Repeated Poles

$$\mathrm{F(s)} = \frac{n(s)}{(s-a)^k d(s)}$$

$(s-a)^k$ specifies a repeated root of order $k$

$$\mathrm{F(s)} = \frac{A}{(s-a)} + \frac{B}{(s-a)^2} + \cdots + \frac{D}{(s-a)^k} + \frac{n_1(s)}{d(s)}$$

$n_1(s)$ and d(s) are whatever remains in the partial fraction expansion of F(s)

Start with obtaining $D$ as:

$$D = (s-a)^k F(s)|s = a$$

## Exercise (p. 571)

Find f(t) for the following function

$$F(s) = \frac{3s^2+10s+9}{(s+2)(s+1)^2}$$

$$F(s) = \frac{A}{(s+2)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$A = 1, B = 2, C = 2$$

$$f(t) = e^{-2t}u(t) + 2e^{-t}u(t) + 2e^{-t}r(t)$$

## Example 1

Find f(t), when $F(s) = \frac{As+B}{s^2+w^2}$

$$F(s) = \frac{As+B}{s^2+w^2} = \frac{A\times s}{s^2+w^2} + \frac{B\times w}{w\times(s^2+w^2)}$$

$$f(t) = A\cos(wt)u(t) + \frac{B}{w}\sin(wt)u(t)$$

# Example 12.16 (p. 574)

Find f(t), when $F(s) = \frac{3s^2+s+3}{(s+1)(s^2+4)}$

$$F(s) = \frac{3s^2+s+3}{(s+1)(s^2+4)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+4)} \qquad \text{Eq (1)}$$

$$A = \frac{3(-1)^2+(-1)+3}{((-1)^2+4)} = 1$$

Determine C, use s = 0 in Eq(1) to get rid of B

$$\frac{3}{(1)(4)} = \frac{1}{(1)} + \frac{0+C}{(4)}$$

$$\mathbf{C = -1}$$

Determine B, use s=1

$$\frac{3(1)^2+1+3}{(1+1)(1^2+4)}=\frac{1}{(1+1)}+\frac{B\times1-1}{(1^2+4)}$$

$$\boldsymbol{B = 2}$$

$$\mathrm{F(s)} = \frac{3s^2+s+3}{(s+1)(s^2+4)}=\frac{1}{(s+1)}+\frac{2s-1}{(s^2+4)}=\frac{1}{(s+1)}+\frac{2s}{(s^2+2^2)}-\frac{0.5\times2}{(s^2+2^2)}$$

$$f(t) = e^{-t}u(t)+2\cos(2t)u(t)-0.5\sin(2t)u(t)$$

# Example 2

Find f(t), when F(s) = $\dfrac{3s+4}{s^2+4s+13}$

Not Fractionable but has two distinct complex roots:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

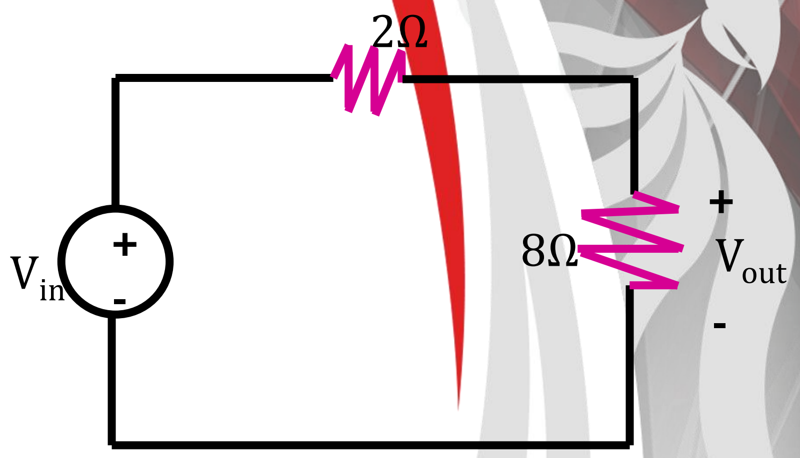
$$= -2 \ \pm j3$$

$$F(s) = \frac{3s+4}{(s+2)^2+3^2} = \frac{3(s+2)-6+4}{(s+2)^2+3^2} = \frac{3(s+2)}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}$$

$$\boxed{f\ (t) \ = 3e^{-2t}\cos(3t)u(t) \ - \frac{2}{3}e^{-2t}\sin(3t)u(t)}$$

# Example 3

**Find $V_{out}(s)$ and $V_{out}(t)$ for the following Circuit** $V_{in}(s) = \frac{10s^2-8s+16}{(s+1)(s^2+16)}$




$$V_{in}(s) = \frac{10s^2-8s+16}{(s+1)(s^2+16)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+16)}$$

$$A = \frac{10(-1)^2-8(-1)+16}{(-1^2+16)} = 2$$

Use s=0 to find the value of C

$$\left.\frac{10s^2-8s+16}{(s+1)(s^2+16)}\right|_{s=0} = A + \frac{C}{16}$$

$$C = -16$$

Use s=1 to find the value of B

$$\frac{10\times1^2-8\times1+16}{(1+1)(1^2+16)}=\frac{2}{(1+1)}+\frac{B\times1-16}{(12+16)}$$

$$\boldsymbol{B=8}$$

$$\mathrm{V_{in}(s)}=\frac{10s^2-8s+16}{(s+1)(s^2+16)}=\frac{2}{(s+1)}+\frac{8s-16}{(s^2+16)}=\frac{\mathbf{2}}{(\boldsymbol{s}+\mathbf{1})}+\frac{\mathbf{8}\boldsymbol{s}}{(\boldsymbol{s}^2+\mathbf{16})}-\frac{\mathbf{16}}{(\boldsymbol{s}^2+\mathbf{16})}$$

$$V_{in}(t) = 2e^{-t}u(t)+8\cos(4t)u(t)-4\sin(4t)u(t)$$

# Time Differentiation Property

$$L[\frac{d}{dt}f(t)]=s\times F(s)-f(0^-)$$

Using the transformation equation and integration by parts

$$L[\frac{d}{dt}f(t)]=\int_{0^-}^{\infty}\left(\frac{d}{dt}f(t)\right)e^{-st}\,dt = f(t)\,e^{-st}\big|_{0^-}^{\infty} - \int_{0^-}^{\infty}(-s)f(t)\,e^{-st}\,dt$$

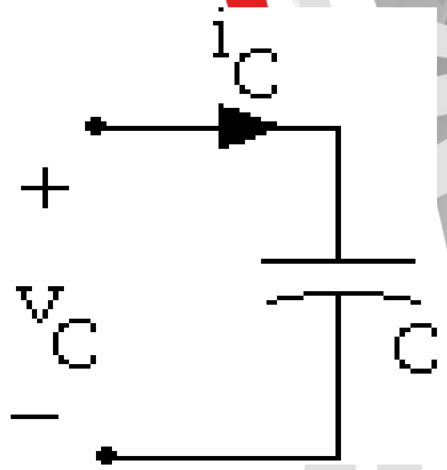
$$=0\text{-}f(0^-)+sF(s)=sF(s)-f(0^-)$$

*Differentiation in the time domain is equivalent to multiplication by s in the s-domain.*

# Example 12.23 (p. 581)

**Find an expression for the current through the capacitor $i_c(s)$ in the S domain.**

$$\text{As } i_C(t) = C\frac{dv_C(t)}{dt}$$




Using the time differentiation property

$$I_C(s) = CsV_c(s) - Cv_c(0^-)$$

# Time Differentiation Formula

$$L[\frac{df}{dt}]=sF(s)-f(0^-)$$

$$L[\frac{d^2f}{dt^2}]=s^2F(s)-sf(0^-)-\dot{f}(0^-)$$

$$L[\frac{d^nf}{dt^n}]=s^n\times F(s)-s^{n-1}\times f(0^-)-s^{n-2}\times \dot{f}(0^-)-\cdots f^{(n-1)}(0^-)$$

# Example 4

Find the solution of the differential equation $\ddot{f}(t) = 2e^{-t}u(t)$

Apply Laplace transform to the given equation

$$s^2F(s) - sf(0^-) - \dot{f}(0^-) = \frac{2}{s+1}$$

***Solve for F(s)***

$$F(s) = \frac{2}{s^2(s+1)} + \frac{f(0^-)}{s} + \frac{\dot{f}(0^-)}{s^2}$$

$$F(s) = \frac{-2}{s} + \frac{2}{s^2} + \frac{2}{s+1} + \frac{f(0^-)}{s} + \frac{\dot{f}(0^-)}{s^2}$$

Apply inverse Laplace Transform to obtain f(t)

$$\boxed{f(t) = -2u(t) + 2tu(t) + 2e^{-t}u(t) + f(0^-)u(t) + \dot{f}(0^-)tu(t)}$$

# Use AI and answer following questions

1. What is the purpose of using partial fraction expansions in circuit analysis?
2. How do you identify if a rational function F(s) has distinct or repeated poles from its denominator?
3. Given a function F(s) with distinct poles, describe the steps you would take to decompose it into partial fractions.
4. What changes in the decomposition process when you have repeated poles in F(s)F(s)F(s)? Provide an example.
5. After decomposing F(s), how is the inverse Laplace transform applied to find the corresponding time-domain function f(t)?

# Summary

- Purpose: Used to decompose a complex rational function F(s) into simpler terms for easier analysis in the time domain.
- **Key Steps**:
  - Express F(s) in a form that can be expanded.
  - Use partial fraction decomposition for distinct and repeated poles.
  - Apply inverse Laplace transform to find f(t).
- **Application**:
  - Finding time-domain solutions such as f(t) for circuits.

## Suggested Additional Problems for Ch. 12:

Example 12.12 (p. 566), 12.13 (p. 567), 12.16 (p.574)
Exercises (p.567), (p.568), (p.575) ?

Example 12.17 (p. 576), 12.18 (p. 577), 12.21 (p. 580), 12.22 (p. 580), Exercise (p. 577)?