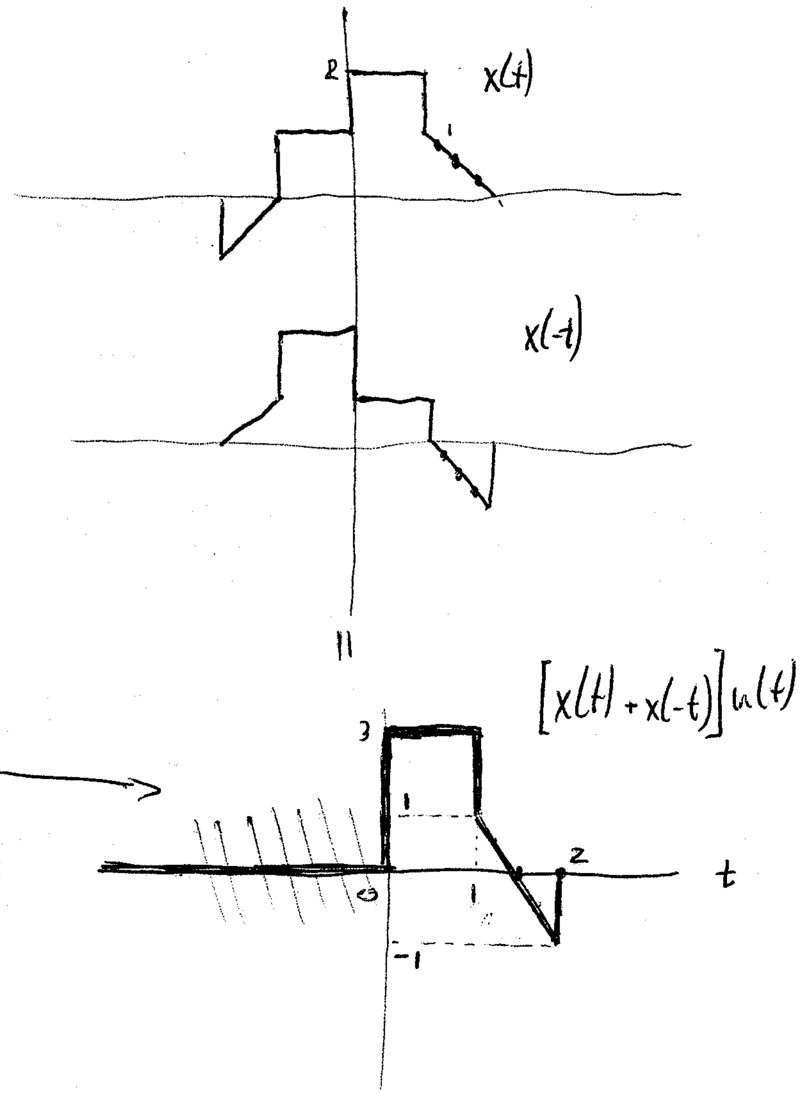
CORRECTED SOLUTIONS

1.21 (e), (f)

$x(t)u(t) + x(-t)u(t)$

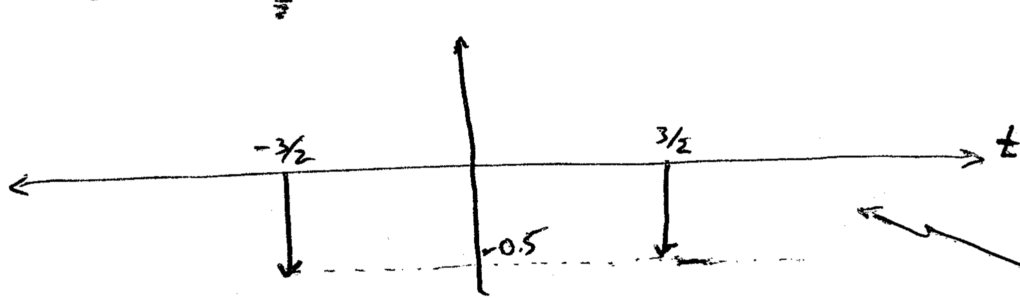
1.21(e)



CORRECT FIGURE

1.21(f) $y(t) = x(t)\left[\delta\left(t+\frac{3}{2}\right) - \delta\left(t-\frac{3}{2}\right)\right] \longrightarrow$ Non-zero at $t = \pm\frac{3}{2}$

$y\left(t = +\frac{3}{2}\right) = -x\left(\frac{3}{2}\right) = \underline{\underline{-0.5}}$ $\quad y\left(t = -\frac{3}{2}\right) = x\left(-\frac{3}{2}\right) = \underline{\underline{-0.5}}$



CORRECT FIGURE

Chapter 1 Answers

1.1. Converting from polar to Cartesian coordinates:

$\frac{1}{2}e^{j\pi} = \frac{1}{2}\cos\pi = -\frac{1}{2}$, $\quad \frac{1}{2}e^{-j\pi} = \frac{1}{2}\cos(-\pi) = -\frac{1}{2}$

$e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$, $\quad e^{-j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$

$e^{j5\frac{\pi}{2}} = e^{j\frac{\pi}{2}} = j$, $\quad \sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) = 1+j$

$\sqrt{2}e^{\frac{9j\pi}{4}} = \sqrt{2}e^{\frac{j\pi}{4}} = 1+j$, $\quad \sqrt{2}e^{\frac{-9j\pi}{4}} = \sqrt{2}e^{\frac{-j\pi}{4}} = 1-j$

$\sqrt{2}e^{\frac{-j\pi}{4}} = 1-j$

1.2. Converting from Cartesian to polar coordinates:

$5 = 5e^{j0}$, $\quad -2 = 2e^{j\pi}$, $\quad -3j = 3e^{-j\frac{\pi}{2}}$

$\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{\pi}{3}}$, $\quad 1+j = \sqrt{2}e^{j\frac{\pi}{4}}$, $\quad (1-j)^2 = 2e^{-j\frac{\pi}{2}}$

$j(1-j) = e^{j\frac{\pi}{4}}$, $\quad \frac{1+j}{1-j} = e^{j\frac{\pi}{2}}$, $\quad \frac{\sqrt{2}+j\sqrt{2}}{1+j\sqrt{3}} = e^{-j\frac{\pi}{12}}$

1.3. **(a)** $E_\infty = \int_0^\infty e^{-4t}dt = \frac{1}{4}$, $P_\infty = 0$, because $E_\infty < \infty$

(b) $x_2(t) = e^{j(2t+\frac{\pi}{4})}$, $|x_2(t)| = 1$. Therefore, $E_\infty = \int_{-\infty}^{\infty} |x_2(t)|^2 dt = \int_{-\infty}^{\infty} dt = \infty$, $P_\infty =$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_2(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt = \lim_{T\to\infty} 1 = 1$$

(c) $x_3(t) = \cos(t)$. Therefore, $E_\infty = \int_{-\infty}^{\infty} |x_3(t)|^2 dt = \int_{-\infty}^{\infty} \cos^2(t) dt = \infty$,

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(t) dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\frac{1+\cos(2t)}{2}\right) dt = \frac{1}{2}$$

(d) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$, $|x_1[n]|^2 = \left(\frac{1}{4}\right)^n u[n]$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4}{3}$,

$P_\infty = 0$, because $E_\infty < \infty$.

(e) $x_2[n] = e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}$, $|x_2[n]|^2 = 1$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \infty$,

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_2[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = 1.$$

(f) $x_3[n] = \cos(\frac{\pi}{4}n)$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=-\infty}^{\infty} \cos^2(\frac{\pi}{4}n) = \infty$,

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2(\frac{\pi}{4}n) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1+\cos(\frac{\pi}{2}n)}{2}\right) = \frac{1}{2}$$

1.4. **(a)** The signal $x[n]$ is shifted by 3 to the right. The shifted signal will be zero for $n < 1$ and $n > 7$.

(b) The signal $x[n]$ is shifted by 4 to the left. The shifted signal will be zero for $n < -6$ and $n > 0$.

(c) The signal $x[n]$ is flipped. The flipped signal will be zero for $n < -4$ and $n > 2$.

(d) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the right. This new signal will be zero for $n < -2$ and $n > 4$.

(e) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the left. This new signal will be zero for $n < -6$ and $n > 0$.

1.5. (a) $x(1-t)$ is obtained by flipping $x(t)$ and shifting the flipped signal by 1 to the right. Therefore, $x(1-t)$ will be zero for $t > -2$.

(b) From (a), we know that $x(1-t)$ is zero for $t > -2$. Similarly, $x(2-t)$ is zero for $t > -1$. Therefore, $x(1-t) + x(2-t)$ will be zero for $t > -2$.

(c) $x(3t)$ is obtained by linearly compressing $x(t)$ by a factor of 3. Therefore, $x(3t)$ will be zero for $t < 1$.

(d) $x(t/3)$ is obtained by linearly stretching $x(t)$ by a factor of 3. Therefore, $x(t/3)$ will be zero for $t < 9$.

1.6. (a) $x_1(t)$ is not periodic because it is zero for $t < 0$.

(b) $x_2[n] = 1$ for all n. Therefore, it is periodic with a fundamental period of 1.

(c) $x_3[n]$ is as shown in the Figure S1.6.

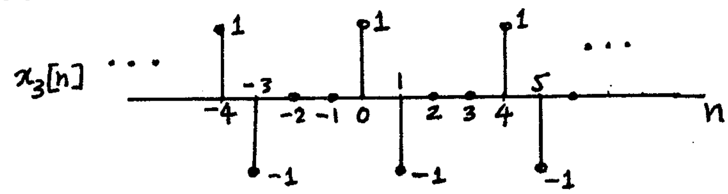


Figure S1.6

Therefore, it is periodic with a fundamental period of 4.

1.7. (a)

$$\mathcal{E}v\{x_1[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}(u[n] - u[n-4] + u[-n] - u[-n-4])$$

Therefore, $\mathcal{E}v\{x_1[n]\}$ is zero for $|n| > 3$.

(b) Since $x_2(t)$ is an odd signal, $\mathcal{E}v\{x_2(t)\}$ is zero for all values of t.

(c)

$$\mathcal{E}v\{x_3[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}[(\frac{1}{2})^n u[n-3] - (\frac{1}{2})^{-n} u[-n-3]]$$

Therefore, $\mathcal{E}v\{x_3[n]\}$ is zero when $|n| < 3$ and when $|n| \to \infty$.

(d)

$$\mathcal{E}v\{x_4(t)\} = \frac{1}{2}(x_4(t) + x_4(-t)) = \frac{1}{2}[e^{-5t}u(t+2) - e^{5t}u(-t+2)$$

Therefore, $\mathcal{E}v\{x_4(t)\}$ is zero only when $|t| \to \infty$.

1.8. (a) $\mathcal{R}e\{x_1(t)\} = -2 = 2e^{0t}\cos(0t + \pi)$

(b) $\mathcal{R}e\{x_2(t)\} = \sqrt{2}\cos(\frac{\pi}{4})\cos(3t + 2\pi) = \cos(3t) = e^{0t}\cos(3t + 0)$

(c) $\mathcal{R}e\{x_3(t)\} = e^{-t}\sin(3t + \pi) = e^{-t}\cos(3t + \frac{\pi}{2})$

(d) $\mathcal{R}e\{x_4(t)\} = -e^{-2t}\sin(100t) = e^{-2t}\sin(100t + \pi) = e^{-2t}\cos(100t + \frac{\pi}{2})$

1.9. (a) $x_1(t)$ is a periodic complex exponential.

$$x_1(t) = je^{j10t} = e^{j(10t + \frac{\pi}{2})}$$

The fundamental period of $x_1(t)$ is $\frac{2\pi}{10} = \frac{\pi}{5}$.

(b) $x_2(t)$ is a complex exponential multiplied by a decaying exponential. Therefore, $x_2(t)$ is not periodic.

(c) $x_3[n]$ is a periodic signal.

$$x_3[n] = e^{j7\pi n} = e^{j\pi n}$$

$x_3[n]$ is a complex exponential with a fundamental period of $\frac{2\pi}{\pi} = 2$.

(d) $x_4[n]$ is a periodic signal. The fundamental period is given by $N = m(\frac{2\pi}{3\pi/5}) = m(\frac{10}{3})$. By choosing $m = 3$, we obtain the fundamental period to be 10.

(e) $x_5[n]$ is not periodic. $x_5[n]$ is a complex exponential with $\omega_0 = 3/5$. We cannot find any integer m such that $m(\frac{2\pi}{\omega_0})$ is also an integer. Therefore, $x_5[n]$ is not periodic.

1.10.

$$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$$

Period of first term in RHS $= \frac{2\pi}{10} = \frac{\pi}{5}$
Period of second term in RHS $= \frac{2\pi}{4} = \frac{\pi}{2}$
Therefore, the overall signal is periodic with a period which is the least common multiple of the periods of the first and second terms. This is equal to π.

1.11.

$$x[n] = 1 + e^{j\frac{4\pi}{7}n} - e^{j\frac{2\pi}{5}n}$$

Period of the first term in the RHS $= 1$
Period of the second term in the RHS $= m(\frac{2\pi}{4\pi/7}) = 7$ (when $m = 2$)
Period of the third term in the RHS $= m(\frac{2\pi}{2\pi/5}) = 5$ (when $m = 1$)
Therefore, the overall signal $x[n]$ is periodic with a period which is the least common multiple of the periods of the three terms in $x[n]$. This is equal to 35.

1.12. The signal $x[n]$ is as shown in Figure S1.12. $x[n]$ can be obtained by flipping $u[n]$ and then shifting the flipped signal by 3 to the right. Therefore, $x[n] = u[-n + 3]$. This implies that $M = -1$ and $n_0 = -3$.

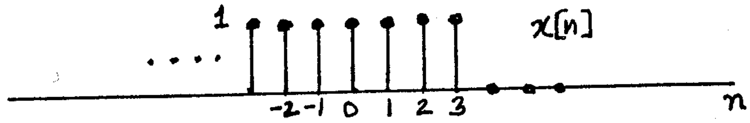


Figure S1.12

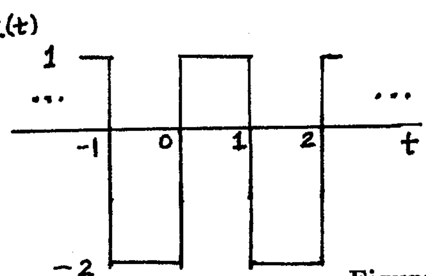
1.13.

$$y(t) = \int_{-\infty}^{t} x(\tau)dt = \int_{-\infty}^{t} (\delta(\tau+2) - \delta(\tau-2))dt = \begin{cases} 0, & t < -2 \\ 1, & -2 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

Therefore,

$$E_\infty = \int_{-2}^{2} dt = 4$$

1.14. The signal $x(t)$ and its derivative $g(t)$ are shown in Figure S1.14.



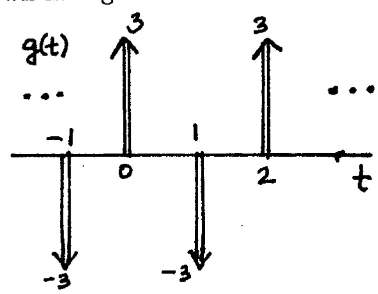


Figure S1.14

Therefore,

$$g(t) = 3 \sum_{k=-\infty}^{\infty} \delta(t-2k) - 3 \sum_{k=-\infty}^{\infty} \delta(t-2k-1)$$

This implies that $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

1.15. **(a)** The signal $x_2[n]$, which is the input to S_2, is the same as $y_1[n]$. Therefore,

$$\begin{aligned} y_2[n] &= x_2[n-2] + \frac{1}{2}x_2[n-3] \\ &= y_1[n-2] + \frac{1}{2}y_1[n-3] \\ &= 2x_1[n-2] + 4x_1[n-3] + \frac{1}{2}(2x_1[n-3] + 4x_1[n-4]) \\ &= 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4] \end{aligned}$$

The input-output relationship for S is

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

(b) The input-output relationship does not change if the order in which S_1 and S_2 are connected in series is reversed. We can easily prove this by assuming that S_1 follows S_2. In this case, the signal $x_1[n]$, which is the input to S_1, is the same as $y_2[n]$. Therefore,

$$\begin{aligned} y_1[n] &= 2x_1[n] + 4x_1[n-1] \\ &= 2y_2[n] + 4y_2[n-1] \\ &= 2(x_2[n-2] + \frac{1}{2}x_2[n-3]) + 4(x_2[n-3] + \frac{1}{2}x_2[n-4]) \\ &= 2x_2[n-2] + 5x_2[n-3] + 2x_2[n-4] \end{aligned}$$

The input-output relationship for S is once again

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

1.16. (a) The system is not memoryless because $y[n]$ depends on past values of $x[n]$.

(b) The output of the system will be $y[n] = \delta[n]\delta[n-2] = 0$.

(c) From the result of part (b), we may conclude that the system output is always zero for inputs of the form $\delta[n-k]$, $k \in \mathcal{I}$. Therefore, the system is not invertible.

1.17. (a) The system is not causal because the output $y(t)$ at some time may depend on future values of $x(t)$. For instance, $y(-\pi) = x(0)$.

(b) Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = x_1(\sin(t))$$

$$x_2(t) \longrightarrow y_2(t) = x_2(\sin(t))$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to the given system, then the corresponding output $y_3(t)$ is

$$\begin{aligned} y_3(t) &= x_3(\sin(t)) \\ &= ax_1(\sin(t)) + bx_2(\sin(t)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is linear.

1.18. (a) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

$$x_2[n] \longrightarrow y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= \sum_{k=n-n_0}^{n+n_0} x_3[k] \\ &= \sum_{k=n-n_0}^{n+n_0} (ax_1[k] + bx_2[k]) = a \sum_{k=n-n_0}^{n+n_0} x_1[k] + b \sum_{k=n-n_0}^{n+n_0} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is linear.

(b) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n - n_1]$$

The output corresponding to this input is

$$y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k] = \sum_{k=n-n_0}^{n+n_0} x_1[k - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k]$$

Also note that

$$y_1[n - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k].$$

Therefore,

$$y_2[n] = y_1[n - n_1]$$

This implies that the system is time-invariant.

(c) If $|x[n]| < B$, then

$$y[n] \leq (2n_0 + 1)B$$

Therefore, $C \leq (2n_0 + 1)B$.

1.19. **(a)** (i) Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = t^2 x_1(t-1)$$

$$x_2(t) \longrightarrow y_2(t) = t^2 x_2(t-1)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = a x_1(t) + b x_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to the given system, then the corresponding output $y_3(t)$ is

$$\begin{aligned} y_3(t) &= t^2 x_3(t-1) \\ &= t^2 (a x_1(t-1) + b x_2(t-1)) \\ &= a y_1(t) + b y_2(t) \end{aligned}$$

Therefore, the system is **linear.**

(ii) Consider an arbitrary input $x_1(t)$. Let

$$y_1(t) = t^2 x_1(t-1)$$

be the corresponding output. Consider a second input $x_2(t)$ obtained by shifting $x_1(t)$ in time:

$$x_2(t) = x_1(t - t_0)$$

The output corresponding to this input is

$$y_2(t) = t^2 x_2(t-1) = t^2 x_1(t-1-t_0)$$

Also note that

$$y_1(t - t_0) = (t - t_0)^2 x_1(t - 1 - t_0) \neq y_2(t)$$

Therefore, the system is **not time-invariant.**

(b) (i) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = x_1^2[n-2]$$

$$x_2[n] \longrightarrow y_2[n] = x_2^2[n-2]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = a x_1[n] + b x_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= x_3^2[n-2] \\ &= (a x_1[n-2] + b x_2[n-2])^2 \\ &= a^2 x_1^2[n-2] + b^2 x_2^2[n-2] + 2ab x_1[n-2] x_2[n-2] \\ &\neq a y_1[n] + b y_2[n] \end{aligned}$$

Therefore, the system is **not linear.**

(ii) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = x_1^2[n-2]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n-n_0]$$

The output corresponding to this input is

$$y_2[n] = x_2^2[n-2] = x_1^2[n-2-n_0]$$

Also note that

$$y_1[n-n_0] = x_1^2[n-2-n_0]$$

Therefore,

$$y_2[n] = y_1[n-n_0]$$

This implies that the system is **time-invariant.**

(c) (i) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = x_1[n+1] - x_1[n-1]$$

$$x_2[n] \longrightarrow y_2[n] = x_2[n+1] - x_2[n-1]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= x_3[n+1] - x_3[n-1] \\ &= ax_1[n+1] + bx_1[n+1] - ax_1[n-1] - bx_2[n-1] \\ &= a(x_1[n+1] - x_1[n-1]) + b(x_2[n+1] - x_2[n-1]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is **linear.**

(ii) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = x_1[n+1] - x_1[n-1]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n-n_0]$$

The output corresponding to this input is

$$y_2[n] = x_2[n+1] - x_2[n-1] = x_1[n+1-n_0] - x_1[n-1-n_0]$$

Also note that

$$y_1[n-n_0] = x_1[n+1-n_0] - x_1[n-1-n_0]$$

Therefore,

$$y_2[n] = y_1[n-n_0]$$

This implies that the system is **time-invariant.**

(d) (i) Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = \mathcal{O}d\{x_1(t)\}$$

$$x_2(t) \longrightarrow y_2(t) = \mathcal{O}d\{x_2(t)\}$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to the given system, then the corresponding output $y_3(t)$ is

$$\begin{aligned} y_3(t) &= \mathcal{O}d\{x_3(t)\} \\ &= \mathcal{O}d\{ax_1(t) + bx_2(t)\} \\ &= a\mathcal{O}d\{x_1(t)\} + b\mathcal{O}d\{x_2(t)\} = ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is **linear**.

(ii) Consider an arbitrary input $x_1(t)$. Let

$$y_1[t) = \mathcal{O}d\{x_1(t)\} = \frac{x_1(t) - x_1(-t)}{2}$$

be the corresponding output. Consider a second input $x_2(t)$ obtained by shifting $x_1[n]$ in time:

$$x_2(t) = x_1(t-t_0)$$

The output corresponding to this input is

$$\begin{aligned} y_2(t) &= \mathcal{O}d\{x_2(t)\} = \frac{x_2(t) - x_2(-t)}{2} \\ &= \frac{x_1(t-t_0) - x_1(-t-t_0)}{2} \end{aligned}$$

Also note that

$$y_1(t-t_0) = \frac{x_1(t-t_0) - x_1(-t+t_0)}{2} \neq y_2(t)$$

Therefore, the system is **not time-invariant**.

1.20. **(a)** Given

$$x(t) = e^{j2t} \longrightarrow y(t) = e^{j3t}$$
$$x(t) = e^{-j2t} \longrightarrow y(t) = e^{-j3t}$$

Since the system is linear,

$$x_1(t) = \frac{1}{2}(e^{j2t} + e^{-j2t}) \longrightarrow y_1(t) = \frac{1}{2}(e^{j3t} + e^{-j3t})$$

Therefore,

$$x_1(t) = \cos(2t) \longrightarrow y_1(t) = \cos(3t)$$

(b) We know that

$$x_2(t) = \cos\left(2(t - \frac{1}{2})\right) = \frac{e^{-j}e^{j2t} + e^{j}e^{-j2t}}{2}$$

Using the linearity property, we may once again write

$$x_1(t) = \frac{1}{2}(e^{-j}e^{j2t} + eje^{-j2t}) \longrightarrow y_1(t) = \frac{1}{2}(e^{-j}e^{j3t} + e^{j}e^{-j3t}) = \cos(3t-1)$$

Therefore,

$$x_1(t) = \cos(2(t-1/2)) \longrightarrow y_1(t) = \cos(3t-1)$$

1.21. The signals are sketched in Figure S1.21.

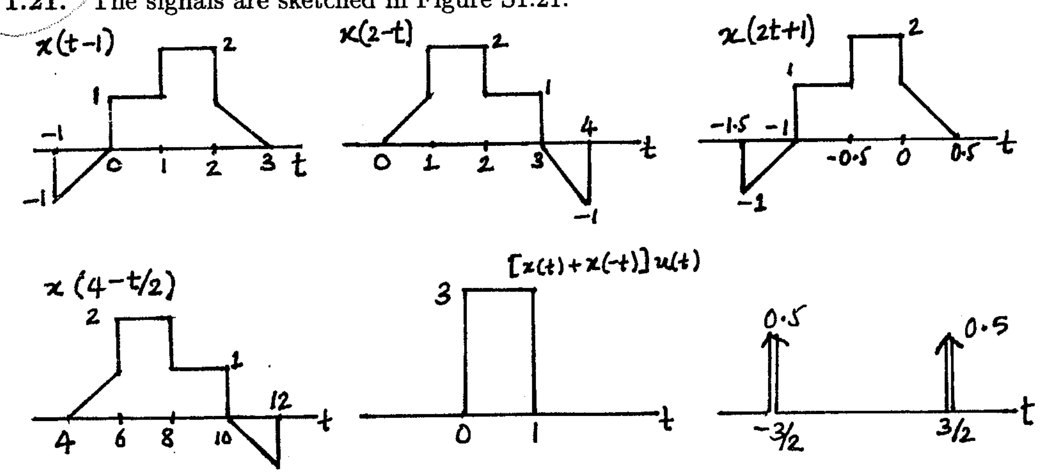


Figure S1.21

1.22. The signals are sketched in Figure S1.22.

1.23. The even and odd parts are sketched in Figure S1.23.

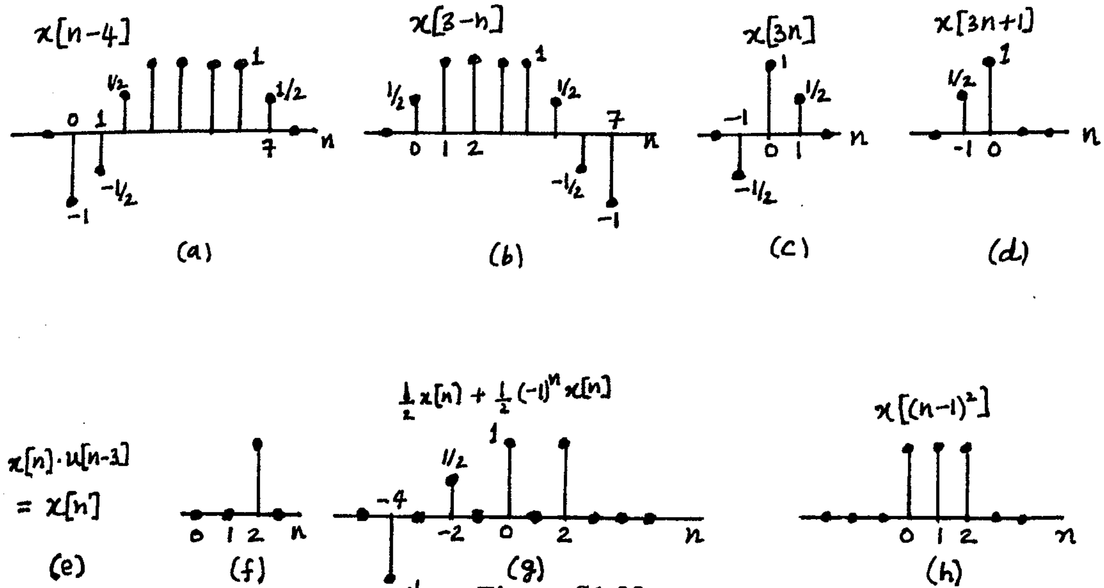


Figure S1.22

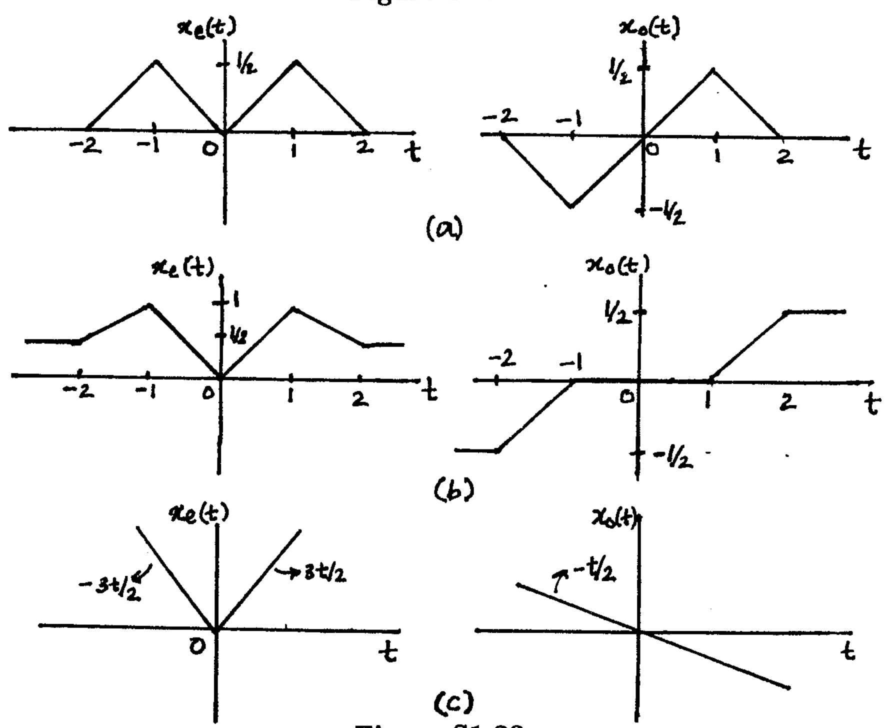


Figure S1.23

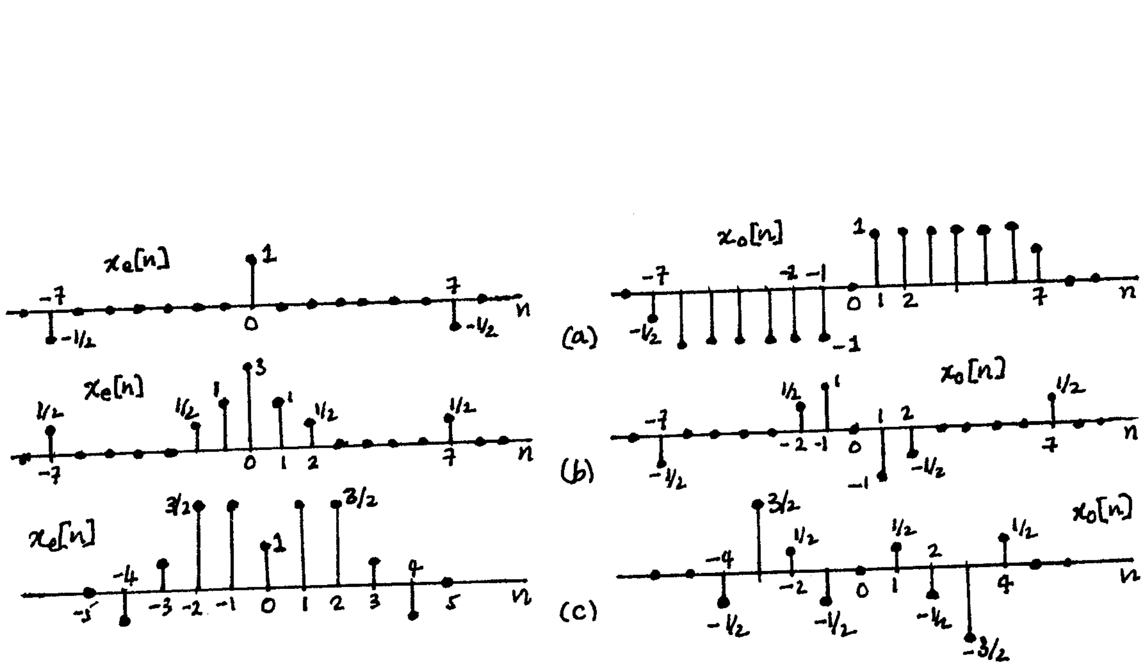


Figure S1.24

1.24. The even and odd parts are sketched in Figure S1.24.

1.25. **(a)** Periodic, period $= 2\pi/(4) = \pi/2$.

(b) Periodic, period $= 2\pi/(\pi) = 2$.

(c) $x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

(d) $x(t) = \cos(4\pi t)/2$. Periodic, period $= 2\pi/(4\pi) = 1/2$.

(e) $x(t) = [\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)]/2$. Not periodic.

(f) Not periodic.

1.26. **(a)** Periodic, period $= 7$.

(b) Not periodic.

(c) Periodic, period $= 8$.

(d) $x[n] = (1/2)[\cos(3\pi n/4) + \cos(\pi n/4)]$. Periodic, period $= 8$.

(e) Periodic, period $= 16$.

1.27. **(a)** Linear, stable.

(b) Memoryless, linear, causal, stable.

(c) Linear

(d) Linear, causal, stable.

(e) Time invariant, linear, causal, stable.

(f) Linear, stable.

(g) Time invariant, linear, causal.

1.30. (a) Invertible. Inverse system: $y(t) = x(t+4)$.

(b) Non invertible. The signals $x(t)$ and $x_1(t) = x(t) + 2\pi$ give the same output.

(c) Non invertible. $\delta[n]$ and $2\delta[n]$ give the same output.

(d) Invertible. Inverse system: $y(t) = dx(t)/dt$.

(e) Invertible. Inverse system: $y[n] = x[n+1]$ for $n \geq 0$ and $y[n] = x[n]$ for $n < 0$.

(f) Non invertible. $x[n]$ and $-x[n]$ give the same result.

(g) Invertible. Inverse system: $y[n] = x[1-n]$.

(h) Invertible. Inverse system: $y(t) = x(t) + dx(t)/dt$.

(i) Invertible. Inverse system: $y[n] = x[n] - (1/2)x[n-1]$.

(j) Non invertible. If $x(t)$ is any constant, then $y(t) = 0$.

(k) Non invertible. $\delta[n]$ and $2\delta[n]$ result in $y[n] = 0$.

(l) Invertible. Inverse system: $y(t) = x(t/2)$.

(m) Non invertible. $x_1[n] = \delta[n] + \delta[n-1]$ and $x_2[n] = \delta[n]$ give $y[n] = \delta[n]$.

(n) Invertible. Inverse system: $y[n] = x[2n]$.

1.31. (a) Note that $x_2(t) = x_1(t) - x_1(t-2)$. Therefore, using linearity we get $y_2(t) = y_1(t) - y_1(t-2)$. This is as shown in Figure S1.31.

(b) Note that $x_3(t) = x_1(t) + x_1(t+1)$. Therefore, using linearity we get $y_3(t) = y_1(t) + y_1(t+1)$. This is as shown in Figure S1.31.

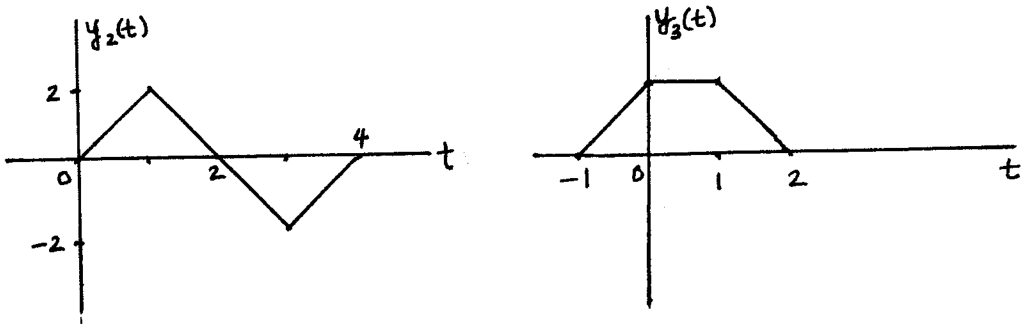


Figure S1.31

1.32. All statements are true.

(1) $x(t)$ periodic with period T; $y_1(t)$ periodic, period $T/2$.

(2) $y_1(t)$ periodic, period T; $x(t)$ periodic, period $2T$.

(3) $x(t)$ periodic, period T; $y_2(t)$ periodic, period $2T$.

(4) $y_2(t)$ periodic, period T; $x(t)$ periodic, period $T/2$.

1.33. (1) True. $x[n] = x[n+N]; y_1[n] = y_1[n+N_0]$. i.e. periodic with $N_0 = N/2$ if N is even, and with period $N_0 = N$ if N is odd.

(2) False. $y_1[n]$ periodic does no imply $x[n]$ is periodic. i.e. let $x[n] = g[n] + h[n]$ where

$$g[n] = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 0, & n \text{ even} \\ (1/2)^n, & n \text{ odd} \end{cases}.$$

Then $y_1[n] = x[2n]$ is periodic but $x[n]$ is clearly not periodic.

(3) True. $x[n+N] = x[n]$; $y_2[n+N_0] = y_2[n]$ where $N_0 = 2N$

(4) True. $y_2[n+N] = y_2[n]$; $x[n+N_0] = x[n]$ where $N_0 = N/2$

1.34. (a) Consider

$$\sum_{n=-\infty}^{\infty} x[n] = x[0] + \sum_{n=1}^{\infty}\{x[n] + x[-n]\}.$$

If $x[n]$ is odd, $x[n] + x[-n] = 0$. Therefore, the given summation evaluates to zero.

(b) Let $y[n] = x_1[n]x_2[n]$. Then

$$y[-n] = x_1[-n]x_2[-n] = -x_1[n]x_2[n] = -y[n].$$

This implies that $y[n]$ is odd.

(c) Consider

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^2[n] &= \sum_{n=-\infty}^{\infty} \{x_e[n] + x_o[n]\}^2 \\ &= \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n] + 2\sum_{n=-\infty}^{\infty} x_e[n]x_o[n]. \end{aligned}$$

Using the result of part (b), we know that $x_e[n]x_o[n]$ is an odd signal. Therefore, using the result of part (a) we may conclude that

$$2\sum_{n=-\infty}^{\infty} x_e[n]x_o[n] = 0.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} x^2[n] == \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n].$$

(d) Consider

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt.$$

Again, since $x_e(t)x_o(t)$ is odd,

$$\int_{-\infty}^{\infty} x_e^2(t)x_o(t)dt = 0.$$

Therefore,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt.$$

1.35. We want to find the smallest N_0 such that $m(2\pi/N)N_0 = 2\pi k$ or $N_0 = kN/m$, where k is an integer. If N_0 has to be an integer, then N must be a multiple of m/k and m/k must be an integer. This implies that m/k is a divisor of both m and N. Also, if we want the smallest possible N_0, then m/k should be the GCD of m and N. Therefore, $N_0 = N/\text{gcd}(m, N)$.

1.36. **(a)** If $x[n]$ is periodic $e^{j\omega_0(n+N)T} = e^{j\omega_0 nT}$, where $\omega_0 = 2\pi/T_0$. This implies that

$$\frac{2\pi}{T_0}NT = 2\pi k \quad \Rightarrow \quad \frac{T}{T_0} = \frac{k}{N} = \text{a rational number.}$$

(b) If $T/T_0 = p/q$ then $x[n] = e^{j2\pi n(p/q)}$. The fundamental period is $q/\text{gcd}(p, q)$ and the fundamental frequency is

$$\frac{2\pi}{q}\text{gcd}(p,q) = \frac{2\pi}{p}\frac{p}{q}\text{gcd}(p,q) = \frac{\omega_0}{p}\text{gcd}(p,q) = \frac{\omega_0 T}{p}\text{gcd}(p,q).$$

(c) $p/\text{gcd}(p, q)$ periods of $x(t)$ are needed.

1.37. **(a)** From the definition of $\phi_{xy}(t)$, we have

$$\begin{aligned}\phi_{xy}(t) &= \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau \\ &= \int_{-\infty}^{\infty} y(-t+\tau)x(\tau)d\tau \\ &= \phi_{yx}(-t).\end{aligned}$$

(b) Note from part (a) that $\phi_{xx}(t) = \phi_{xx}(-t)$. This implies that $\phi_{xx}(t)$ is even. Therefore, the odd part of $\phi_{xx}(t)$ is zero.

(c) Here, $\phi_{xy}(t) = \phi_{xx}(t - T)$ and $\phi_{yy}(t) = \phi_{xx}(t)$.

1.38. **(a)** We know that $2\delta_\Delta(2t) = \delta_{\Delta/2}(t)$. Therefore,

$$\lim_{\Delta\to 0}\delta_\Delta(2t) = \lim_{\Delta\to 0}\frac{1}{2}\delta_{\Delta/2}(t).$$

This implies that

$$\delta(2t) = \frac{1}{2}\delta(t).$$

(b) The plots are as shown in Figure S1.38.

1.39. We have

$$\lim_{\Delta\to 0} u_\Delta(t)\delta(t) = \lim_{\Delta\to 0} u_\Delta(0)\delta(t) = 0.$$

Also,

$$\lim_{\Delta\to 0} u_\Delta(t)\delta_\Delta(t) = \frac{1}{2}\delta(t).$$

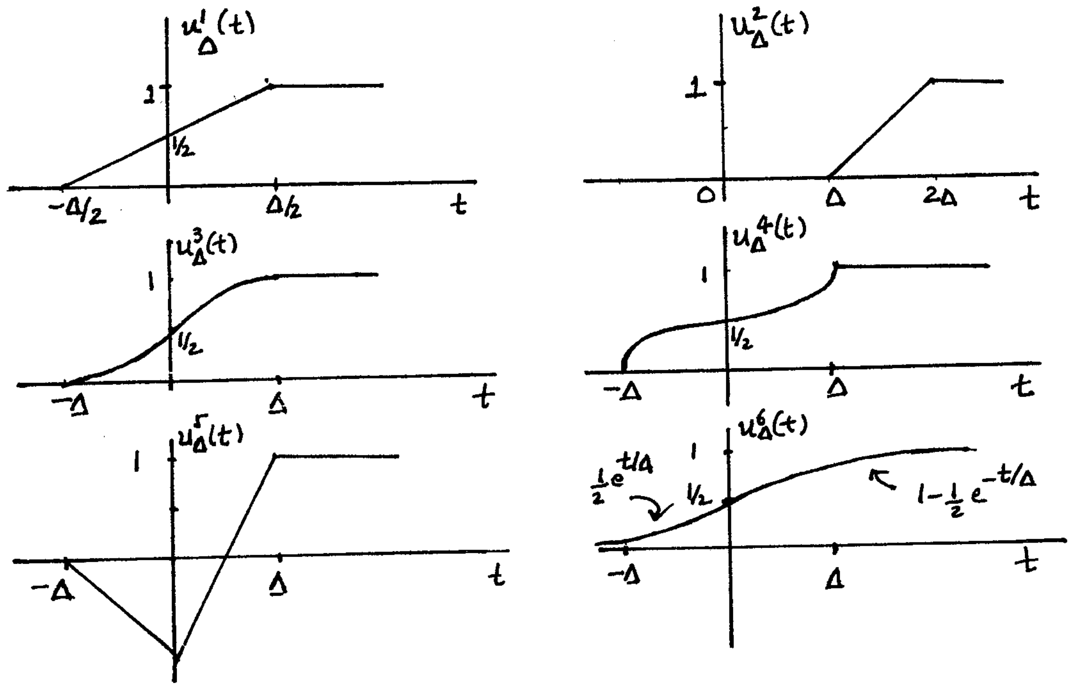


Figure S1.38

We have

$$g(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} u(\tau)\delta(t-\tau)d\tau.$$

Therefore,

$$g(t) = \begin{cases} 0, & t<0 \quad \because \delta(t-\tau)=0 \\ 1, & t>0 \quad \because u(\tau)\delta(t-\tau)=\delta(t-\tau) \\ \text{undefined} & \text{for } t=0 \end{cases}.$$

1.40. **(a)** If a system is additive, then

$$0 = x(t) - x(t) \longrightarrow y(t) - y(t) = 0.$$

Also, if a system is homogeneous, then

$$0 = 0.x(t) \longrightarrow y(t).0 = 0.$$

(b) $y(t) = x^2(t)$ is such a system.

(c) No. For example, consider $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ with $x(t) = u(t) - u(t-1)$. Then $x(t) = 0$ for $t > 1$, but $y(t) = 1$ for $t > 1$.

(g) Since $r_1 > 0, r_2 > 0$ and $-1 \le \cos(\theta_1 - \theta_2) \le 1$,

$$\begin{aligned}(|z_1| - |z_2|)^2 &= r_1^2 + r_2^2 - 2r_1r_2 \\ &\le r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) \\ &= |z_1 + z_2|^2\end{aligned}$$

and

$$(|z_1| + |z_2|)^2 = r_1^2 + r_2^2 + 2r_1r_2 \ge |z_1 + z_2|^2.$$

1.54. (a) For $\alpha = 1$, it is fairly obvious that

$$\sum_{n=0}^{N-1} \alpha^n = N.$$

For $\alpha \ne 1$, we may write

$$(1-\alpha)\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{N-1} \alpha^n - \sum_{n=0}^{N-1} \alpha^{n+1} = 1 - \alpha^N.$$

Therefore,

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}.$$

(b) For $|\alpha| < 1$,

$$\lim_{N\to\infty} \alpha^N = 0.$$

Therefore, from the result of the previous part,

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

(c) Differentiating both sides of the result of part (b) wrt α, we get

$$\begin{aligned}\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty} \alpha^N\right) &= \frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right) \\ \sum_{n=0}^{\infty} n\alpha^{n-1} &= \frac{1}{(1-\alpha)^2}\end{aligned}$$

(d) We may write

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \text{ for } |\alpha| < 1.$$

1.55. (a) The desired sum is

$$\sum_{n=0}^{9} e^{j\pi n/2} = \frac{1 - e^{j\pi 10/2}}{1 - e^{j\pi/2}} = 1 + j.$$

Chapter 2 Answers

2.1. **(a)** We know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad \text{(S2.1-1)}$$

The signals $x[n]$ and $h[n]$ are as shown in Figure S2.1.

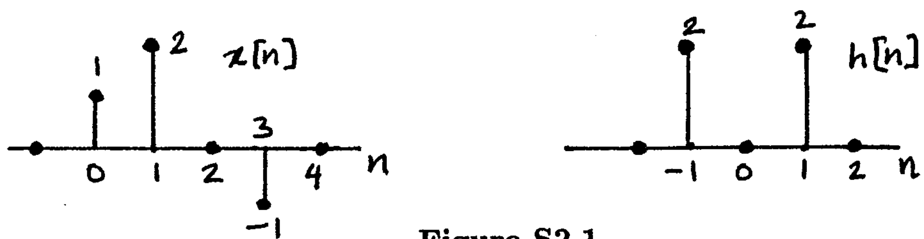


Figure S2.1

From this figure, we can easily see that the above convolution sum reduces to

$$\begin{aligned} y_1[n] &= h[-1]x[n+1] + h[1]x[n-1] \\ &= 2x[n+1] + 2x[n-1] \end{aligned}$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

(b) We know that

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$$

Comparing with eq. (S2.1-1), we see that

$$y_2[n] = y_1[n+2]$$

(c) We may rewrite eq. (S2.1-1) as

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similarly, we may write

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k]h[n+2-k]$$

Comparing this with eq. (S2.1), we see that

$$y_3[n] = y_1[n+2]$$

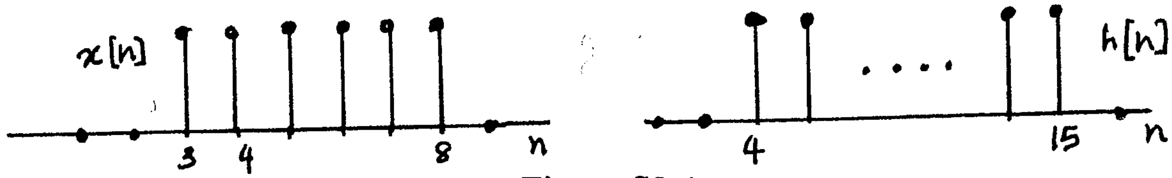


Figure S2.4

2.5. The signal $y[n]$ is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, this summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

From this it is clear that $y[n]$ is a summation of shifted replicas of $h[n]$. Since the last replica will begin at $n = 9$ and $h[n]$ is zero for $n > N$, $y[n]$ is zero for $n > N + 9$. Using this and the fact that $y[14] = 0$, we may conclude that N can *at most* be 4. Furthermore, since $y[4] = 5$, we can conclude that $h[n]$ has *at least* 5 non-zero points. The only value of N which satifies both these conditions is 4.

2.6. From the given information, we have:

$$\begin{aligned} y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} (\frac{1}{3})^{-k} u[-k-1]u[n-k-1] \\ &= \sum_{k=-\infty}^{-1} (\frac{1}{3})^{-k} u[n-k-1] \\ &= \sum_{k=1}^{\infty} (\frac{1}{3})^{k} u[n+k-1] \end{aligned}$$

Replacing k by $p-1$,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} u[n+p] \qquad \text{(S2.6–1)}$$

For $n \geq 0$ the above equation reduces to,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$$

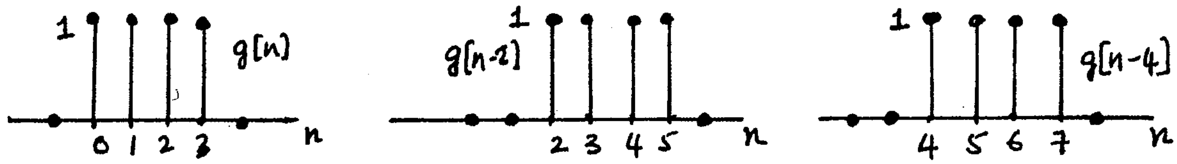


Figure S2.7

2.8. Using the convolution integral,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Given that $h(t) = \delta(t+2) + 2\delta(t+1)$, the above integral reduces to

$$x(t) * y(t) = x(t+2) + 2x(t+1)$$

The signals $x(t+2)$ and $2x(t+1)$ are plotted in Figure S2.8.

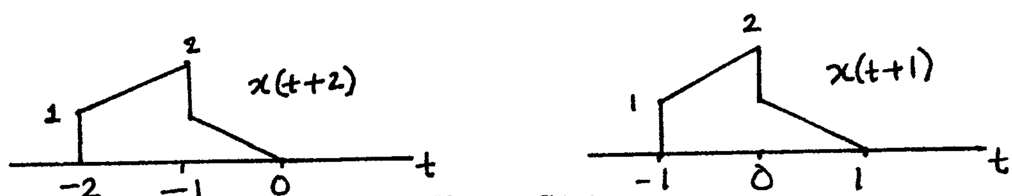


Figure S2.8

Using these plots, we can easily show that

$$y(t) = \begin{cases} t+3, & -2 < t \le -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

2.9. Using the given definition for the signal $h(t)$, we may write

$$h(\tau) = e^{2\tau}u(-\tau+4) + e^{-2\tau}u(\tau-5) = \begin{cases} e^{-2\tau}, & \tau > 5 \\ e^{2\tau}, & \tau < 4 \\ 0, & 4 < \tau < 5 \end{cases}$$

Therefore,

$$h(-\tau) = \begin{cases} e^{2\tau}, & \tau < -5 \\ e^{-2\tau}, & \tau > -4 \\ 0, & -5 < \tau < -4 \end{cases}$$

If we now shift the signal $h(-\tau)$ by t to the right, then the resultant signal $h(t-\tau)$ will be

$$h(t-\tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < t-5 \\ e^{2(t-\tau)}, & \tau > t-4 \\ 0, & (t-5) < \tau < (t-4) \end{cases}$$

Therefore,

$$A = t - 5, \qquad B = t - 4.$$

2.10. From the given information, we may sketch $x(t)$ and $h(t)$ as shown in Figure S2.10.

(a) With the aid of the plots in Figure S2.10, we can show that $y(t) = x(t) * h(t)$ is as shown in Figure S2.10.

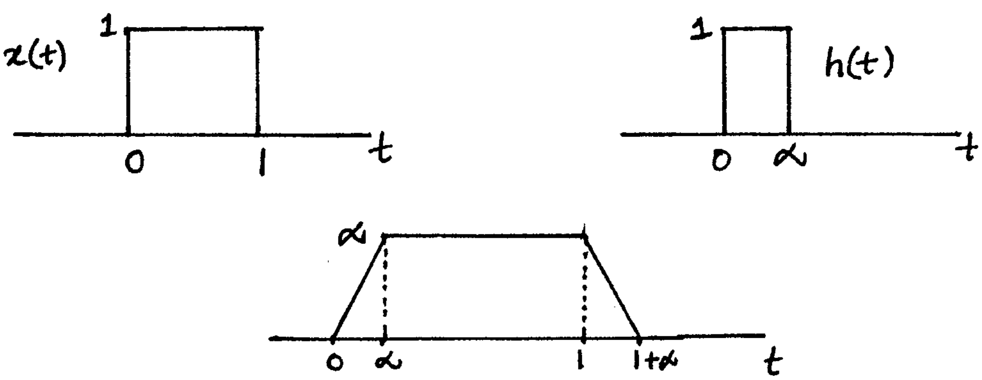


Figure S2.10

Therefore,

$$y(t) = \begin{cases} t, & 0 \le t \le \alpha \\ \alpha, & \alpha \le t \le 1 \\ 1 + \alpha - t, & 1 \le t \le (1+\alpha) \\ 0, & \text{otherwise} \end{cases}$$

(b) From the plot of $y(t)$, it is clear that $\frac{dy(t)}{dt}$ has discontinuities at 0, α, 1, and $1 + \alpha$. If we want $\frac{dy(t)}{dt}$ to have only three discontinuities, then we need to ensure that $\alpha = 1$.

2.11. (a) From the given information, we see that $h(t)$ is non zero only for $0 \le t \le \infty$. Therefore,

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-3\tau}(u(t-\tau-3) - u(t-\tau-5))d\tau \end{aligned}$$

We can easily show that $(u(t-\tau-3) - u(t-\tau-5))$ is non zero only in the range $(t-5) < \tau < (t-3)$. Therefore, for $t \le 3$, the above integral evaluates to zero. For $3 < t \le 5$, the above integral is

$$y(t) = \int_{0}^{t-3} e^{-3\tau} d\tau = \frac{1 - e^{-3(t-3)}}{3}$$

For $t > 5$, the integral is

$$y(t) = \int_{t-5}^{t-3} e^{-3\tau} d\tau = \frac{(1 - e^{-6})e^{-3(t-5)}}{3}$$

Therefore, the result of this convolution may be expressed as

$$y(t) = \begin{cases} 0, & -\infty < t \le 3 \\ \frac{1-e^{-3(t-3)}}{3}, & 3 < t \le 5 \\ \frac{(1-e^{-6})e^{-3(t-5)}}{3}, & 5 < t \le \infty \end{cases}$$

(b) By differentiating $x(t)$ with respect to time we get

$$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$$

Therefore,

$$g(t) = \frac{dx(t)}{dt} * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5).$$

(c) From the result of part (a), we may compute the derivative of $y(t)$ to be

$$\frac{dy(t)}{dt} = \begin{cases} 0, & -\infty < t \le 3 \\ e^{-3(t-3)}, & 3 < t \le 5 \\ (e^{-6}-1)e^{-3(t-5)}, & 5 < t \le \infty \end{cases}$$

This is exactly equal to $g(t)$. Therefore, $g(t) = \frac{dy(t)}{dt}$.

2.12. The signal $y(t)$ may be written as

$$y(t) = \cdots + e^{-(t+6)}u(t+6) + e^{-(t+3)}u(t+3) + e^{-t}u(t) + e^{-(t-3)}u(t-3) + e^{-(t-6)}u(t-6) + \cdots$$

In the range $0 \le t < 3$, we may write $y(t)$ as

$$\begin{aligned} y(t) &= \cdots + e^{-(t+6)}u(t+6) + e^{-(t+3)}u(t+3) + e^{-t}u(t) \\ &= e^{-t} + e^{-(t+3)} + e^{-(t+6)} + \cdots \\ &= e^{-t}(1 + e^{-3} + e^{-6} + \cdots) \\ &= e^{-t}\frac{1}{1-e^{-3}} \end{aligned}$$

Therefore, $A = \frac{1}{1-e^{-3}}$.

2.13. **(a)** We require that

$$\left(\frac{1}{5}\right)^n u[n] - A\left(\frac{1}{5}\right)^{(n-1)} u[n-1] = \delta[n]$$

Putting $n = 1$ and solving for A gives $A = \frac{1}{5}$.

(b) From part (a), we know that

$$\begin{aligned} h[n] - \frac{1}{5}h[n-1] &= \delta[n] \\ h[n] * (\delta[n] - \frac{1}{5}\delta[n-1]) &= \delta[n] \end{aligned}$$

From the definition of an inverse system, we may argue that

$$g[n] = \delta[n] - \frac{1}{5}\delta[n-1].$$

2.14. (a) We first determine if $h_1(t)$ is absolutely integrable as follows

$$\int_{-\infty}^{\infty} |h_1(\tau)| d\tau = \int_{0}^{\infty} e^{-t} d\tau = 1$$

Therefore, $h_1(t)$ is the impulse response of a stable LTI system.

(b) We determine if $h_2(t)$ is absolutely integrable as follows

$$\int_{-\infty}^{\infty} |h_2(\tau)| d\tau = \int_{0}^{\infty} e^{-t} |\cos(2t)| d\tau$$

This integral is clearly finite-valued because $e^{-t}|\cos(2t)|$ is an exponentially decaying function in the range $0 \le t \le \infty$. Therefore, $h_2(t)$ is the impulse response of a stable LTI system.

2.15. (a) We determine if $h_1[n]$ is absolutely summable as follows

$$\sum_{k=-\infty}^{\infty} |h_1[k]| = \sum_{k=0}^{\infty} k|\cos(\frac{\pi}{4}k)|$$

This sum does not have a finite value because the function $k|\cos(\frac{\pi}{4}k)|$ increases as the value of k increases. Therefore, $h_1[n]$ cannot be the impulse response of a stable LTI system.

(b) We determine if $h_2[n]$ is absolutely summable as follows

$$\sum_{k=-\infty}^{\infty} |h_2[k]| = \sum_{k=-\infty}^{10} 3^k \approx 3^{11}/2$$

Therefore, $h_2[n]$ is the impulse response of a stable LTI system.

2.16. (a) **True**. This may be easily argued by noting that convolution may be viewed as the process of carrying out the superposition of a number of echos of $h[n]$. The first such echo will occur at the location of the first non zero sample of $x[n]$. In this case, the first echo will occur at N_1. The echo of $h[n]$ which occurs at $n = N_1$ will have its first non zero sample at the time location $N_1 + N_2$. Therefore, for all values of n which are lesser that $N_1 + N_2$, the output $y[n]$ is zero.

(b) **False**. Consider

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

From this,

$$\begin{aligned} y[n-1] &= \sum_{k=-\infty}^{\infty} x[k]h[n-1-k] \\ &= x[n] * h[n-1] \end{aligned}$$

This shows that the given statement is false.

(c) True. Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

From this,

$$\begin{aligned} y(-t) &= \int_{-\infty}^{\infty} x(\tau)h(-t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(-\tau)h(-t+\tau)d\tau \\ &= x(-t) * h(-t) \end{aligned}$$

This shows that the given statement is true.

(d) True. This may be argued by considering

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

In Figure S2.16, we plot $x(\tau)$ and $h(t-\tau)$ under the assumptions that (1) $x(t) = 0$ for $t > T_1$ and (2) $h(t) = 0$ for $t > T_2$. Clearly, the product $x(\tau)h(t-\tau)$ is zero if

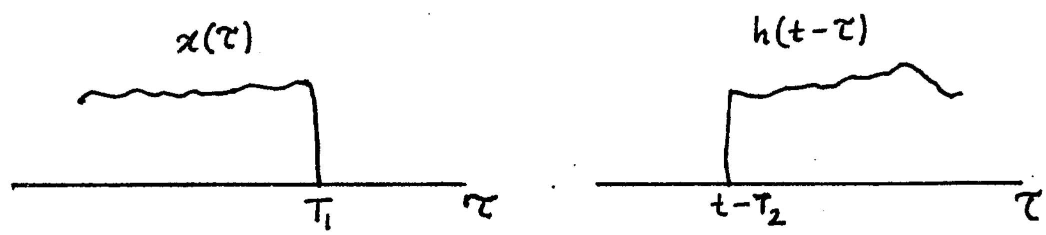


Figure S2.16

$t - T_2 > T_1$. Therefore, $y(t) = 0$ for $t > T_1 + T_2$.

2.17. **(a)** We know that $y(t)$ is the sum of the particular and homogeneous solutions to the given differential equation. We first determine the particular solution $y_p(t)$ by using the method specified in Example 2.14. Since we are given that the input is $x(t) = e^{(-1+3j)t}u(t)$ for $t > 0$, we hypothesize that for $t > 0$

$$y_p(t) = Ke^{(-1+3j)t}.$$

Substituting for $x(t)$ and $y(t)$ in the given differential equation,

$$(-1+3j)Ke^{(-1+3j)t} + 4Ke^{(-1+3j)t} = e^{(-1+3j)t}$$

This gives

$$(-1+3j)K + 4K = 1, \qquad \Rightarrow K = \frac{1}{3(1+j)}$$

Therefore,

$$y_p(t) = \frac{1}{3(1+j)} e^{(-1+3j)t}, \qquad t > 0$$

In order to determine the homogeneous solution, we hypothesize that

$$y_h(t) = Ae^{st}$$

Since the homogeneous solution has to satisfy the following differential equation

$$\frac{dy_h(t)}{dt} + 4y_h(t) = 0,$$

we obtain

$$Ase^{st} + 4Ae^{st} = Ae^{st}(s+4) = 0.$$

This implies that $s = -4$ for any A. The overall solution to the differential equation now becomes

$$y(t) = Ae^{-4t} + \frac{1}{3(1+j)} e^{(-1+3j)t}, \qquad t > 0$$

Now in order to determine the constant A, we use the fact that the system satisfies the condition of initial rest. Given that $y(0) = 0$, we may conclude that

$$A + \frac{1}{3(1+j)} = 0, \qquad A = \frac{-1}{3(1+j)}$$

Therefore for $t > 0$,

$$y(t) = \frac{1}{3(1+j)} \left[-e^{-4t} + e^{(-1+3j)t} \right], \qquad t > 0$$

Since the system satisfies the condition of initial rest, $y(t) = 0$ for $t < 0$. Therefore,

$$y(t) = \frac{1-j}{6} \left[-e^{-4t} + e^{(-1+3j)t} \right] u(t)$$

(b) The output will now be the real part of the answer obtained in part (a).

$$y(t) = \frac{1}{6} \left[e^{-t} \cos 3t + e^{-t} \sin 3t - e^{-4t} \right] u(t).$$

2.18. Since the system is causal, $y[n] = 0$ for $n < 1$. Now,

$$\begin{aligned}
y[1] &= \frac{1}{4}y[0] + x[1] = 0 + 1 = 1 \\
y[2] &= \frac{1}{4}y[1] + x[2] = \frac{1}{4} + 0 = \frac{1}{4} \\
y[3] &= \frac{1}{4}y[2] + x[3] = \frac{1}{16} + 0 = \frac{1}{16} \\
&\vdots \\
y[m] &= (\frac{1}{4})^{m-1} \\
&\vdots
\end{aligned}$$

Therefore,

$$y[n] = (\frac{1}{4})^{n-1}u[n-1]$$

2.19. **(a)** Consider the difference equation relating $y[n]$ and $w[n]$ for S_2:

$$y[n] = \alpha y[n-1] + \beta w[n]$$

From this we may write

$$w[n] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1]$$

and

$$w[n-1] = \frac{1}{\beta}y[n-1] - \frac{\alpha}{\beta}y[n-2]$$

Weighting the previous equation by 1/2 and subtracting from the one before, we obtain

$$w[n] - \frac{1}{2}w[n-1] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1] - \frac{1}{2\beta}y[n-1] + \frac{\alpha}{2\beta}y[n-2]$$

Substituting this in the difference equation relating $w[n]$ and $x[n]$ for S_1,

$$\frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1] - \frac{1}{2\beta}y[n-1] + \frac{\alpha}{2\beta}y[n-2] = x[n]$$

That is,

$$y[n] = (\alpha + \frac{1}{2})y[n-1] - \frac{\alpha}{2}y[n-2] + \beta x[n]$$

Comparing with the given equation relating $y[n]$ and $x[n]$, we obtain

$$\alpha = \frac{1}{4}, \qquad \beta = 1$$

Now,

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau$$

which is the desired integral. We now evaluate the value of the integral as

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \sin(2\pi t)|_{t=1} = 0.$$

2.21. **(a)** The desired convolution is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \beta^n \sum_{k=0}^{n} (\alpha/\beta)^k \text{ for } n \geq 0 \\ &= [\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}]u[n] \text{ for } \alpha \neq \beta. \end{aligned}$$

(b) From (a),

$$y[n] = \alpha^n \left[\sum_{k=0}^{n} 1\right] u[n] = (n+1)\alpha^n u[n].$$

(c) For $n \leq 6$,

$$y[n] = 4^n \left\{\sum_{k=0}^{\infty}(-\frac{1}{8})^k - \sum_{k=0}^{3}(-\frac{1}{8})^k\right\}.$$

For $n > 6$,

$$y[n] = 4^n \left\{\sum_{k=0}^{\infty}(-\frac{1}{8})^k - \sum_{k=0}^{n-1}(-\frac{1}{8})^k\right\}.$$

Therefore,

$$y[n] = \begin{cases} (8/9)(-1/8)^4 4^n, & n \leq 6 \\ (8/9)(-1/2)^n, & n > 6 \end{cases}$$

(d) The desired convolution is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] + x[4]h[n-4] \\ &= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]. \end{aligned}$$

This is as shown in Figure S2.21.

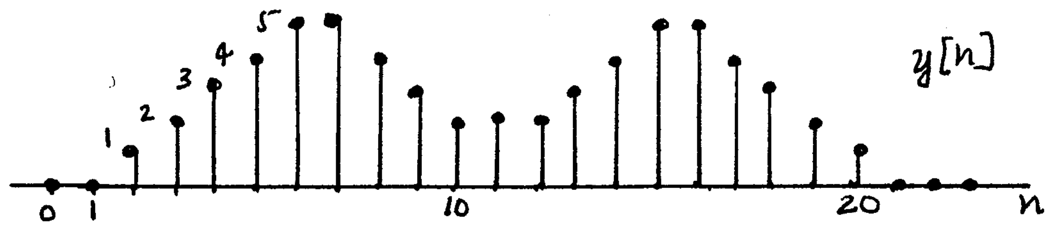


Figure S2.21

2.22. **(a)** The desired convolution is

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{0}^{t} e^{-\alpha t} e^{-\beta(t-\tau)} d\tau, \ t \geq 0
\end{aligned}
$$

Then

$$
y(t) = \begin{cases} \frac{e^{-\beta t}\{e^{-(\alpha-\beta)t}-1\}}{\beta-\alpha} u(t) & \alpha \neq \beta \\ te^{-\beta t}u(t) & \alpha = \beta \end{cases}.
$$

(b) The desired convolution is

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{0}^{2} h(t-\tau)d\tau - \int_{2}^{5} h(t-\tau)d\tau.
\end{aligned}
$$

This may be written as

$$
y(t) = \begin{cases} \int_{0}^{2} e^{2(t-\tau)}d\tau - \int_{2}^{5} e^{2(t-\tau)}d\tau, & t \leq 1 \\ \int_{t-1}^{2} e^{2(t-\tau)}d\tau - \int_{2}^{5} e^{2(t-\tau)}d\tau, & 1 \leq t \leq 3 \\ -\int_{t-1}^{5} e^{2(t-\tau)}d\tau, & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}
$$

Therefore,

$$
y(t) = \begin{cases} (1/2)[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}], & t \leq 1 \\ (1/2)[e^{2} + e^{2(t-5)} - 2e^{2(t-2)}], & 1 \leq t \leq 3 \\ (1/2)[e^{2(t-5)} - e^{2}], & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}
$$

(c) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^2 \sin(\pi\tau)h(t-\tau)d\tau. \end{aligned}$$

This gives us

$$y(t) = \begin{cases} 0, & t<1 \\ (2/\pi)[1-\cos\{\pi(t-1)\}], & 1<t<3 \\ (2/\pi)[\cos\{\pi(t-3)\}-1], & 3<t<5 \\ 0, & 5<t \end{cases}$$

(d) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2),$$

where

$$h_1(t) = \begin{cases} 4/3, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Now,

$$y(t) = h(t) * x(t) = [h_1(t) * x(t)] - \frac{1}{3}x(t-2).$$

We have

$$h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)].$$

Therefore,

$$y(t) = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)] - \frac{1}{3}[a(t-2)+b] = at + b = x(t).$$

(e) $x(t)$ periodic implies $y(t)$ periodic. $\therefore$ determine 1 period only. We have

$$y(t) = \begin{cases} \int_{t-1}^{-\frac{1}{2}} (t-\tau-1)d\tau + \int_{-\frac{1}{2}}^{t} (1-t+\tau)d\tau = \frac{1}{4} + t - t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-1}^{\frac{1}{2}} (1-t+\tau)d\tau + \int_{\frac{1}{2}}^{t} (t-1-\tau)d\tau = t^2 - 3t + 7/4, & \frac{1}{2} < t < \frac{3}{2} \end{cases}.$$

The period of $y(t)$ is 2.

2.23. $y(t)$ is sketched in Figure S2.23 for the different values of T.

2.24. (a) We are given that $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

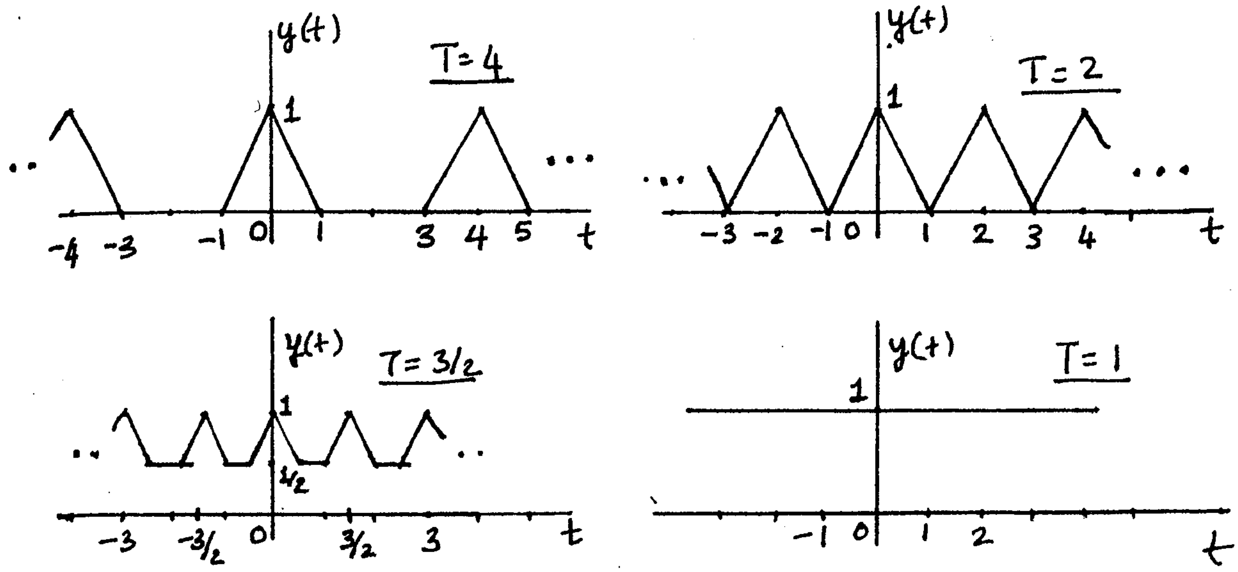


Figure S2.23

Since

$$h[n] = h_1[n] * [h_2[n] * h_2[n]],$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2].$$

Therefore,

$$\begin{aligned}
h[0] &= h_1[0] & \Rightarrow \quad & h_1[0] = 1, \\
h[1] &= h_1[1] + 2h_1[0] & \Rightarrow \quad & h_1[1] = 3, \\
h[2] &= h_1[2] + 2h_1[1] + h_1[0] & \Rightarrow \quad & h_1[2] = 3, \\
h[3] &= h_1[3] + 2h_1[2] + h_1[1] & \Rightarrow \quad & h_1[3] = 2, \\
h[4] &= h_1[4] + 2h_1[3] + h_1[2] & \Rightarrow \quad & h_1[4] = 1 \\
h[5] &= h_1[5] + 2h_1[4] + h_1[3] & \Rightarrow \quad & h_1[5] = 0.
\end{aligned}$$

$h_1[n] = 0$ for $n < 0$ and $n \geq 5$.

(b) In this case,

$$y[n] = x[n] * h[n] = h[n] - h[n-1].$$

2.25. **(a)** We may write $x[n]$ as

$$x[n] = \left(\frac{1}{3}\right)^{|n|}.$$

(b) Now,

$$y[n] = x_3[n] * y_1[n] = y_1[n] - y_1[n-1].$$

Therefore,

$$y[n] = \begin{cases} 2\left\{1-(1/2)^{n+3}\right\} + 2\left\{1-(1/2)^{n+4}\right\} = (1/2)^{n+3}, & n \geq -2 \\ 1, & n = -3 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, $y[n] = (1/2)^{n+3}u[n+3]$.

(c) We have

$$y_2[n] = x_2[n] * x_3[n] = u[n+3] - u[n+2] = \delta[n+3].$$

(d) From the result of part (c), we get

$$y[n] = y_2[n] * x_1[n] = x_1[n+3] = (1/2)^{n+3}u[n+3].$$

2.27. The proof is as follows.

$$\begin{aligned} A_y &= \int_{-\infty}^{\infty} y(t)dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)dtd\tau \\ &= \int_{-\infty}^{\infty} x(\tau)A_h d\tau \\ &= A_x A_h \end{aligned}$$

2.28. (a) Causal because $h[n] = 0$ for $n < 0$. Stable because $\sum_{n=0}^{\infty}(\frac{1}{5})^n = 5/4 < \infty$.

(b) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-2}^{\infty}(0.8)^n = 5 < \infty$.

(c) Anti-causal because $h[n] = 0$ for $n > 0$. Unstable because $\sum_{n=-\infty}^{0}(1/2)^n = \infty$

(d) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-\infty}^{3} 5^n = \frac{625}{4} < \infty$

(e) Causal because $h[n] = 0$ for $n < 0$. Unstable because the second term becomes infinite as $n \to \infty$.

(f) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n-\infty}^{\infty}|h[n]| = 305/3 < \infty$

(g) Causal because $h[n] = 0$ for $n < 0$. Stable because $\sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty$.

2.29. (a) Causal because $h(t) = 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{-8}/4 < \infty$.

(b) Not causal because $h(t) \neq 0$ for $t < 0$. Unstable because $\int_{-\infty}^{\infty} |h(t)| = \infty$.

(c) Not causal because $h(t) \neq 0$ for $t < 0$. a Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{100}/2 < \infty$.

(d) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{-2}/2 < \infty$.

(e) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = 1/3 < \infty$.

(f) Causal because $h(t) = 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = 1 < \infty$.

(g) Causal because $h(t) = 0$ for $t < 0$. Unstable because $\int_{-\infty}^{\infty} |h(t)|dt = \infty$.

2.30. We need to find the output of the system when the input is $x[n] = \delta[n]$. Since we are asked to assume initial rest, we may conclude that $y[n] = 0$ for $n < 0$. Now,

$$y[n] = x[n] - 2y[n-1].$$

Therefore,

$$y[0] = x[0] - 2y[-1] = 1, \qquad y[1] = x[1] - 2y[0] = -2, \qquad y[2] = x[2] + 2y[2] = -4$$

and so on. In closed form,

$$y[n] = (-2)^n u[n].$$

This is the impulse response of the system.

2.31. Initial rest implies that $y[n] = 0$ for $n < -2$. Now

$$y[n] = x[n] + 2x[n-2] - 2y[n-1].$$

Therefore,

$$y[-2] \;=\; 1, \quad y[-1] = 0, \quad y[0] = 5, \quad y[1] =$$

$$y[4] = 56, y[5] = -110, \quad y[n] = -110(-2)^{n-5} \quad \text{for} n \geq 5.$$

2.32. (a) If $y_h[n] = A(1/2)^n$, then we need to verify

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0.$$

Clearly this is true.

(b) We now require that for $n \geq 0$

$$B\left(\frac{1}{3}\right)^n - \frac{1}{2}B\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n.$$

Therefore, $B = -2$.

(c) From eq. (P2.32-1), we know that $y[0] = x[0] + (1/2)y[-1] = x[0] = 1$. Now we also have

$$y[0] = A + B \qquad \Rightarrow \qquad A = 1 - B = 3.$$

2.33. (a) (i) From Example 2.14, we know that

$$y_1(t) = \left[\frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}\right]u(t).$$

(ii) We solve this along the lines of Example 2.14. First assume that $y_p(t)$ is of the form Ke^{2t} for $t > 0$. Then using eq. (P2.33-1), we get for $t > 0$

$$2Ke^{2t} + 2Ke^{2t} = e^{2t} \qquad \Rightarrow \qquad K = \frac{1}{4}.$$

We now know that $y_p(t) = \frac{1}{4}e^{2t}$ for $t > 0$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}.$$

Therefore,

$$y_2(t) = Ae^{-2t} + \frac{1}{4}e^{2t}, \qquad \text{for } t > 0.$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq 0$. Therefore,

$$y_2(0) = 0 = A + \frac{1}{4} \qquad \Rightarrow \qquad A = -\frac{1}{4}.$$

Then,

$$y_2(t) = \left[-\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t}\right]u(t).$$

(iii) Let the input be $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$. Assume that the particular solution $y_p(t)$ is of the form

$$y_p(t) = K_1\alpha e^{3t} + K_2\beta e^{2t}$$

for $t > 0$. Using eq. (P2.33-1), we get

$$3K_1\alpha e^{3t} + 2K_2\beta e^{2t} + 2K_1\alpha e^{3t} + 2K_2\beta e^{2t} = \alpha^{3t} + \beta e^{2t}.$$

Equating the coefficients of e^{3t} and e^{2t} on both sides, we get

$$K_1 = \frac{1}{5} \qquad \text{and} \qquad K_2 = \frac{1}{4}.$$

2.37. Let us consider two inputs

$$x_1(t) = 0, \qquad \text{for all } t$$

and

$$x_2(t) = e^t[u(t) - u(t-1)].$$

Since the system is linear, the response $y_1(t) = 0$ for all t.

Now let us find the output $y_2(t)$ when the input is $x_2(t)$. The particular solution is of the form

$$y_p(t) = Ye^t, \qquad 0 < t < 1.$$

Substituting in eq. (P2.33-1), we get

$$3Y = 1.$$

Now, including the homogeneous solution which is of the form $y_h(t) = Ae^{-2t}$, we get the overall solution:

$$y_2(t) = Ae^{-2t} + \frac{1}{3}e^t, \qquad 0 < t < 1.$$

Assuming final rest, we have $y(1) = 0$. Using this we get $A = -e^3/3$. Therefore,

$$y_2(t) = -\frac{1}{3}e^{-2t+3} + \frac{1}{3}e^t, \qquad 0 < t < 1. \qquad \text{(S2.37–1)}$$

For $t < 0$, we note that $x_2(t) = 0$. Thus the particular solution is zero in this range and

$$y_2(t) = Be^{-2t}, \qquad t < 0. \qquad \text{(S2.37–2)}$$

Since the two pieces of the solution for $y_2(t)$ in eqs. (S2.37-1) and (S2.37-2) must match at $t = 0$, we can determine B from the equation

$$\frac{1}{3} - \frac{1}{3}e^3 = B$$

which yields

$$y_2(t) = \left(\frac{1}{3} - \frac{1}{3}e^3\right)e^{-2t}, \qquad t < 0.$$

Now note that since $x_1(t) = x_2(t)$ for $t < 0$, it must be true that for a causal system $y_1(t) = y_2(t)$ for $t < 0$. However, the results of obtained above show that this is not true. Therefore, the system is not causal.

2.38. The block diagrams are as shown in Figure S2.38.

2.39. The block diagrams are as shown in Figure S2.39.

2.40. (a) Note that

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau - 2)d\tau = \int_{-\infty}^{t-2} e^{-(t-2-\tau')}x(\tau')d\tau'.$$

Therefore,

$$h(t) = e^{-(t-2)}u(t-2).$$

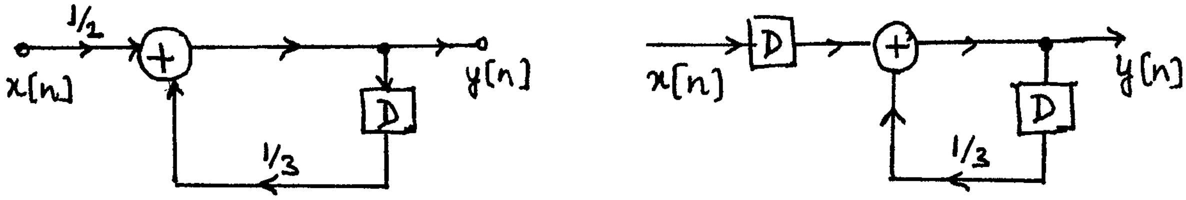


Figure S2.38

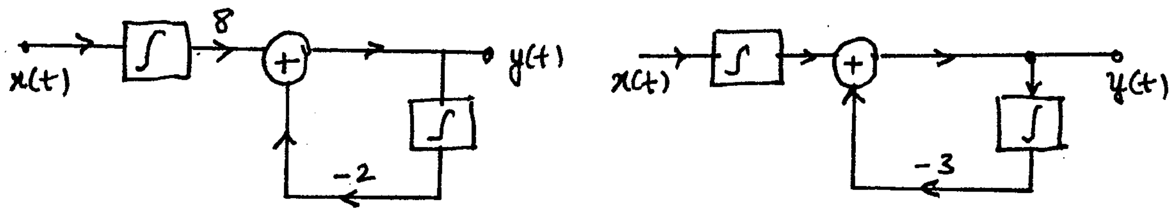


Figure S2.39

(b) We have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{2}^{\infty} e^{-(\tau-2)}[u(t-\tau+1)-u(t-\tau-2)) \end{aligned}$$

$h(\tau)$ and $x(t-\tau)$ are as shown in the figure below.

Using this figure, we may write

$$y(t) = \begin{cases} 0, & t < 1 \\ \int_{2}^{t+1} e^{-(\tau-2)}d\tau = 1 - e^{-(t-1)}, & 1 < t < 4 \\ \int_{t-2}^{t+1} e^{-(\tau-2)}d\tau = e^{-(t-4)}[1-e^{-3}], & t > 4 \end{cases}$$

2.41. (a) We may write

$$\begin{aligned} g[n] &= x[n] - \alpha x[n-1] \\ &= \alpha^n u[n] - \alpha^n u[n-1] \\ &= \delta[n]. \end{aligned}$$

2.47. (a) $y(t) = 2y_0(t)$.

(b) $y(t) = y_0(t) - y_0(t-2)$.

(c) $y(t) = y_0(t-1)$.

(d) Not enough information.

(e) $y(t) = y_0(-t)$.

(f) $y(t) = y_0''(t)$.

The signals for all parts of this problem are plotted in the Figure S2.47.

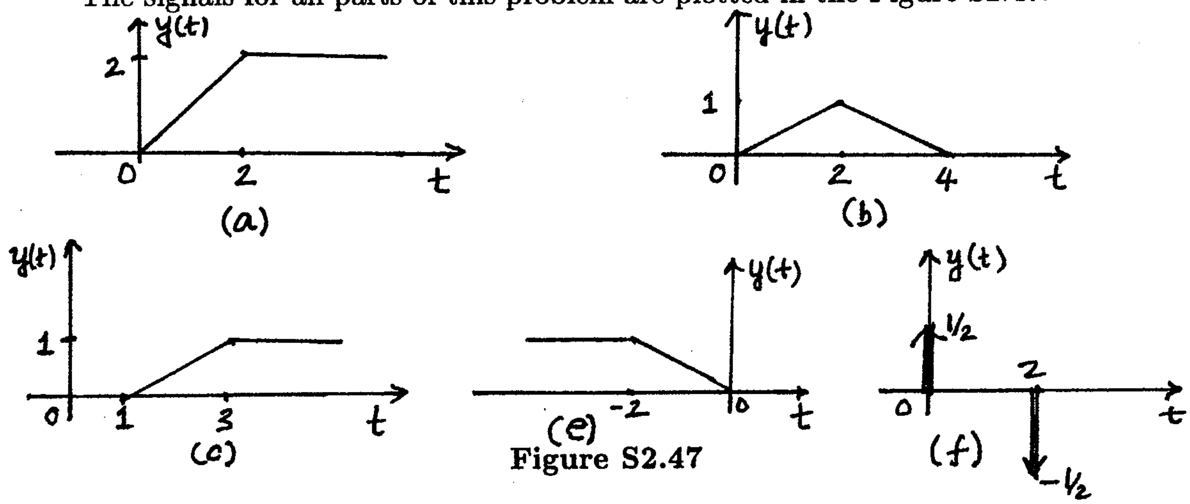


Figure S2.47

2.48. (a) True. If $h(t)$ periodic and nonzero, then

$$\int_{-\infty}^{\infty} |h(t)|dt = \infty.$$

Therefore, $h(t)$ is unstable.

(b) False. For example, inverse of $h[n] = \delta[n-k]$ is $g[n] = \delta[n+k]$ which is noncausal.

(c) False. For example $h[n] = u[n]$ implies that

$$\sum_{n=-\infty}^{\infty} |h[n]| = \infty.$$

This is an unstable system.

(d) True. Assuming that $h[n]$ is bounded and nonzero in the range $n_1 \le n \le n_2$,

$$\sum_{k=n_1} n_2 |h[k]| < \infty.$$

This implies that the system is stable.

(e) False. For example, $h(t) = e^t u(t)$ is causal but not stable.

(f) False. For example, the cascade of a causal system with impulse response $h_1[n] = \delta[n-1]$ and a non-causal system with impulse response $h_2[n] = \delta[n+1]$ leads to a system with overall impulse response given by $h[n] = h_1[n] * h_2[n] = \delta[n]$.

(g) False. For example, if $h(t) = e^{-t}u(t)$, then $s(t) = (1 - e^{-t})u(t)$ and

$$\int_0^\infty |1 - e^{-t}|dt = t + e^{-t}\Big|_0^\infty = \infty.$$

Although the system is stable, the step response is not absolutely integrable.

(h) True. We may write $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$. Therefore,

$$s[n] = \sum_{k=0}^{\infty} h[n-k].$$

If $s[n] = 0$ for $n < 0$, then $h[n] = 0$ for $n < 0$ and the system is causal.

2.49. (a) It is a bounded input. $|x[n]| \leq 1 = B_x$ for all n.

(b) Consider

$$\begin{aligned} y[0] &= \sum_{k=-\infty}^{\infty} x[-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} \frac{h^2[k]}{|h[k]|} \\ &= \sum_{k=-\infty}^{\infty} |h[k]| \to \infty \end{aligned}$$

Therefore, the output is not bounded. Thus, the system is not stable and absolute summability is necessary.

(c) Let

$$x(t) = \begin{cases} 0, & \text{if } h(-t) = 0 \\ \frac{h(-t)}{|h(-t)|}, & \text{if } h(-t) \neq 0 \end{cases}.$$

Now, $|x(t)| \leq 1$ for all t. Therefore, $x(t)$ is a bounded input Now,

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} x(-\tau)h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{h^2(\tau)}{|h(\tau)|}d\tau \\ &= \int_{-\infty}^{\infty} |h(t)|dt = \infty \end{aligned}$$

Therefore, the system is unstable if the impulse response is not absolutely integrable.

2.50. (a) The output will be $ax_1(t) + bx_2(t)$.

(b) The output will be $x_1(t - \tau)$.

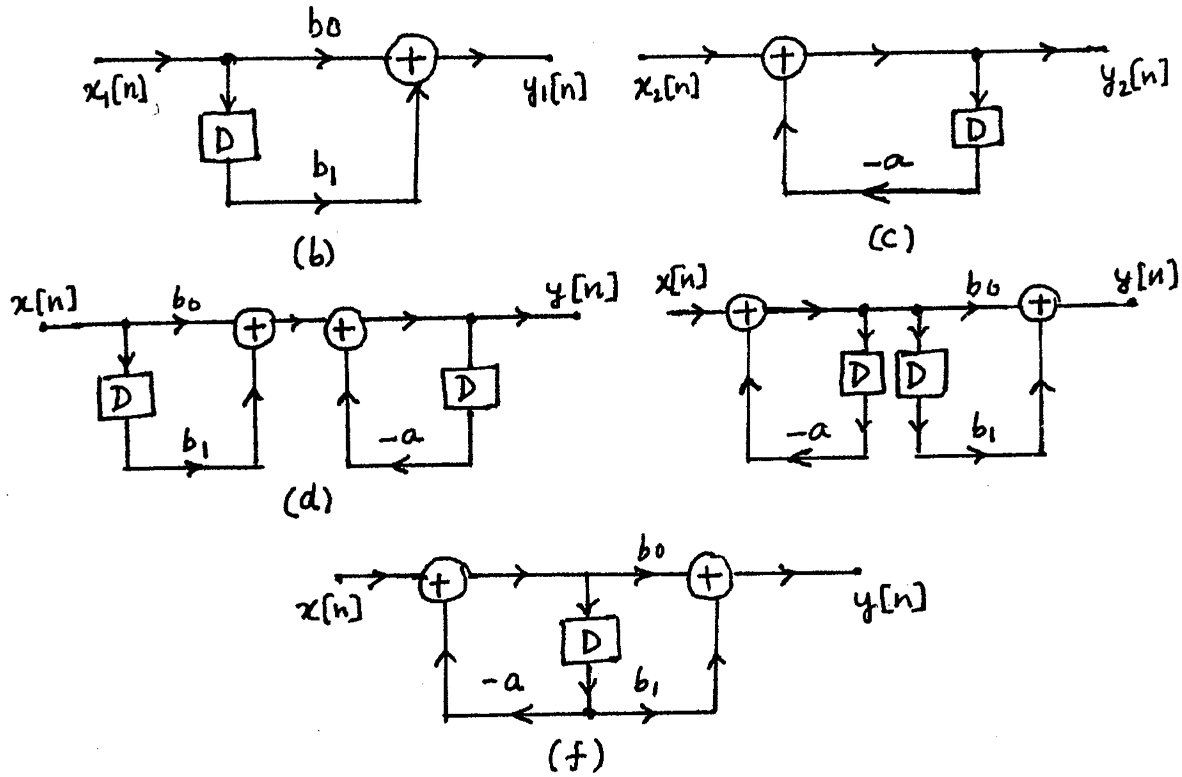


Figure S2.57

(b) The figures corresponding to the remaining parts of this problem are shown in the Figure S2.57.

2.58. (a) Realizing that $x_2[n] = y_1[n]$, we may eliminate these from the two given difference equations. This would give us

$$2y_2[n] - y_2[n-1] + y_2[n-3] = x_1[n] - 5x_1[n-4].$$

This is the same as the overall difference equation.

(b) The figures corresponding to the remaining parts of this problem are shown in Figure S2.58.

2.59. (a) Integrating the given differential equation once and simplifying, we get

$$y(t) = -\frac{a_0}{a_1}\int_{-\infty}^{t} y(\tau)d\tau + \frac{b_0}{a_1}\int_{-\infty}^{t} x(\tau)d\tau + \frac{b_1}{a_1}x(t).$$

Therefore, $A = -a_0/a_1$, $B = b_1/a_1$, $C = b_0/a_1$.

(b) Realizing that $x_2(t) = y_1(t)$, we may eliminate these from the two given integral equations. This would give us

$$y_2(t) = A\int_{-\infty}^{t} y_2(\tau)d\tau + B\int_{-\infty}^{t} x_1(\tau)d\tau + Cx_1(t).$$

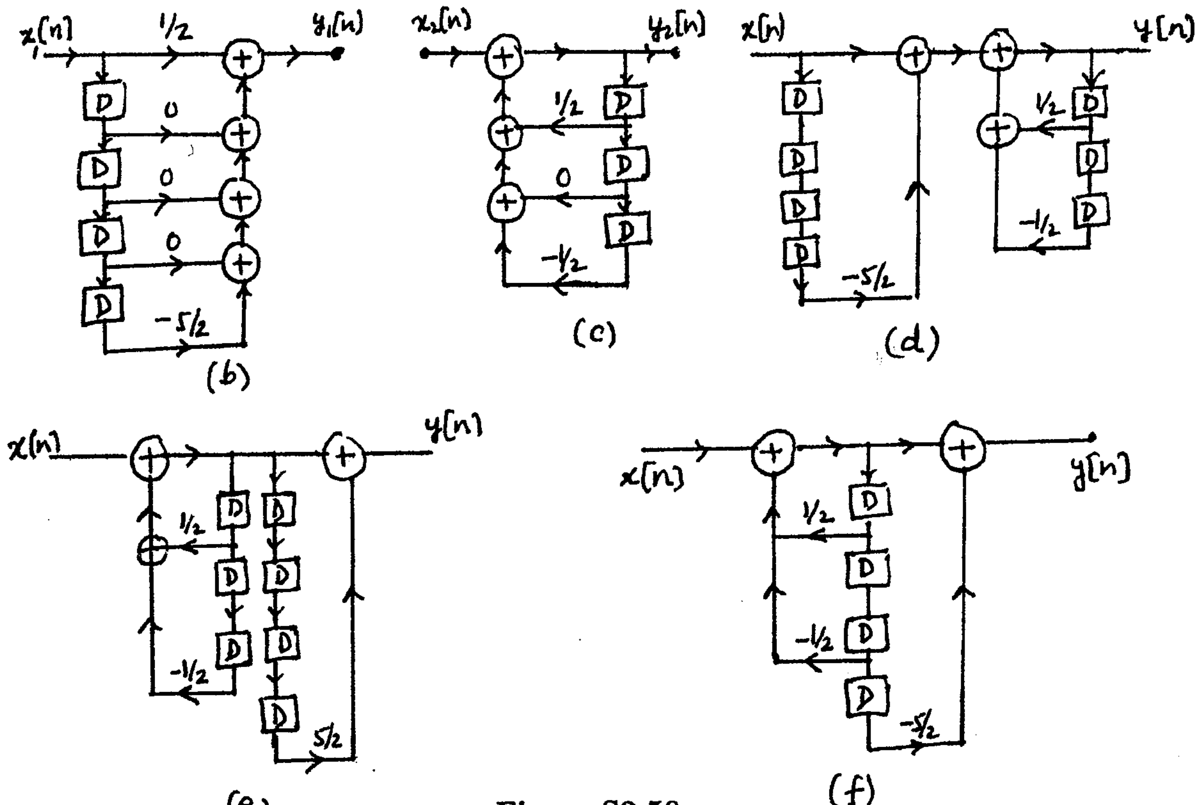


Figure S2.58

(c) The figures corresponding to the remaining parts of this problem are shown in Figure S2.59.

2.60. (a) Integrating the given differential equation once and simplifying, we get

$$\begin{aligned} y(t) &= -\frac{-a_1}{a_2}\int_{-\infty}^{t} y(\tau)d\tau - \frac{a_0}{a_2}\int_{-\infty}^{t}\int_{-\infty}^{\tau} y(\sigma)d\sigma d\tau \\ &+ \frac{b_0}{a_2}\int_{-\infty}^{t}\int_{-\infty}^{\tau} x(\sigma)d\sigma d\tau + \frac{b_1}{a_2}\int_{-\infty}^{t} x(\tau)d\tau + \frac{b_2}{a_1}x(t). \end{aligned}$$

Therefore, $A = -a_1/a_2$, $B = -a_0/a_2$, $C = b_2/a_1$, $D = b_1/a_2$, $E = b_0/a_2$.

(b) Realizing that $x_2(t) = y_1(t)$, we may eliminate these from the two given integral equations.

(c) The figures corresponding to the remaining parts of this problem are shown in Figure S2.60.

2.61. (a) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the inductor and capacitor. Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + y(t).$$

Chapter 3 Answers

3.1. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned}
x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3}e^{-j3(2\pi/T)t} \\
&= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4je^{j3(2\pi/8)t} - 4je^{-j3(2\pi/8)t} \\
&= 4\cos(\frac{\pi}{4}t) - 8\sin(\frac{6\pi}{8}t) \\
&= 4\cos(\frac{\pi}{4}t) + 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2})
\end{aligned}$$

3.2. Using the Fourier series synthesis eq. (3.95).

$$\begin{aligned}
x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\
&= 1 + e^{j(\pi/4)}e^{j2(2\pi/5)n} + e^{-j(\pi/4)}e^{-2j(2\pi/5)n} \\
&\quad +2e^{j(\pi/3)}e^{j4(2\pi/N)n} + 2e^{-j(\pi/3)}a_{-4}e^{-j4(2\pi/N)n} \\
&= 1 + 2\cos(\frac{4\pi}{5}n + \frac{\pi}{4}) + 4\cos(\frac{8\pi}{5}n + \frac{\pi}{3}) \\
&= 1 + 2\sin(\frac{4\pi}{5}n + \frac{3\pi}{4}) + 4\sin(\frac{8\pi}{5}n + \frac{5\pi}{6})
\end{aligned}$$

3.3. The given signal is

$$\begin{aligned}
x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\
&= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t}
\end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \qquad a_2 = a_{-2} = \frac{1}{2}, \qquad a_5 = a_{-5}^* = -2j$$

3.4. Since $\omega_0 = \pi$, $T = 2\pi/\omega_0 = 2$. Therefore,

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt$$

Now,

$$a_0 = \frac{1}{2}\int_0^1 1.5dt - \frac{1}{2}\int_1^2 1.5dt = 0$$

and for $k \neq 0$

$$\begin{aligned}
a_k &= \frac{1}{2}\int_0^1 1.5e^{-jk\pi t}dt - \frac{1}{2}\int_1^2 1.5e^{-jk\pi t}dt \\
&= \frac{3}{2k\pi j}[1 - e{-}jk\pi] \\
&= \frac{3}{k\pi}e^{-jk(\pi/2)}\sin(\frac{k\pi}{2})
\end{aligned}$$

3.5. Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundemental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundemental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \stackrel{FS}{\longleftrightarrow} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \stackrel{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \stackrel{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \stackrel{FS}{\longleftrightarrow} a_{-k} e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \stackrel{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)} + a_{-k} e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k}(a_k + a_{-k})$$

3.6. (a) Comparing $x_1(t)$ with the Fourier series synthesis eq. (3.38), we obtain the Fourier series coefficients of $x_1(t)$ to be

$$a_k = \begin{cases} \left(\frac{1}{2}\right)^k, & 0 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_1(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is not true for $x_1(t)$, the signal is **not real valued.**

Similarly, the Fourier series coefficients of $x_2(t)$ are

$$a_k = \begin{cases} \cos(k\pi), & 100 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_2(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is true for $x_2(t)$, the signal is **real valued.**

Similarly, the Fourier series coefficients of $x_3(t)$ are

$$a_k = \begin{cases} j\sin(k\pi/2), & 100 \le k \le 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_3(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is true for $x_3(t)$, the signal is **real valued.**

(b) For a signal to be even, its Fourier series coefficients must be even. This is true only for $x_2(t)$.

3.7. Given that

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k$$

we have

$$g(t) = \frac{dx(t)}{dt} \stackrel{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T}a_k.$$

Therefore,

$$a_k = \frac{b_k}{j(2\pi/T)k}, \qquad k \neq 0$$

When $k = 0$,

$$a_k = \frac{1}{T}\int_{<T>} x(t)dt = \frac{2}{T} \quad \text{using given information}$$

Therefore,

$$a_k = \begin{cases} \frac{2}{T}, & k = 0 \\ \frac{b_k}{j(2\pi/T)k}, & k \neq 0 \end{cases}.$$

3.8. Since $x(t)$ is real and odd (clue 1), its Fourier series coefficients a_k are purely imaginary and odd (See Table 3.1). Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. Also, since it is given that $a_k = 0$ for $|k| > 1$, the only unknown Fourier series coefficients are a_1 and a_{-1}. Using Parseval's relation,

$$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

for the given signal we have

$$\frac{1}{2}\int_0^2 |x(t)|^2 dt = \sum_{k=-1}^{1} |a_k|^2.$$

Using the information given in clue (4) along with the above equation,

$$|a_1|^2 + |a_{-1}|^2 = 1 \quad \Rightarrow \quad 2|a_1|^2 = 1$$

Therefore,

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}j} \quad \text{or} \quad a_1 = -a_{-1} = -\frac{1}{\sqrt{2}j}$$

The two possible signals which satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} - \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = -\sqrt{2}\sin(\pi t)$$

and

$$x_2(t) = -\frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} + \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = \sqrt{2}\sin(\pi t)$$

3.9. The period of the given signal is 4. Therefore,

$$\begin{aligned} a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}[4 + 8e^{-j\frac{\pi}{2}k}] \end{aligned}$$

This gives

$$a_0 = 3, \quad a_1 = 1 - 2j, \quad a_2 = -1, \quad a_3 = 1 + 2j$$

3.10. Since the Fourier series coeffiecients repeat every N, we have

$$a_1 = a_{15}, \quad a_2 = a_{16} \quad \text{, and} \quad a_3 = a_{17}$$

Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd. Therefore, $a_0 = 0$ and

$$a_1 = -a_{-1}, \quad a_2 = -a_{-2} \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j, \quad a_{-2} = -2j, \quad a_{-3} = -3j$$

3.11. Since the Fourier series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$. Furthermore, since $x[n]$ is real and even, a_k is also real and even. Therefore, $a_1 = a_{-1} = 5$. We are also given that

$$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50.$$

Using Parseval's relation,

$$\begin{aligned}
\sum_{k=<N>} |a_k|^2 &= 50 \\
\sum_{k=-1}^{8} |a_k|^2 &= 50 \\
|a_{-1}|^2 + |a_1|^2 + a_0^2 + \sum_{k=2}^{8} |a_k|^2 &= 50 \\
a_0^2 + \sum_{k=2}^{8} |a_k|^2 &= 0
\end{aligned}$$

Therefore, $a_k = 0$ for $k = 2, \cdots, 8$. Now using the synthesis eq.(3.94), we have

$$\begin{aligned}
x[n] &= \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^{8} a_k e^{j\frac{2\pi}{10}kn} \\
&= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n} \\
&= 10\cos(\frac{\pi}{5}n)
\end{aligned}$$

3.12. Using the multiplication property (see Table 3.2), we have

$$\begin{aligned}
x_1[n]x_2[n] &\stackrel{FS}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l} = \sum_{k=0}^{3} a_l b_{k-l} \\
&\stackrel{FS}{\longleftrightarrow} a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} \\
&\stackrel{FS}{\longleftrightarrow} b_k + 2b_{k-1} + 2b_{k-2} + 2b_{k-3}
\end{aligned}$$

Since b_k is 1 for all values of k, it is clear that $b_k + 2b_{k-1} + 2b_{k-3} + 2b_{k-3}$ will be 6 for all values of k. Therefore,

$$x_1[n]x_2[n] \overset{FS}{\longleftrightarrow} 6, \qquad \text{for all } k.$$

3.13. Let us first evaluate the Fourier series coefficients of $x(t)$. Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd. Therefore, $a_0 = 0$. Now,

$$\begin{aligned} a_k &= \frac{1}{8}\int_0^8 x(t)e^{-j(2\pi/8)kt}dt \\ &= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}dt \\ &= \frac{1}{j\pi k}[1 - e^{-j\pi k}] \end{aligned}$$

Clearly, the above expression evaluates to zero for all even values of k. Therefore,

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \cdots \\ \frac{2}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by (see Section 3.8)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Since a_k is non zero only for odd values of k, we need to evaluate the above summation only for odd k. Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

is always zero for odd values of k. Therefore,

$$y(t) = 0.$$

3.14. The signal $x[n]$ is periodic with period $N = 4$. Its Fourier series coefficients are

$$\begin{aligned} a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}, \qquad \text{for all } k \end{aligned}$$

From the results presented in Section 3.8, we know that the output $y[n]$ is given by

$$\begin{aligned} y[n] &= \sum_{k=0}^{3} a_k H(e^{j(2\pi/4)k})e^{jk(2\pi/4)n} \\ &= \tfrac{1}{4}H(e^{j0})e^{j0} + \tfrac{1}{4}H(e^{j(\pi/2)})e^{j(\pi/2)} \\ &\quad + \tfrac{1}{4}H(e^{j(3\pi/2)})e^{j(3\pi/2)} + \tfrac{1}{4}H(e^{j(\pi)})e^{j(\pi)} \end{aligned} \qquad \text{(S3.14-1)}$$

From the given information, we know that $y[n]$ is

$$\begin{aligned} y[n] &= \cos(\frac{5\pi}{2}n + \frac{\pi}{4}) \\ &= \cos(\frac{\pi}{2}n + \frac{\pi}{4}) \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n+\frac{\pi}{4})} + \frac{1}{2}e^{-j(\frac{\pi}{2}n+\frac{\pi}{4})} \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n+\frac{\pi}{4})} + \frac{1}{2}e^{j(3\frac{\pi}{2}n-\frac{\pi}{4})} \end{aligned}$$

Comparing this with eq. (S3.14-1), we have

$$H(e^{j0}) = H(e^{j\pi}) = 0$$

and

$$H(e^{j\frac{\pi}{2}}) = 2e^{j\frac{\pi}{4}}, \quad \text{and} \quad H(e^{3j\frac{\pi}{2}}) = 2e^{-j\frac{\pi}{4}}$$

3.15. From the results of Section 3.8,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = 12$. Since $H(j\omega)$ is zero for $|\omega| > 100$, the largest value of $|k|$ for which a_k is nonzero should be such that

$$|k|\omega_0 \leq 100$$

This implies that $|k| \leq 8$. Therefore, for $|k| > 8$, a_k is guaranteed to be zero.

3.16. (a) The given signal $x_1[n]$ is

$$x_1[n] = (-1)^n = e^{j\pi n} = e^{j(2\pi/2)n}$$

Therefore, $x_1[n]$ is periodic with period $N = 2$ and it's Fourier series coefficients in the range $0 \leq k \leq 1$ are

$$a_0 = 0, \quad \text{and} \quad a_1 = 1$$

Using the results derived in Section 3.8, the output $y_1[n]$ is given by

$$\begin{aligned} y_1[n] &= \sum_{k=0}^{1} a_k H(e^{j2\pi k/2})e^{k(2\pi/2)} \\ &= 0 + a_1 H(e^{j\pi})e^{j\pi} \\ &= 0 \end{aligned}$$

(b) The signal $x_2[n]$ is periodic with period $N = 16$. The signal $x_2[n]$ may be written as

$$\begin{aligned} x_2[n] &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{-j(2\pi/16)(3)n} \\ &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n} \end{aligned}$$

Therefore, the non-zero Fourier series coefficients of $x_2[n]$ in the range $0 \le k \le 15$ are

$$a_0 = 1, \qquad a_3 = -(j/2)e^{j(\pi/4)}, \qquad a_{13} = (j/2)e^{-j(\pi/4)}$$

Using the results derived in Section 3.8, the output $y_2[n]$ is given by

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{15} a_k H(e^{j2\pi k/16}) e^{k(2\pi/16)} \\ &= 0 - (j/2)e^{j(\pi/4)} e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)} e^{j(2\pi/16)(13)n} \\ &= \sin(\frac{3\pi}{8}n + \frac{\pi}{4}) \end{aligned}$$

(c) The signal $x_3[n]$ may be written as

$$x_3[n] = \left[\left(\frac{1}{2}\right)^n u[n]\right] * \sum_{k=-\infty}^{\infty} \delta[n-4k] = g[n] * r[n]$$

where $g[n] = \left(\frac{1}{2}\right)^n u[n]$ and $r[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$. Therefore, $y_3[n]$ may be obtained by passing the signal $r[n]$ through the filter with frequency response $H(e^{j\omega})$, and then convolving the result with $g[n]$.

The signal $r[n]$ is periodic with period 4 and its Fourier series coeffients are

$$a_k = \frac{1}{4}, \qquad \text{for all } k \text{ (See Problem 3.14)}$$

The output $q[n]$ obtained by passing $r[n]$ through the filter with frequency response $H(e^{j\omega})$ is

$$\begin{aligned} q[n] &= \sum_{k=0}^{3} a_k H(e^{j2\pi k/4}) e^{k(2\pi/4)} \\ &= (1/4)(H(e^{j0})e^{j0} + H(e^{j(\pi/2)})e^{j(\pi/2)} + H(e^{j\pi})e^{j\pi} + H(e^{j3(\pi/2)})e^{j3(\pi/2)}) \\ &= 0 \end{aligned}$$

Therefore, the final output $y_3[n] = q[n] * g[n] = 0$.

3.17. **(a)** Since complex exponentials are Eigen functions of LTI systems, the input $x_1(t) = e^{j5t}$ has to produce an output of the form Ae^{j5t}, where A is a complex constant. But clearly, in this case the output is not of this form. Therefore, system S_1 is definitely **not LTI**.

(b) This system may be LTI because it satisifies the Eigen function property of LTI systems.

(c) In this case, the output is of the form $y_3(t) = (1/2)e^{j5t} + (1/2)e^{-j5t}$. Clearly, the output contains a complex exponential with frequency -5 which was not present in the input $x_3(t)$. We know that an LTI system can never produce a complex exponential of frequency -5 unless there was complex exponential of the same frequency at its input. Since this is not the case in this problem, S_3 is definitely **not LTI**.

3.18. (a) By using an argument similar to the one used in part (a) of the previous problem, we conclude that S_1 is defintely not LTI.

(b) The output in this case is $y_2[n] = e^{j(3\pi/2)n} = e^{-j(\pi/2)n}$. Clearly this violates the eigen function property of LTI systems. Therefore, S_2 is definitely **not LTI**.

(c) The output in this case is $y_3[n] = 2e^{j(5\pi/2)n} = 2e^{j(\pi/2)n}$. This does not violate the eigen function property of LTI systems. Therefore, S_3 could possibly be an LTI system.

3.19. (a) Voltage across inductor $= L\frac{dy(t)}{dt}$.
Current through resistor $= \frac{L}{R}\frac{dy(t)}{dt}$.
Input current $x(t)$ = current through resistor + current through inductor
Therefore,

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t).$$

Substituting for R and L we obtain

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

(b) Using the approach outlined in Section 3.10.1, we know that the output of this system will be $H(j\omega)e^{j\omega t}$ when the input is $e^{j\omega t}$. Substituting in the differential equation of part (a),

$$j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

Therefore,

$$H(j\omega) = \frac{1}{1+j\omega}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2}e^{j(2\pi/2\pi)t} + \frac{1}{2}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1} = \frac{1}{2}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} + a_{-1}H(-j)e^{-jt} \\ &= (1/2)(\frac{1}{1+j}ejt + \frac{1}{1-j}e^{-jt}) \\ &= (1/2\sqrt{2})(e^{-j\pi/4}e^{jt} + e^{j\pi/4}e^{-jt}) \\ &= (1/\sqrt{2})\cos(t - \frac{\pi}{4}) \end{aligned}$$

3.22. (a) (i) $T = 1$, $a_0 = 0$, $a_k = \frac{j(-1)^k}{k\pi}$, $k \neq 0$.

(ii) Here,

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

$T = 6$, $a_0 = 1/2$, and

$$a_k = \begin{cases} 0, & k \text{ even} \\ \frac{6}{\pi^2 k^2}\sin(\frac{\pi k}{2})\sin(\frac{\pi k}{6}), & k \text{ odd} \end{cases}.$$

(iii) $T = 3$, $a_0 = 1$, and

$$a_k = \frac{3j}{2\pi^2 k^2}[e^{jk2\pi/3}\sin(k2\pi/3) + 2e^{jk\pi/3}\sin(k\pi/3)], \qquad k \neq 0.$$

(iv) $T = 2$, $a_0 = -1/2$, $a_k = \frac{1}{2} - (-1)^k$, $k \neq 0$.

(v) $T = 6$, $\omega_0 = \pi/3$, and

$$a_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi/3}.$$

Note that $a_0 = 0$ and $a_{k\ even} = 0$.

(vi) $T = 4$, $\omega_0 = \pi/2$, $a_0 = 3/4$ and

$$a_k = \frac{e^{-jk\pi/2}\sin(k\pi/2) + e^{-jk\pi/4}\sin(k\pi/4)}{k\pi}, \qquad \forall k.$$

(b) $T = 2$, $a_k = \frac{-1^k}{2(1+jk\pi)}[e - e^{-1}]$ for all k.

(c) $T = 3$, $\omega_0 = 2\pi/3$, $a_0 = 1$ and

$$a_k = \frac{2e^{-j\pi k/3}}{\pi k}\sin(2\pi k/3) + \frac{e^{-j\pi k}}{\pi k}\sin(\pi k).$$

3.23. (a) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/4)}{k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & 1/2 < |t| < 2 \end{cases}.$$

Now, note that $b_0 = 1/4$. Let us define another signal $z(t) = -1/4$ whose only nonzero FS coefficient is $c_0 = -1/4$. The signal $p(t) = y(t) + z(t)$ will have FS coefficients

$$d_k = a_k + c_k = \begin{cases} 0, & k = 0 \\ \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise.} \end{cases}$$

Now note that $a_k = d_k e^{j(\pi/2)k}$. Therefore, the signal $x(t) = p(t+1)$ which is as shown in Figure S2.23(a).

Therefore,

$$x(t) = y(t) + p(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k) + \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t}\delta(t-2k).$$

3.24. (a) We have

$$a_0 = \frac{1}{2}\int_0^1 t dt + \frac{1}{2}\int_1^2 (2-t)dt = 1/2.$$

(b) The signal $g(t) = dx(t)/dt$ is as shown in Figure S3.24.

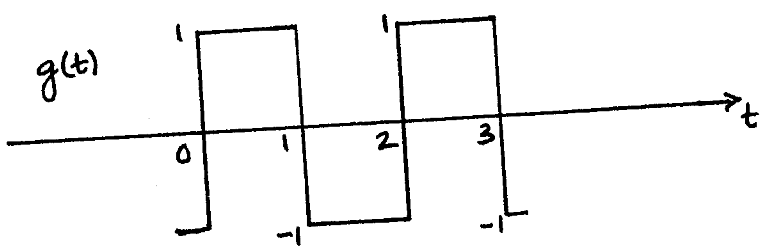


Figure S3.24

The FS coefficients b_k of $g(t)$ may be found as follows:

$$b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

and

$$\begin{aligned} b_k &= \frac{1}{2}\int_0^1 e^{-j\pi k t}dt - \frac{1}{2}\int_1^2 e^{-j\pi k t}dt \\ &= \frac{1}{j\pi k}[1 - e^{-j\pi k}]. \end{aligned}$$

(c) Note that

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\pi a_k.$$

Therefore,

$$a_k = \frac{1}{jk\pi}b_k = -\frac{1}{\pi^2 k^2}\{1 - e^{-j\pi k}\}.$$

3.25. (a) The nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1}^* = 1/2j$.

(c) Using the multiplication property, we know that

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Therefore,

$$c_k = a_k * b_k = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = (1/4j)$.

(d) We have

$$z(t) = \sin(4t)\cos(4t) = \frac{1}{2}\sin(8t).$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2} = (1/4j)$.

3.26. **(a)** If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a_{-k}^*$. Since this is not true in this case problem, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T_0}a_k.$$

Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T_0), & \text{otherwise} \end{cases}.$$

Since b_k is not even, $g(t)$ is not even.

3.27. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\ &= 2 + 2e^{j\pi/6}e^{j(4\pi/5)n} + 2e^{-j\pi/6}e^{-j(4\pi/5)n} + e^{j\pi/3}e^{j(8\pi/5)n} + e^{-j\pi/3}e^{-j(8\pi/5)n} \\ &= 2 + 4\cos[(4\pi n/5) + \pi/6] + 2\cos[(8\pi n/5) + \pi/3] \\ &= 2 + 4\sin[(4\pi n/5) + 2\pi/3] + 2\sin[(8\pi n/5) + 5\pi/6] \end{aligned}$$

3.28. **(a)** $N = 7$,

$$a_k = \frac{1}{7}\frac{e^{-j4\pi k/7}\sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period $(0 \le k \le 5)$ may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6}e^{-j\pi k/2}\frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \qquad 1 \le k \le 5.$$

3.33. We will first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. From the discussion in Section 3.9.2 we know that the response to this input will be $y(t) = H(j\omega)e^{j\omega t}$. Therefore, substituing these in the given differential equation, we get

$$H(j\omega)j\omega e^{j\omega t} + 4e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

From eq. (3.124), we know that

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

when the input is $x(t)$. $x(t)$ has the Fourier series coefficients a_k and fundamental frequency ω_0. Therefore, the Fourier series coefficients of $y(t)$ are $a_k H(jk\omega_0)$.

(a) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(4+j2\pi)}, \qquad b_{-1} = a_{-1}H(-j2\pi) = \frac{1}{2(4-j2\pi)}.$$

(b) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_2 = a_{-2}^* = 1/2j$ and $a_3 = a_{-3}^* = e^{j\pi/4}/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_2 = a_2 H(j4\pi) = \frac{1}{2j(4+j4\pi)}, \qquad b_{-2} = a_{-2}H(-j4\pi) = -\frac{1}{2j(4-j4\pi)},$$

$$b_3 = a_3 H(j6\pi) = \frac{e^{j\pi/4}}{2(4+j6\pi)}, \qquad b_{-3} = a_{-3}H(-j6\pi) = -\frac{e^{-j\pi/4}}{2(4-j6\pi)}.$$

3.34. The frequency response of the system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|t|}e^{-j\omega t}dt = \frac{1}{4+j\omega} + \frac{1}{4-j\omega}.$$

(a) Here, $T = 1$ and $\omega_0 = 2\pi$ and $a_k = 1$ for all k. The FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}.$$

(b) Here, $T = 2$ and $\omega_0 = \pi$ and

$$a_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 0, & k \text{ even} \\ \frac{1}{4+j\pi k} + \frac{1}{4-j\pi k}, & k \text{ odd} \end{cases}.$$

(c) Here, $T = 1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}\left[\frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}\right], & k \text{ odd} \end{cases}.$$

3.35. We know that the Fourier series coefficient of $y(t)$ are $b_k = H(jk\omega_0)a_k$, where ω_0 is the fundamental frequency of $x(t)$ and a_k are the FS coefficients of $x(t)$.

If $y(t)$ is identical to $x(t)$, then $b_k = a_k$ for all k. Noting that $H(j\omega) = 0$ for $|\omega| \geq 250$, we know that $H(jk\omega_0) = 0$ for $|k| \geq 18$ (because $\omega_0 = 14$). Therefore, a_k must be zero for $|k| \geq 18$.

3.36. We will first evaluate the frequency response of the system. Consider an input $x[n]$ of the form $e^{j\omega n}$. From the discussion in Section 3.9 we know that the response to this input will be $y[n] = H(e^{j\omega})e^{j\omega n}$. Therefore, substituing these in the given difference equation, we get

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}e^{-j\omega}e^{j\omega n}H(e^{j\omega}) = e^{j\omega n}.$$

Therefore,

$$H(j\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

From eq. (3.131), we know that

$$y[n] = \sum_{k=<N>} a_k H(e^{j2\pi k/N})e^{jk(2\pi/N)n}$$

when the input is $x[n]$. $x[n]$ has the Fourier series coefficients a_k and fundamental frequency $2\pi/N$. Therefore, the Fourier series coefficients of $y[n]$ are $a_k H(e^{j2\pi k/N})$.

(a) Here, $N = 4$ and the nonzero FS coefficients of $x[n]$ are $a_3 = a_{-3}^* = 1/2j$. Therefore, the nonzero FS coefficients of $y[n]$ are

$$b_3 = a_1 H(e^{3j\pi/4}) = \frac{1}{2j(1-(1/4)e^{-j3\pi/4})}, \qquad b_{-3} = a_{-1}H(e^{-3j\pi/4}) = \frac{-1}{2j(1-(1/4)e^{j3\pi/4})}.$$

(b) Here, $N = 8$ and the nonzero FS coefficients of $x[n]$ are $a_1 = a_{-1} = 1/2$ and $a_2 = a_{-2} = 1$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(e^{j\pi/4}) = \frac{1}{2(1-(1/4)e^{-j\pi/4})}, \qquad b_{-1} = a_{-1}H(e^{-j\pi/4}) = \frac{1}{2(1-(1/4)e^{j\pi/4})},$$

$$b_2 = a_2 H(e^{j\pi/2}) = \frac{1}{(1-(1/4)e^{-j\pi/2})}, \qquad b_{-2} = a_{-2}H(e^{-j\pi/2}) = \frac{1}{(1-(1/4)e^{j\pi/2})}.$$

3.37. The frequency response of the system may be easily shown to be

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{1-2e^{-j\omega}}.$$

(a) The Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{4}, \qquad \text{for all } k.$$

Also, $N = 4$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2k\pi/N}) = \frac{1}{4}\left[\frac{1}{1-\frac{1}{2}e^{-j\pi k/2}} - \frac{1}{1-2e^{-j\pi k/2}}\right].$$

(b) In this case, the Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{6}[1+2\cos(k\pi/3)], \qquad \text{for all} k.$$

Also, $N = 6$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2k\pi/N}) = \frac{1}{6}[1+2\cos(k\pi/3)]\left[\frac{1}{1-\frac{1}{2}e^{-j\pi k/3}} - \frac{1}{1-2e^{-j\pi k/3}}\right].$$

3.38. The frequency response of the system may be evaluated as

$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}.$$

For $x[n]$, $N = 4$ and $\omega_0 = \pi/2$. The FS coefficients of the input $x[n]$ are

$$a_k = \frac{1}{4}, \qquad \text{for all } n.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4}[1 - e^{jk\pi/2} + e^{-jk\pi/2}].$$

3.39. Let the FS coefficients of the input be a_k. The FS coeffients of the output are of the form

$$b_k = a_k H(e^{jk\omega_0}),$$

where $\omega_0 = 2\pi/3$. Note that in the range $0 \le k \le 2$, $H(e^{jk\omega_0}) = 0$ for $k = 1, 2$. Therefore, only b_0 has a nonzero value among b_k in the range $0 \le k \le 2$.

3.40. Let the Fourier series coefficients of $x(t)$ be a_k.

(a) $x(t-t_0)$ is also periodic with period T. The Fourier series coefficients b_k of $x(t-t_0)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0)e^{-jk(2\pi/T)t}dt \\ &= \frac{e^{-jk(2\pi/T)t_0}}{T}\int_T x(\tau)e^{-jk(2\pi/T)\tau}d\tau \\ &= e^{-jk(2\pi/T)t_0}a_k \end{aligned}$$

Similarly, the Fourier series coefficients of $x(t+t_0)$ are

$$c_k = e^{jk(2\pi/T)t_0}a_k.$$

Finally, the Fourier series coefficients of $x(t-t_0)+x(t+t_0)$ are

$$d_k = b_k + c_k = e^{-jk(2\pi/T)t_0}a_k + e^{jk(2\pi/T)t_0}a_k = 2\cos(k2\pi t_0/T)a_k.$$

(b) Note that $\mathcal{E}v\{x(t)\} = [x(t)+x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(-t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_T x(\tau)e^{jk(2\pi/T)\tau}d\tau \\ &= a_{-k} \end{aligned}$$

Therefore, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are

$$c_k = \frac{a_k+b_k}{2} = \frac{a_k+a_{-k}}{2}.$$

(c) Note that $\mathcal{R}e\{x(t)\} = [x(t)+x^*(t)]/2$. The FS coefficients of $x^*(t)$ are

$$b_k = \frac{1}{T}\int_T x^*(t)e^{-jk(2\pi/T)t}dt.$$

Conjugating both sides, we get

$$b_k^* = \frac{1}{T}\int_T x(t)e^{jk(2\pi/T)t}dt = a_{-k}.$$

Therefore, the FS coefficients of $\mathcal{R}e\{x(t)\}$ are

$$c_k = \frac{a_k+b_k}{2} = \frac{a_k+a_{-k}^*}{2}.$$

(d) The Fourier series synthesis equation gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}.$$

Differentiating both sides wrt t twice, we get

$$\frac{d^2x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} -k^2\frac{4\pi^2}{T^2}a_k e^{j(2\pi/T)kt}.$$

By inspection, we know that the Fourier series coefficients of $d^2x(t)/dt^2$ are $-k\frac{4\pi^2}{T^2}a_k$.

(e) The period of $x(3t)$ is a third of the period of $x(t)$. Therefore, the signal $x(3t-1)$ is periodic with period $T/3$. The Fourier series coefficents of $x(3t)$ are still a_k. Using the analysis of part (a), we know that the Fourier series coefficients of $x(3t-1)$ is $e^{-jk(6\pi/T)}a_k$.

3.41. Since $a_k = a_{-k}$, we require that $x(t) = x(-t)$. Also, note that since $a_k = a_{k+2}$, we require that

$$x(t) = x(t)e^{-j(4\pi/3)t}.$$

This in turn implies that $x(t)$ may have nonzero values only for $t = 0, \pm1.5, \pm3, \pm4.5, \cdots$. Since $\int_{-0.5}^{0.5} x(t) = 1$, we may conclude that $x(t) = \delta(t)$ for $-0.5 \le t \le 0.5$. Also, since $\int_{0.5}^{1.5} x(t)dt = 2$, we may conclude that $x(t) = 2\delta(t-3/2)$ in the range $0.5 \le t \le 3/2$. Therefore, $x(t)$ may be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k3) + 2\sum_{k=-\infty}^{\infty} \delta(t-3k-3/2).$$

3.42. (a) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x^*(t)$ are a^*_{-k}. Now, we know that is $x(t)$ is real, then $x(t) = x^*(t)$. Therefore, $a_k = a^*_{-k}$. Note that this implies $a_0 = a^*_0$. Therefore, a_0 must be real.

(b) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k}. If $x(t)$ is even, then $x(t) = x(-t)$. This implies that

$$a_k = a_{-k}. \qquad \text{(S3.42–1)}$$

This implies that the FS coefficients are even. From the previous part, we know that if $x(t)$ is real, then

$$a_k = a^*_{-k}. \qquad \text{(S3.42–2)}$$

Using eqs. (S3.42-1) and (S3.42-2), we know that $a_k = a^*_k$. Therefore, a_k is real for all k. Hence, we may conclude that a_k is real and even.

(c) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k}. If $x(t)$ is odd, then $x(t) = -x(-t)$. This implies that

$$a_k = -a_{-k}. \qquad \text{(S3.42–3)}$$

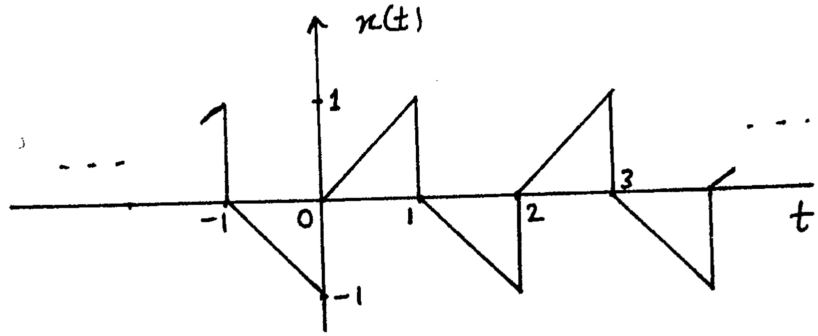


Figure S3.43

(d) (1) If a_1 or a_{-1} is nonzero, then

$$x(t) = a_{\pm 1} e^{\pm j2\pi t/T} + \cdots .$$

and

$$x(t+t_0) = a_{\pm 1} e^{\pm j\frac{2\pi}{T}(t+t_0)} + \cdots$$

The smallest value of $|t_0|$ (other than $|t_0| = 0$ for which $e^{\pm j\frac{2\pi}{T}t_0} = 1$ is the fundamental period. Only then is

$$x(t+t_0) = a_{\pm 1} e^{\pm j2\pi t/T} + \cdots = x(t).$$

Therefore, t_0 has to be the fundamental period.

(2) The period of $x(t)$ is the least common multiple of the periods of $e^{jk(2\pi/T)t}$ and $e^{jl(2\pi/T)t}$. The period of $e^{jk(2\pi/T)t}$ is T/k and the period of $e^{jl(2\pi/T)t}$ and T/l. Since k and l have no common factors, the least common multiple of T/k and T/l is T.

3.44. The only unknown FS coefficients are a_1, a_{-1}, a_2, and a_{-2}. Since $x(t)$ is real, $a_1 = a_{-1}^*$ and $a_2 = a_{-2}^*$. Since a_1 is real, $a_1 = a_{-1}$. Now, $x(t)$ is of the form

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t + \theta),$$

where $\omega_0 = 2\pi/6$. From this we get

$$x(t-3) = A_1 \cos(\omega_0 t - 3\omega_0) + A_2 \cos(2\omega_0 t + \theta - 6\omega_0).$$

Now if we need $x(t) = -x(t-3)$, then $3\omega_0$ and $6\omega_0$ should both be odd multiples of π. Clearly, this is impossible. Therefore, $a_2 = a_{-2} = 0$ and

$$x(t) = A_1 \cos(\omega_0 t).$$

Now, using Parseval's relation on Clue 5, we get

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = \frac{1}{2}.$$

Therefore, $|a_1| = 1/2$. Since a_1 is positive, we have $a_1 = a_{-1} = 1/2$. Therefore, $x(t) = \cos(\pi t/3)$.

3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

3.48. **(a)** The FS coefficients of $x[n-n_0]$ are

$$\begin{aligned}\hat{a}_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]e^{-j2\pi nk/N} \\ &= \frac{1}{N}e^{-j\frac{2\pi n_0 k}{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} \\ &= e^{-j2\pi k n_0/N}a_k\end{aligned}$$

(b) Using the results of part (a), the FS coefficients of $x[n]-x[n-1]$ are given by

$$\hat{a}_k = a_k - e^{-j2\pi k/n}a_k = [1-e^{-j2\pi k/n}]a_k.$$

(c) Using the results of part (a), the FS coefficients of $x[n]-x[n-N/2]$ are given by

$$\hat{a}_k = a_k[1-e^{-jk\pi}] = \begin{cases} 0, & k \text{ even} \\ 2a_k, & k \text{ odd}\end{cases}.$$

(d) Note that $x[n]+x[n+N/2]$ has a period of $N/2$. The FS coefficients of $x[n]+x[n-N/2]$ are given by

$$\hat{a}_k = \frac{2}{N}\sum_{n=0}^{\frac{N}{2}-1}\left[x[n]+x[n+\frac{N}{2}]\right]e^{-j4\pi nk/N} = 2a_{2k}$$

for $0 \le k \le (N/2-1)$.

(e) The FS coefficients of $x^*[-n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x^*[-n]e^{-j2\pi nk/N} = a_k^*.$$

(f) With N even the FS coefficients of $(-1)^n x[n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi n/N)(k-\frac{N}{2})} = a_{k-N/2}$$

(g) With N odd, the period of $(-1)^n x[n]$ is $2N$. Therefore, the FS coefficients are

$$\hat{a}_k = \frac{1}{2N}\left[\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})} + \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})}e^{-j\pi(k-N)}\right].$$

Note that for k odd $\frac{k-N}{2}$ is an integer and $k-N$ is an even integer. Also, for k even, $k-N$ is an odd integer and $e^{-j\pi(k-N)} = -1$. Therefore,

$$\hat{a}_k = \begin{cases} a_{\frac{k-N}{2}}, & k \text{ odd} \\ 0, & k \text{ even}\end{cases}.$$

(h) Here,

$$y[n] = \frac{1}{2}[x[n] + (-1)^n x[n]].$$

For N even,

$$\hat{a}_k = \frac{1}{2}[a_k + a_{k-\frac{N}{2}}].$$

For N odd,

$$\hat{a}_(k) = \begin{cases} \frac{1}{2}[a_k + a_{\frac{k-N}{2}}], & k \text{ even} \\ \frac{1}{2}a_k, & k \text{ odd} \end{cases}.$$

3.49. **(a)** The FS coefficients are given by

$$\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} + \frac{1}{N}\sum_{n=N/2}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} + \frac{e^{-j\pi k}}{N}\sum_{n=0}^{(N/2)-1} x[n+N/2]e^{-j\frac{2\pi nk}{N}} = 0 \\
&= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} - \frac{e^{-j\pi k}}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} \\
&= 0, \qquad \text{for } k \text{ even.}
\end{aligned}$$

(b) By adopting an approach similar to part (a), we may show that

$$\begin{aligned}
a_k &= \frac{1}{N}\left[\sum_{n=0}^{\frac{N}{4}-1}\{1 - e^{-jk\pi/2} + e^{-j\pi k} - e^{-j\frac{3\pi k}{2}}\}x[n]e^{-j\frac{2\pi nk}{N}}\right] \\
&= 0, \qquad \text{for } k = 4r, r \in \mathcal{I}
\end{aligned}$$

(c) If N/M is an integer, we may generalize the approach of part (a) to show that

$$a_k = \frac{1}{N}\left[\sum_{k=0}^{B-1}\{1 - e^{-j2\pi r} + e^{-j4\pi r} - \cdots + e^{-j2\pi(M-1)r}\}x[n]e^{-j\frac{2\pi nk}{N}}\right].$$

where $B = N/M$ and $r = k/m$. From the above equation, it is clear that

$$a_k = 0, \qquad \text{if } k = rM, r \in \mathcal{I}.$$

3.50. From Table 3.2, we know that if

$$x[n] \xleftrightarrow{FS} a_k,$$

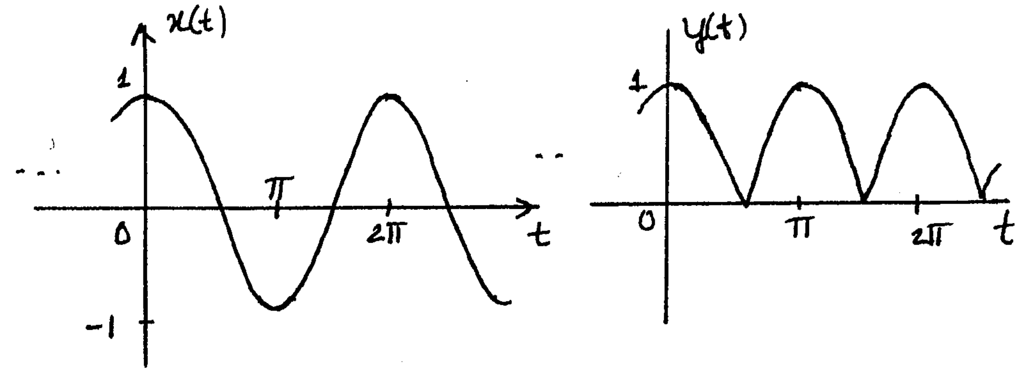


Figure S3.62

(c) The dc component of the input is 0. The dc component of the output is $2/\pi$.

3.63. The average energy per period is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |\alpha_k|^2 = \sum_k \alpha^{2|k|} = \frac{1+\alpha^2}{1-\alpha^2}.$$

We want N such that

$$\sum_{-N+1}^{N-1} |\alpha_k|^2 = 0.9\frac{1+\alpha^2}{1-\alpha^2}.$$

This implies that

$$\frac{1-2\alpha^{2N}+2\alpha^2}{1-\alpha^2} = \frac{1+\alpha^2}{1-\alpha^2}.$$

Solving,

$$N = \frac{\log[1.45\alpha^2+0.95]}{2\log\alpha},$$

and

$$\frac{\pi N}{4} < W < \frac{(N-1)\pi}{4}.$$

3.64. **(a)** Due to linearity, we have

$$y(t) = \sum_k c_k \lambda_k \phi_k(t).$$

(b) Let

$$x_1(t) \longrightarrow y_1(t) \quad \text{and} \quad x_2(t) \longrightarrow y_2(t).$$

Also, let

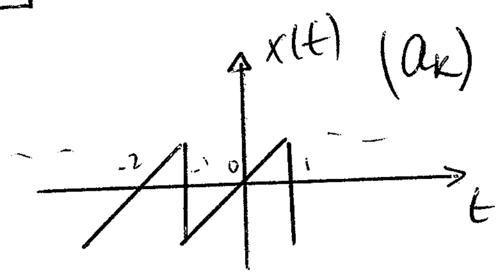
$$x_3(t) = ax_1(t) + bx_2(t) \longrightarrow y_3(t).$$

Then,

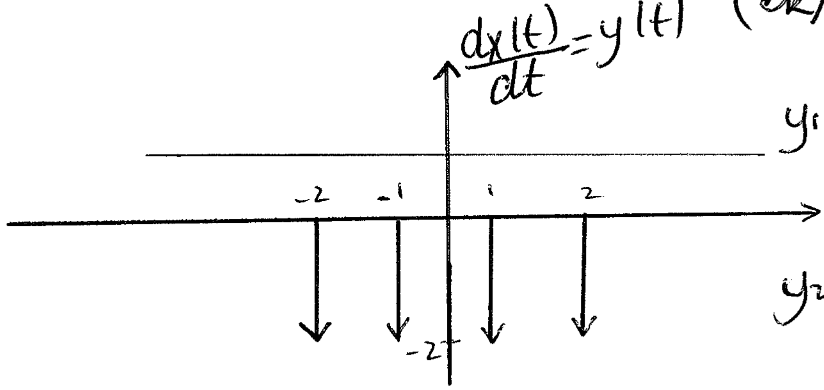
$$\begin{aligned} y_3(t) &= t^2[a{x_1}''(t) + b{x_2}''(t)] + t[ax_1'(t) + b{x_2}'(t)] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Problem 3.22

(a)



$T=2 \Rightarrow \omega_0=\pi$



$y_1(t)=1 \quad (C_k)$

$y_2(t)=-2\sum_{k=-\infty}^{+\infty}\delta(t-(2k+1))$

(d_k)

- $y(t)$ is periodic with $T=2$
- $y(t) = y_1(t)+y_2(t)$

$$= 1-2\delta(t-1) \qquad (-1<t\leq 1)$$

- $y_1(t)=1 \Rightarrow C_k = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$ (Table 4.2)

- $y_2(t)=-2\delta(t-1) \Rightarrow d_k = -2\times\frac{1}{T}\times e^{+jk\pi}$ (for all k)

(Using time shifting property)

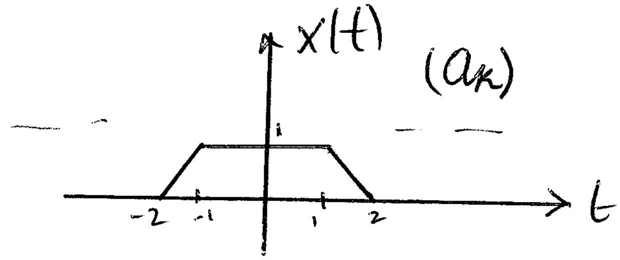
$$= -e^{+jk\pi} = \begin{cases} -1 & k=0 \\ -(-1)^k & k\neq 0 \end{cases}$$

$\Rightarrow b_k = C_k + d_k$ (Using linearity property, Table 3.1)

$$= \begin{cases} 0 & k=0 \\ -(-1)^k & k\neq 0 \end{cases}$$

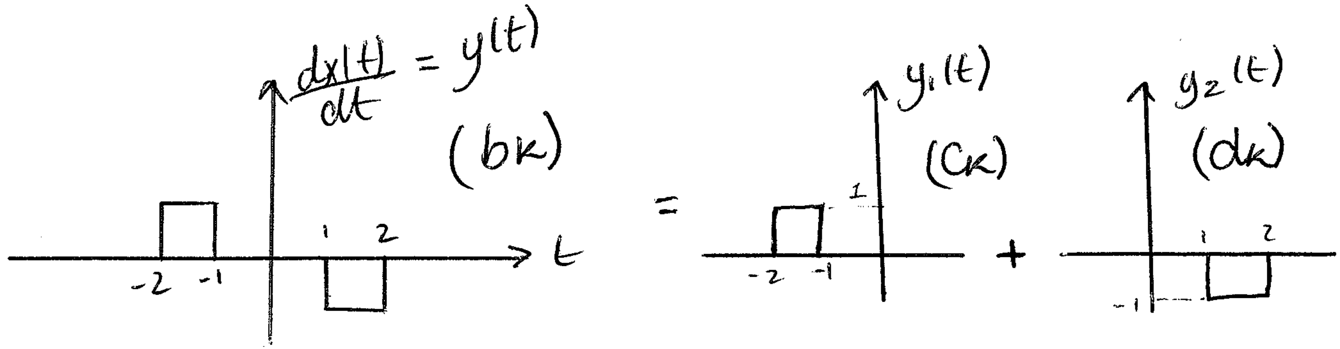
$$a_k = \frac{b_k}{jk\omega_0} = \frac{b_k}{jk\pi} = \begin{cases} 0 & k=0 \\ \frac{j(-1)^k}{k\pi} & k\neq 0 \end{cases}$$

(Integration property, Table 3.1)

(b) 

$T = 6 \Rightarrow \omega_0 = \frac{\pi}{3}$

$a_0 = \frac{1}{6} \times \overbrace{3}^{\text{area}} = \frac{1}{2}$



- $y_1(t)$ is the periodic square wave given in table 4.2 with $T_1 = \frac{1}{2}$ & shifted by $t_0 = 3/2$
- Similarly for $y_2(t)$, with $T_1 = \frac{1}{2}$ & $t_0 = -\frac{3}{2}$ (amplitude $= -1$)
- Using the fourier coefficients derived in table 4.2, & using shifting property:

$c_k = \frac{\sin\left(\frac{k\pi}{6}\right)}{k\pi} e^{+j\frac{k\pi}{2}}$ & $d_k = \frac{-\sin\left(k\pi/6\right)}{k\pi} e^{-j\frac{k\pi}{2}}$

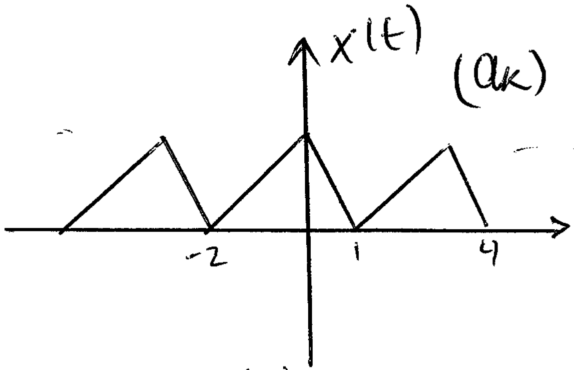
- $b_k = c_k + d_k = \frac{\sin\left(k\pi/6\right)}{k\pi} \left[e^{+j\frac{k\pi}{2}} - e^{-j\frac{k\pi}{2}}\right]$

$= 2j \frac{\sin\left(k\pi/6\right)}{k\pi} \sin\left(k\pi/2\right)$

- Using integration property, $a_k = \frac{b_k}{jk\omega_0} = \frac{b_k}{j\frac{k\pi}{3}}$

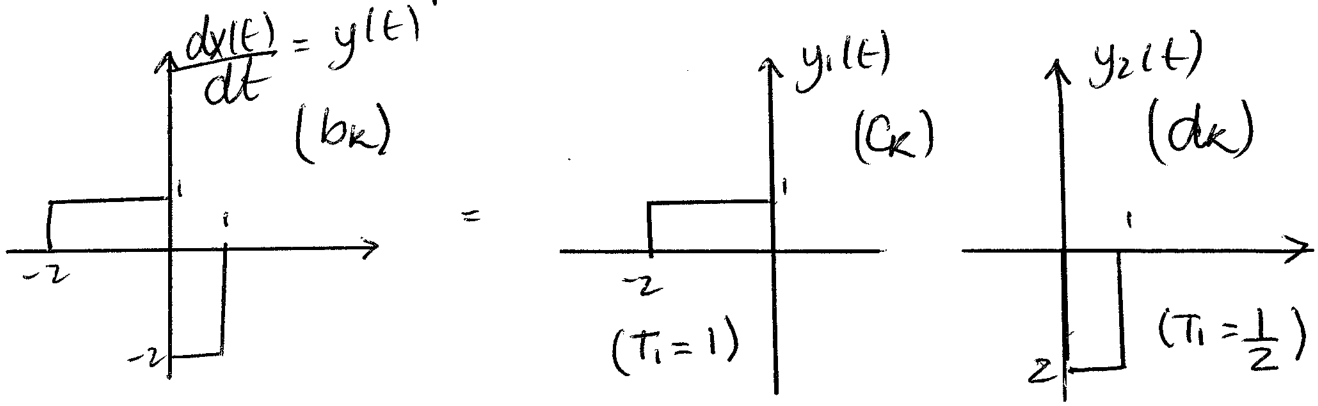
$\Rightarrow a_k = \frac{6}{k^2\pi^2} \sin\left(\frac{k\pi}{6}\right) \sin\left(\frac{k\pi}{2}\right)$

(c)



$T = 3 \Rightarrow \omega_0 = \frac{2\pi}{3}$

$a_0 = \frac{1}{3} \times 3 = 1$



- Using formula for fourier coefficients of periodic square wave given in table 4.2 & shifting property in table 3.1.

$$C_k = \frac{\sin\left(\frac{2\pi k}{3}\right)}{k\pi} e^{+\frac{j2k\pi}{3}} \quad \& \quad d_k = \frac{-2\sin\left(\frac{\pi k}{3}\right)}{k\pi} e^{-j\frac{k\pi}{3}}$$

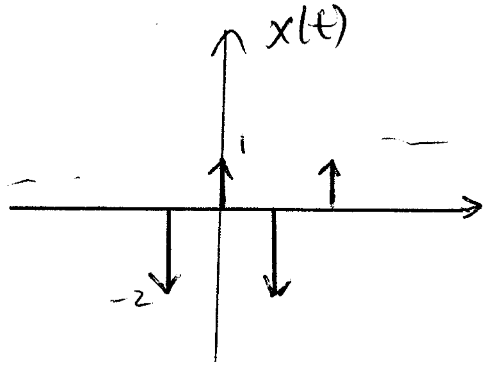
$$b_k = C_k + d_k \qquad \text{(Linearity property)}$$

$$a_k = \frac{b_k}{jk\omega_0} = \frac{b_k}{j\frac{2\pi k}{3}} \qquad \text{(Integration property)}$$

$$\Rightarrow a_k = \frac{-3j}{2\pi k}\left[\frac{\sin\left(\frac{2\pi k}{3}\right)}{k\pi} e^{+j\frac{2k\pi}{3}} - \frac{2\sin\left(\pi k/3\right)}{k\pi} e^{-j\frac{k\pi}{3}}\right]$$

$$= \frac{3j}{2\pi^2 k^2}\left[2\sin\left(\frac{\pi k}{3}\right)e^{-j\frac{k\pi}{3}} - \sin\left(\frac{2\pi k}{3}\right)e^{+j\frac{2k\pi}{3}}\right]$$

(d)



$$T = 2 \Rightarrow \omega_0 = \pi$$

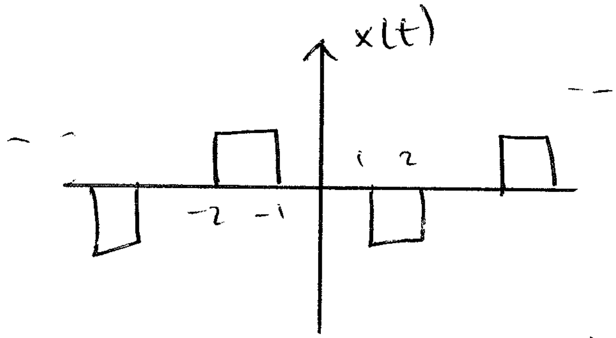
$$a_0 = \frac{1}{2}\int_0^1 [\delta(t) - 2\delta(t-1)] = \frac{1}{2} \times -1 = -\frac{1}{2}$$

$$x(t) = \delta(t) + 2\delta(t-1) \qquad (0 \le t < 2)$$

- Using fourier coefficients for $\delta(t)$ given in table 4.2 & shifting property, we get:

$$a_k = \frac{1}{T} - 2 \times \frac{1}{T} e^{-jk\pi} = \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - (-1)^k$$

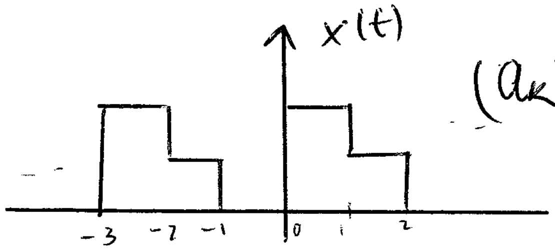
(e)



$x(t)$ here equals $y(t)$ of part (b)

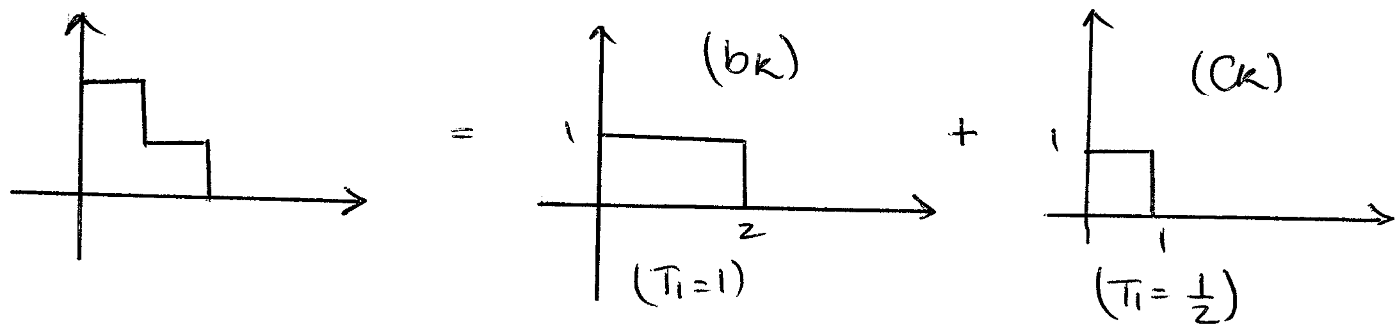
$$\Rightarrow a_k = b_k \text{ (part(b))} = \frac{2j}{k\pi} \sin\left(\frac{k\pi}{6}\right) \sin\left(\frac{k\pi}{2}\right)$$

(f)



$T=3 \Rightarrow \omega_0 = \frac{2\pi}{3}$

$a_0 = \frac{1}{3} \times 3 = 1$



- (b_k) & (c_k) can be obtained using fourier coefficient for periodic square wave given in table 4.2 & shifting property in table 3.1.

$$b_k = \frac{\sin\left(\frac{2k\pi}{3}\right)}{k\pi} e^{-j\frac{2\pi k}{3}}$$

$$\& \quad c_k = \frac{\sin(k\pi/3)}{k\pi} e^{-j\frac{\pi k}{3}}$$

$a_k = b_k + c_k$ (Linearity property)

$$= \frac{\sin\left(\frac{2k\pi}{3}\right)}{k\pi} e^{-j\frac{2\pi k}{3}} + \frac{\sin(k\pi/3)}{k\pi} e^{-j\frac{\pi k}{3}}$$

Chapter 4 Answers

4.1. (a) Let $x(t) = e^{-2(t-1)}u(t-1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

(b) Let $x(t) = e^{-2|t-1|}$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2|t-1|}e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt + \int_{-\infty}^{1} e^{2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) + e^{-j\omega}/(2-j\omega) \\ &= 4e^{-j\omega}/(4+\omega^2) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

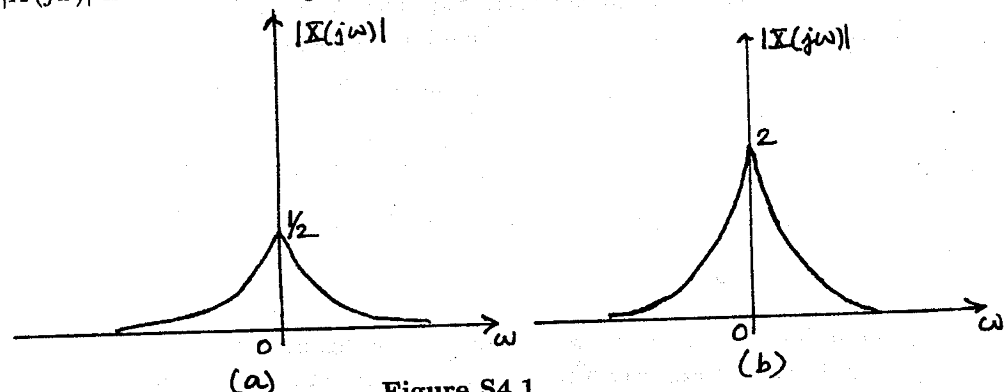


Figure S4.1

4.2. (a) Let $x_1(t) = \delta(t+1) + \delta(t-1)$. Then the Fourier transform $X_1(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)]e^{-j\omega t}dt \\ &= e^{j\omega} + e^{-j\omega} = 2\cos\omega \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

(b) The signal $x_2(t) = u(-2-t) + u(t-2)$ is as shown in the figure below. Clearly,

$$\frac{d}{dt}\{u(-2-t) + u(t-2)\} = \delta(t-2) - \delta(t+2)$$

Therefore,

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} [\delta(t-2) - \delta(t+2)] e^{-j\omega t} dt \\ &= e^{-2j\omega} - e^{2j\omega} = -2j\sin(2\omega) \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

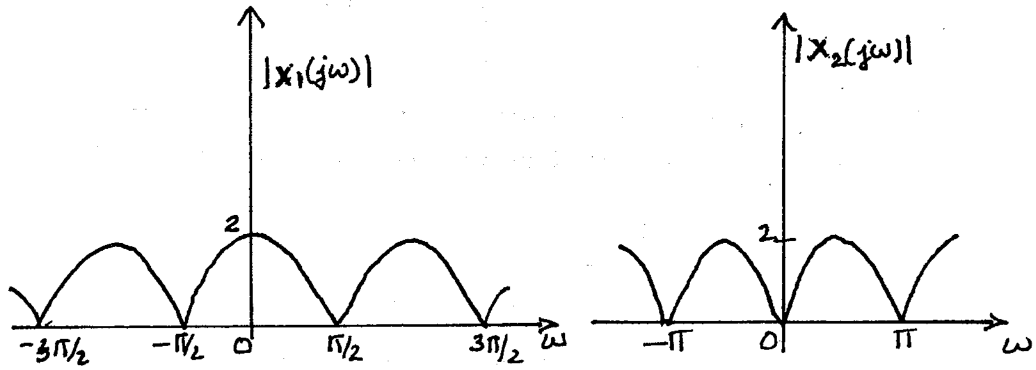


Figure S4.2

4.3. **(a)** The signal $x_1(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_1(t) &= \frac{1}{2j}\left(e^{j(2\pi t+\pi/4)} - e^{-j(2\pi t+\pi/4)}\right) \\ &= \frac{1}{2j}e^{j\pi/4}e^{j2\pi t} - \frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_1(t)$ are

$$a_1 = \frac{1}{2j}e^{j\pi/4}e^{j2\pi t}, \qquad a_{-1} = -\frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_1(t)$, the corresponding Fourier transform $X_1(j\omega)$ is given by

$$\begin{aligned} X_1(j\omega) &= 2\pi a_1 \delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= (\pi/j)e^{j\pi/4}\delta(\omega - 2\pi) - (\pi/j)e^{-j\pi/4}\delta(\omega + 2\pi) \end{aligned}$$

(b) The signal $x_2(t) = 1 + \cos(6\pi t + \pi/8)$ is periodic with a fundamental period of $T = 1/3$. This translates to a fundamental frequency of $\omega_0 = 6\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_2(t) &= 1 + \frac{1}{2}\left(e^{j(6\pi t+\pi/8)} - e^{-j(6\pi t+\pi/8)}\right) \\ &= 1 + \frac{1}{2}e^{j\pi/8}e^{j6\pi t} + \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_2(t)$ are

$$a_0 = 1, \qquad a_1 = \frac{1}{2}e^{j\pi/8}e^{j6\pi t}, \qquad a_{-1} = \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_2(t)$, the corresponding Fourier transform $X_2(j\omega)$ is given by

$$\begin{aligned} X_2(j\omega) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= 2\pi\delta(\omega) + \pi e^{j\pi/8}\delta(\omega - 6\pi) + \pi e^{-j\pi/8}\delta(\omega + 6\pi) \end{aligned}$$

4.4. (a) The inverse Fourier transform is

$$\begin{aligned} x_1(t) &= (1/2\pi)\int_{-\infty}^{\infty}[2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)]e^{j\omega t}d\omega \\ &= (1/2\pi)[2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}] \\ &= 1 + (1/2)e^{j4\pi t} + (1/2)e^{-j4\pi t} = 1 + \cos(4\pi t) \end{aligned}$$

(b) The inverse Fourier transform is

$$\begin{aligned} x_2(t) &= (1/2\pi)\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_0^2 2e^{j\omega t}d\omega + (1/2\pi)\int_{-2}^{0}(-2)e^{j\omega t}d\omega \\ &= (e^{j2t} - 1)/(\pi j t) - (1 - e^{-j2t})/(\pi j t) \\ &= -(4j\sin^2 t)/(\pi t) \end{aligned}$$

4.5. From the given information,

$$\begin{aligned} x(t) &= (1/2\pi)\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-\infty}^{\infty}|X(j\omega)|e^{j\sphericalangle\{X(j\omega)\}}e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-3}^{3}2e^{-\frac{3}{2}\omega + \pi}e^{j\omega t}d\omega \\ &= \frac{-2}{\pi(t - 3/2)}\sin[3(t - 3/2)] \end{aligned}$$

The signal $x(t)$ is zero when $3(t - 3/2)$ is a nonzero integer multiple of π. This gives

$$t = \frac{k\pi}{2} + \frac{3}{2}, \qquad \text{for } k \in \mathcal{I}, \text{ and } k \neq 0.$$

Therefore, the desired result is

$$\mathcal{F}T\{\text{Odd part of} x(t)\} = \frac{\sin\omega}{j\omega^2} - \frac{\cos\omega}{j\omega}$$

4.10. **(a)** We know from Table 4.2 that

$$\frac{\sin t}{\pi t} \stackrel{FT}{\longleftrightarrow} \text{Rectangular function } Y(j\omega) \text{ [See Figure S4.10]}$$

Therefore

$$\left(\frac{\sin t}{\pi t}\right)^2 \stackrel{FT}{\longleftrightarrow} (1/2\pi)\,[\text{Rectangular function } Y(j\omega) * \text{Rectangular function } Y(j\omega)]$$

This is a triangular function $Y_1(j\omega)$ as shown in the Figure S4.10.

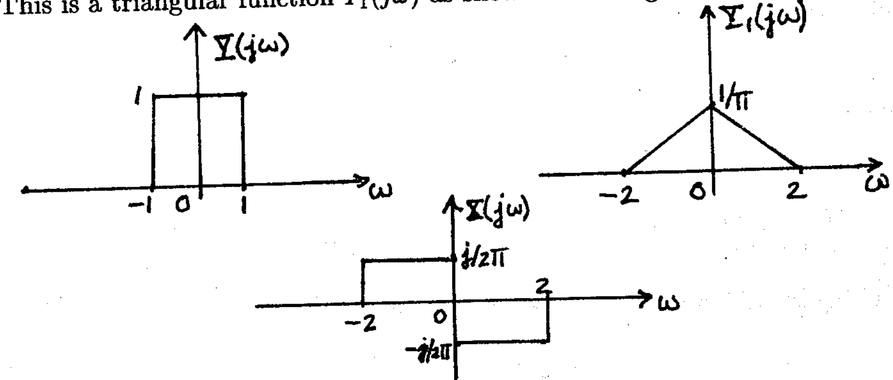


Figure S4.10

Using Table 4.1, we may write

$$t\left(\frac{\sin t}{\pi t}\right)^2 \stackrel{FT}{\longleftrightarrow} X(j\omega) = j\frac{d}{d\omega}Y_1(j\omega)$$

This is as shown in the figure above. $X(j\omega)$ may be expressed mathematically as

$$X(j\omega) = \begin{cases} j/2\pi, & -2 \le \omega < 0 \\ -j/2\pi, & 0 \le \omega 2 \\ 0. & \text{otherwise} \end{cases}$$

(b) Using Parseval's relation,

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi^3}$$

4.11. We know that

$$x(3t) \stackrel{FT}{\longleftrightarrow} \frac{1}{3}X(j\frac{\omega}{3}), \qquad h(3t) \stackrel{FT}{\longleftrightarrow} \frac{1}{3}H(j\frac{\omega}{3})$$

Therefore,

$$G(j\omega) = \mathcal{F}T\{x(3t) * h(3t)\} = \frac{1}{9}X(j\frac{\omega}{3})H(j\frac{\omega}{3})$$

Now note that

$$Y(j\omega) = \mathcal{F}T\{x(t) * h(t)\} = X(j\omega)H(j\omega)$$

From this, we may write

$$Y(j\frac{\omega}{3}) = X\left(j\frac{\omega}{3}\right) H\left(j\frac{\omega}{3}\right)$$

Using this in eq. (**), we have

$$G(j\omega) = \frac{1}{9}Y(j\frac{\omega}{3})$$

and

$$g(t) = \frac{1}{3}y(3t).$$

Therefore, $A = \frac{1}{3}$ and $B = 3$.

4.12. **(a)** From Example 4.2 we know that

$$e^{-|t|} \stackrel{FT}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

Using the differentiation in frequency property, we have

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} j\frac{d}{d\omega}\left\{\frac{2}{1+\omega^2}\right\} = -\frac{4j\omega}{(1+\omega^2)^2}.$$

(b) The duality property states that if

$$g(t) \stackrel{FT}{\longleftrightarrow} G(j\omega)$$

then

$$G(t) \stackrel{FT}{\longleftrightarrow} 2\pi g(j\omega).$$

Now, since

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} -\frac{4j\omega}{(1+\omega^2)^2}$$

we may use duality to write

$$-\frac{4jt}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} 2\pi\omega e^{-|\omega|}$$

Multiplying both sides by j, we obtain

$$\frac{4t}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} j2\pi\omega e^{-|\omega|}.$$

4.13. (a) Taking the inverse Fourier transform of $X(j\omega)$, we obtain

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi t} + \frac{1}{2\pi}e^{j5t}$$

The signal $x(t)$ is therefore a constant summed with two complex exponentials whose fundamental frequencies are $2\pi/5$ rad/sec and 2 rad/sec. These two complex exponentials are not harmonically related. That is, the fundamental frequencies of these complex complex exponentials can never be integral multiples of a common fundamental frequency. Therefore, the signal is **not periodic.**

(b) Consider the signal $y(t) = x(t) * h(t)$. From the convolution property, we know that $Y(j\omega) = X(j\omega)H(j\omega)$. Also, from $h(t)$, we know that

$$H(j\omega) = e^{-j\omega}\frac{2\sin\omega}{\omega}.$$

The function $H(j\omega)$ is zero when $\omega = k\pi$, where k is a nonzero integer. Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \delta(\omega) + \delta(\omega - 5)$$

This gives

$$y(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j5t}$$

Therefore, $y(t)$ is a complex exponential summed with a constant. We know that a complex exponential is periodic. Adding a constant to a complex exponential does not affect its periodicity. Therefore, $y(t)$ will be a signal with a fundamental frequency of $2\pi/5$.

(c) From the results of parts (a) and (b), we see that the answer is yes.

4.14. Taking the Fourier transform of both sides of the equation

$$\mathcal{F}^{-1}\{(1 + j\omega)X(j\omega)\} = A2^{-2t}u(t),$$

we obtain

$$X(j\omega) = \frac{A}{(1 + j\omega)(2 + j\omega)} = A\left\{\frac{1}{1 + j\omega} - \frac{1}{2 + j\omega}\right\}.$$

Taking the inverse Fourier transform of the above equation

$$x(t) = Ae^{-t}u(t) - Ae^{-2t}u(t)$$

Using Parseval's relation, we have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Using the fact that $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi$, we have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$$

Substituting the previously obtained expression for $x(t)$ in the above equation, we have

$$\int_{-\infty}^{\infty}[A^2e^{-2t} + A^2e^{-4t} - 2A^2e^{-3t}]u(t)dt = 1$$
$$\int_{0}^{\infty}[A^2e^{-2t} + A^2e^{-4t} - 2A^2e^{-3t}]dt = 1$$
$$A^2/12 = 1$$
$$\Rightarrow A = \sqrt{12}$$

We choose A to be $\sqrt{12}$ instead of $-\sqrt{12}$ because we know that $x(t)$ is non negative.

4.15. Since $x(t)$ is real,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} \overset{FT}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\}.$$

We are given that

$$\mathcal{IFT}\{\mathcal{R}e\{X(j\omega)\}\} = |t|e^{-|t|}.$$

Therefore,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} = |t|e^{-|t|}.$$

We also know that $x(t) = 0$ for $t \leq 0$. This implies that $x(-t)$ is zero for $t > 0$. We may conclude that

$$x(t) = 2|t|e^{-|t|} \qquad \text{for } t \geq 0$$

Therefore,

$$x(t) = 2te^{-t}u(t)$$

4.16. **(a)** We may write

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/4)}{k\pi/4}\delta(t - k\pi/4) \\ &= \frac{\sin t}{\pi t}\sum_{k=-\infty}^{\infty} \pi\delta(t - k\pi/4) \end{aligned}$$

Therefore, $g(t) = \sum_{k=-\infty}^{\infty} \pi\delta(t - k\pi/4)$.

(b) Since $g(t)$ is an impulse train, its Fourier transform $G(j\omega)$ is also an impulse train. From Table 4.2,

$$\begin{aligned} G(j\omega) &= \pi\frac{2\pi}{\pi/4}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{\pi/4}\right) \\ &= 8\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 8k) \end{aligned}$$

We see that $G(j\omega)$ is periodic with a period of 8. Using the multiplication property, we know that

$$X(j\omega) = \frac{1}{2\pi}\left[\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\} * G(j\omega)\right]$$

If we denote $\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\}$ by $A(j\omega)$, then

$$\begin{aligned} X(j\omega) &= (1/2\pi)[A(j\omega) * 8\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 8k) \\ &= 4 \sum_{k=-\infty}^{\infty} A(j\omega - 8k) \end{aligned}$$

$X(j\omega)$ may thus be viewed as a replication of $4A(j\omega)$ every 8 rad/sec. This is obviously periodic.

Using Table 4.2, we obtain

$$A(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, we may specify $X(j\omega)$ over one period as

$$X(j\omega) = \begin{cases} 4, & |\omega| \le 1 \\ 0, & 1 < |\omega \le 4 \end{cases}$$

4.17. **(a)** From Table 4.1, we know that a real and odd signal signal $x(t)$ has a purely imaginary and odd Fourier transform $X(j\omega)$. Let us now consider the purely imaginary and odd signal $jx(t)$. Using linearity, we obtain the Fourier transform of this signal to be $jX(j\omega)$. The function $jX(j\omega)$ will clearly be real and odd. Therefore, the given statement is **false.**

(b) An odd Fourier transform corresponds to an odd signal, while an even Fourier transform corresponds to an even signal. The convolution of an even Fourier transform with an odd Fourier may be viewed in the time domain as a multiplication of an even and odd signal. Such a multiplication will always result in an odd time signal. The Fourier transform of this odd signal will always be odd. Therefore, the given statement is **true.**

4.18. Using Table 4.2, we see that the rectangular pulse $x_1(t)$ shown in Figure S4.18 has a Fourier transform $X_1(j\omega) = \sin(3\omega)/\omega$. Using the convolution property of the Fourier transform, we may write

$$x_2(t) = x_1(t) * x_1(t) \xleftrightarrow{FT} X_2(j\omega) = X_1(j\omega)X_1(j\omega) = \left(\frac{\sin(3\omega)}{\omega}\right)^2$$

The signal $x_2(t)$ is shown in Figure S4.18. Using the shifting property, we also note that

$$\frac{1}{2}x_2(t+1) \xleftrightarrow{FT} \frac{1}{2}e^{j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2$$

and

$$\frac{1}{2}x_2(t-1) \stackrel{FT}{\longleftrightarrow} \frac{1}{2}e^{-j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

Adding the two above equations, we obtain

$$h(t) = \frac{1}{2}x_2(t+1) + \frac{1}{2}x_2(t-1) \stackrel{FT}{\longleftrightarrow} \cos(\omega)\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

The signal $h(t)$ is as shown in Figure S4.18. We note that $h(t)$ has the given Fourier transform $H(j\omega)$.

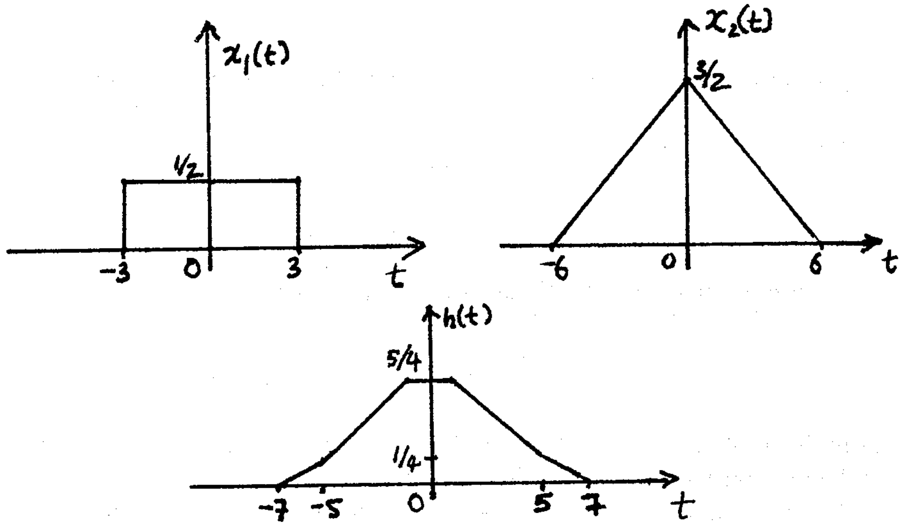


Figure S4.18

Mathematically $h(t)$ may be expressed as

$$h(t) = \begin{cases} \frac{5}{4}, & |t| < 1 \\ -\frac{|t|}{4} + \frac{3}{2}, & 1 \le |t| \le 5 \\ -\frac{|t|}{8} + \frac{7}{8}, & 5 < |t| \le 7 \\ 0, & \text{otherwise} \end{cases}$$

4.19. We know that

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}.$$

Since it is given that $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, we can compute $Y(j\omega)$ to be

$$Y(j\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{1}{(3+j\omega)(4+j\omega)}.$$

Since, $H(j\omega) = 1/(3 + j\omega)$, we have

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = 1/(4 + j\omega)$$

Taking the inverse Fourier transform of $X(j\omega)$, we have

$$x(t) = e^{-4t}u(t).$$

4.20. From the answer to Problem 3.20, we know that the frequency response of the circuit is

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}.$$

Breaking this up into partial fractions, we may write

$$H(j\omega) = -\frac{1}{j\sqrt{3}}\left[\frac{-1}{\frac{1}{2} - \frac{\sqrt{3}}{2}j + j\omega} + \frac{-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}j + j\omega}\right]$$

Using the Fourier transform pairs provided in Table 4.2, we obtain the Fourier transform of $H(j\omega)$ to be

$$h(t) = -\frac{1}{j\sqrt{3}}\left[-e^{(-\frac{1}{2}+\frac{\sqrt{3}}{2}j)t} + +e^{(-\frac{1}{2}-\frac{\sqrt{3}}{2}j)t}\right]u(t).$$

Simplifying,

$$h(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t).$$

4.21. **(a)** The given signal is

$$e^{-\alpha t}\cos(\omega_0 t)u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} - \frac{1}{2(\alpha - j\omega_0 + j\omega)}.$$

(b) The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t).$$

We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}.$$

Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{1/2j}{3 - j2 - j\omega} - \frac{1/2j}{3 + j2 - j\omega}.$$

Therefore,

$$X(j\omega) = X_1(j\omega + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}.$$

(c) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi-\omega} - \frac{\sin\omega}{\pi+\omega}.$$

(d) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{1}{1-\alpha e^{-j\omega T}}.$$

(e) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)te^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1/2j}{(2-j4+j\omega)^2} - \frac{1/2j}{(2+j4-j\omega)^2}.$$

(f) We have

$$x_1(t) = \frac{\sin\pi t}{\pi t} \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}.$$

Also

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = \begin{cases} e^{-2\omega}, & |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

$$x(t) = x_1(t)x_2(t) \stackrel{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{2\pi}\{X_1(j\omega) * X_2(j\omega)\}.$$

Therefore,

$$X(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < \pi \\ (1/2\pi)(3\pi+\omega)e^{-j\omega}, & -3\pi < \omega < -\pi \\ (1/2\pi)(3\pi-\omega)e^{-j\omega}, & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(g) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{2j}{\omega}\left[\cos 2\omega - \frac{\sin\omega}{\omega}\right].$$

(h) If

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k),$$

then

$$x(t) = 2x_1(t) + x_1(t-1).$$

Therefore,

$$X(j\omega) = X_1(j\omega)[2+e^{-\omega}] = \pi\sum_{k=-\infty}^{\infty}\delta(\omega-k\pi)[2+(-1)^k].$$

(i) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega}-2}{j\omega^2}.$$

(j) $x(t)$ is periodic with period 2. Therefore,

$$X(j\omega) = \pi \sum_{k=-\infty}^{\infty} \tilde{X}(jk\pi)\delta(\omega - k\pi),$$

where $\tilde{X}(j\omega)$ is the Fourier transform of one period of $x(t)$. That is,

$$\tilde{X}(j\omega) = \frac{1}{1-e^{-2}}\left[\frac{1-e^{-2(1+j\omega)}}{1+j\omega} - \frac{e^{-2}[1-e^{-2(1+j\omega)}]}{1-j\omega}\right].$$

4.22. (a) $x(t) = \begin{cases} e^{j2\pi t}, & |t| < 3 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = \frac{1}{2}e^{-j\pi/3}\delta(t-4) + \frac{1}{2}e^{j\pi/3}\delta(t+4)$.

(c) The Fourier transform synthesis eq. (4.8) may be written as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)| e^{j\sphericalangle X(j\omega)} e^{j\omega t} d\omega.$$

From the given figure we have

$$x(t) = \frac{1}{\pi}\left[\frac{\sin(t-3)}{t-3} + \frac{\cos(t-3)-1}{(t-3)^2}\right].$$

(d) $x(t) = \frac{2j}{\pi}\sin t + \frac{3}{\pi}\cos(2\pi t)$

(e) Using the Fourier transform synthesis equation (4.8),

$$x(t) = \frac{\cos 3t}{j\pi t} + \frac{\sin t - \sin 2t}{j\pi t^2}.$$

4.23. For the given signal $x_0(t)$, we use the Fourier transform analysis eq. (4.8) to evaluate the corresponding Fourier transform

$$X_0(j\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}.$$

(i) We know that

$$x_1(t) = x_0(t) + x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2 - 2e^{-1}\cos\omega - 2\omega e^{-1}\sin\omega}{1+\omega^2}.$$

(ii) We know that

$$x_2(t) = x_0(t) - x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = j\left[\frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1+\omega^2}\right].$$

(iii) We know that

$$x_3(t) = x_0(t) + x_0(t+1).$$

Using the linearity and time shifting properties of the Fourier transform we have

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega}X_0(-j\omega) = \frac{1 + e^{j\omega} - e^{-1}(1 + e^{-j\omega})}{1+j\omega}.$$

(iv) We know that

$$x_4(t) = tx_0(t).$$

Using the differentiation in frequency property

$$X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega).$$

Therefore,

$$X_4(j\omega) = \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}.$$

4.24. **(a)** (i) For $\mathcal{R}e\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and odd. Therefore, signals in figures (a) and (c) have this property.

(ii) For $\mathcal{I}m\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and even. Therefore, signals in figures (e) and (f) have this property.

(iii) For there to exist a real α such that $e^{j\alpha\omega}X(j\omega)$ is real, we require that $x(t+\alpha)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

(iv) For this condition to be true, $x(0) = 0$. Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.

(v) For this condition to be true the derivative of $x(t)$ has to be zero at $t = 0$. Therefore, signals in figures (b), (c), (e), and (f) have this property.

(vi) For this to be true, the signal $x(t)$ has to be periodic. Only the signal in figure (a) has this property.

(b) For a signal to satisfy only properties (i), (iv), and (v), it must be real and odd, and

$$x(t) = 0, \qquad x'(0) = 0.$$

The signal shown below is an example of that.

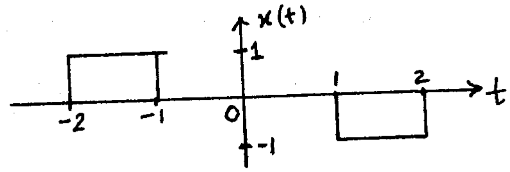


Figure S4.24

4.25. **(a)** Note that $y(t) = x(t+1)$ is a real and even signal. Therefore, $Y(j\omega)$ is also real and even. This implies that $\sphericalangle Y(j\omega) = 0$. Also, since $Y(j\omega) = e^{j\omega}X(j\omega)$, we know that $\sphericalangle X(j\omega) = -\omega$.

(b) We have

$$X(j0) = \int_{-\infty}^{\infty} x(t)dt = 7.$$

(c) We have

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 4\pi.$$

(d) Let $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{2j\omega}$. The corresponding signal $y(t)$ is

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases}.$$

Then the given integral is

$$\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)d\omega = 2\pi\{x(t) * y(t)\}_{t=0} = 7\pi.$$

(e) We have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = 26\pi.$$

(f) The inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$ is the $\mathcal{E}v\{x(t)\}$ which is $[x(t) + x(-t)]/2$. This is as shown in the figure below.

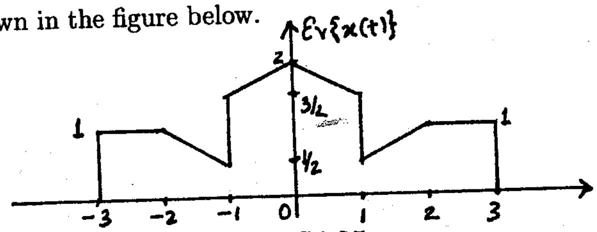


Figure S4.25

4.26. **(a)** (i) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{4+j\omega}\right] \\ &= \frac{(1/4)}{4+j\omega} - \frac{(1/4)}{2+j\omega} + \frac{(1/2)}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t).$$

(ii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{(4+j\omega)^2}\right] \\ &= \frac{(1/4)}{2+j\omega} + \frac{(1/4)}{(2+j\omega)^2} - \frac{(1/4)}{4+j\omega} + \frac{(1/4)}{(4+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t).$$

(iii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \left[\frac{1}{1+j\omega}\right]\left[\frac{1}{1-j\omega}\right] \\ &= \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \frac{1}{2}e^{-|t|}.$$

(b) By direct convolution of $x(t)$ with $h(t)$ we obtain

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 < t \leq 5 \\ e^{-(t-5)} - e^{-(t-1)}, & t > 5 \end{cases}.$$

Taking the Fourier transform of $y(t)$,

$$\begin{aligned} Y(j\omega) &= \frac{2e^{-j3\omega}\sin(2\omega)}{\omega(1+j\omega)} \\ &= \left[\frac{e^{-j2\omega}}{1+j\omega}\right]\frac{e^{-j\omega}2\sin(2\omega)}{\omega} \\ &= X(j\omega)H(j\omega) \end{aligned}$$

4.27. **(a)** The Fourier transform $X(j\omega)$ is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_1^2 e^{-j\omega t}dt - \int_2^3 e^{-j\omega t}dt \\ &= 2\frac{\sin(\omega/2)}{\omega}\{1 - e^{-j\omega}\}e^{-j3\omega/2} \end{aligned}$$

(b) The Fourier series coefficients a_k are

$$\begin{aligned} a_k &= \frac{1}{T}\int_{<T>} \tilde{x}(t)e^{-j\frac{2\pi}{T}kt} \\ &= \frac{1}{2}\{\int_1^2 e^{-j\frac{2\pi}{T}kt}dt - \int_2^3 e^{-j\frac{2\pi}{T}kt}dt\} \\ &= \frac{\sin(k\pi/2)}{k\pi}\{1 - e^{-jk\pi}\}e^{-j3k\pi/2} \end{aligned}$$

Comparing the answers to parts (a) and (b), it is clear that

$$a_k = \frac{1}{T}X(j\frac{2\pi k}{T}),$$

where $T = 2$.

4.28. **(a)** From Table 4.2 we know that

$$p(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \xleftrightarrow{FT} P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0).$$

From this,

$$Y(j\omega) = \frac{1}{2\pi}\{X(j\omega) * H(j\omega)\} = \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\omega_0)).$$

(b) The spectra are sketched in Figure S4.28.

4.29. (i) We have

$$X_a(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega) - ja\omega} = X(j\omega)e^{-ja\omega}.$$

From the time shifting property we know that

$$x_a(t) = x(t-a).$$

(ii) We have

$$X_b(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega) + jb\omega} = X(j\omega)e^{jb\omega}.$$

From the time shifting property we know that

$$x_b(t) = x(t+b).$$

(iii) We have

$$X_c(j\omega) = |X(j\omega)|e^{-j\sphericalangle X(j\omega)} = X^*(j\omega).$$

From the conjugation and time reversal properties we know that

$$x_c(t) = x^*(-t).$$

Since $x(t)$ is real, $x_c(t) = x(-t)$.

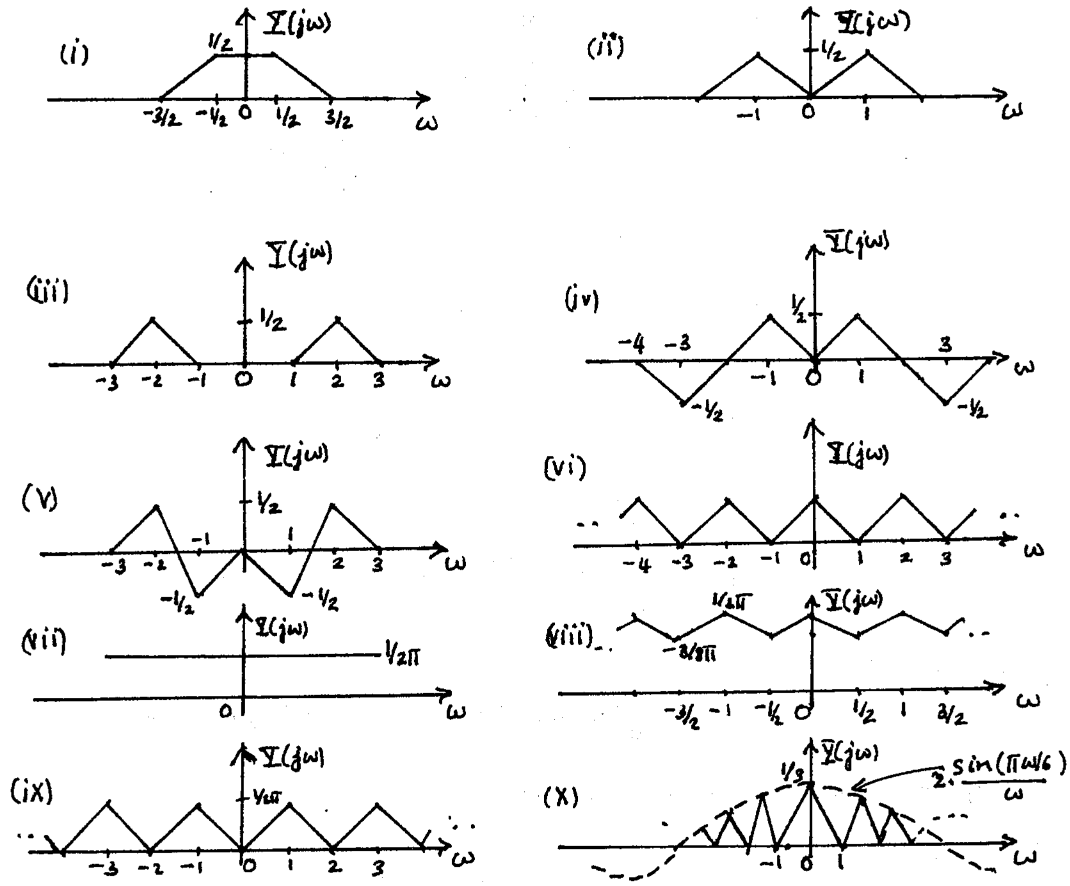


Figure S4.28

(iv) We have

$$X_d(j\omega) = |X(j\omega)|e^{-j\sphericalangle X(j\omega)+jd\omega} = X^*(j\omega)e^{jb\omega}.$$

From the conjugation, time reversal, and time shifting properties, we know that

$$x_d(t) = x^*(-t-d).$$

Since $x(t)$ is real, $x_d(t) = x(-t-d)$.

4.30. **(a)** We know that

$$w(t) = \cos t \stackrel{FT}{\longleftrightarrow} W(j\omega) = \pi[\delta(\omega-1) + \delta(\omega+1)$$

and

$$g(t) = x(t)\cos t \stackrel{FT}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}\{X(j\omega) * W(j\omega)\}.$$

Therefore,

$$G(j\omega) = \frac{1}{2}X(j(\omega-1)) + \frac{1}{2}X(j(\omega+1)).$$

Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

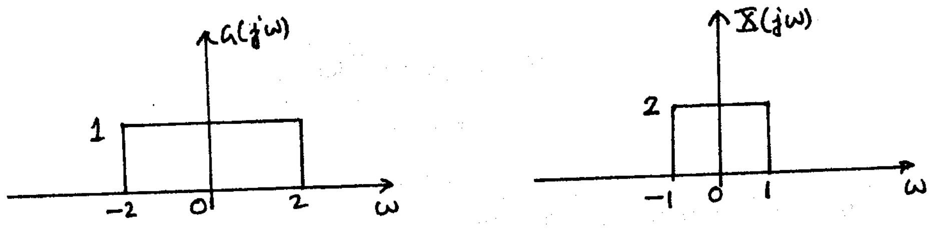


Figure S4.30

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$

(b) $X_1(j\omega)$ is as shown in Figure S4.30.

4.31. **(a)** We have

$$x(t) = \cos t \overset{FT}{\longleftrightarrow} X(j\omega) = \pi[\delta(\omega+1) + \delta(\omega-1)].$$

(i) We have

$$h_1(t) = u(t) \overset{FT}{\longleftrightarrow} H_1(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(ii) We have

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \overset{FT}{\longleftrightarrow} H_2(j\omega) = -2 + \frac{5}{2+j\omega}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(iii) We have

$$h_3(t) = 2te^{-t}u(t) \stackrel{FT}{\longleftrightarrow} H_2(j\omega) = \frac{2}{(1+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(b) An LTI system with impulse response

$$h_4(t) = \frac{1}{2}\left[h_1(t) + h_2(t)\right]$$

will have the same response to $x(t) = \cos(t)$. We can find other such impulse responses by suitably scaling and linearly combining $h_1(t)$, $h_2(t)$, and $h_3(t)$.

4.32. Note that $h(t) = h_1(t-1)$, where

$$h_1(t) = \frac{\sin 4t}{\pi t}.$$

The Fourier transform $H_1(j\omega)$ of $h_1(t)$ is as shown in Figure S4.32.

From the above figure it is clear that $h_1(t)$ is the impulse response of an ideal lowpass filter whose passband is in the range $|\omega| < 4$. Therefore, $h(t)$ is the impulse response of an ideal lowpass filter shifted by one to the right. Using the shift property,

$$H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

(a) We have

$$X_1(j\omega) = \pi e^{j\frac{\pi}{12}}\delta(\omega-6) + \pi e^{j\frac{\pi}{12}}\delta(\omega+6).$$

It is clear that

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0.$$

This result is equivalent to saying that $X_1(j\omega)$ is zero in the passband of $H(j\omega)$.

(b) We have

$$X_2(j\omega) = \frac{\pi}{j}\left[\sum_{k=0}^{\infty}(\frac{1}{2})^k\{\delta(\omega-3k) - \delta(\omega+3k)\}\right].$$

Therefore,

$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{\pi}{j}\left[(1/2)\{\delta(\omega-3) - \delta(\omega+3)\}e^{-j\omega}\right].$$

This implies that

$$y_2(t) = \frac{1}{2}\sin(3t-1).$$

We may have obtained the same result by noting that only the sinusoid with frequency 3 in $X_2(j\omega)$ lies in the passband of $H(j\omega)$.

(c) We have

$$X_3(j\omega) = \begin{cases} e^{j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

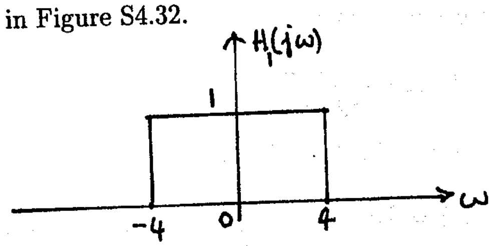
$$Y_3(j\omega) = X_3(j\omega)H(j\omega) = X_3(j\omega)e^{-j\omega}.$$

This implies that

$$y_3(t) = x_3(t-1) = \frac{\sin(4t)}{\pi t}.$$

We may have obtained the same result by noting that $X_3(j\omega)$ lies entirely in the passband of $H(j\omega)$.

(d) $X_4(j\omega)$ is as shown in Figure S4.32.



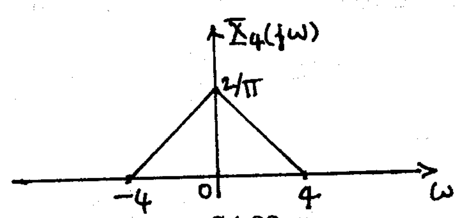


Figure S4.32

Therefore,

$$Y_4(j\omega) = X_4(j\omega)H(j\omega) = X_4(j\omega)e^{-j\omega}.$$

This implies that

$$y_4(t) = x_4(t-1) = \left(\frac{\sin(2(t-1))}{\pi(t-1)}\right)^2.$$

We may have obtained the same result by noting that $X_4(j\omega)$ lies entirely in the passband of $H(j\omega)$.

4.33. (a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{-\omega^2 + 2j\omega + 8}.$$

Using partial fraction expansion, we obtain

$$H(j\omega) = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 4}.$$

Taking the inverse Fourier transform,

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t).$$

(b) For the given signal $x(t)$, we have

$$X(j\omega) = \frac{1}{(2+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{2}{(-\omega^2 + 2j\omega + 8)}\frac{1}{(2+j\omega)^2}.$$

Using partial fraction expansion, we obtain

$$Y(j\omega) = \frac{1/4}{j\omega+2} - \frac{1/2}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^3} - \frac{1/4}{j\omega+4}.$$

Taking the inverse Fourier transform,

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t).$$

(c) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2(-\omega^2-1)}{-\omega^2+\sqrt{2}j\omega+1}.$$

Using partial fraction expansion, we obtain

$$H(j\omega) = 2 + \frac{-\sqrt{2}-2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}+j\sqrt{2}}{2}} + \frac{-\sqrt{2}+2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}-j\sqrt{2}}{2}}.$$

Taking the inverse Fourier transform,

$$h(t) = 2\delta(t) - \sqrt{2}(1+2j)e^{-(1+j)t/\sqrt{2}}u(t) - \sqrt{2}(1-2j)e^{-(1-j)t/\sqrt{2}}u(t).$$

4.34. (a) We have

$$\frac{Y(j\omega}{X(j\omega)} = \frac{j\omega+4}{6-\omega^2+5j\omega}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t).$$

(b) We have

$$H(j\omega) = \frac{2}{2+j\omega} - \frac{1}{3+j\omega}.$$

Taking the inverse Fourier transform we obtain,

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t).$$

(c) We have

$$X(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(4+j\omega)(2+j\omega)}.$$

Finding the partial fraction expansion of $Y(j\omega)$ and taking the inverse Fourier transform,

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t).$$

4.35. (a) From the given information,

$$|H(j\omega)| = \frac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}} = 1.$$

Also,

$$\sphericalangle H(j\omega) = -\tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{a} = -2\tan^{-1}\frac{\omega}{a}.$$

Also,

$$H(j\omega) = -1 + \frac{2a}{a+j\omega} \qquad \Rightarrow \qquad h(t) = -\delta(t) + 2ae^{-at}u(t).$$

(b) If $a = 1$, we have

$$|H(j\omega)| = 1, \qquad \sphericalangle H(j\omega) = -2\tan^{-1}\omega.$$

Therefore,

$$y(t) = \cos\left(\frac{t}{\sqrt{3}} - \frac{\pi}{3}\right) - \cos\left(t - \frac{\pi}{2}\right) + \cos\left(\sqrt{3}t - \frac{2\pi}{3}\right).$$

4.36. (a) The frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}.$$

(b) Finding the partial fraction expansion of answer in part (a) and taking its inverse Fourier transform, we obtain

$$h(t) = \frac{3}{2}\left[e^{-4t} + e^{-2t}\right]u(t).$$

(c) We have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{(9+3j\omega)}{8+6j\omega-\omega^2}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t).$$

4.37. **(a)** Note that

$$x(t) = x_1(t) * x_1(t),$$

where

$$x_1(t) = \begin{cases} 1, & |\omega| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Also, the Fourier transform $X_1(j\omega)$ of $x_1(t)$ is

$$X_1(j\omega) = 2\frac{\sin(\omega/2)}{\omega}.$$

Using the convolution property we have

$$X(j\omega) = X_1(j\omega)X_1(j\omega) = \left[2\frac{\sin(\omega/2)}{\omega}\right]^2.$$

(b) The signal $\tilde{x}(t)$ is as shown in Figure S4.37

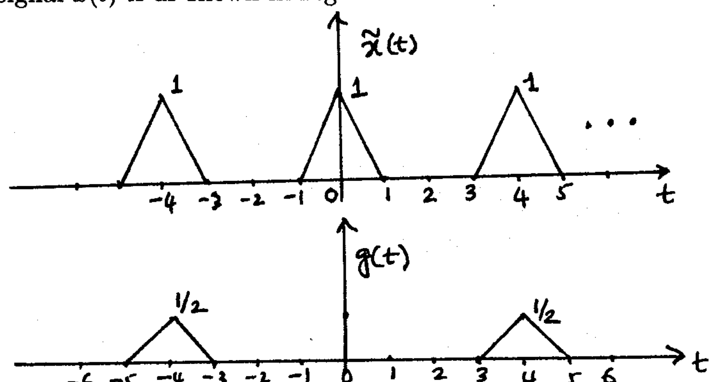


Figure S4.37

(c) One possible choice of $g(t)$ is as shown in Figure S4.37.

(d) Note that

$$\tilde{X}(j\omega) = X(j\omega)\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(j(\omega - k\frac{\pi}{2})) = G(j\omega)\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(j(\omega - k\frac{\pi}{2})).$$

This may also be written as

$$\tilde{X}(j\omega) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}X(j\pi k/2)\delta(j(\omega - k\frac{\pi}{2})) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}G(j\pi k/2)\delta(j(\omega - k\frac{\pi}{2})).$$

Clearly, this is possible only if

$$G(j\pi k/2) = X(j\pi k/2).$$

Therefore, an LTI system with impulse response $h(t) = \frac{1}{2}\delta(t)$ may be used to obtain $g(t)$ from $x(t)$.

4.44. **(a)** Taking the Fourier transform of both sides of the given differential equation, we have

$$Y(j\omega)[10 + j\omega] = X(j\omega)[Z(j\omega) - 1].$$

Since, $Z(j\omega) = \frac{1}{1+j\omega} + 3$, we obtain from the above equation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + 2j\omega}{(1 + j\omega)(10 + j\omega)}.$$

(b) Finding the partial fraction expansion of $H(j\omega)$ and then taking its inverse Fourier transform we obtain

$$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t).$$

4.45. We have

$$y(t) = x(t) * h(t) \quad \Rightarrow \quad Y(j\omega) = X(j\omega)H(j\omega).$$

From Parseval's relation the total energy in $y(t)$ is

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 |H(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_0-\Delta/2}^{-\omega_0+\Delta/2} |X(j\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_0-\Delta/2}^{\omega_0+\Delta/2} |X(j\omega)|^2 d\omega \\
&\approx \frac{1}{2\pi}|X(-j\omega_0)|^2\Delta + \frac{1}{2\pi}|X(j\omega_0)|^2\Delta
\end{aligned}$$

For real $x(t)$, $|X(-j\omega_0)|^2 = |X(j\omega_0)|^2$. Therefore,

$$E = \frac{1}{\pi}|X(j\omega_0)|^2\Delta.$$

4.46. Let $g_1(t)$ be the response of $H_1(j\omega)$ to $x(t)\cos\omega_c t$. Let $g_2(t)$ be the response of $H_2(j\omega)$ to $x(t)\sin\omega_c t$. Then, with reference to Figure 4.30,

$$y(t) = x(t)e^{j\omega_c t} = x(t)\cos\omega_c t + jx(t)\sin\omega_c t,$$

and

$$w(t) = g_1(t) + jg_2(t).$$

Also,

$$f(t) = e^{-j\omega_c t}w(t) = [\cos\omega_c t - j\sin\omega_c t][g_1(t) + jg_2(t)].$$

Therefore,

$$\mathcal{R}e\{f(t)\} = g_1(t)\cos\omega_c t + g_2(t)\sin\omega_c t.$$

This is exactly what Figure P4.46 implements.

4.47. **(a)** We have

$$h_e(t) = \frac{h(t) + h(-t)}{2}.$$

Since $h(t)$ is causal, the non-zero portions of $h(t)$ and $h(-t)$ overlap *only* at $t = 0$. Therefore,

$$h(t) = \begin{cases} 0, & t < 0 \\ h_e(t), & t = 0 \\ 2h_e(t), & t > 0 \end{cases}. \qquad \text{(S4.47–1)}$$

Also, from Table 4.1 we have

$$h_e(t) \stackrel{FT}{\longleftrightarrow} \mathcal{R}e\{H(j\omega)\}.$$

Given $\mathcal{R}e\{H(j\omega)$, we can obtain $h_e(t)$. From $h_e(t)$, we can recover $h(t)$ (and consequently $H(j\omega)$) by using eq. (S4.47-1). Therefore, $H(j\omega)$ is completely specified by $\mathcal{R}e\{H(j\omega)$.

(b) If

$$\mathcal{R}e\{H(j\omega)\} = \cos t = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

then,

$$h_e(t) = \frac{1}{2}\delta(t+1) + \frac{1}{2}\delta(t-1).$$

Therefore from eq. (S4.47-1),

$$h(t) = \delta(t-1).$$

(c) We have

$$h_o(t) = \frac{h(t) + h(-t)}{2}.$$

Since $h(t)$ is causal, the non-zero portions of $h(t)$ and $h(-t)$ overlap *only* at $t = 0$ and $h_o(t)$ will be zero at $t = 0$. Therefore,

$$h(t) = \begin{cases} 0, & t < 0 \\ \text{unknown}, & t = 0 \\ 2h_o(t), & t > 0 \end{cases}. \qquad \text{(S4.47–2)}$$

Also, from Table 4.1 we have

$$h_o(t) \stackrel{FT}{\longleftrightarrow} \mathcal{I}m\{H(j\omega)\}.$$

Given $\mathcal{I}m\{H(j\omega)$, we can obtain $h_o(t)$. From $h_o(t)$, we can recover $h(t)$ except for $t = 0$ by using eq. (S4.47-1). If there are no singularities in $h(t)$ at $t = 0$, then $H(j\omega)$ can be recovered from $h(t)$ even if $h(0)$ is unknown. Therefore $H(j\omega)$ is completely specified by $\mathcal{I}m\{H(j\omega)$ in this case.