American University Of The Middle East

Inverse Laplace Transform

Linear Circuit Analysis II
EECE 202

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Announcement

1. Project Proposal released
2. Quiz next Week 4

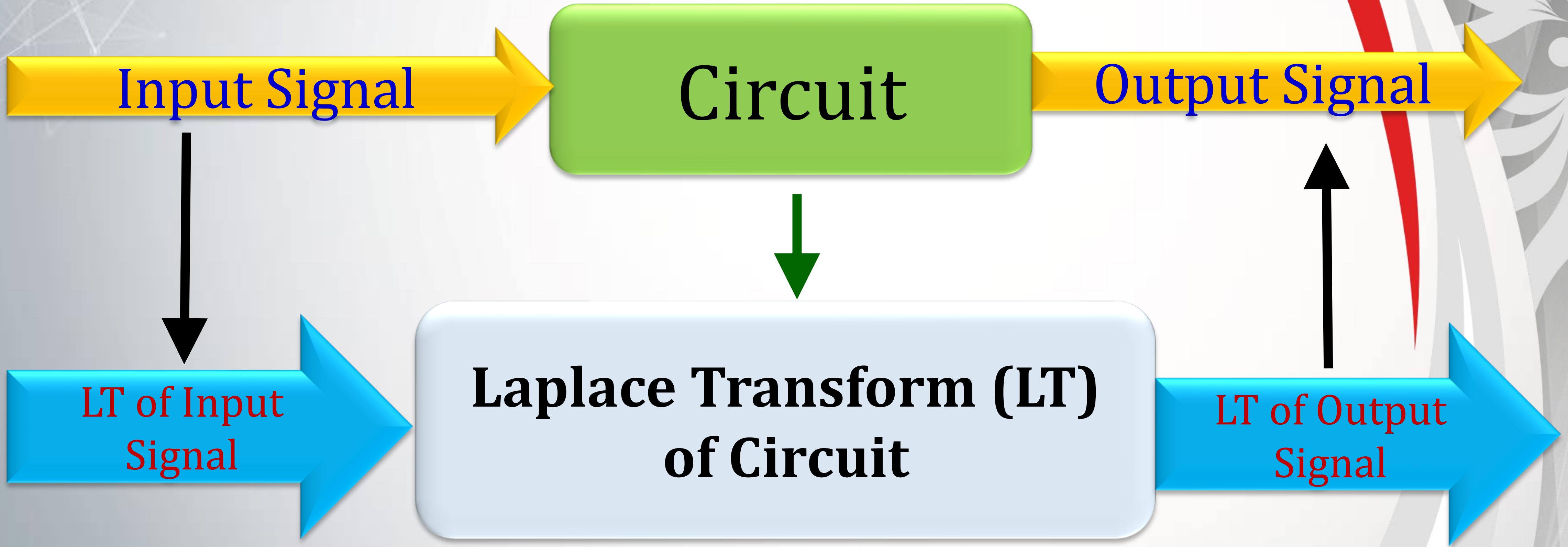
Recap

1. Properties of Laplace transform
2. Laplace transform of sinusoidal signals

New Material

1. Inverse Laplace transform analysis
2. LT using partial fraction
3. Applications of LT

Laplace Transform Analysis



Inverse Laplace Transform

The inverse Laplace Transform of a Signal, a Function, or an Excitation is given by:

$$f(t) = L^{-1}\,[F(s)] = \frac{1}{2\pi j}\int_{\Gamma} F(s)e^{st}ds$$

$\Gamma = \sigma_1 + j\omega$ (is the particular path in a complex plane)

$j = \sqrt{-1}$

ω ranges from $-\infty$ to $+\infty$

σ_1 is a real number

Example 1

Determine the inverse Laplace transform of each of the following functions:

$$\text{(a)}\quad F(s) = \frac{1}{s} + \frac{2}{s+1}$$

$$\text{(b)}\quad G(s) = \frac{3s+1}{s+4}$$

Answer

$$\text{(a)}\qquad f(t) = u(t) + 2e^{-t}u(t)$$

$$\text{(b)}\qquad G(s) = \frac{3(s+4) - 11}{s+4} = 3 - \frac{11}{s+4}$$

Can be also obtained by applying partial fraction!

$$g(t) = 3\delta(t) - 11e^{-4t}u(t)$$

Example 2

Find f (t) when F (s)= $\frac{20s^2+30s+20}{s(s+2)}$

Step 1. Use partial fraction $\frac{20s^2+30s+20}{s(s+2)} = \mathrm{K}+\frac{A}{s} + \frac{B}{(s+2)}$

Step 2. Determine K.

$$K = \lim_{s\to\infty}\left(\frac{20s^2+30s+20}{s(s+2)}\right) = 20$$

K = 20

Step 3. Determine A

$$A = \left(\frac{20s^2 + 30s + 20}{(s+2)}\right)_{s=0}$$

A = 10

Step 4. Determine B

$$B = \left(\frac{20s^2 + 30s + 20}{s}\right)_{s=-2}$$

B = -20

Step 5. Obtain f(t) from $F(s)$ “apply inverse Laplace transform”

$$F(s) = \frac{20s^2+30s+20}{s(s+2)} = 20+\frac{10}{s} - \frac{20}{(s+2)}$$

$$\Downarrow$$

$$f(t) = 20\delta(t)+10u(t) - 20e^{-2t}u(t)$$

Exercise (p. 571)

Find f (t) when F (s)= $\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}$

Step 1. Use partial fraction

$$\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}=\mathrm{K}+\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+1)}+\frac{D}{(s+1)^2} \qquad \mathrm{Eq(1)}$$

Step 2. Determine K.

$$K = \lim_{s\to\infty}\left(\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}\right)=2$$

Step 3. Determine *B*

$$B= \left[s^2\left(\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}\right)\right]_{s=0}=2$$

Step 4. Determine *D*

$$D= \left[(s+1)^2\left(\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}\right)\right]_{s=-1}=2$$

Step 5. Determine *A and C*

We need to make 2 equations with 2 unknowns (A and C) using Eq(1)

Use s = 1 in Eq(1)

$$\frac{2\times1^4+2\times1^3+3\times1^2+3\times1+2}{1^2(1+1)^2}=2+\frac{A}{1}+\frac{2}{1^2}+\frac{C}{(1+1)}+\frac{2}{(1+1)^2}$$

A+0.5C=-1.5 ....(1)

Use s = -2 in Eq(1)

$$\frac{2\times-2^4+2\times-2^3+3\times-2^2+3\times-2+2}{-2^2(-2+1)^2}=2+\frac{A}{-2}+\frac{2}{-2^2}+\frac{C}{(-2+1)}+\frac{2}{(-2+1)^2}$$

0.5A+C=-1.5 ....(2)

By solving both equations, A=-1 and C=-1

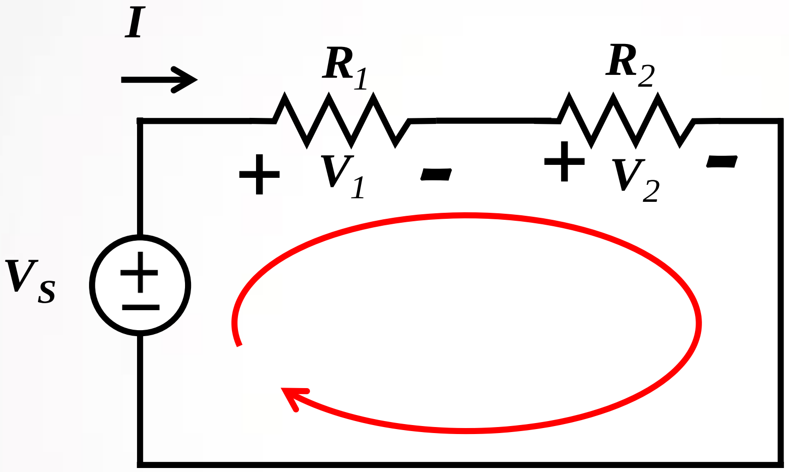
Step 6. Obtain f(t) from $F(s)$ "apply inverse Laplace transform"

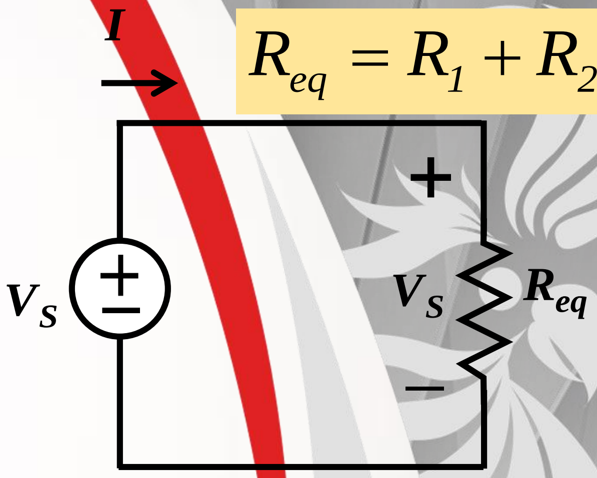
$$\mathrm{F(s)}=\frac{2s^4+2s^3+3s^2+3s+2}{s^2(s+1)^2}= 2-\frac{1}{s}+\frac{2}{s^2}-\frac{1}{(s+1)}+\frac{2}{(s+1)^2}$$

$$\downarrow$$

$$\boxed{\mathrm{f(t)} = 2\delta(\mathrm{t})\text{-}\mathrm{u(t)} + 2\mathrm{r(t)} - e^{-t}u(t) + 2e^{-t}r(t)}$$

Reminder Series Resistors and Voltage division





$$R_{eq} = R_1 + R_2$$

Ohm's law:

$$V_1 = I\ R_1$$

and

$$V_2 = I\ R_2$$

The equivalent resistance of any number of resistors connected in series is the sum of the individual resistances.

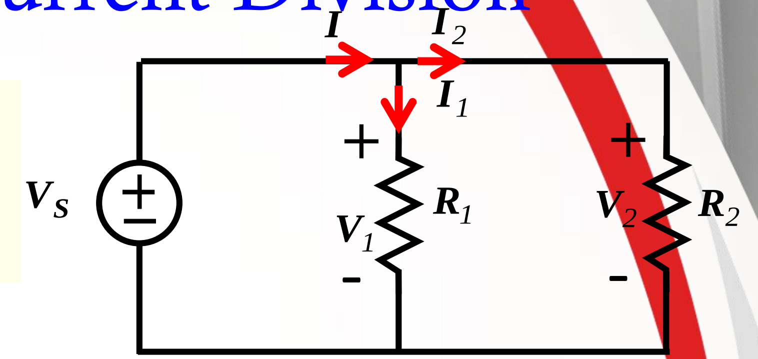
KVL:

$$V_s = V_1 + V_2$$
$$V_s = IR_1 + IR_2$$
$$V_s = I(R_1 + R_2)$$
$$V_s = I(R_{eq})$$

$$V_s = I(R_1 + R_2)$$
$$I = \frac{V_s}{(R_1 + R_2)}$$

$$V_1 = IR_1 = \frac{R_1}{(R_1 + R_2)} V_s$$
$$V_2 = IR_2 = \frac{R_2}{(R_1 + R_2)} V_s$$

Reminder: Parallel Resistors And Current Division



Ohm's law:

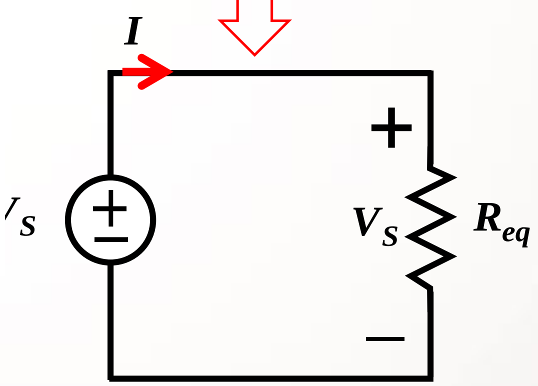
$$I_1 = \frac{V_1}{R_1} = \frac{V_s}{R_1} \quad \& \quad I_2 = \frac{V_2}{R_2} = \frac{V_s}{R_2}$$

KCL:

$$I = I_1 + I_2 \Rightarrow I = V_S\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \Rightarrow I = V_S \frac{1}{R_{eq}}$$

$$V_S = V_1 = V_2$$

$$\Rightarrow V_S = I\, R_{eq}$$



Substitute

$$I_1 = I \frac{R_2}{(R_1 + R_2)}$$

$$I_2 = I \frac{R_1}{(R_1 + R_2)}$$

$$R_{eq} = \frac{R_1 R_2}{(R_1 + R_2)}$$

NOTE: This equation is only valid for 2 resistors connected in parallel

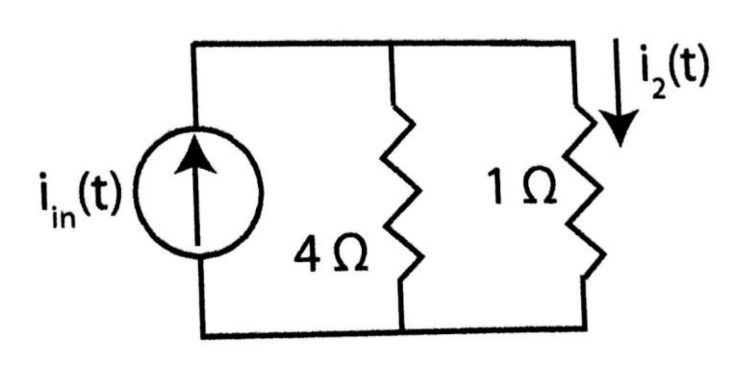
Circuit applications 1

Find i2(t) in the following circuit, where $I_{in}(s) = \frac{2s^3+12s^2+23s+17}{(s+1)(s+2)(s+4)}$

$$\frac{2s^3+12s^2+23s+17}{(s+1)(s+2)(s+4)} = K + \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$i_2(t) = \left(\frac{4}{4+1}\right) i_{in}(t)$$

K=2, A=4/3, B=-1.5, C=-11/6



$$i_{in}(t) = 2\delta(t) + 1.334e^{-t}u(t) - 1.5e^{-2t}u(t) - \left(\frac{11}{6}\right)e^{-4t}u(t)\ A$$

Circuit applications 2 (Ex. 12-13 page 567)

Find $V_{out}(t)$ in the following circuit, where $V_{in}(s) = \frac{24s-72}{s^2+4s+40}$

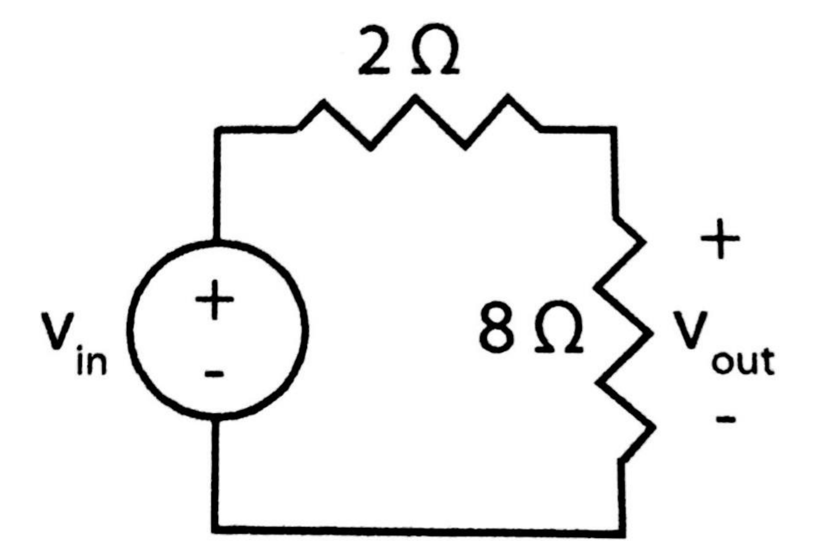
$$V_{in}(s) = \frac{24s-72}{s^2+4s+40} = \frac{24(s-3)}{(s+a)^2+b^2}$$

$\boldsymbol{s = -2 \pm j6}$ $\qquad$ $\boldsymbol{a = 2\,,\, b = 6}$

$$V_{out}(t) = \left(\frac{8}{8+2}\right) V_{in}(t)$$

$$V_{in}(s) = \frac{24(s-3)}{(s+2)^2+6^2} = \frac{24(s+2-2-3)}{(s+2)^2+6^2}$$

$$= \frac{24([s+2]-5)}{(s+2)^2+6^2} = \frac{24[s+2]}{(s+2)^2+6^2} - \frac{24(5)}{(s+2)^2+6^2}$$



$$v_{in}(t) = 24e^{-2t}\cos(6t)u(t) - \frac{24x5}{6}e^{-2t}\sin(6t)u(t)\ V$$

Practicing problems

Find f(t) of the following functions

A) $F(s) = \frac{2s+16}{s^2+16}$

B) $F(s) = \frac{2s^2+88s}{(s+4)(s^2+64)}$

C) $F(s) = \frac{2s^3+12s^2+22s+8}{(s^2+2s+1)(s+2)}$

Practice

1. Why is partial fraction decomposition useful in finding the inverse Laplace transform of complex rational functions?

2. Explain how the inverse Laplace transform helps in solving time-domain circuit problems from their s-domain equivalents.

3. How does transforming a function from the frequency domain (s-domain) to the time domain help in analyzing real-world signals and systems?

4. Solve activity on Moodle

Summary

- **Inverse Laplace Transform Basics**
  - Converts a function from the s-domain back to the time domain.
- **Key Methods**
  - Partial Fraction Decomposition : Used to break down complex rational functions to make inverse transformation easier.
  - Direct Inverse Laplace: Using standard Laplace pairs to convert back to time-domain functions.
- **Applications in Circuit Analysis**
  - Helps analyze circuits with initial conditions and transient responses.
  - Used to solve differential equations in circuit systems

Suggested Additional Problems for Ch. 12:

Example 12.13 (p. 567), 12.15 (p. 570), Exercises (p. 575) ?