# Draft of the examination set in MAT-2201 Numerical methods (December 4, 2019)

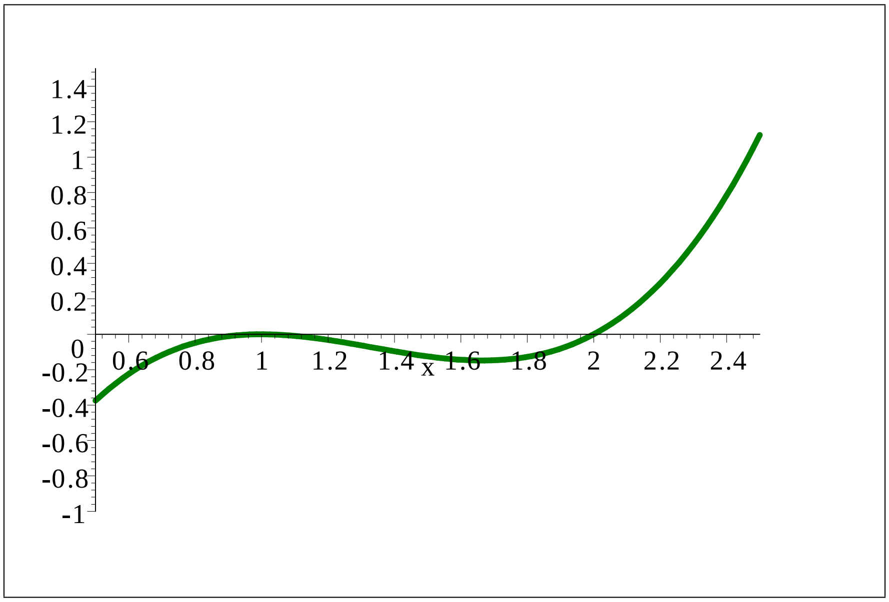
November 21, 2019

## Contents

## 1 Problem

Given an equation

$$f(x) = x^3 - 4x^2 + 5x - 2 = 0.$$




**a)** Investigate which of the following interation processes converge locally to the root $r = 2$:

1. $x_{n+1} = g(x_n)$ where $g(x) = x^3 - 4x^2 + 6x - 2$.
2. $x_{n+1} = h(x_n)$ where $h(x) = -x^3 + 4x^2 - 4x + 2$.

**Remark**. It is **not** enough to prove that an iteration process converges locally to **something**. You should explore whether it can converge to the root $r = 2$.

**b)** Formulate Newton's iteration method for the equation. Perform 2 iterations of this method with the initial value $x_0 = 0$.

**c)** Find the rate

$$S = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|}$$

of the linear convergence to the root $r = 1$ for Newton's method.

**d)** Formulate an iteration method $x_{n+1} = \varphi(x_n)$ that converges to $r = 1$ **quadratically**:

$$M = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} < \infty.$$

## 2 Problem

Given the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix}.$$

**a)** Perform a $PA = LU$ factorization. Remember that $L$ should have the form

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

with $|l_{ij}| \leq 1$.

**b)** Solve the system using the factorization you have obtained in **a)**.

# 3 Problem

A quantity $y(t)$ depending on the time variable $t$, is measured in four time points. The measurements are given in the table below:

| $t_i, i = 1, \ldots, 4$ | | $y_i, i = 1, \ldots, 4$ |
|---|---|---|
| 0 | | 1 |
| 1 | | 2 |
| 2 | | 7 |
| 3 | | 5 |

**a)** Find a polynomial $P(t)$ of degree $\leq 2$ that goes through the **first three** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3).$$

**b)** Find a polynomial $S(t)$ of degree $\leq 2$ that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (S(t_i) - y_i)^2.$$

**Hint**: use normal equations.

**c)** Given the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find the **reduced** $QR$ factorization of $B$, i.e. express $B$ as

$$B = QR$$

where

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}$$

is a $4 \times 2$ matrix with ortho**normal** columns ($\mathbf{q}_i \bullet \mathbf{q}_j = 0$ if $i \neq j$ and 1 if $i = j$), and $R$ is an upper-triangular $2 \times 2$ matrix.

**d)** Find a polynomial $U(t)$ of degree $\leq 1$ that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (U(t_i) - y_i)^2.$$

Use the $QR$ factorization you have obtained in **c)**.

## 4 Problem

Given the initial value problem (IVP):

$$\begin{aligned} y' &= (t+1)\,y = f(t,y)\,, \\ y(0) &= 1. \end{aligned}$$

**a)** Show that the function

$$y(t) = e^{\frac{1}{2}t^2+t}$$

is the exact solution of the IVP.

**b)** Formulate the Trapezoid method for the numerical solution of the IVP.

**c)** Choose step size $h = \frac{1}{2}$, and perform 2 steps of the Trapezoid method, producing the estimates:

$$\begin{aligned} w_0 &= y(0)\,, \\ w_1 &\approx y\left(\frac{1}{2}\right), \\ w_2 &\approx y(1)\,. \end{aligned}$$

Compare with the exact solution from **a)**, and find the global error of the method.

# GOOD LUCK!

# 5 Answers

## 5.1 Problem

$$f(x) = x^3 - 4x^2 + 5x - 2 = 0.$$

**a)**

1. $x_{n+1} = g(x_n)$ where $g(x) = x^3 - 4x^2 + 6x - 2$, does **not** converge to $r = 2$.
2. $x_{n+1} = h(x_n)$ where $h(x) = -x^3 + 4x^2 - 4x + 2$, **does** converge to $r = 2$.

**b)**

$$x_{n+1} = p(x_n),$$

$$\begin{aligned} p(x) &= x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 4x^2 + 5x - 2}{3x^2 - 8x + 5} = q(x) = 2\frac{-x + x^2 - 1}{3x - 5} \\ x_1 &= q(0) = 0.4 \\ x_2 &= q(0.4) = 0.652\,632 \end{aligned}$$

**c)**

$$S = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = \frac{1}{2}.$$

**d)**

$$\begin{aligned} x_{n+1} &= \varphi(x_n), \\ \varphi(x) &= x - 2\frac{f(x)}{f'(x)} = x - 2\frac{x^3 - 4x^2 + 5x - 2}{3x^2 - 8x + 5} = \frac{x + x^2 - 4}{3x - 5}. \end{aligned}$$

## 5.2 Problem

$$A = \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix}.$$

**a)**

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 6 & -6 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}.$$

**b)**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

## 5.3 Problem

| $t_i, i = 1, \ldots, 4$ | | $y_i, i = 1, \ldots, 4$ |
|---|---|---|
| 0 | | 1 |
| 1 | | 2 |
| 2 | | 7 |
| 3 | | 5 |

**a)**

$$P(t) = 2t^2 - t + 1.$$

**b)**

$$S(t) = -\frac{3}{4}t^2 + \frac{79}{20}t + \frac{9}{20}.$$

**c)**

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = QR = \begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix} \approx \\ &\approx \begin{bmatrix} 0.5 & -0.670\,820\,393\,249\,937 \\ 0.5 & -0.223\,606\,797\,749\,979 \\ 0.5 & 0.223\,606\,797\,749\,979 \\ 0.5 & 0.670\,820\,393\,249\,937 \end{bmatrix} \begin{bmatrix} 2.0 & 3.0 \\ 0 & 2.\,236\,067\,977\,499\,79 \end{bmatrix}. \end{aligned}$$

**d)**

$$U(t) = \frac{6}{5} + \frac{17}{10}t.$$

## 5.4 Problem

$$\begin{aligned} y' &= (t+1)\,y = f(t, y), \\ y(0) &= 1. \end{aligned}$$

**b)**

$$\begin{aligned} w_0 &= y_0 \text{ (given)}, \\ w_1 &= w_0 + \frac{h}{2}\left(f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))\right), \\ &\ldots, \\ w_{k+1} &= w_k + \frac{h}{2}\left(f(t_k, w_k) + f(t_{k+1}, w_k + hf(t_k, w_k))\right), \\ &\ldots \end{aligned}$$

**c)**

$$\begin{aligned} w_0 &= 1, \\ w_1 &= g\left(0, 1, \frac{1}{2}\right) = \frac{29}{16} = 1.8125, \\ w_2 &= g\left(\frac{1}{2}, \frac{29}{16}, \frac{1}{2}\right) = \frac{261}{64} = 4.078125. \end{aligned}$$

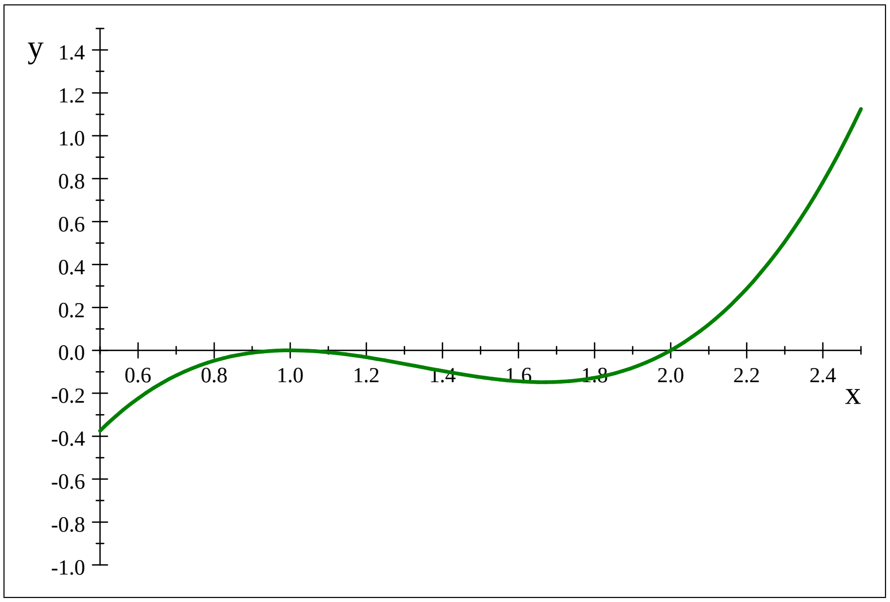
$$\begin{aligned} |w_2 - y_2| &= \left|4.078125 - \left[e^{\frac{1}{2}t^2+t}\right]_{t=1}\right| = \left|4.078125 - e^{\frac{3}{2}}\right| \approx \\ &\approx 0.403564070338065. \end{aligned}$$

# 6 Solutions

## 6.1 Problem

Given an equation

$$f(x) = x^3 - 4x^2 + 5x - 2 = 0.$$




**a)** Investigate which of the following interation processes converge locally to the root $r = 2$:

1. $x_{n+1} = g(x_n)$ where $g(x) = x^3 - 4x^2 + 6x - 2$.
2. $x_{n+1} = h(x_n)$ where $h(x) = -x^3 + 4x^2 - 4x + 2$.

**Remark**. It is **not** enough to prove that an iteration process converges locally to **something** but namely to the root $r = 2$.

**Solution 6.1**

*1.*

$$|g'(2)| = \left[\left|3x^2 - 8x + 6\right|\right]_{x=2} = |2| = 2 > 1.$$

*The process does **not** converge to $r = 2$.*

*2. If the process converges to $s$, then*

$$\begin{aligned} s &= h(s) \iff s = -s^3 + 4s^2 - 4s + 2 \iff \\ &\iff -s^3 + 4s^2 - 5s + 2 = 0 \iff f(s) = 0, \end{aligned}$$

*therefore $s$ is a root of $f$. Can the process converge to $r=2$?*

$$|h'(2)| = \left[\left|-3x^2+8x-4\right|\right]_{x=2} = |0| = 0 < 1.$$

*Yes, the process **does** converge to $r=2$.*

**b)** Formulate Newton's iteration method that converges locally to the root $r=1$. Perform 2 iterations of this method with the initial value $x_0=0$.

**Solution 6.2** *Newton's method is*

$$x_{n+1} = p(x_n)$$

*where (only $x_1$ and $x_2$ are required).*

$$\begin{aligned}
p(x) &= x - \frac{f(x)}{f'(x)} = x - \frac{x^3-4x^2+5x-2}{3x^2-8x+5} = q(x) = 2\frac{-x+x^2-1}{3x-5} \\
x_1 &= q(0) = 0.4 \\
x_2 &= q(0.4) = 0.652\,632 \\
x_3 &= q(0.652\,632) = 0.806\,485 \\
x_4 &= q(0.806\,485) = 0.895\,987 \\
x_5 &= q(0.895\,987) = 0.945\,654
\end{aligned}$$

*The convergence is too slow!*

**c)** Find the rate

$$S = \lim_{n\to\infty} \frac{|x_{n+1}-r|}{|x_n-r|}$$

of the linear convergence to the root $r=1$ for Newton's method.

**Solution 6.3** *Let us find the multiplicity $m$ of the root $r=1$:*

$$\begin{aligned}
f(r) &= 0, \\
f'(r) &= \left[3x^2-8x+5\right]_{x=1} = 0, \\
f''(r) &= [6x-8x]_{x=1} = -2 \neq 0.
\end{aligned}$$

*Therefore, $m=2$, and*

$$S = \lim_{n\to\infty} \frac{|x_{n+1}-r|}{|x_n-r|} = \frac{m-1}{m} = \frac{1}{2}.$$

**d)** Formulate an iteration method $x_{n+1} = \varphi(x_n)$ that converges to $r = 1$ **quadratically**:

$$M = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} < \infty.$$

**Solution 6.4** *The modified Newton's method*

$$\begin{aligned} x_{n+1} &= \varphi(x_n), \\ \varphi(x) &= x - 2\frac{f(x)}{f'(x)} = x - 2\frac{x^3 - 4x^2 + 5x - 2}{3x^2 - 8x + 5} = \frac{x + x^2 - 4}{3x - 5} \end{aligned}$$

*gives* ***quadratic*** *convergence. Let us check the convergence (****not required****):*

$$\begin{aligned} x_1 &= \varphi(0) = 0.8, \\ x_2 &= \varphi(0.8) = 0.984\,615, \\ x_3 &= \varphi(0.984\,615) = 0.999\,884, \\ x_4 &= \varphi(0.999\,884) = 0.999\,999\,993. \end{aligned}$$

## 6.2 Problem

Given the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix}.$$

**a)** Perform a $PA = LU$ factorization. Remember that $L$ should have the form

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

with $|l_{ij}| \leq 1$.

**Solution 6.5** *In the beginning*

$$P = L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix}.$$

*Then:*

*1.*

(a) *Pivoting:*

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 6 & -6 \\ 2 & -1 & -2 \\ 3 & 9 & 9 \end{bmatrix}.$$

(b) *Elementary row operations:*

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 6 & -6 \\ 0 & -3 & 0 \\ 0 & 6 & 12 \end{bmatrix}.$$

*2.*

(a) *Pivoting:*

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ L &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 6 & -6 \\ 0 & 6 & 12 \\ 0 & -3 & 0 \end{bmatrix}. \end{aligned}$$

(b) *Elementary row operations:*

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 6 & -6 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}.$$

*Let us check the factorization:*

$$\begin{aligned} PA &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -6 \\ 3 & 9 & 9 \\ 2 & -1 & -2 \end{bmatrix}, \\ LU &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & -6 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -6 \\ 3 & 9 & 9 \\ 2 & -1 & -2 \end{bmatrix}. \end{aligned}$$

**b**) Solve the system using the factorization you have obtained in **a)**.

**Solution 6.6**

*1.*

$$L\mathbf{y} = LU\mathbf{x} = PA\mathbf{x} = P\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \\ 2 \end{bmatrix}.$$

*2.*

$$
\begin{aligned}
L\mathbf{y} &= P\mathbf{b} \\
\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -12 \\ -6 \\ 2 \end{bmatrix}, \\
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -12 \\ -6 - \frac{1}{2}(-12) \\ x_3 \end{bmatrix} = \\
\begin{bmatrix} -12 \\ 0 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -12 \\ 0 \\ 2 - \frac{1}{3}(-12) + \frac{1}{2} \cdot 0 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ 6 \end{bmatrix}.
\end{aligned}
$$

*3.*

$$
\begin{aligned}
U\mathbf{x} &= \mathbf{y} \\
\begin{bmatrix} 6 & 6 & -6 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -12 \\ 0 \\ 6 \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_1 \\ \frac{-12 \cdot 1}{6} \\ 1 \end{bmatrix} = \\
\begin{bmatrix} \frac{-12-6(-2)+6\cdot 1}{6} \\ -2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.
\end{aligned}
$$

*4. Let us check the solution:*

$$
\begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix} = \mathbf{b}.
$$

## 6.3 Problem

A quantity $y(t)$ depending on the time variable $t$, is measured in four time points. The measurements are given in the table below:

| $t_i, i = 1, \ldots, 4$ | | $y_i, i = 1, \ldots, 4$ |
|---|---|---|
| 0 | | 1 |
| 1 | | 2 |
| 2 | | 7 |
| 3 | | 5 |

**a)** Find a polynomial $P(t)$ of degree $\leq 2$ that goes through the **first three** points

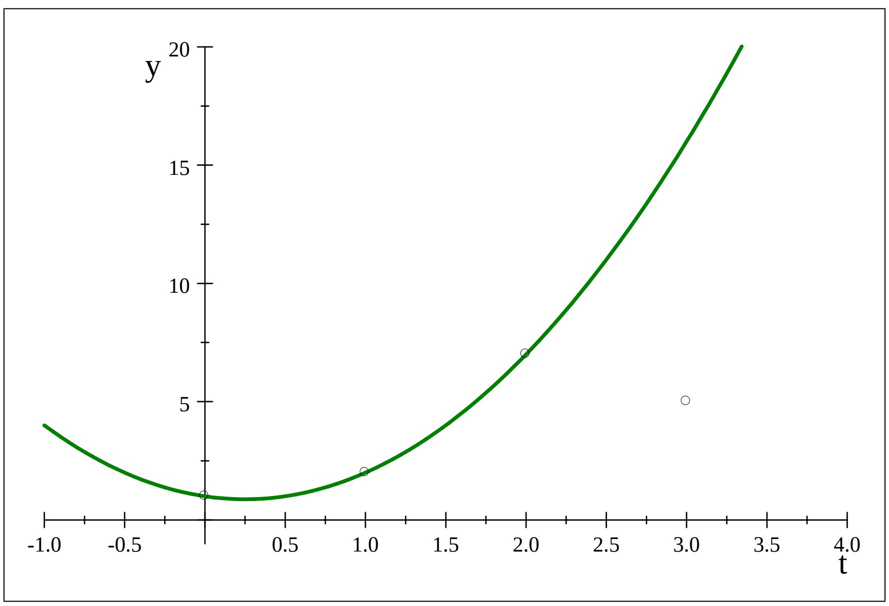
$$(t_1, y_1), (t_2, y_2), (t_3, y_3).$$

**Solution 6.7** *Use Newton's divided differences:*

$$\begin{array}{cccc}
t & y & [y]_{ij} & [y]_{ijk} \\
 & & & \\
0 & \mathbf{1} & & \\
 & & \frac{2-1}{1-0} = \mathbf{1} & \\
1 & 2 & & \frac{5-1}{2-0} = \mathbf{2} \\
 & & \frac{7-2}{2-1} = 5 & \\
2 & 7 & &
\end{array}$$

*Finally:*

$$P(t) = 1 + 1(t-0) + 2(t-0)(t-1) = 2t^2 - t + 1:$$




**b)** Find a polynomial $S(t)$ of degree $\leq 2$ that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (S(t_i) - y_i)^2.$$

**Hint**: use normal equations.

**Solution 6.8** *Let*

$$S(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2.$$

*The polynomial that goes through the four points, should satisfy the (**overdetermined**) system of equtaions*

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix} = \mathbf{b}.$$

*To find a solution that minimizes*

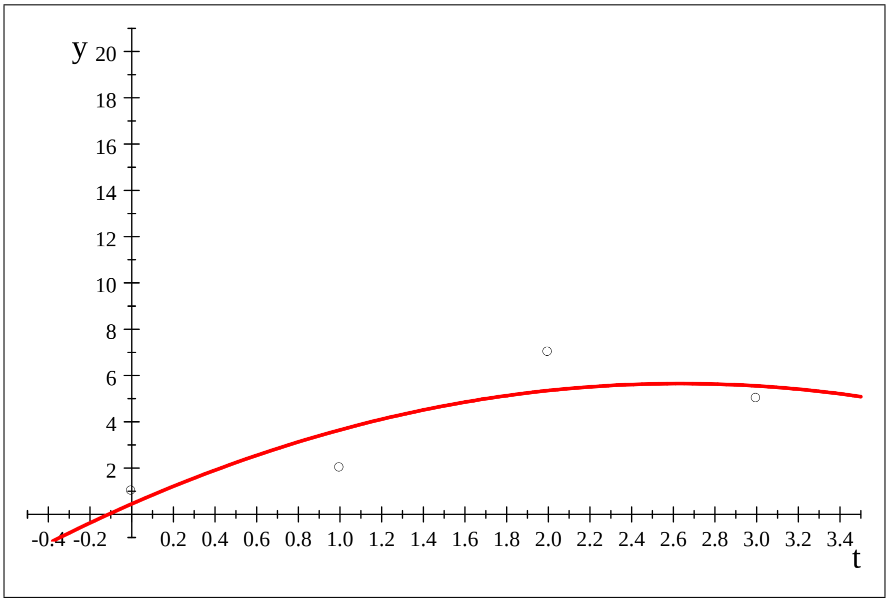
$$\sum_{i=1}^{4} \left(S(t_i) - y_i\right)^2 = \|\mathbf{b} - A\mathbf{x}\|_2\,,$$

*we write the normal equations:*

$$\begin{aligned} A^T A\mathbf{x} &= A^T\mathbf{b}, \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix}, \\ \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 15 \\ 31 \\ 75 \end{bmatrix}, \\ \mathbf{x} &= \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 31 \\ 75 \end{bmatrix} = \begin{bmatrix} \frac{9}{20} \\ \frac{79}{20} \\ -\frac{3}{4} \end{bmatrix}. \end{aligned}$$

*Therefore,*

$$S(t) = \begin{bmatrix} \frac{9}{20} \\ \frac{79}{20} \\ -\frac{3}{4} \end{bmatrix}^T \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} = -\frac{3}{4}t^2 + \frac{79}{20}t + \frac{9}{20}.$$




**c)** Given the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find the **reduced** $QR$ factorization of $B$, i.e. express $B$ as

$$B = QR$$

where

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}$$

is a $4 \times 2$ matrix with orthonormal columns, i.e. with

$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and $R$ is an upper-triangular $2 \times 2$ matrix.

**Solution 6.9**

*1.*

$$\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

*2.*

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{\|\mathbf{y}_1\|}\mathbf{y}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \\
r_{11} &= \|\mathbf{y}_1\| = 2.
\end{aligned}$$

*3.*

$$\begin{aligned}
r_{12} &= \mathbf{y}_2 \bullet \mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 3, \\
\mathbf{y}_{curr} &= \mathbf{y}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
r_{22} &= \|\mathbf{y}_{curr}\| = \left\| \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \right\| = \sqrt{5}, \\
\mathbf{q}_2 &= \frac{1}{r_{22}}\mathbf{y}_{curr} = \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{10}\sqrt{5} \\ -\frac{1}{10}\sqrt{5} \\ \frac{1}{10}\sqrt{5} \\ \frac{3}{10}\sqrt{5} \end{bmatrix}.
\end{aligned}$$

*4. Finally:*

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 0.5 & -0.670\,820\,393\,249\,937 \\ 0.5 & -0.223\,606\,797\,749\,979 \\ 0.5 & 0.223\,606\,797\,749\,979 \\ 0.5 & 0.670\,820\,393\,249\,937 \end{bmatrix}.$$

$$Q^TQ = \begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

*and the columns of $Q$ are indeed orthonormal.*

*5.*

$$\begin{aligned}
R &= \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix}, \\
QR &= \begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = B.
\end{aligned}$$

**d)** Find a polynomial $U(t)$ of degree $\leq 1$ that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (U(t_i) - y_i)^2.$$

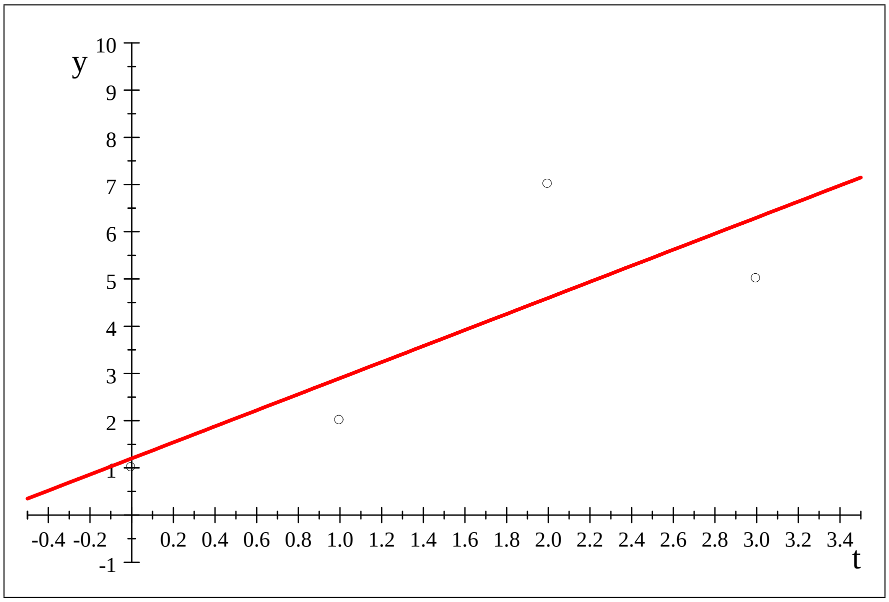
Use the $QR$ factorization you have obtained in **d)**.

**Solution 6.10** *We solve the system ($U(t) = \beta_0 + \beta_1 t$):*

$$\begin{aligned}
R\mathbf{x} &= Q^T\mathbf{b}, \\
\begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} &= Q^T\mathbf{b} = \\
\begin{bmatrix} \frac{1}{2} & -\frac{3}{10}\sqrt{5} \\ \frac{1}{2} & -\frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{1}{10}\sqrt{5} \\ \frac{1}{2} & \frac{3}{10}\sqrt{5} \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix} &= \begin{bmatrix} \frac{15}{2} \\ \frac{17}{10}\sqrt{5} \end{bmatrix}, \\
\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} &= \begin{bmatrix} \frac{\frac{15}{2} - 3\cdot\frac{17}{10}}{2} \\ \frac{17}{10} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{17}{10} \end{bmatrix}.
\end{aligned}$$

*and the best line that fits the four points, is*

$$y = \begin{bmatrix} \frac{6}{5} \\ \frac{17}{10} \end{bmatrix}^T \begin{bmatrix} 1 \\ t \end{bmatrix} = \frac{6}{5} + \frac{17}{10}t:$$




*Compare to the previous result (not required!):*

1. *The **parabola** that fits the four points, gives the error*

$$\left\| \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} \frac{9}{20} \\ \frac{79}{20} \\ -\frac{3}{4} \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \frac{11}{20} \\ -\frac{33}{20} \\ \frac{33}{20} \\ -\frac{11}{20} \end{bmatrix} \right\| = \frac{11}{10}\sqrt{5} \approx 2.459\,674\,775\,249\,77.$$

2. *The **line** that fits the four points, gives the error*

$$\left\| \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{6}{5} \\ \frac{17}{10} \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -\frac{1}{5} \\ -\frac{9}{10} \\ \frac{12}{5} \\ -\frac{13}{10} \end{bmatrix} \right\| = \frac{1}{10}\sqrt{830} \approx 2.880\,972\,058\,177\,59.$$

## 6.4 Problem

Given the initial value problem (IVP):

$$\begin{aligned} y' &= (t+1)\,y = f(t,y), \\ y(0) &= 1. \end{aligned}$$

**a)** Show that the function

$$y(t) = e^{\frac{1}{2}t^2+t}$$

is the exact solution of the IVP.

**Solution 6.11**

$$\begin{aligned} y' &= \left(e^{\frac{1}{2}t^2+t}\right)' = \left(\frac{1}{2}t^2+t\right)' e^{\frac{1}{2}t^2+t} = \\ &= (t+1)\,e^{\frac{1}{2}t^2+t} = (t+1)\,y, \\ y(0) &= e^0 = 1. \end{aligned}$$

**b)** Formulate the Trapezoid method for the numerical solution of the IVP.

**Solution 6.12** *Choose a stepsize h, and use the estimates*

$$\begin{aligned} y(t+h) &\approx y(t) + hf(t,y) \text{ (Euler)}, \\ y(t+h) &= y(t) + \int_t^{t+h} f(s,y)\,ds \approx y(t) + \frac{h}{2}\left(f(t,y(t)) + f(t+h,y(t+h))\right) \approx \\ &\approx y(t) + \frac{h}{2}\left(f(t,y(t)) + f(t+h,y(t)+hf(t,y))\right). \end{aligned}$$

*The Trapezoid method is the following: let*

$$\begin{aligned} t_k &= t_0 + kh, \\ y_k &= y(t_k). \end{aligned}$$

*Obtain the following estimates for $w_k \approx y_k$:*

$$\begin{aligned} w_0 &= y_0 \text{ (given)}, \\ w_1 &= w_0 + \frac{h}{2}\left(f(t_0,w_0) + f(t_1,w_0+hf(t_0,w_0))\right), \\ &\ldots, \\ w_{k+1} &= w_k + \frac{h}{2}\left(f(t_k,w_k) + f(t_{k+1},w_k+hf(t_k,w_k))\right), \\ &\ldots \end{aligned}$$

**c)** Choose step size $h = \frac{1}{2}$, and perform 2 steps of the Trapezoid method, producing the estimates:

$$\begin{aligned} w_0 &= y(0), \\ w_1 &\approx y\left(\frac{1}{2}\right), \\ w_2 &\approx y(1). \end{aligned}$$

Compare with the exact solution from **a)**, and find the global error of the method.

**Solution 6.13** *Define*

$$\begin{aligned} f(t,y) &= (t+1)\,y \\ g(t,y,h) &= y + \frac{h}{2}\left(f(t,y) + f(t+h, y+hf(t,y))\right). \end{aligned}$$

*Then:*

$$\begin{aligned} w_0 &= 1, \\ w_1 &= g\left(0,1,\frac{1}{2}\right) = \frac{29}{16} = 1.812\,5, \\ w_2 &= g\left(\frac{1}{2},\frac{29}{16},\frac{1}{2}\right) = \frac{261}{64} = 4.078\,125. \end{aligned}$$

*The global error is*

$$\begin{aligned} |w_2 - y_2| &= \left|4.078\,125 - \left[e^{\frac{1}{2}t^2+t}\right]_{t=1}\right| = \left|4.078\,125 - e^{\frac{3}{2}}\right| \approx \\ &\approx 0.403\,564\,070\,338\,065. \end{aligned}$$