UiT THE ARCTIC UNIVERSITY OF NORWAY

Faculty of Science and Technology

EXAMINATION QUESTION PAPER

Exam in:	**Mat-2201 Numerical Methods**
Date:	**Wednesday December 4, 2019**
Time:	**15:00-19:00**
Place:	**Hall del 1 Kraft sportssenter (Tromsø)**
Approved aids:	Approved calculator; Rottmann's Tables; Two A4 sheets of personal notes on both sides (four A4 pages): handwritten, printed or mixed.
Type of sheets (sqares/lines):	
Number of pages incl. cover page:	**5 (1+4)** **There are 13 subproblems (1abcd, 2ab, 3abcd, 4abc)**
Contact person during the exam:	**Andrei Prasolov**
Phone:	**93 67 58 32**
	Will a lecturer/person in charge visit the venue? **YES, approx. time: 16:30.**

NB! It is not allowed to submit rough paper along with the answer sheets.

If you do submit rough paper it will not be evaluated.

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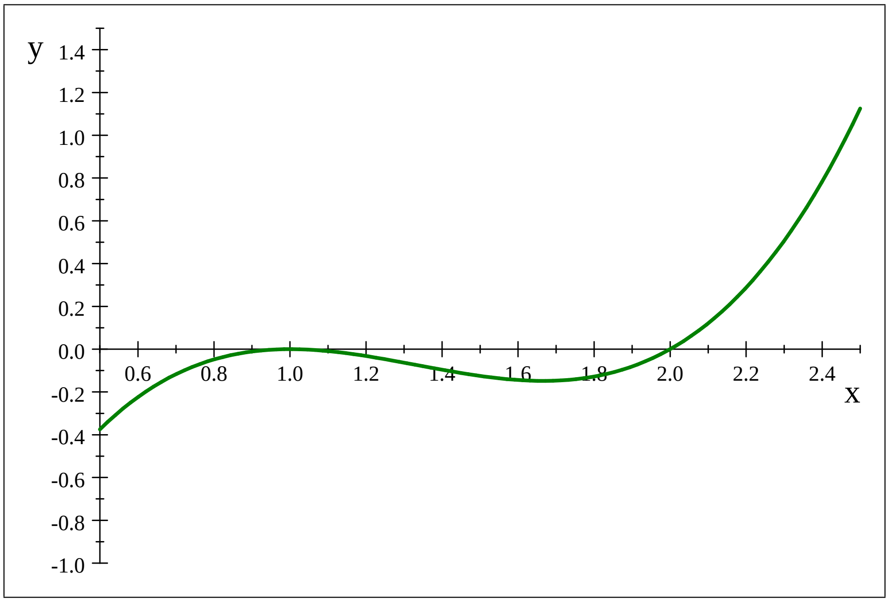
1 Problem

Given an equation

$$f(x) = x^3 - 4x^2 + 5x - 2 = 0.$$



a) Investigate which of the following interation processes converge locally to the root $r = 2$:

1. $x_{n+1} = g(x_n)$ where $g(x) = x^3 - 4x^2 + 6x - 2$.
2. $x_{n+1} = h(x_n)$ where $h(x) = -x^3 + 4x^2 - 4x + 2$.

Remark. It is **not** enough to prove that an iteration process converges locally to **something**. You should explore whether it can converge to the root $r = 2$.

b) Formulate Newton's iteration method for the equation. Perform 2 iterations of this method with the initial value $x_0 = 0$.

c) Find the rate

$$S = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|}$$

of the linear convergence to the root $r = 1$ for Newton's method.

d) Formulate an iteration method $x_{n+1} = \varphi(x_n)$ that converges to $r = 1$ **quadratically**:

$$M = \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} < \infty.$$

2 Problem

Given the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 9 & 9 \\ 2 & -1 & -2 \\ 6 & 6 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix}.$$

a) Perform a $PA = LU$ factorization. Remember that L should have the form

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

with $|l_{ij}| \leq 1$.

b) Solve the system using the factorization you have obtained in **a)**.

3 Problem

A quantity $y(t)$ depending on the time variable t, is measured in four time points. The measurements are given in the table below:

$t_i, i = 1, \ldots, 4$		$y_i, i = 1, \ldots, 4$
0		1
1		2
2		7
3		5

a) Find a polynomial $P(t)$ of degree ≤ 2 that goes through the **first three** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3).$$

b) Find a polynomial $S(t)$ of degree ≤ 2 that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (S(t_i) - y_i)^2.$$

Hint: use normal equations.

c) Given the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find the **reduced** QR factorization of B, i.e. express B as

$$B = QR$$

where

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}$$

is a 4×2 matrix with ortho**normal** columns ($\mathbf{q}_i \bullet \mathbf{q}_j = 0$ if $i \neq j$ and 1 if $i = j$), and R is an upper-triangular 2×2 matrix.

d) Find a polynomial $U(t)$ of degree ≤ 1 that fits the **four** points

$$(t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4),$$

i.e. that **minimizes** the sum

$$\sum_{i=1}^{4} (U(t_i) - y_i)^2.$$

Use the QR factorization you have obtained in **c)**.

4 Problem

Given the initial value problem (IVP):

$$
\begin{aligned}
y' &= (t+1)\,y = f(t,y)\,, \\
y(0) &= 1.
\end{aligned}
$$

a) Show that the function

$$
y(t) = e^{\frac{1}{2}t^2+t}
$$

is the exact solution of the IVP.

b) Formulate the Trapezoid method for the numerical solution of the IVP.

c) Choose step size $h = \frac{1}{2}$, and perform 2 steps of the Trapezoid method, producing the estimates:

$$
\begin{aligned}
w_0 &= y(0)\,, \\
w_1 &\approx y\left(\frac{1}{2}\right), \\
w_2 &\approx y(1)\,.
\end{aligned}
$$

Compare with the exact solution from **a)**, and find the global error of the method.

GOOD LUCK!