American University Of The Middle East

# Equivalent Circuits for L and C with Initial Conditions

## Linear Circuit Analysis II
## EECE 202

www.aum.edu.kw

# Announcement

1. GCA2 – Group Week 11 Lecture 2
2. PD2 due Week 12

# Recap

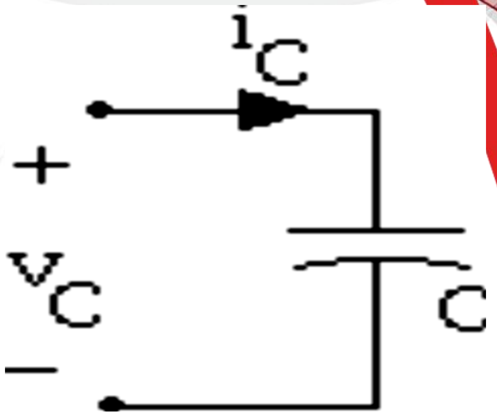
1. Transfer function revision
2. Stability in s-plane
3. Effect of pole location

# New Material

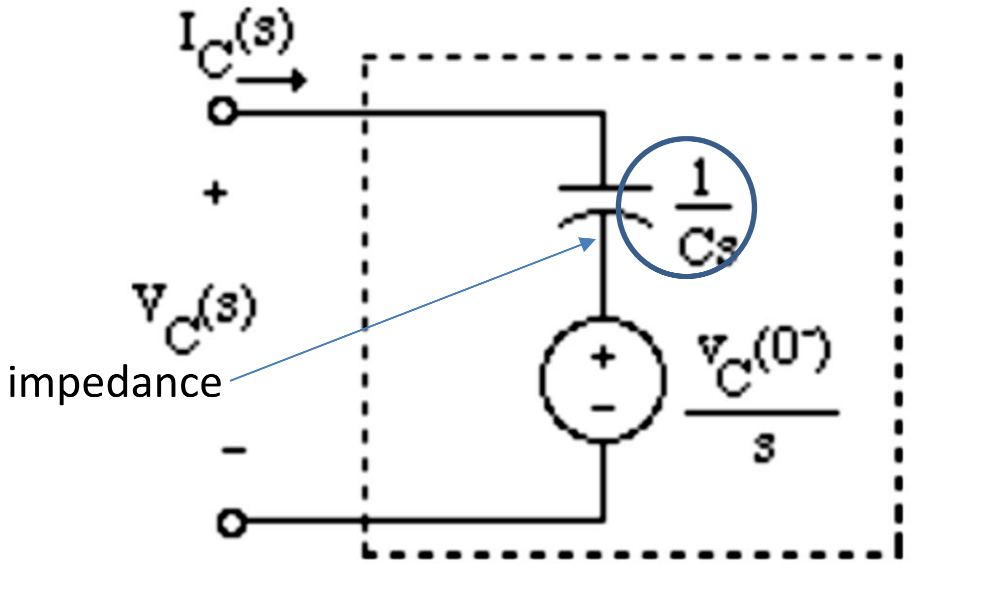
1. Equivalent circuit for capacitors in s domain
2. Equivalent circuit for Inductors in s domain
3. Examples

# Equivalent circuit in s domain for a Capacitor

In case of non-zero initial condition $v_c(0^-)$




$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c\,(t)d\tau$$

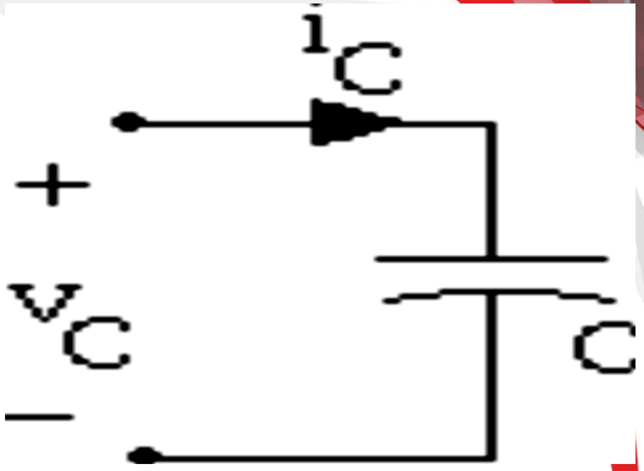



$$V_c(s) = \frac{1}{Cs} I_c(s) + \frac{V_c(0^-)}{s}$$

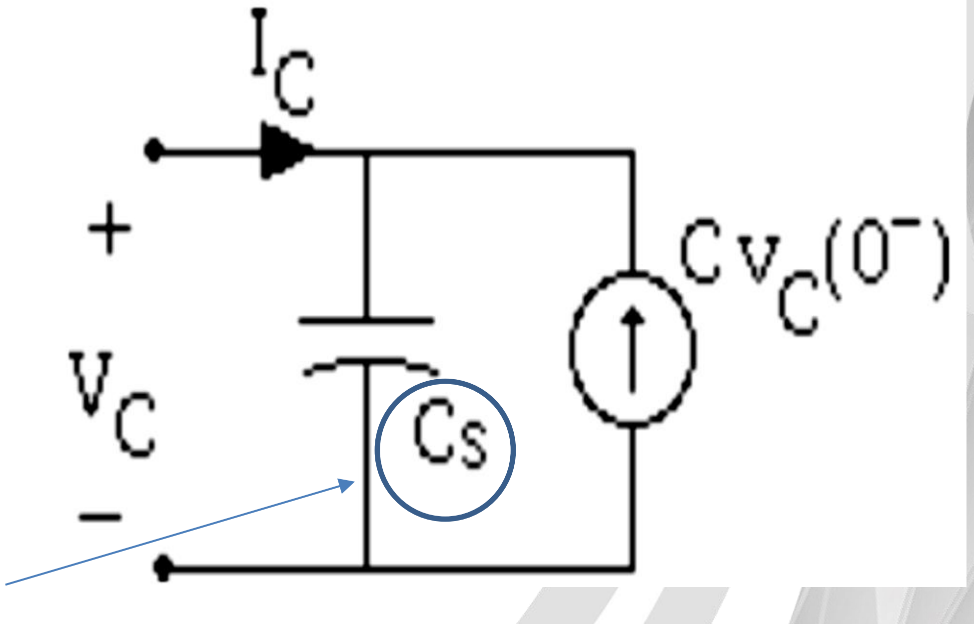
# Equivalent circuit in s domain for a Capacitor

In case of non-zero initial condition $v_c(0^-)$

$$i_c(t) = C\frac{dv_c}{dt}$$




$$I_c(s) = CsV_c(s) - Cv_c(0^-)$$



# The Capacitor

**Meaning of the initial conditions:**

- For the circuit shown in Figure 1, the capacitor was already charged before the switch (S1) closes at time (t=0).
- This charge makes the voltage across the capacitor equals 3V.
- Mathematically, we write this voltage as $V_c(0^-) = 3V$, where "$0^-$" means the time just before "t=0".

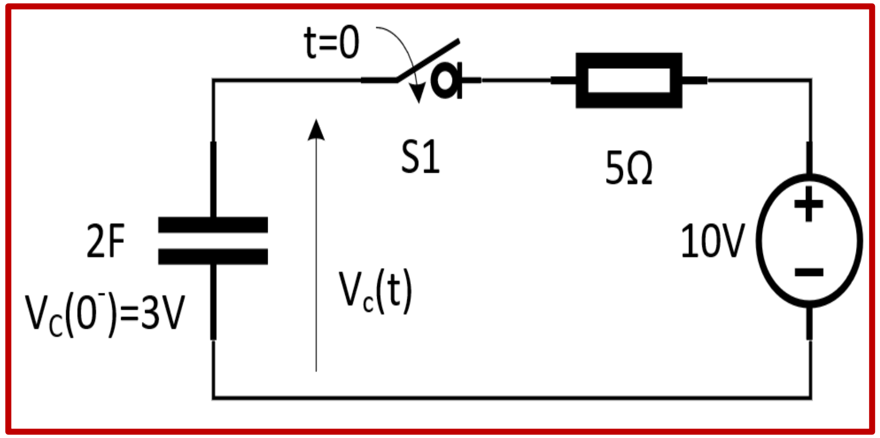



Figure 1

# The Capacitor

**Representation of the Capacitor in the Laplace "S" domain:**

•*The first way "Series":*

We put a voltage source in series with the capacitor as shown in Figure 2. The value of this voltage source, in this case, equals $V_c(0^-)$. Accordingly, this voltage source is represented in the "S" domain as $V_c(0^-)/s$. The capacitor is represented as we learned before as *1/Cs* .

•*The second way "Parallel":*

We put a current source in parallel with the capacitor as shown in Figure 3. The value of this current source, in this case, equals "$CV_c(0^-)$" in the "S" domain. Additionally, the capacitor is represented as we learned before as *1/Cs*

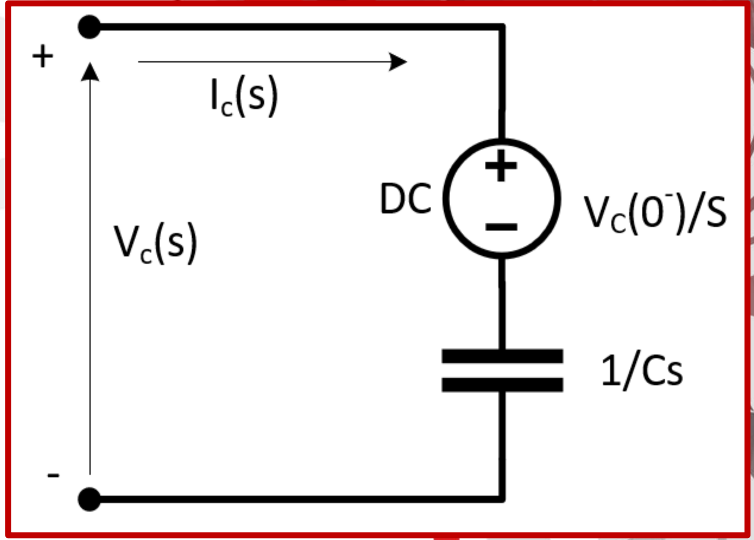



Figure 2

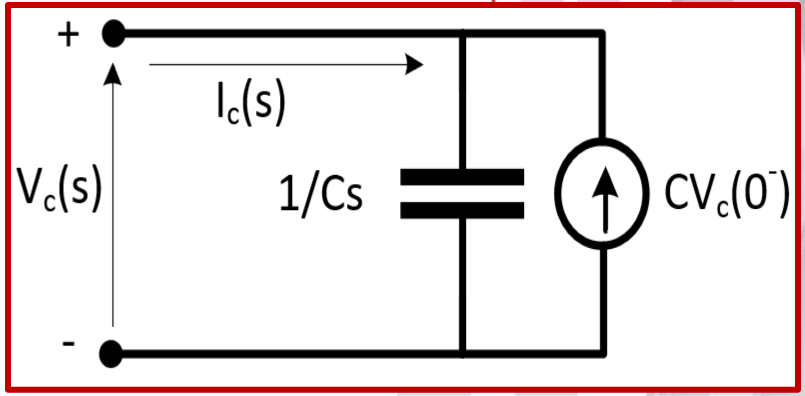



Figure 3

The circuit in Figure 4 will be represented in the "S" domain as either one of the below circuits:

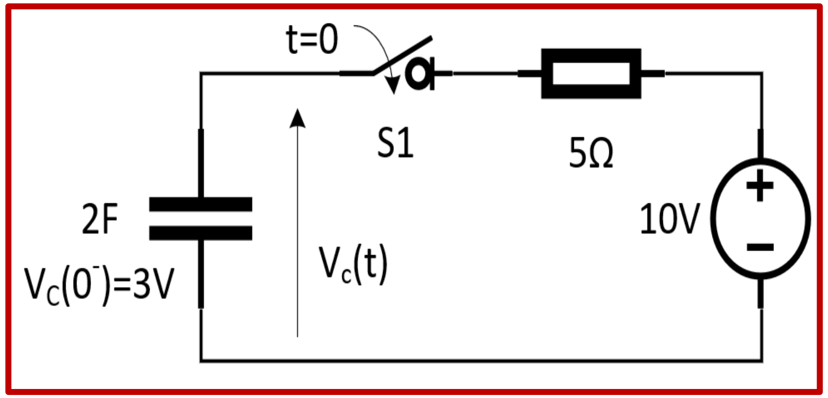
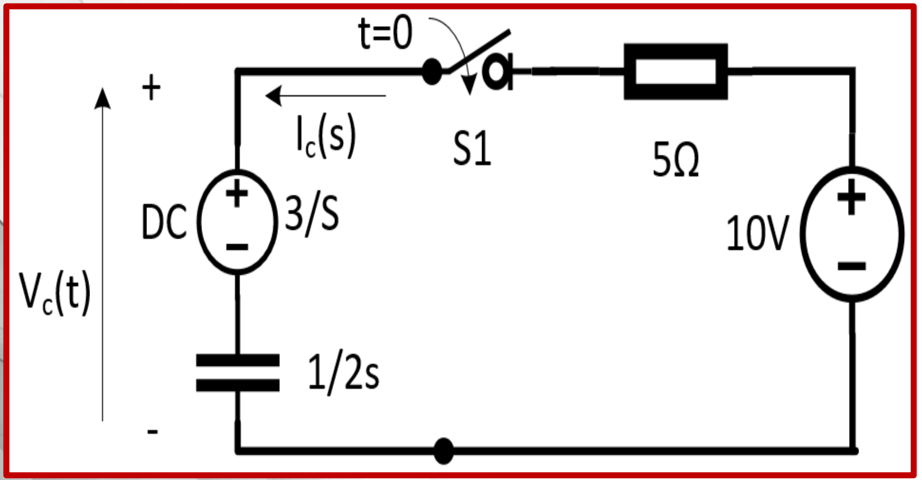
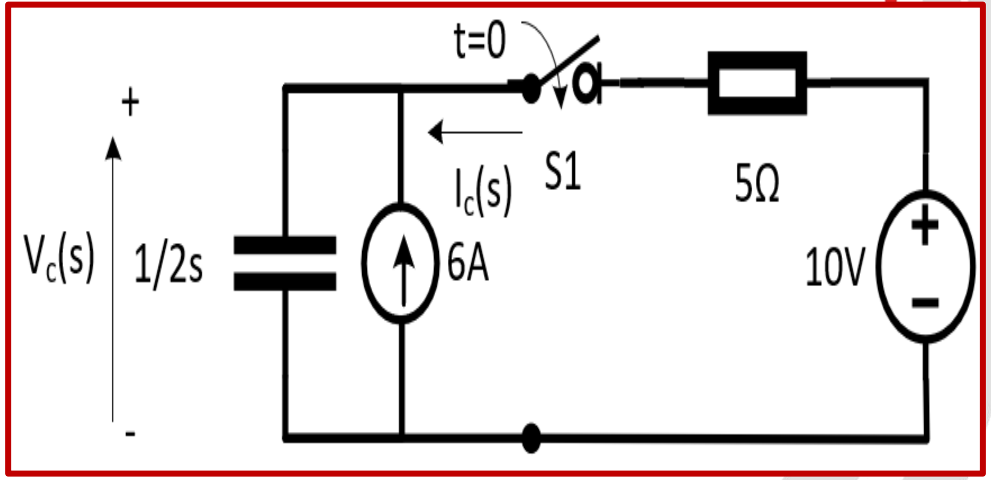



**Figure 4**

Or



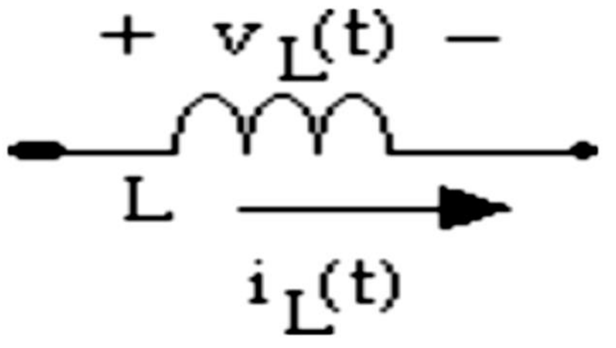



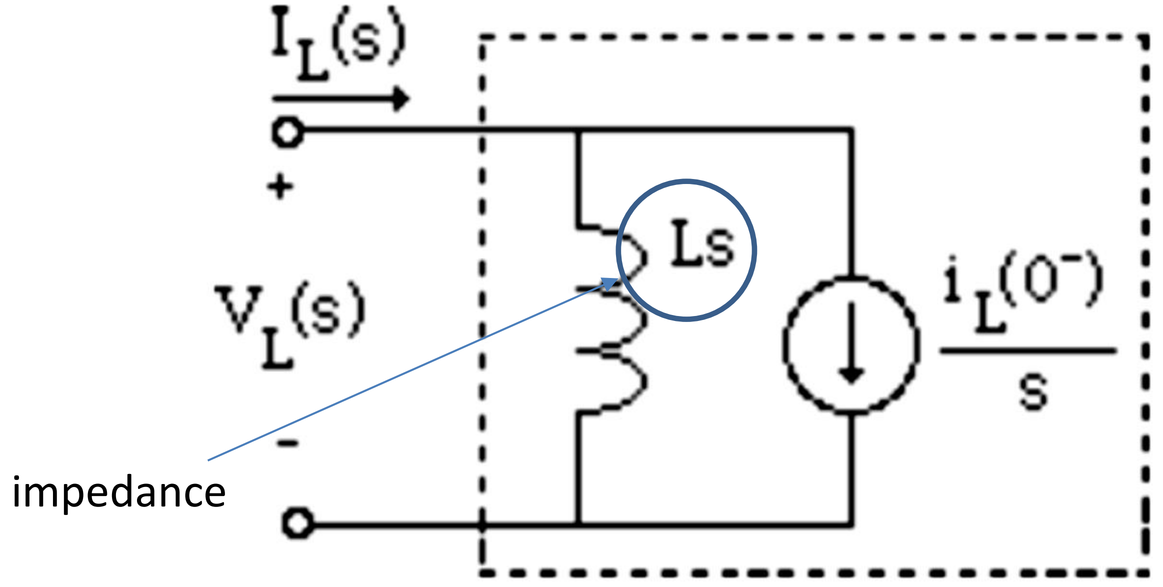
# Equivalent circuit in s domain for an Inductor

In case of non-zero initial condition $i_L(0^-)$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$$

$$I_L(s) = \frac{V_L(s)}{Ls} + \frac{i_L(0^-)}{s}$$



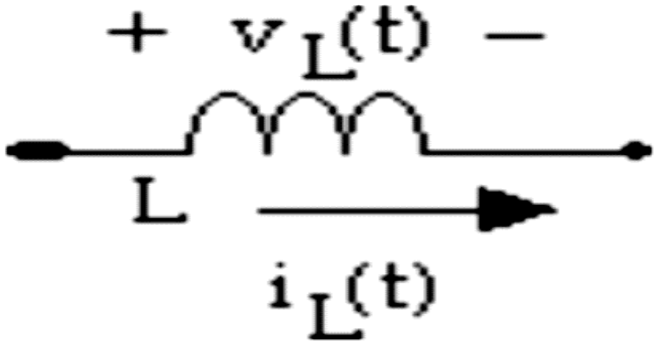



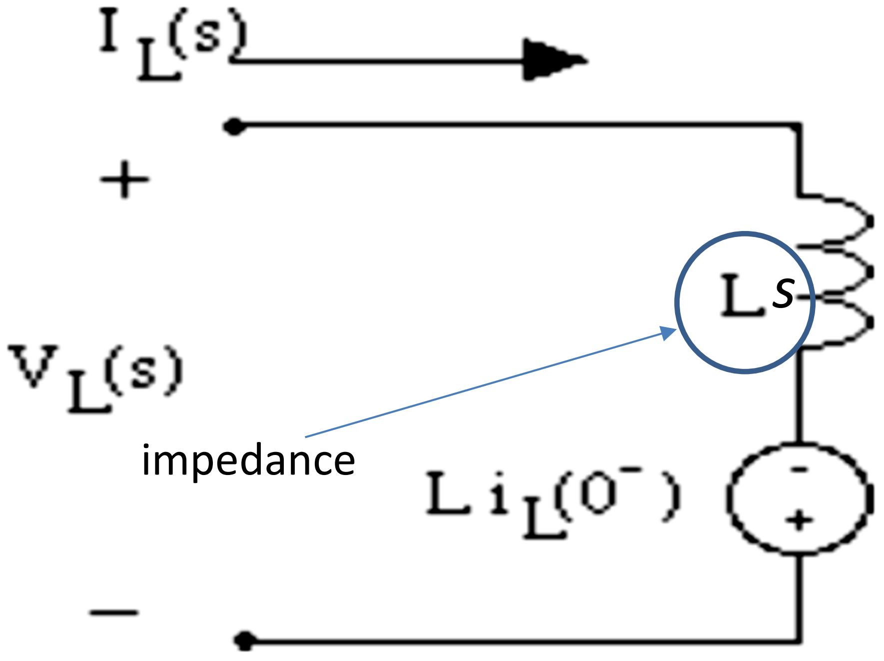
# Equivalent circuit in s domain for an Inductor

In case of non-zero initial condition $i_L(0^-)$

$$v_L(t) = L\frac{di_L}{dt}$$

$$V_L(s) = LsI_L(s) - LI_L(0^-)$$



# The inductor

**Meaning of the initial conditions:**

- For the circuit shown in Figure 5, there was a current passing through the inductor before the switch (S1) closes at time (t=0).
- The value of this current equals 3V.
- Mathematically, we write this current as $i_L(0^-) = 3A$, where "$0^-$" means the time just before "t=0"

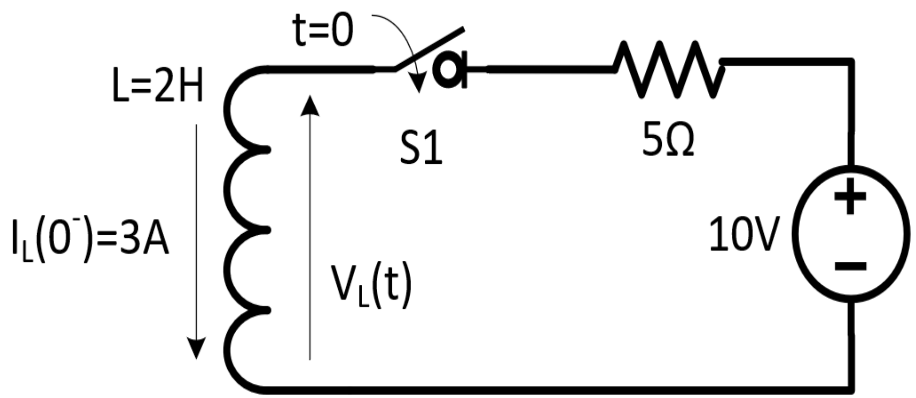



Figure 5

# The inductor

**<u>Representation of the Inductor in the Laplace "S" domain:</u>**

•*<u>The first way "Series":</u>*

We put a voltage source in series with the inductor as shown in Figure 6. The value of this voltage source, in this case, equals "$LI_L(0^-)$" in the "S" domain. The inductor itself is represented as we learned before as "Ls" .

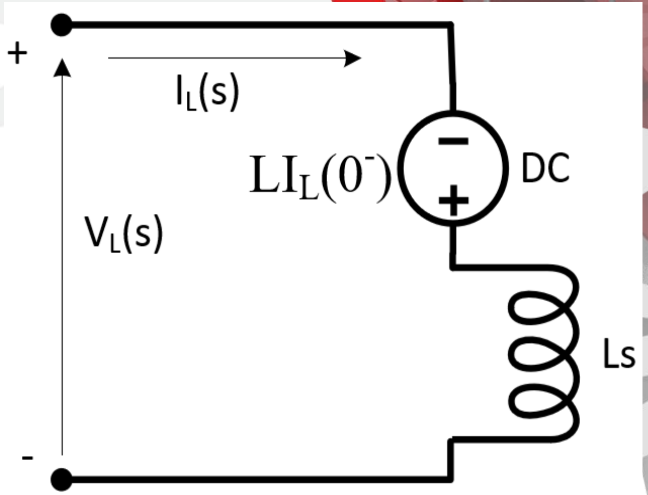



Figure 6

•*<u>The second way "Parallel":</u>*

We put a current source in parallel with the capacitor as shown in Figure 7. The value of this current source, in this case, equals $I_L(0^-)/s$ in the "S" domain. Additionally, the inductor is represented as we learned before as "*Ls*".

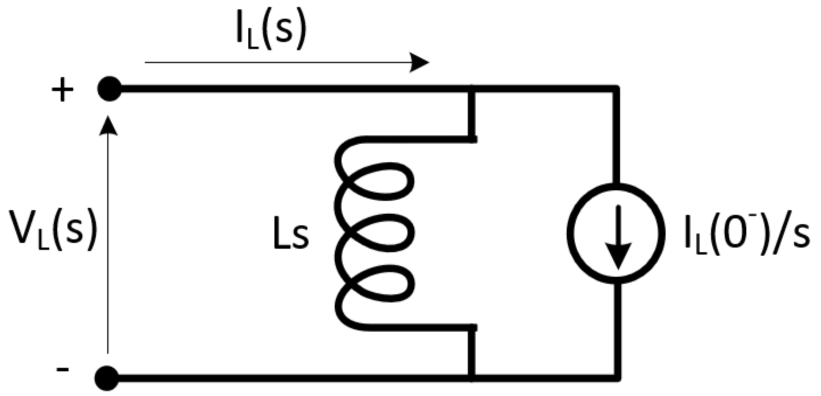



Figure 7

The circuit in Figure 8 will be represented in the “S” domain as either one of the below circuits:

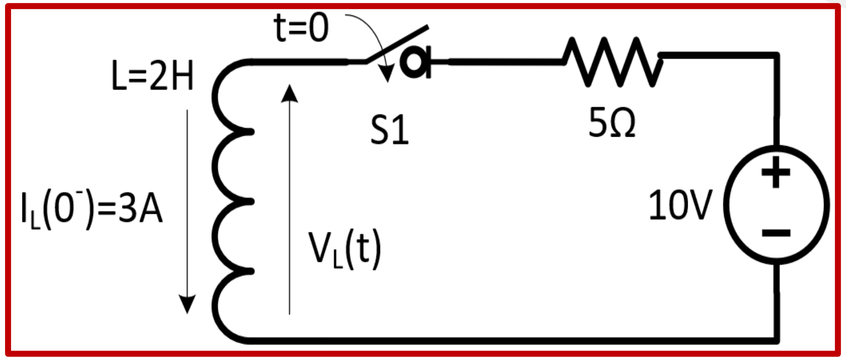
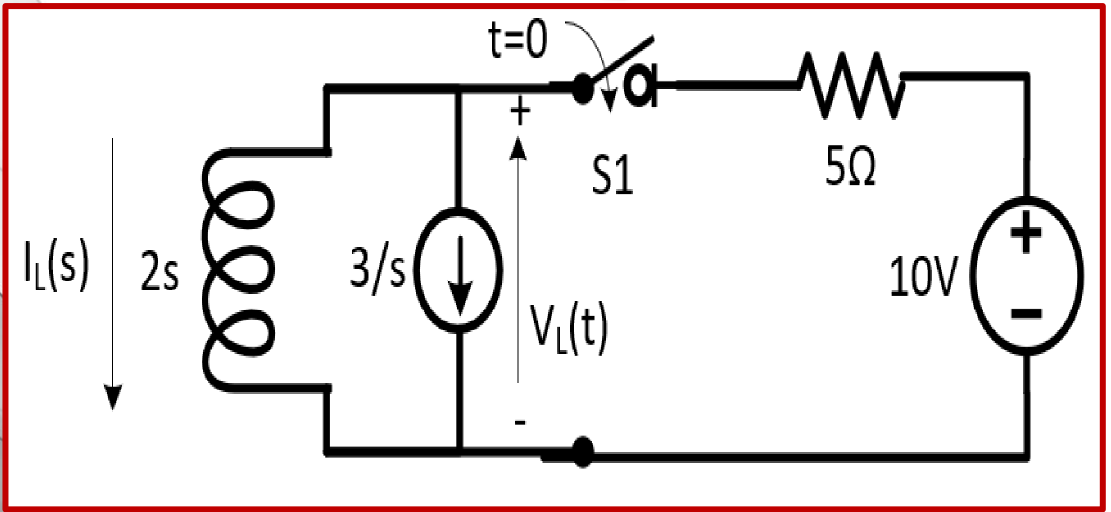
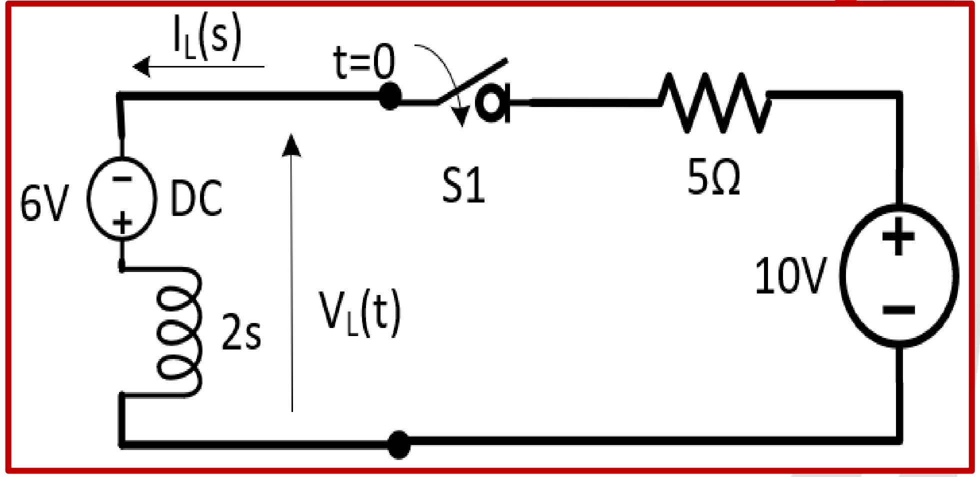



Figure 8



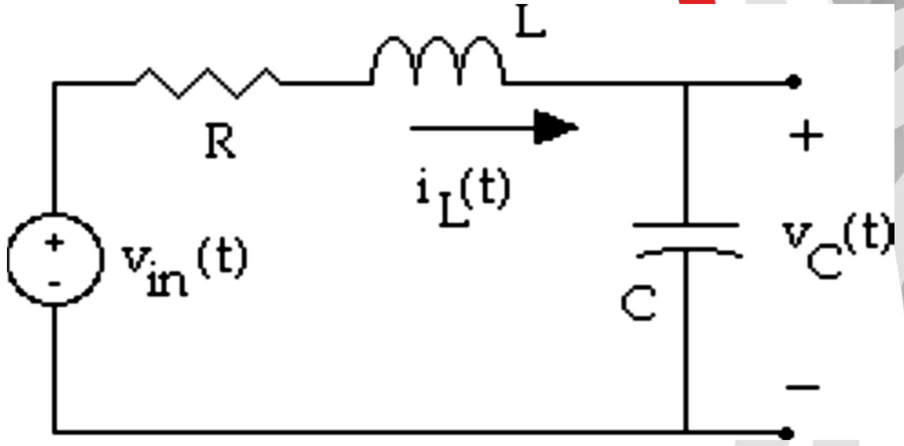
# Example -1

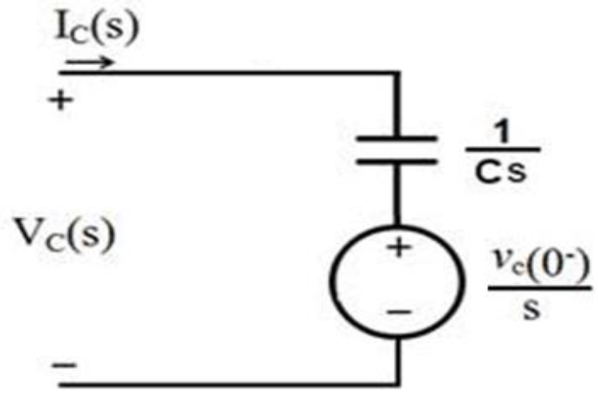
For the Circuit shown, given that R=2 Ω, C=0.25F, L=1H, $Vin(t)=e^{-4t}u(t)V$, $I_L(0^-)=0A$ and $V_C(0^-)=1V$, do the following:

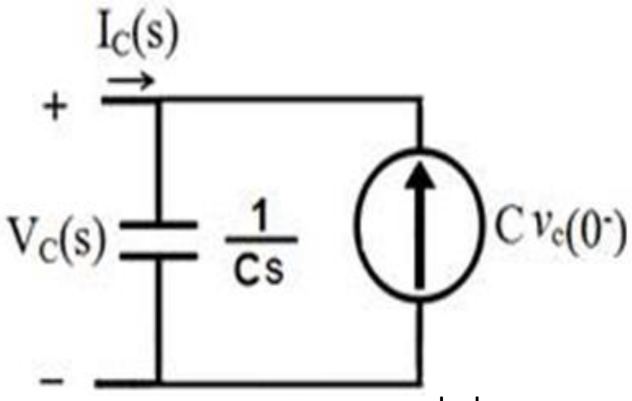
1- Redraw the circuit in the S-domain, include the initial conditions.

2- Find $V_c(s)$.

3- Find $v_c(t)$ if $Vc(s)=\frac{s+6}{s^2+4s+13}\ V$



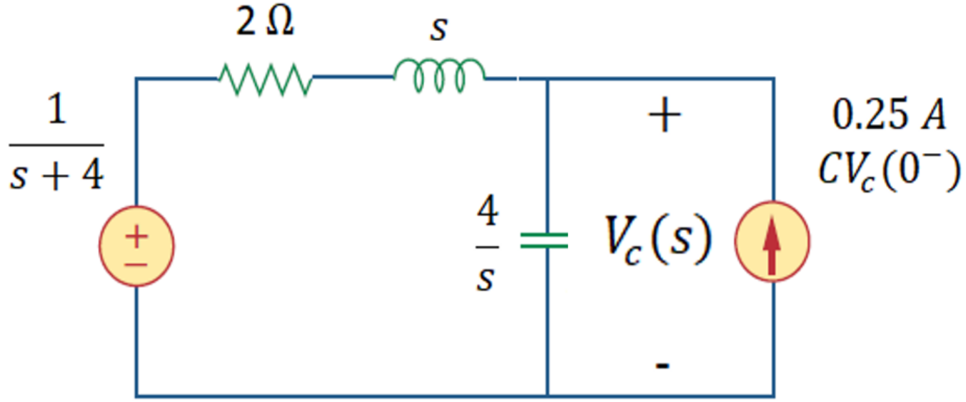



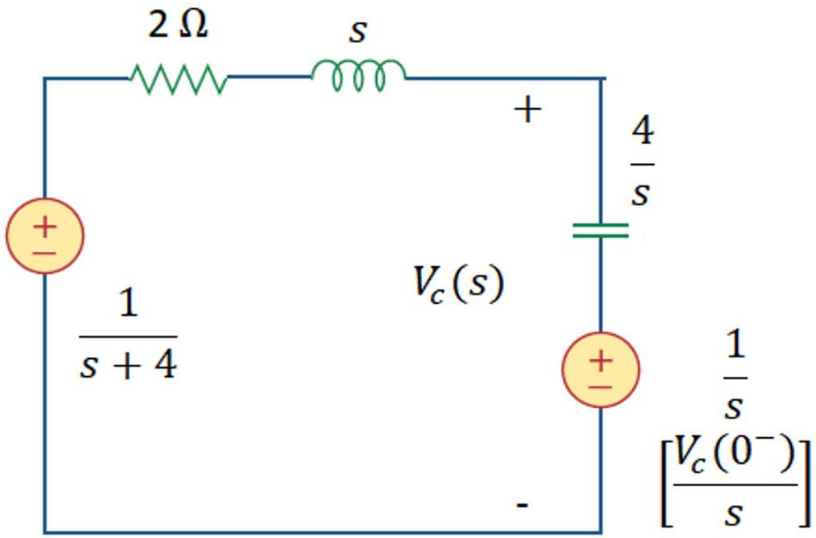



**Step 1.**

- We have zero initial condition for the inductor ($I_L(0^-) = 0A$) as a result we are not going to replace it with the initial condition equivalent circuit. (refer to the previous slide)

- We have two options to redraw the circuit, we can use either the parallel combination or the series combination for the initial condition equivalent circuit. (refer to the previous slide)

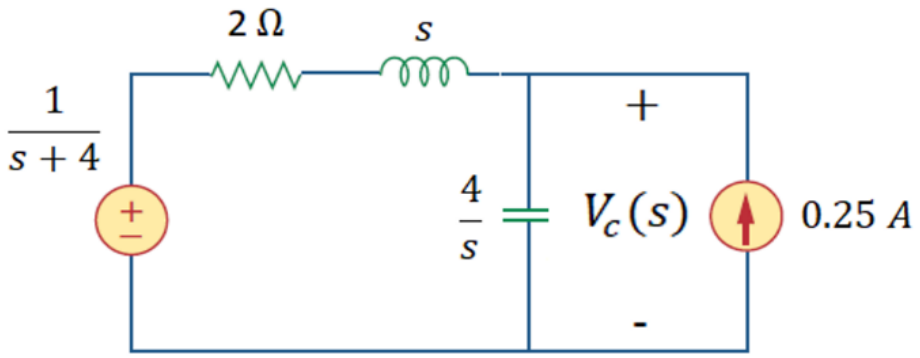



It is based on your preference to choose which circuit to work on. Please choose the circuit that makes you analysis easier and short.

## Step 2.

**Let's use nodal analysis as the required signal is Vc(s), we have one node to analyze Vc(s), which results in one equation. Write a single node equation and solve for $V_C$ (s)**




$$\frac{V_c(s) - \frac{1}{s+4}}{(s+2)} + \frac{V_c(s)}{\frac{4}{s}} - 0.25 = 0$$

Isolate $V_c(s)$ on one side, the rest of terms should go to the other side.

$$\frac{V_c(s)}{(s+2)} - \frac{1}{(s+2)(s+4)} + \frac{sV_c(s)}{4} - 0.25 = 0$$

$$V_c(s)\left[\frac{1}{(s+2)} + \frac{s}{4}\right] = \frac{1}{(s+2)(s+4)} + 0.25$$

Apply common denominator to the left-hand side, no need to do it to the right-hand side as we will be doing partial fraction later on anyways (you can do it if you want).

$$V_c(s)\left[\frac{s^2+2s+4}{4(s+2)}\right] = \frac{1}{(s+2)(s+4)} + 0.25$$

Solve for $V_c(s)$:

$$V_c(s) = \left[\frac{4(s+2)}{(s^2+2s+4)(s+2)(s+4)} + \frac{0.25 * 4(s+2)}{(s^2+2s+4)}\right]$$

$$V_c(s) = \left[\frac{4}{(s^2+2s+4)(s+4)} + \frac{(s+2)}{(s^2+2s+4)}\right]V$$

## Step 3.

By factorizing the denominator $(s^2 + 4s + 13)$ using the calculator, use mode 5 then 3

$S = -2 \pm j3$ Since we have complex roots

a=2 b=3

$$Vc(s) = \frac{s+6}{(s+a)^2 + b^2} = \frac{s+6}{(s+2)^2 + (3)^2}$$

$$\boxed{Vc(s) = \frac{s+6}{s^2 + 4s + 13} V}$$

$$= \frac{s+2+4}{(s+2)^2 + (3)^2} = \frac{(s+2)}{(s+2)^2 + (3)^2} + \frac{4}{(s+2)^2 + (3)^2}$$

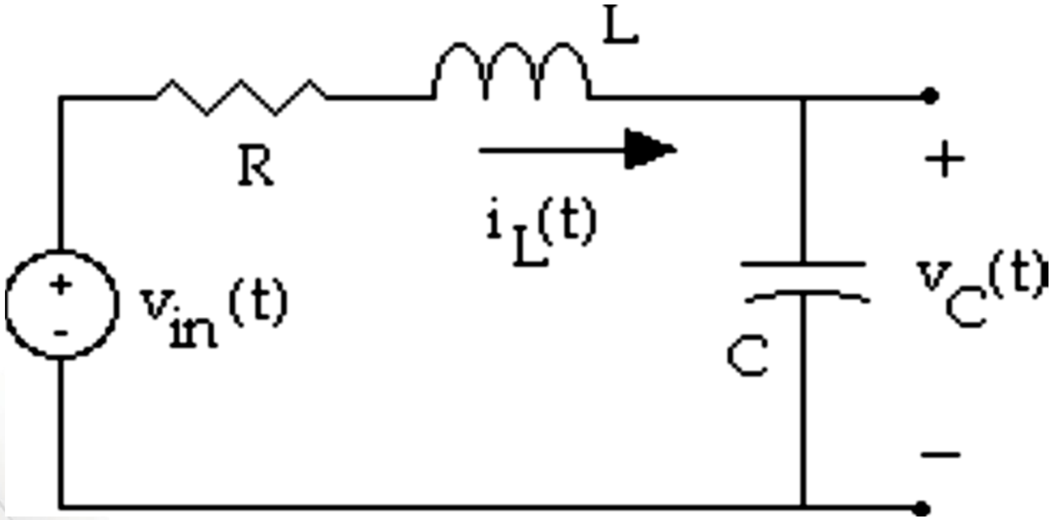
$$Vc(s) = \frac{(s+2)}{(s+2)^2 + (3)^2} + \frac{4}{(s+2)^2 + (3)^2} * \frac{3}{3}$$

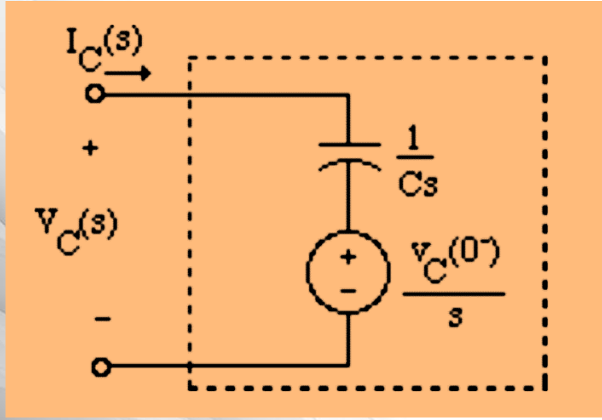
Apply inverse Laplace Transform

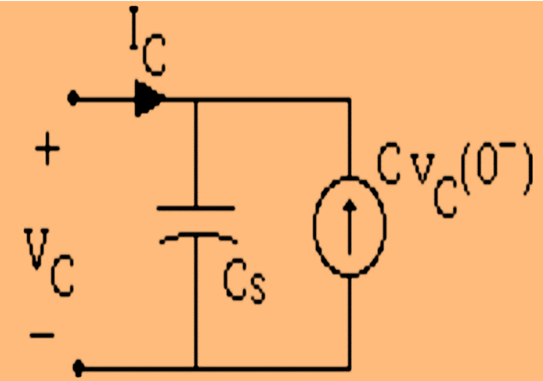
$$v_c(t) = e^{-2t} Cos(3t)u(t) + \frac{4}{3} e^{-2t} Sin(3t)u(t)\ V$$

# Example -2

Find $V_C$ (s) in the following circuit in terms of R, L, C and Vin(s), consider the initial conditions.

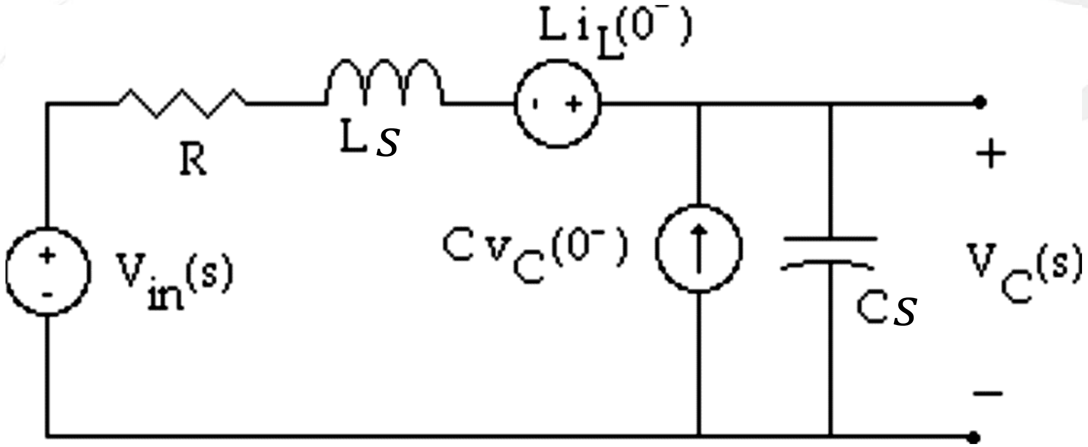


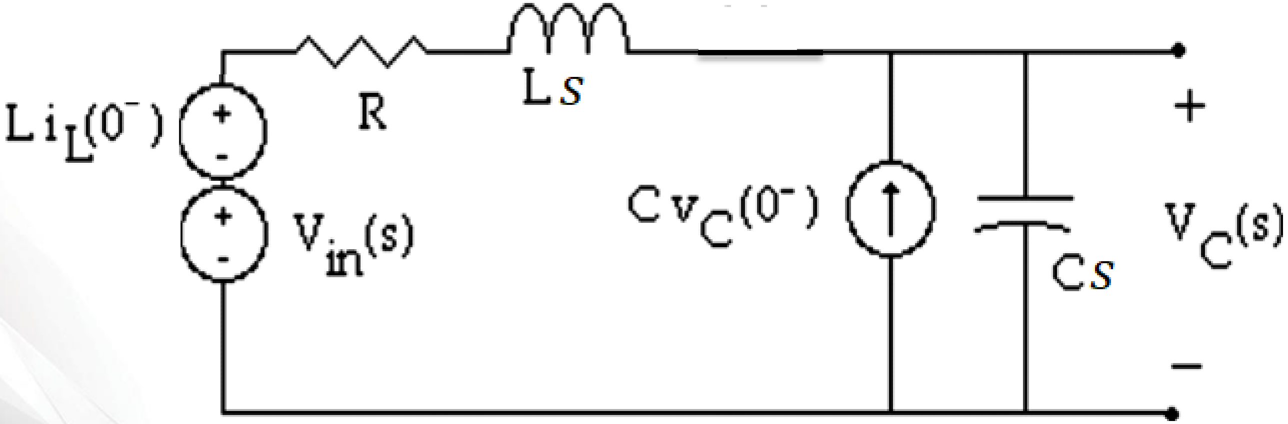



Step 1.

- Draw equivalent circuits accounting for Initial conditions in the *s* domain.

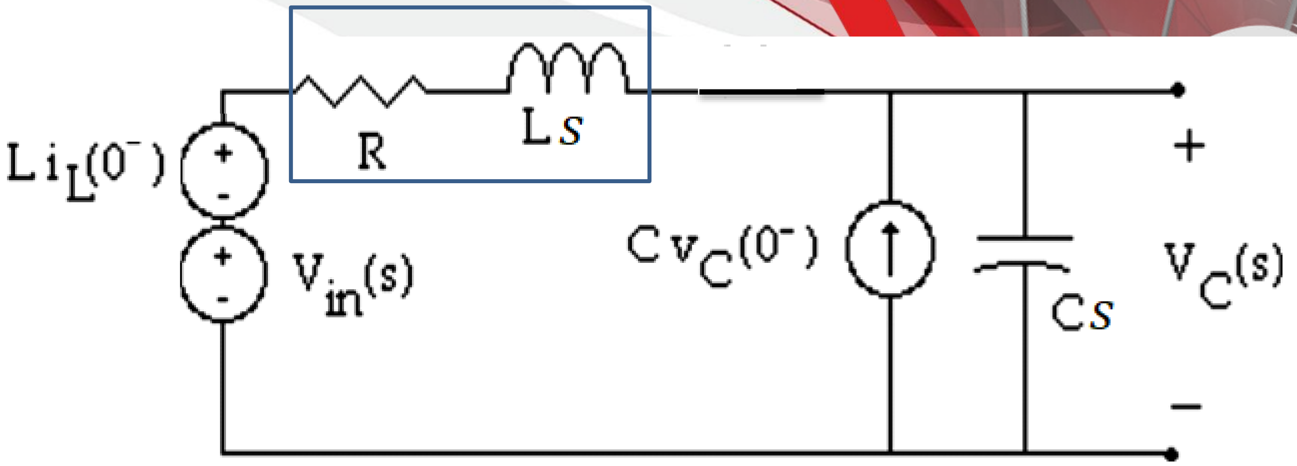



- Move the voltage source associated with the inductor to the left so that there are two voltage sources driving the circuit.



**Step 2.**

Write a single node equation and solve for $V_C$ (s)




$$\frac{1}{Ls+R}\left(V_C - V_{in} - Li_L(0^-)\right) - Cv_C(0^-) + CsV_C = 0$$

$$\left[\frac{LCs^2+CRs+1}{Ls+R}\right]V_C = \frac{1}{Ls+R}V_{in} + \frac{1}{Ls+R}Li_L(0^-) + Cv_C(0^-)$$

$$V_C = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} V_{in} + \frac{\frac{1}{C}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} i_L(0^-) + \frac{s + \frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} v_C(0^-)$$

# Example -2 cont.

Given that R = 2 Ω, C = 0.25 F, L = 0.25 H, $v_{in}(t) = (1 - e^{-4t})u(t)$,

$I_L(0)$=1A and $V_C(0)$=1V. Compute $v_C(t)$.

$$V_C = \frac{16}{(s+4)^2}\left[\frac{1}{s} - \frac{1}{s+4}\right] + \frac{4}{(s+4)^2} + \frac{(s+8)}{(s+4)^2}$$

$$= \frac{16}{s(s+4)^2} - \frac{16}{(s+4)^3} + \frac{(s+12)}{(s+4)^2}$$

$$V_C = \frac{16}{s(s+4)^2} - \frac{16}{(s+4)^3} + \frac{(s+4)}{(s+4)^2} + \frac{8}{(s+4)^2}$$

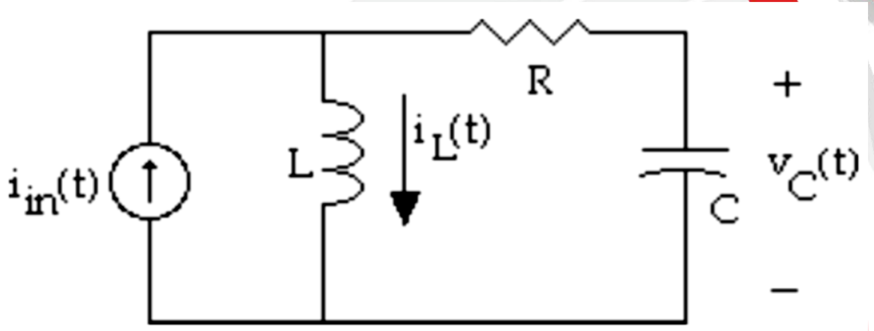
$$V_C = \frac{1}{s} - \cancel{\frac{1}{s+4}} - \frac{4}{(s+4)^2} - \frac{16}{(s+4)^3} + \cancel{\frac{1}{(s+4)}} + \frac{8}{(s+4)^2}$$

$$\boldsymbol{V_C(t) = u(t) + 4e^{-4t}r(t) - 8t^2e^{-4t}u(t)}$$

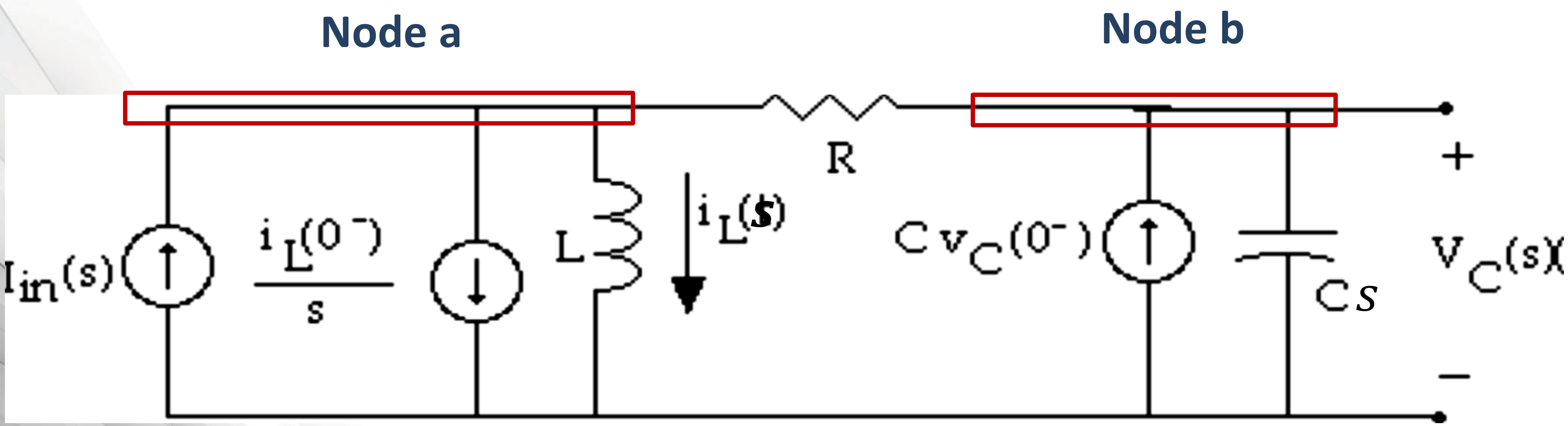
# Example -3

Find $v_C$ (t) when $v_C$ (0− ) = 1 V, $i_L$(0− ) = 1A,

$i_{in}$(t) = 2u(t) A, L = 2 H,R = 2 Ω, and C = 1 F.




*Step 1*. Draw the equivalent s domain circuit accounting for all initial conditions.



*Step 2*. Write 2 nodal equations in the s domain

$$-I_{in}(s)+\frac{i_L(0^-)}{s}+\frac{V_L(s)}{Ls}+\frac{(V_L(s)-V_C(s))}{R}=0$$

**For node a**

$$\boxed{\frac{s+1}{2s}V_L(s)-0.5\,V_C(s)=\frac{1}{s}}$$

$$-C\,V_C(0^-)+\frac{V_C(s)}{\frac{1}{Cs}}+\frac{(V_C(s)-V_L(s))}{R}=0$$

**For node b**

$$\boxed{-0.5\,V_L(s)+(s+0.5)\,V_C(s)=1}$$

*Step 3*. Solve the two equations for Vc

$$\begin{bmatrix} V_L \\ V_C \end{bmatrix} = \frac{1}{(s+0.5)^2+(0.5)^2}\begin{bmatrix} 3s+1 \\ s+2 \end{bmatrix}$$

$$V_C(s) = \frac{s+0.5+3\times 0.5}{(s+0.5)^2+(0.5)^2}$$

$$v_C(t) = e^{-0.5t}\cos(0.5t)u(t) + 3e^{-0.5t}\sin(0.5t)u(t)\ \text{V}.$$

Example 2

$$\begin{bmatrix} \frac{s+1}{2s} & -0.5 \\ -0.5 & s+0.5 \end{bmatrix} \begin{bmatrix} V_L \\ V_c \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$M \quad V_1 = V_2 \qquad M V_1 = V_2$$

$$\rightarrow V_1 = M^{-1} V_2$$

$$\rightarrow \begin{bmatrix} V_L \\ V_c \end{bmatrix} = \begin{bmatrix} \frac{s+1}{2s} & -0.5 \\ -0.5 & s+0.5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} \qquad \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\begin{bmatrix} V_L \\ V_c \end{bmatrix} = \frac{1}{\frac{(s+1)}{2s}(s+0.5) - (0.5)^2} \begin{bmatrix} s+0.5 & +0.5 \\ +0.5 & \frac{s+1}{2s} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$= \frac{1}{\frac{s^2+1.5s+0.5}{2s} - 0.25} \begin{bmatrix} s+0.5 & 0.5 \\ 0.5 & \frac{s+1}{2s} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$= \frac{2s}{s^2+s+0.5} \begin{bmatrix} 1 + \frac{0.5}{s} + 0.5 \\ \frac{0.5}{s} + \frac{s+1}{2s} \end{bmatrix}$$

$$= \frac{2s}{s^2+s+0.5} \begin{bmatrix} \frac{1.5s+0.5}{s} \\ \frac{s+2}{2s} \end{bmatrix} = \frac{2s}{(s+0.5)^2+0.5^2} \begin{bmatrix} \frac{3s+1}{2s} \\ \frac{s+2}{2s} \end{bmatrix}$$

$$= \frac{1}{(s+0.5)^2+0.5^2} \begin{bmatrix} 3s+1 \\ s+2 \end{bmatrix}$$

$$\rightarrow V_c(s) = \frac{s+2}{s^2+s+0.5} = \frac{s+2}{(s+0.5)^2+0.5^2}$$

Another approach

$$\frac{s+1}{2s} V_L(s) - 0.5 V_c(s) = \frac{1}{s} \quad (1)$$

$$-0.5 V_L(s) + (s+0.5) V_c(s) = 1 \quad \left(\times \frac{s+1}{s}\right)$$

$$\rightarrow -0.5\left(\frac{s+1}{s}\right) V_L(s) + (s+0.5)\left(\frac{s+1}{s}\right) V_c(s) = 1 \times \frac{s+1}{s}$$

$$\rightarrow -\frac{s+1}{2s} V_L(s) + \frac{s^2+1.5s+0.5}{s} V_c(s) = \frac{s+1}{s} \quad (2)$$

$$(2) + (1) \rightarrow V_c(s)\left(\frac{s^2+1.5s+0.5}{s} - 0.5\right) = \frac{s+1}{s} + \frac{1}{s}$$

$$V_c(s)\left(\frac{s^2+s+0.5}{s}\right) = \frac{s+2}{s}$$

$$\rightarrow V_c(s) = \frac{s+2}{s^2+s+0.5}$$

# Answer the following questions using ChatGPT

1. What does $v_c(0^-)$ represent in the context of a capacitor, and why is it significant in circuit analysis?
2. Explain the physical significance of $i_L(0^-)$ for an inductor and its role in determining the circuit behaviour at $(t = 0)$.
3. Why is the Laplace transform used to handle initial conditions in circuit analysis?
4. How do initial conditions modify the equivalent impedance and admittance in the s-domain?
5. In practical scenarios, why is it necessary to consider non-zero initial conditions when analysing RLC circuits?
6. How do initial conditions influence energy storage elements in transient circuit analysis?

# Summary

- Initial voltage $v_c(0^-)$ affects equivalent circuits.
    - Store energy as an electric field, initial conditions reflect pre-existing charge.

- Initial current $i_L(0^-)$ influences equivalent circuits.
    - Store energy as a magnetic field, initial conditions reflect pre-existing current flow.
- Enables accurate modeling of circuits with energy-storing components during switching events.
- Vital for understanding system stability and response in the sss-domain.

**Suggested examples**

- Page 620, example 13.6
- Exercise 1, page 623
- Exercise 2, page 623