# $\mathfrak{EE} - 210$. $\mathfrak{Signals\ and\ Systems}$

## Solutions of homework 2*

Spring 2010

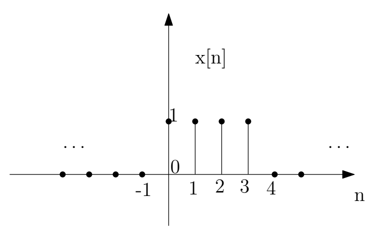
### Exercise Due Date

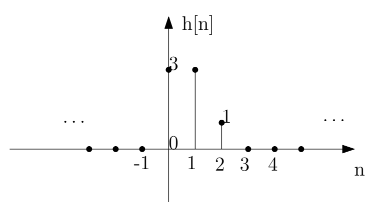
Week of $22^{nd}$ Feb.

### Problems

Q1 Compute and sketch the output y[n] of each discrete-time LTI system below with impulse response h[n] and input signal x[n]. Use the graphical method to compute the discrete-time convolutions.

(a) The impulse response h[n] and input signal x[n] are as depicted below.




ANSWER The output of the system will be the sum of system response to signal h[0], h[1] and h[2]. The three signals are: $h[0]x[n] = 3x[n]$, $h[1]x[n-1] = 3x[n-1]$, $h[2]x[n-2] = x[n-2]$. Thus, the output y[n] of the system will be $y[n] = [0, 0, 3_{\uparrow}, 6, 7, 7, 4, 1, 0, 0]$ ($\uparrow$ indicates the $n = 0$ sample)

(b) $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ , $h[n] = \delta[n+1] - \delta[n] + \delta[n-1]$

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ANSWER $y[n] = [0, 0, 1, 1_{\uparrow}, 0, 1, 1, 0, 0]$ ($\uparrow$ indicates the $n = 0$ sample)

Q2 Compute the convolutions $y[n] = x[n] * h[n]$

(a) $x[n] = \alpha^n u[n]$ , $h[n] = \beta^n u[n]$ , $\alpha \neq \beta$. Sketch the output signal $y[n]$ for the case $\alpha = 0.8$ , $\beta = 0.9$.

ANSWER

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{+\infty} \alpha^k u[k]\beta^{n-k}u[n-k] \\
&= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k , \quad n \geq 0 \\
&= \beta^n \left( \frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}}{1 - \left(\frac{\alpha}{\beta}\right)} \right) u[n] \\
&= \left( \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \right) u[n], \quad \alpha \neq \beta
\end{aligned}$$

you can plug in the values of $\alpha$ and $\beta$ to plot $y[n]$

(b) $x[n] = \delta[n] - \delta[n-1]$ , $h[n] = u[n]$

ANSWER

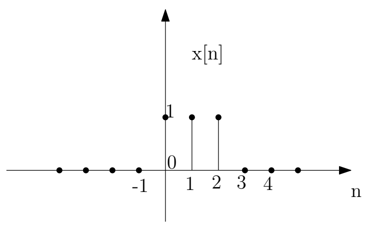
$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} (\delta[k] - \delta[k-1])u[n-k] \\
&= \sum_{k=-\infty}^{+\infty} (\delta[k]u[n-k]) - \sum_{k=-\infty}^{+\infty} (\delta[k-1]u[n-k]) \\
&= u[n] \sum_{k=-\infty}^{+\infty} \delta[k] - u[n-1] \sum_{k=-\infty}^{+\infty} \delta[k-1] \\
&= u[n] - u[n-1] = \delta[n]
\end{aligned}$$

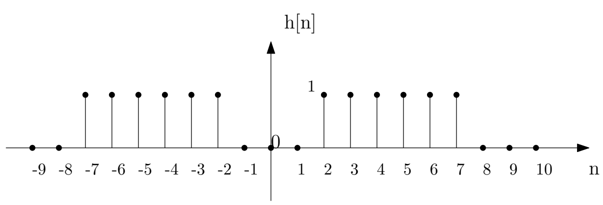
(c) $x[n] = u[n]$ , $h[n] = u[n]$

ANSWER

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} u[k]u[n-k] \\
&= \sum_{k=-\infty}^{+\infty} (u[k]u[-(k-n)]) \\
&= \begin{cases} \sum_{k=0}^{n} 1 = n+1, & n \geq 0 \\ 0, \quad n < 0 \end{cases} \\
&= (n+1)u[n]
\end{aligned}$$

(d) The input signal and impulse response depicted below. Sketch the output signal $y[n]$




ANSWER $\quad y[n] = [0, 0, 0, 0, 1, 2, 3, 3, 3, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, 3, 3, 3, 2, 1, 0, 0]$

Q3 Compute and sketch the output y(t) of the continuous-time LTI system $S_0$ with:
Impulse response $h(t) = e^{1-t}u(t-1)$
Input signal $x(t) = u(t+1) - u(t-3)$

ANSWER

$$y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & 0 \leq t < 4 \\ (e^4 - 1)e^{-t}, & t \geq 4 \end{cases}$$

Q4 Compute the response y(t) of a coninuous-time LTI system described by its impulse response $h(t) = e^{(-\sqrt{3}+j)t}u(t)$ to the input signal $x(t) = u(t)$. Is the system BIBO stable? Is it causal?

ANSWER The step response of the system is 0 for $t < 0$ and for $t > 0$ is given by:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{0}^{t} e^{(-\sqrt{3}+j)(t-\tau)}d\tau = \frac{1}{\sqrt{3}-j}e^{(-\sqrt{3}+j)(t-\tau)}|_0^t, \quad t > 0$$
$$= (e^{-2(t-4)} - e^{-2t}) = (e^8 - 1)e^{-2t}$$

Hence, $y(t) = \dfrac{1}{\sqrt{3}-j}(1 - e^{(-\sqrt{3}+j)t})u(t)$

The system is BIBO stable since

$$\int_{-\infty}^{+\infty} |h(t)|dt = \int_{0}^{+\infty} e^{-\sqrt{3}t}dt = -\frac{1}{\sqrt{3}}\left[e^{-\sqrt{3}t}\right]_0^{+\infty} = \frac{1}{\sqrt{3}} < +\infty.$$

It is also causal since $h(t) = 0, t < 0$

Q5 Consider the following first-order, causal LTI differential system $S_1$ initially at rest:

$$S_1 : \frac{dy(t)}{dt} + ay(t) = \frac{dx(t)}{dt} - 2x(t), \ a > 0 \ is \ real$$

(a) Calculate the impulse response $h_1(t)$ of the system $S_1$. Sketch it for a = 2.

(b) Is the system $S_1$ BIBO stable? Justify your answer.

ANSWER *Step 1*: Set up the problem to calculate the impulse response of the left-hand side of the equation

$$\frac{dh_a(t)}{dt} + ah_a(t) = \delta(t) \tag{1}$$

*Step 2:* Find the initial condition of the corresponding homogeneous equation $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in term $\dfrac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{dh_a(\tau)}{d\tau}d\tau = h_a(0^+) = 1$$

hence $h_1(0^+) = 1$ is our initial condition for the homogeneous equation for $t > 0$

$$\frac{dh_a(t)}{dt} + ah_a(t) = 0$$

*Step 3:* The characteristic polynomial is $p(s) = s + a$ and it has one zero at $s = -a$, which means that the homogeneous response has the form $h_a(t) = Ae^{-at}$ for $t > 0$. The initial condition allows us to determine the constant $A$:

$$h_a(0^+) = A = 1,$$

so that,

$$h_a(t) = e^{at}u(t)$$

Now since LTI systems are commutative. This means that:

$$\begin{aligned}
h_1(t) &= \frac{dh-a(t)}{dt} - 2h_a(t) \\
&= \frac{d}{dt}\left(e^{-at}u(t)\right) - 2e^{-at}u(t) \\
&= -(2+a)e^{-at}u(t) + \delta(t)
\end{aligned}$$

The system is BIBO stable. The single zero of its characteristic polynomial is $s = -a < 0$

Q6 Consider the following second-order, causal LTI differential system $S_2$ initially at rest:

$$S_2 : \frac{d^2y(t)}{dt^2} + 5\frac{dx(y)}{dt} + 6y(t) = x(t)$$

Calculate the impulse response $h_2(t)$ of the system $S_2$.

ANSWER <u>*Step 1*</u>: Set up the problem to calculate the impulse response of the left-hand side of the equation

$$\frac{d^2h_2(t)}{dt^2} + 5\frac{dh_2(t)}{dt} + 6h_2(t) = \delta(t) \tag{2}$$

<u>*Step 2:*</u> Find the initial condition of the corresponding homogeneous equation $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in term $\frac{d^2h_2(t)}{dt^2}$, so $\frac{dh_2(t)}{dt}$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2h_2(\tau)}{d\tau^2} d\tau = \frac{dh_2(0^+)}{dt} = 1$$

hence $\frac{dh_2(0^+)}{dt} = 1$ is one of our initial conditions for the homogeneous equation for $t > 0$

$$\frac{d^2h_2(t)}{dt^2} + 5\frac{dh_2(t)}{dt} + 6h_2(t) = 0$$

Since $\frac{dh_2(t)}{dt}$ has a finite jump at $t = 0^-$ to $t = 0^+$, the other initial condition is $h_2(0^+) =)$.
<u>*Step 3:*</u> The characteristic polynomial is $p(s) = s^2 + 5s + 6$ and it has zeros at $s_1 = -2$, $s_2 = -3$, which means that the homogeneous response has the form $h_a(t) = Ae^{-2t} + Be^{-3t}$ for $t > 0$. The initial condition allows us to determine the constant $A$:

$$\begin{aligned}
h_2(0^+) &= 0 = A + B, \\
\frac{dh_2(0^+)}{dt} &= 1 = -2A - 3B
\end{aligned}$$

so that, $A = 1$, $B = -1$ and finally

$$h_2(t) = (e^{-2t} - e^{-3t})u(t) \tag{3}$$

Q7 Consider the following second-order, causal difference LTI system $S_3$ initially at rest:

$$S_3 : y[n] + 0.9y[n-1] + 0.2y[n-2] = x[n]$$

(a) What is the characteristic polynomial of $S_3$? What are its zeros?

ANSWER

$$p(z) = z^2 + 0.9z + 0.2 = (z + 0.4)(z + 0.5)$$

The zeros are $z_1 = -0.4$, $z_2 = -0.5$.

(b) Compute the impulse response of $S_3$ for all n.

ANSWER The homogeneous response is given by

$$y[n] = A(-0.4)^n + B(-0.5)^n, \; n > 0$$

The initial conditions for the homogeneous equation for $n > 0$ are $y[-1] = 0$ and $y[0] = \delta[0] = 1$. Now we can compute the coefficients A and B:

$$\begin{aligned} y[-1] &= A(-0.4)^{-1} + B(-0.5)^{-1} = -2.5A - 2B = 0 \\ y[0] &= A + B = 1 \end{aligned}$$

Hence,

$$A = -4, B = 5$$

and the impulse response is

$$h_3[n] = \left[-4(-0.4)^n + 5(-0.5)^n\right] u[n]$$