Lecture Computer Graphics

Raytracing - Intersections

Prof. Dr. David Bommes
Computer Graphics Group

Part I: Raytracing

Image Source

Part I: Raytracing

- Goal: How to generate realistic digital images of synthetic 3D scenes?

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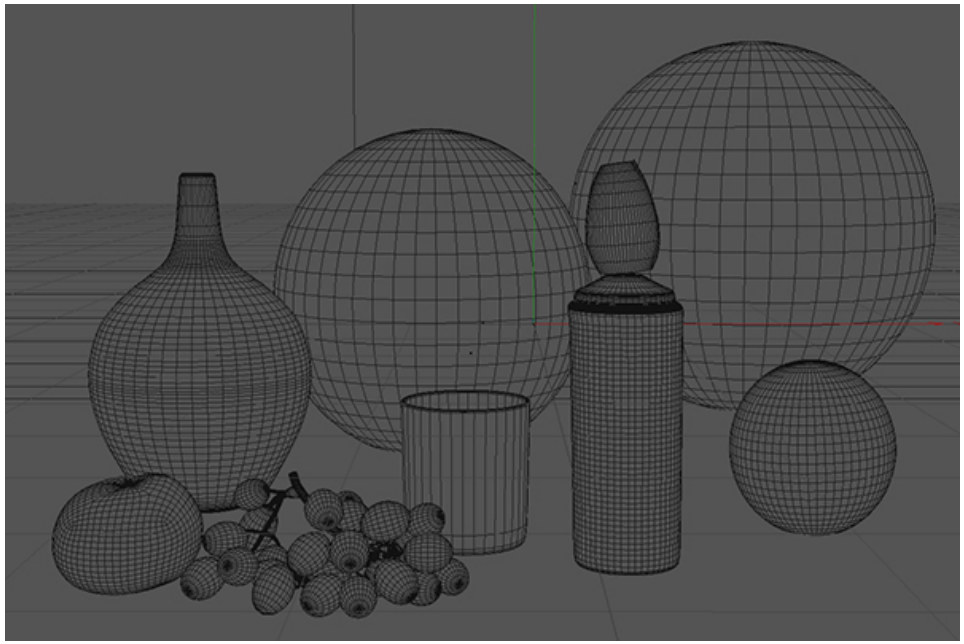


Image Source

Part I: Raytracing

- Goal: How to generate realistic digital images of synthetic 3D scenes?

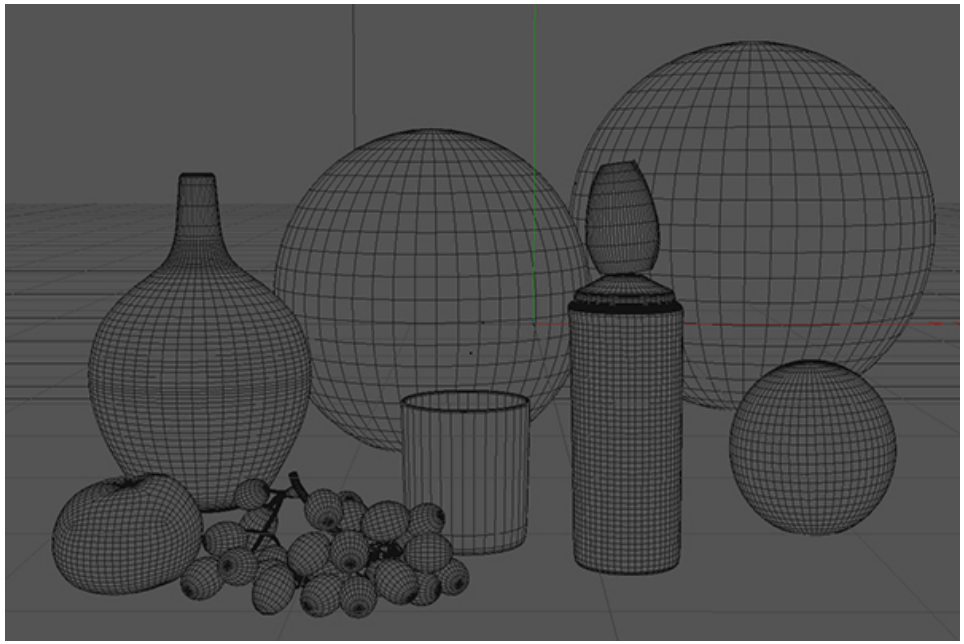
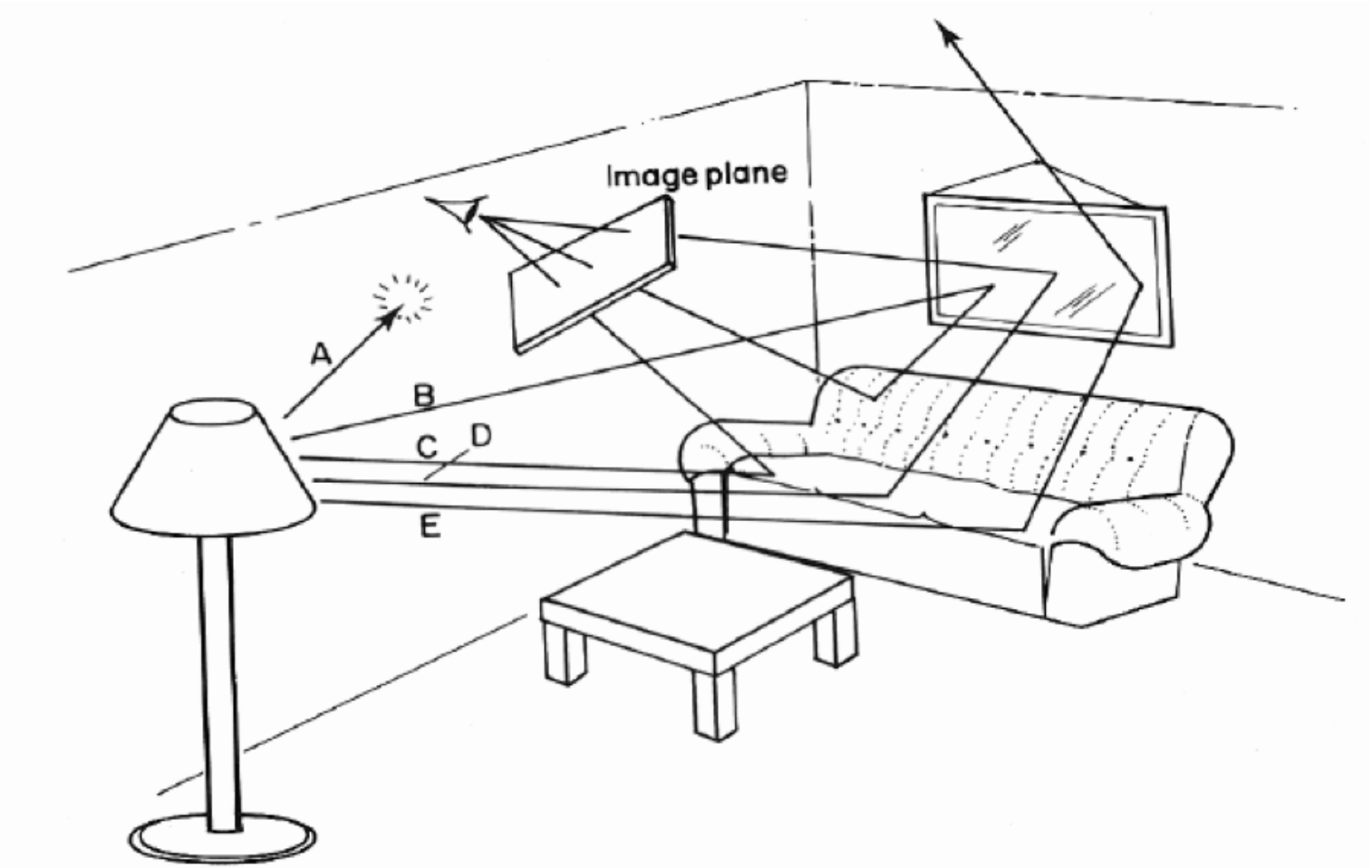


Image Source

Part I: Raytracing

- Goal: How to generate realistic digital images of synthetic 3D scenes?
- Several questions arise:
  - What is light?
  - How is light propagated?
  - How does light interact with objects?
  - How does a camera work?

Light Transport



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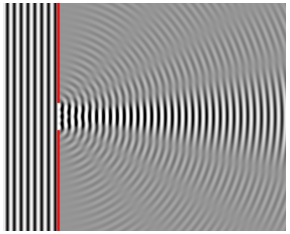
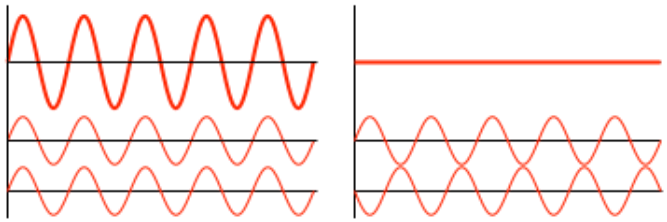
Light Transport Assumptions

- **Geometric optics**
  Light propagates along rays, not waves
  ⇒ no diffraction, no interference, no polarization



diffraction



interference

polarization

Images from Wikipedia

Light Transport Assumptions

- **Geometric optics**
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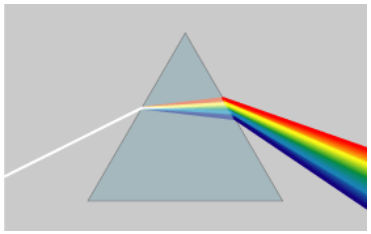
- **RGB approximation**
  Approximate light spectrum by discrete RGB wavelengths
  ⇒ no dispersion, no fluorescence



dispersion

fluorescence

Images from Wikipedia

Light Transport Assumptions

- **Geometric optics**
  Light propagates along rays, not waves
  ⇒ no diffraction, no interference, no polarization
- **RGB approximation**
  Approximate light spectrum by discrete RGB wavelengths
  ⇒ no dispersion, no fluorescence
- **No participating media**
  Light travels along straight rays through vacuum
  ⇒ no atmospheric scattering, no gravity effects
- **Superposition**
  Simply add multiple light contributions
  ⇒ No nonlinearly reflecting materials

Quiz Setup

http://cgg.unibe.ch:8080

Rendering or Photograph?

cgtrader.com

A: Rendering

A: Photo

Rendering or Photograph?

cgtrader.com

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Autodesk

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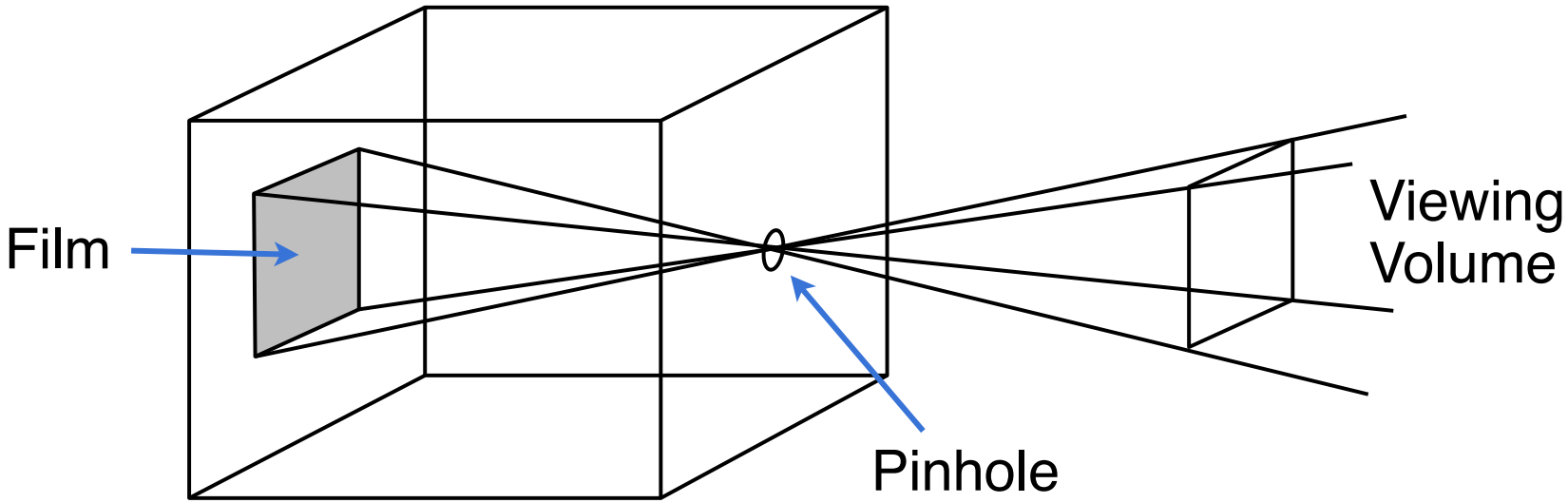
Pinhole Camera

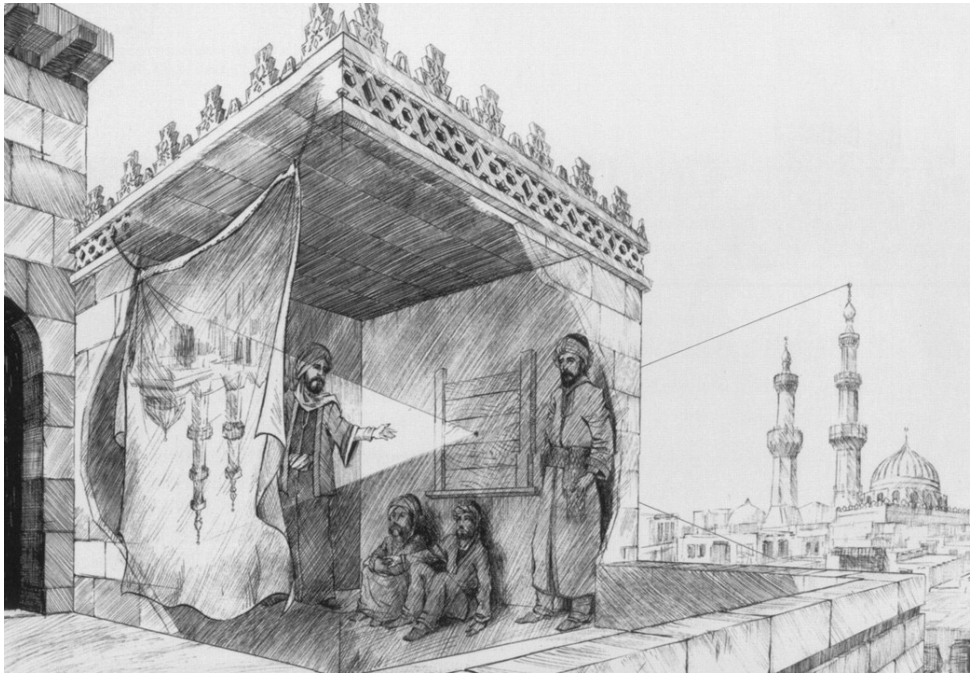
- We assume a classical pinhole camera



Pinhole Camera

- Studied extensively throughout history
- Application: Camera Obscura

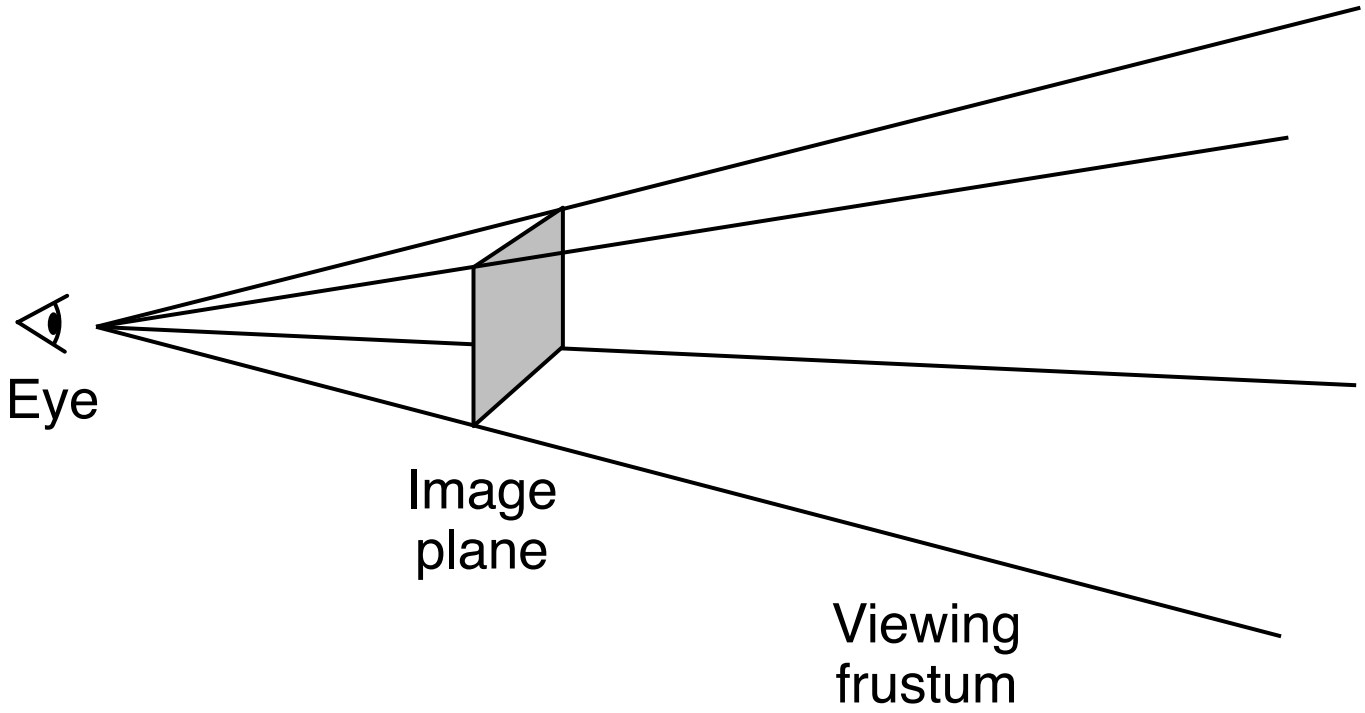
  - Mo Tzu (c. 470–c. 390 BC)
  - Aristotle (384–322 BC)
  - **Ibn al-Haytham (965–1040)**
  - Shen Kuo (1031–1095)
  - Roger Bacon (c. 1214–1294)
  - Johannes Kepler (1571–1630)



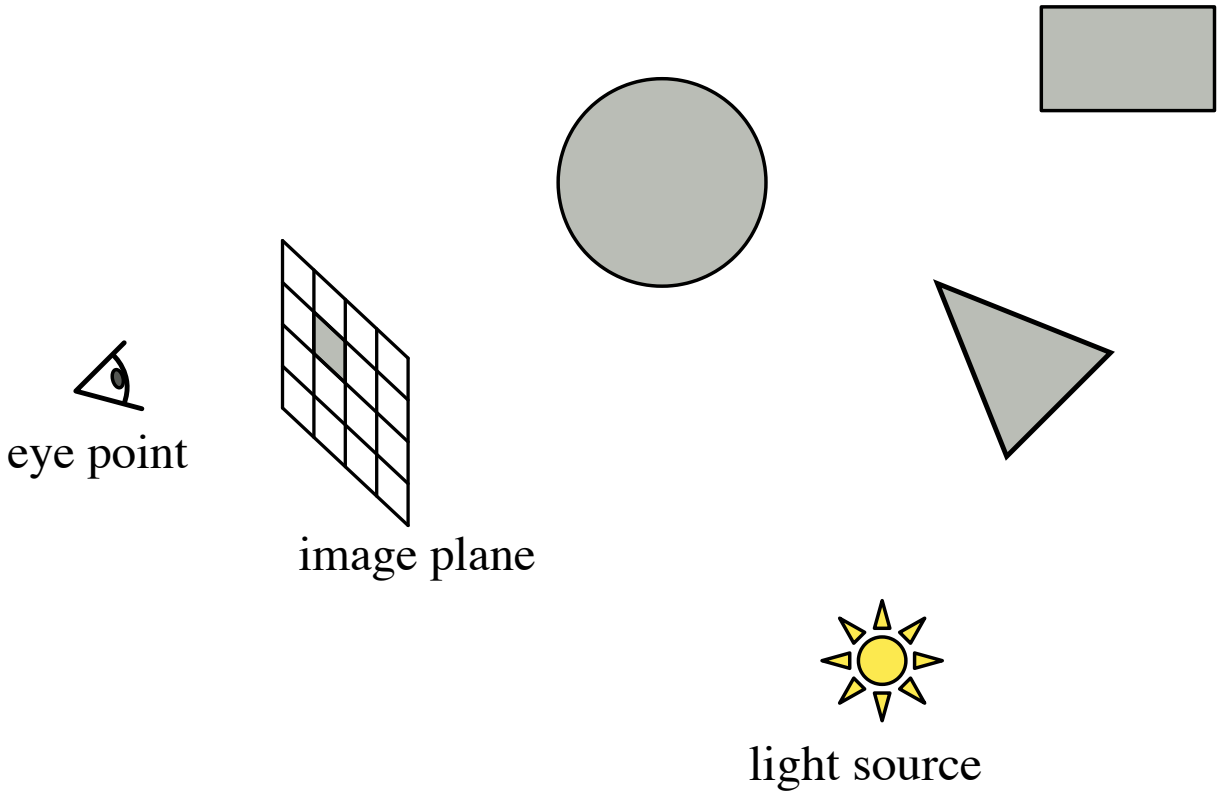
Source: A. H. Zewail,Phil. Trans. R. Soc. A 2010;368:1191-1204

Pinhole Camera

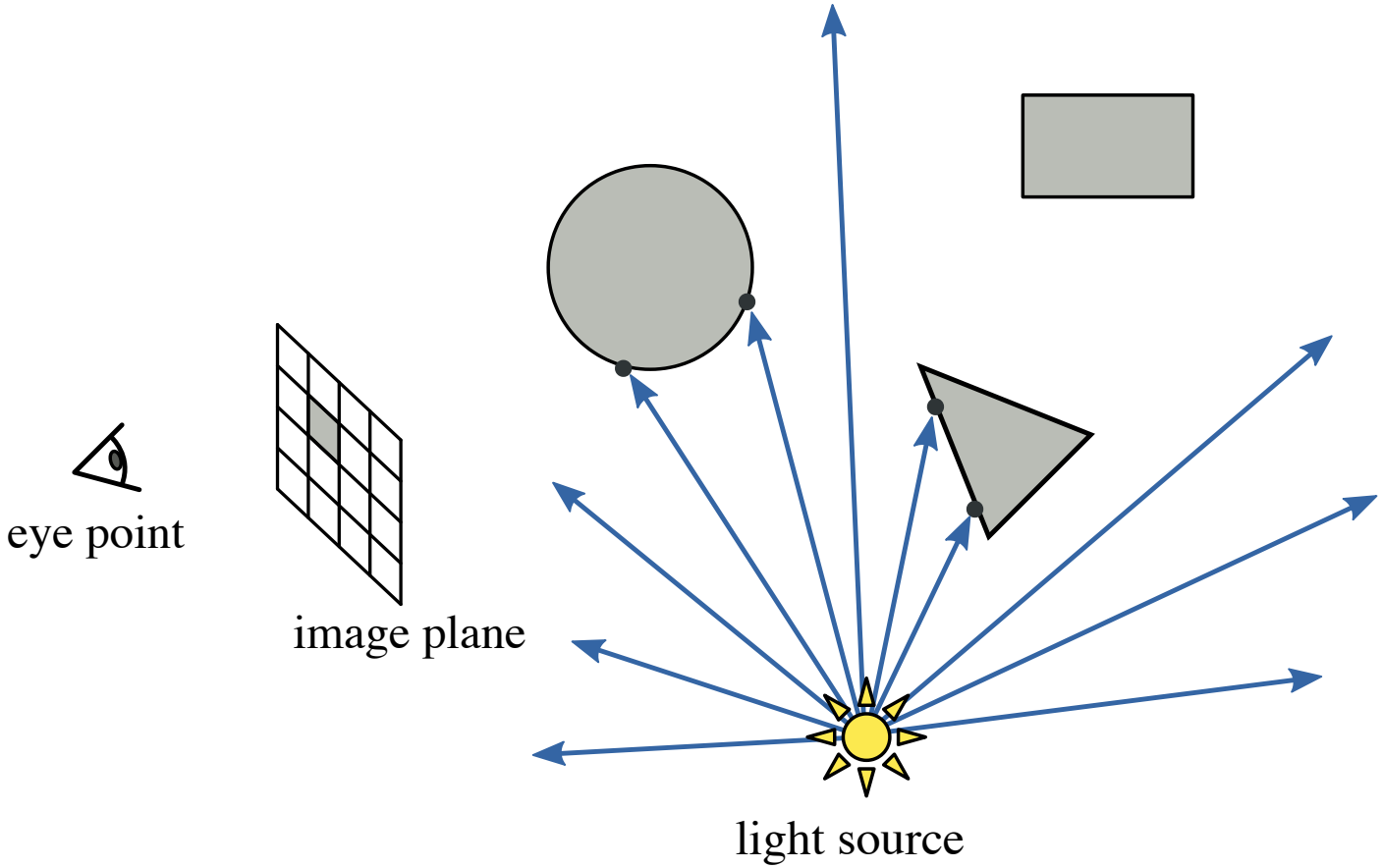
- We represent the camera using eye point (pinhole) and image plane (mirrored film)



Forward Ray Tracing



Forward Ray Tracing



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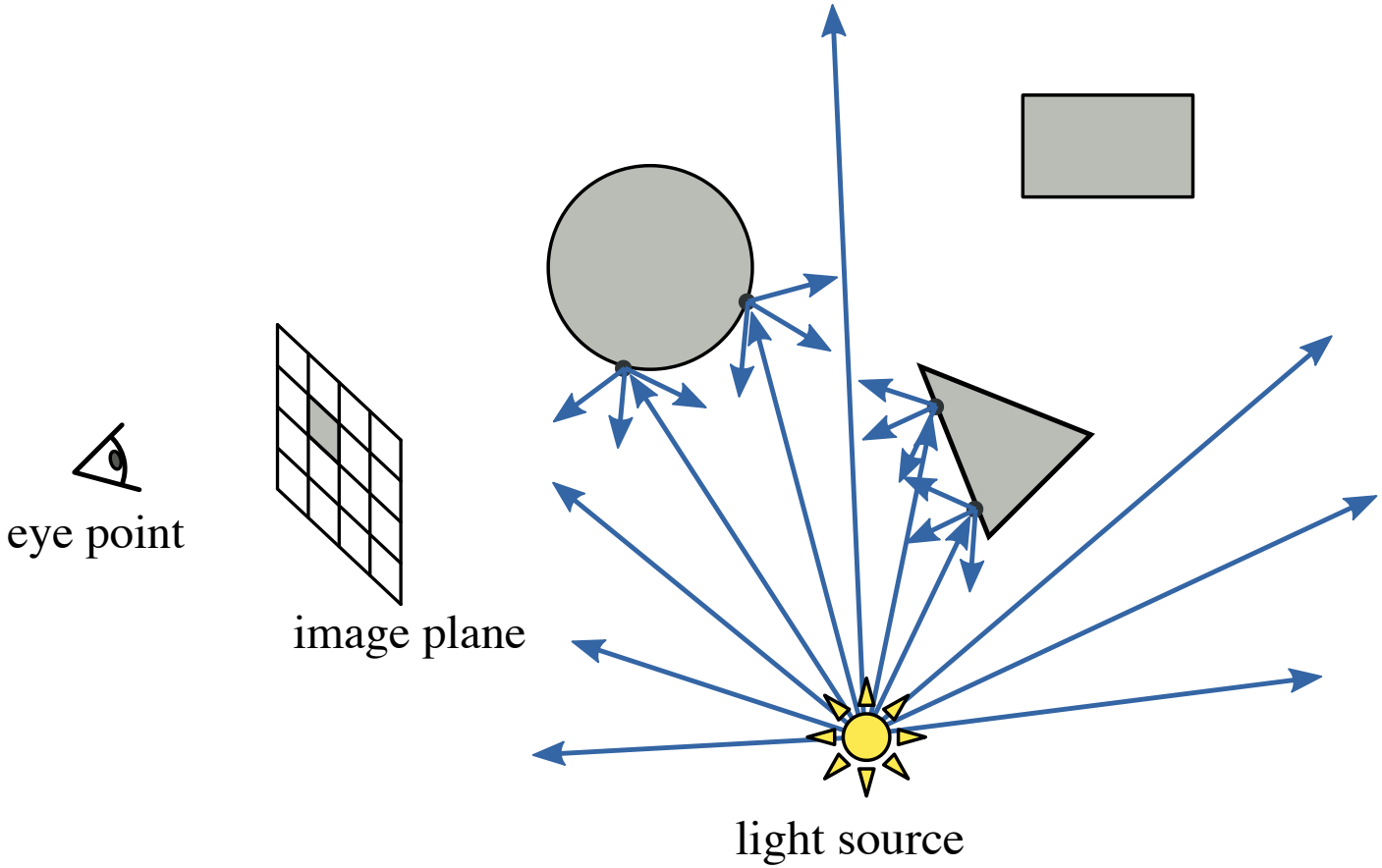

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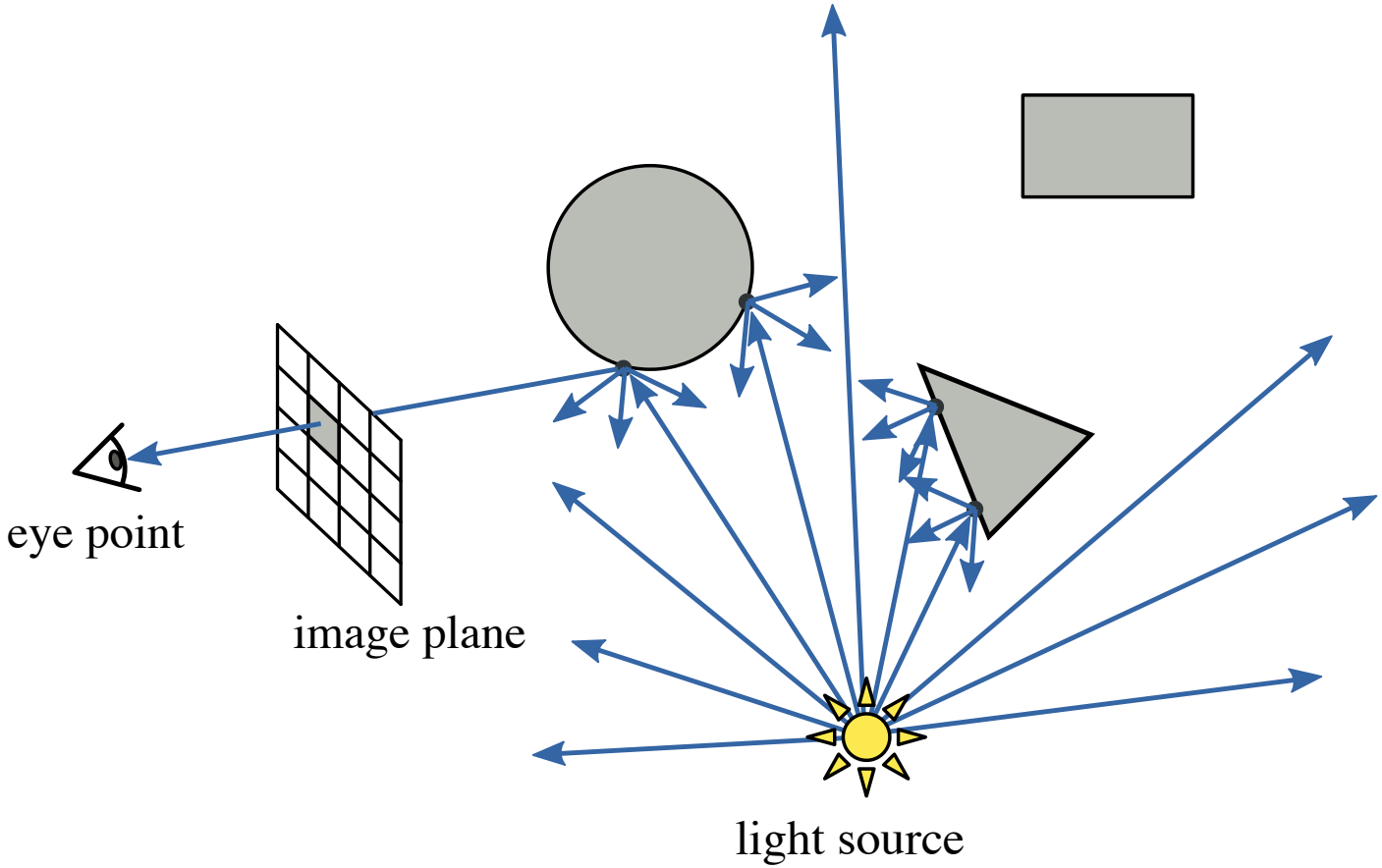

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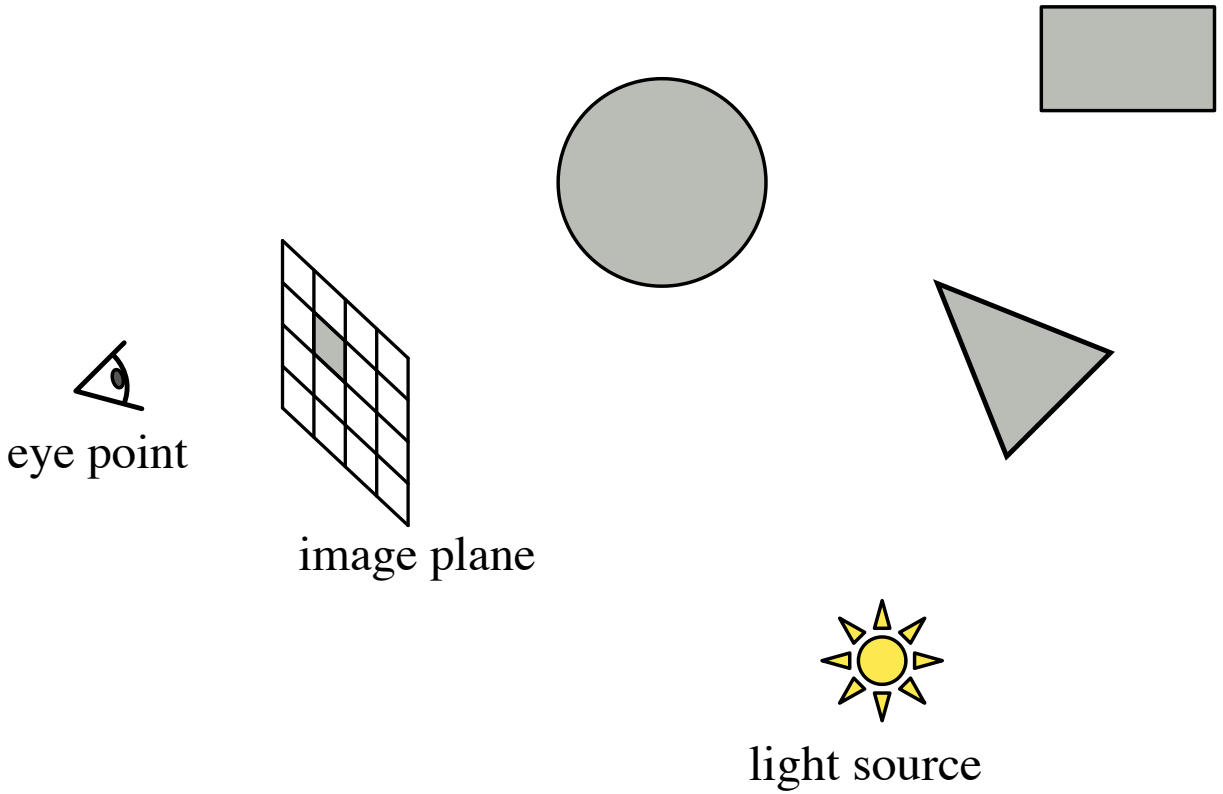

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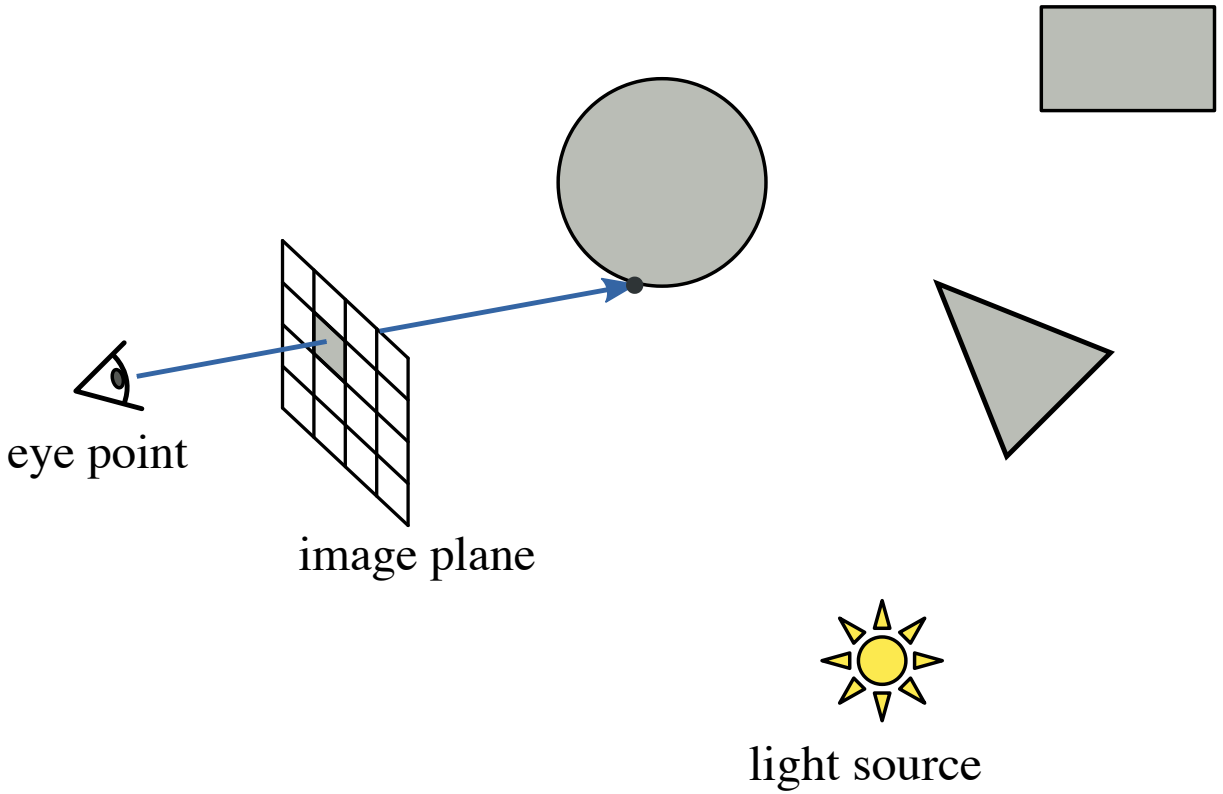

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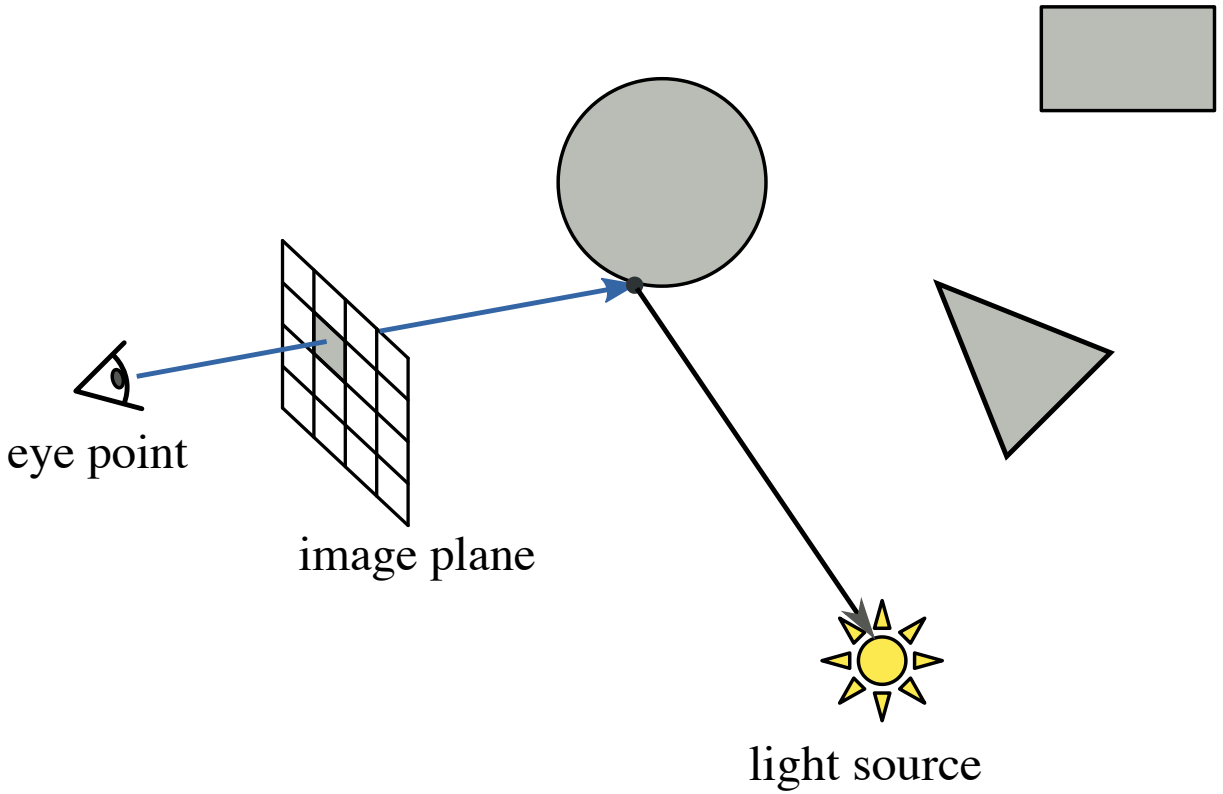

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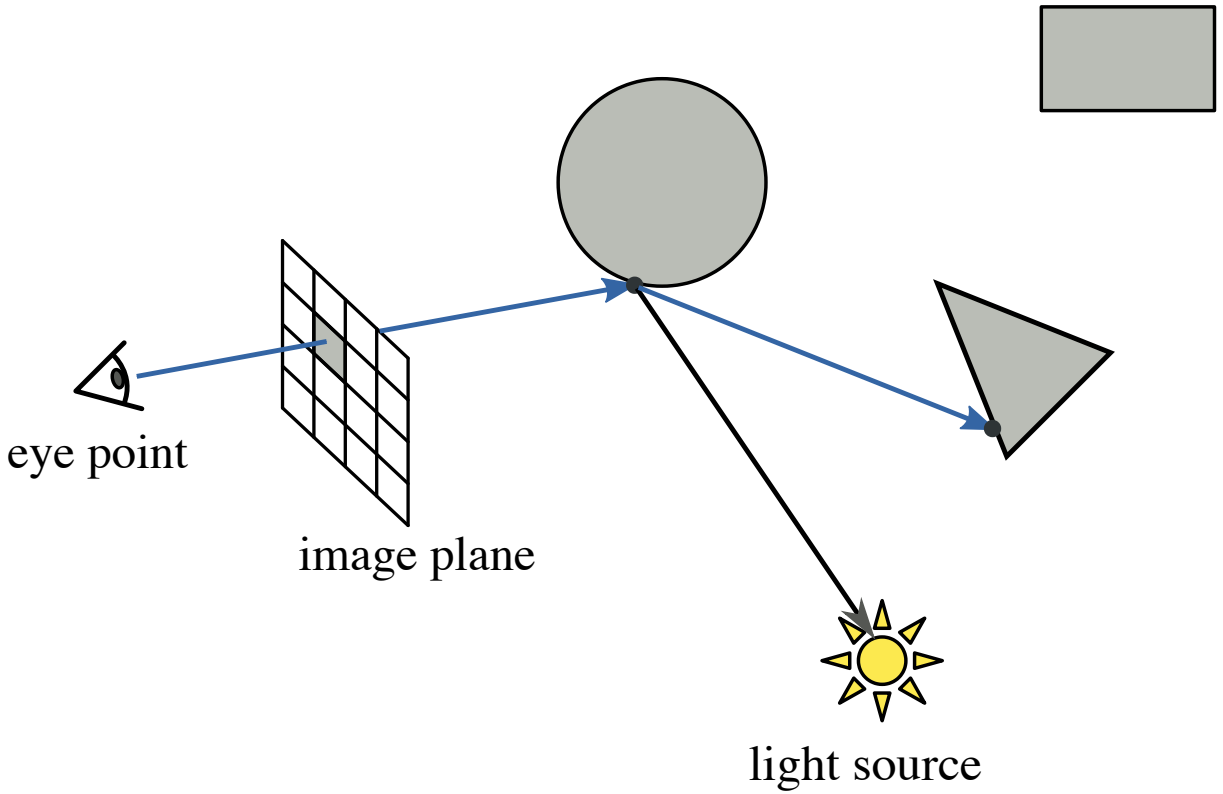

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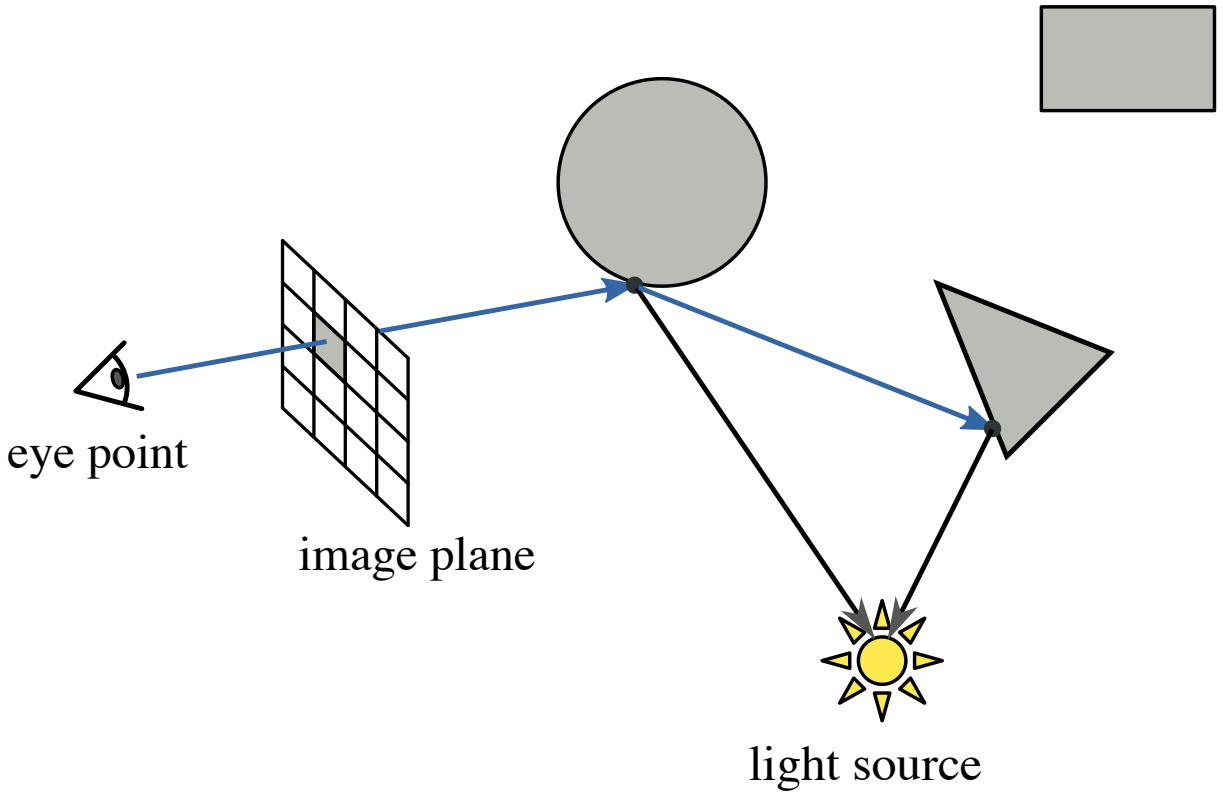

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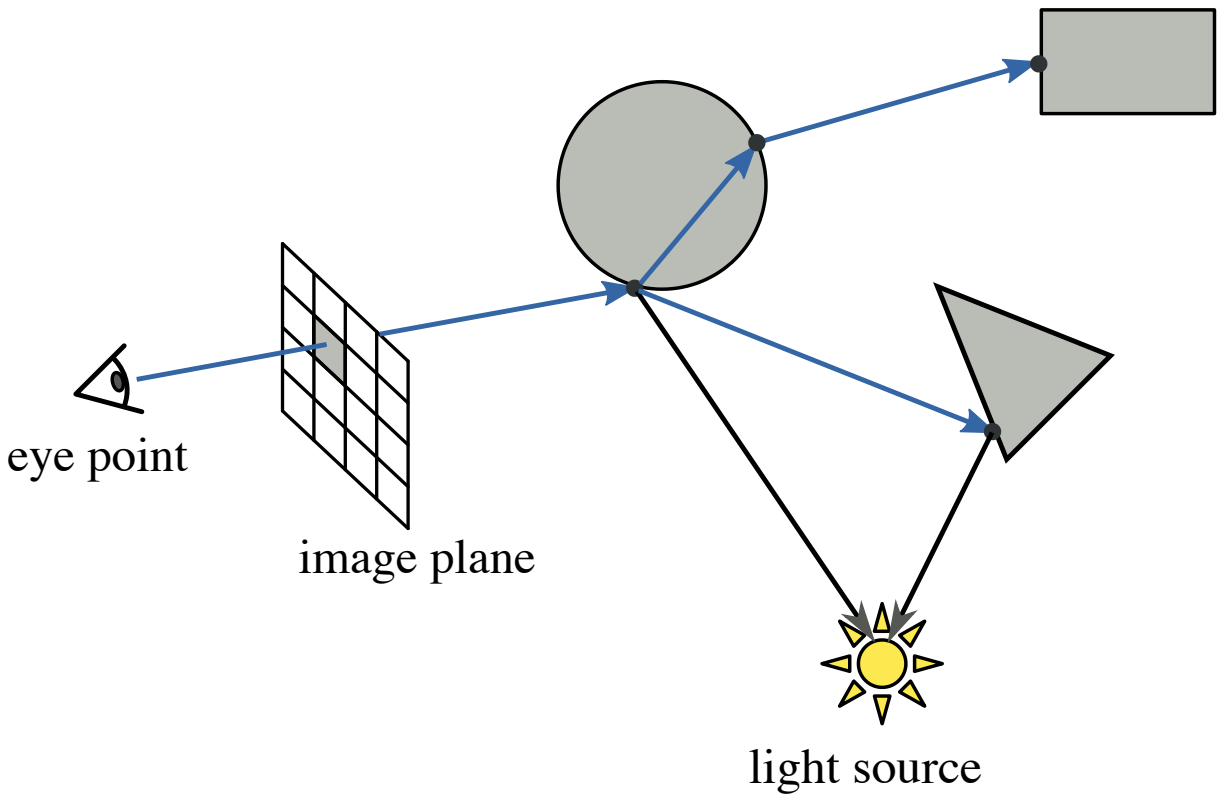

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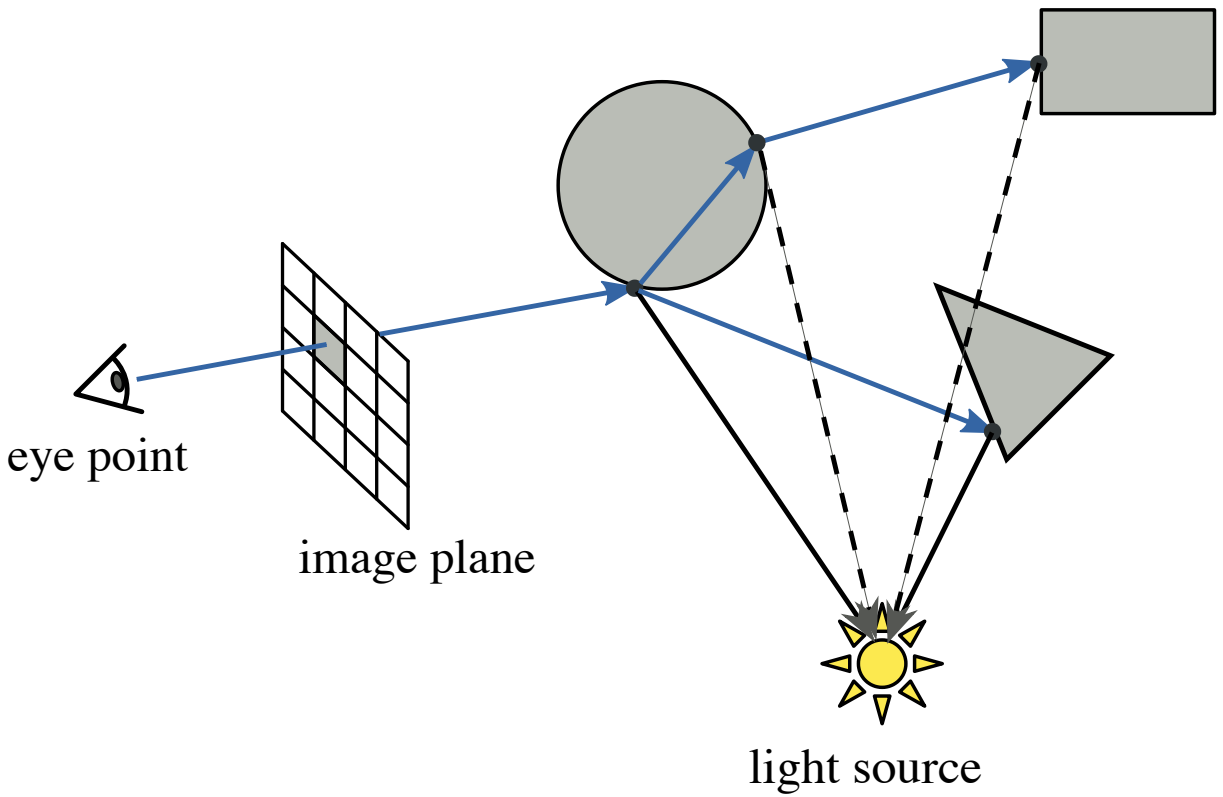

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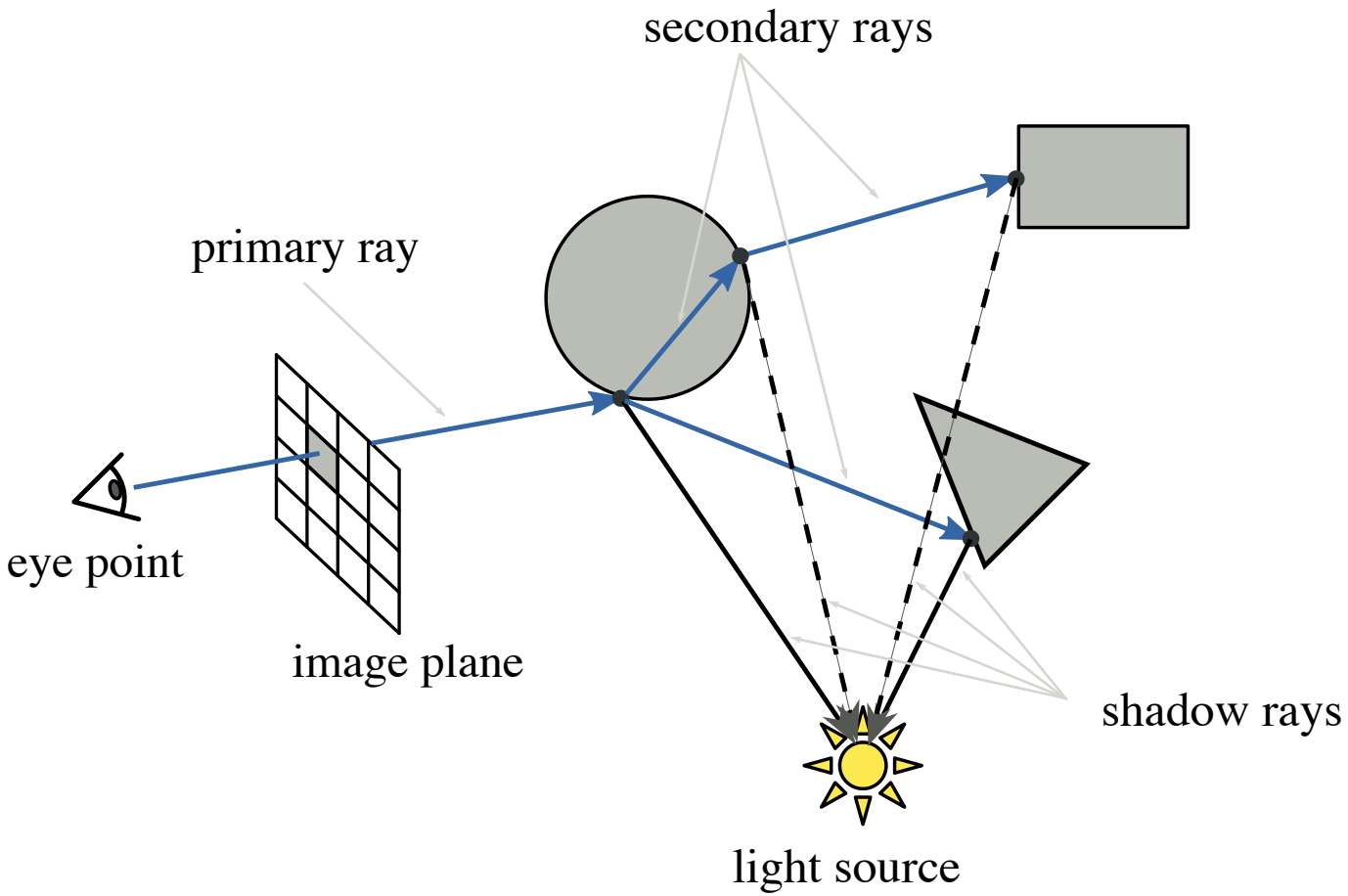
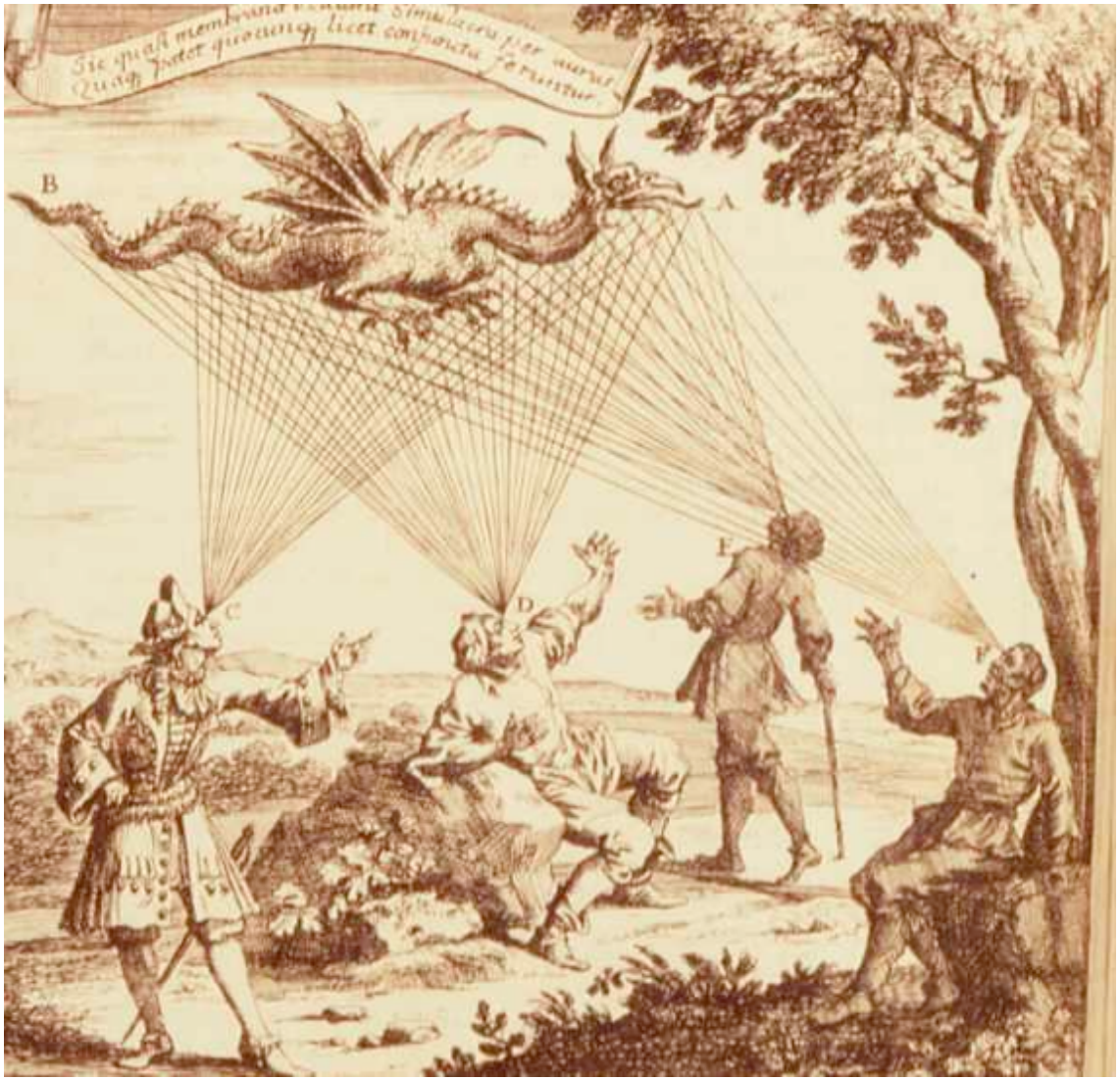
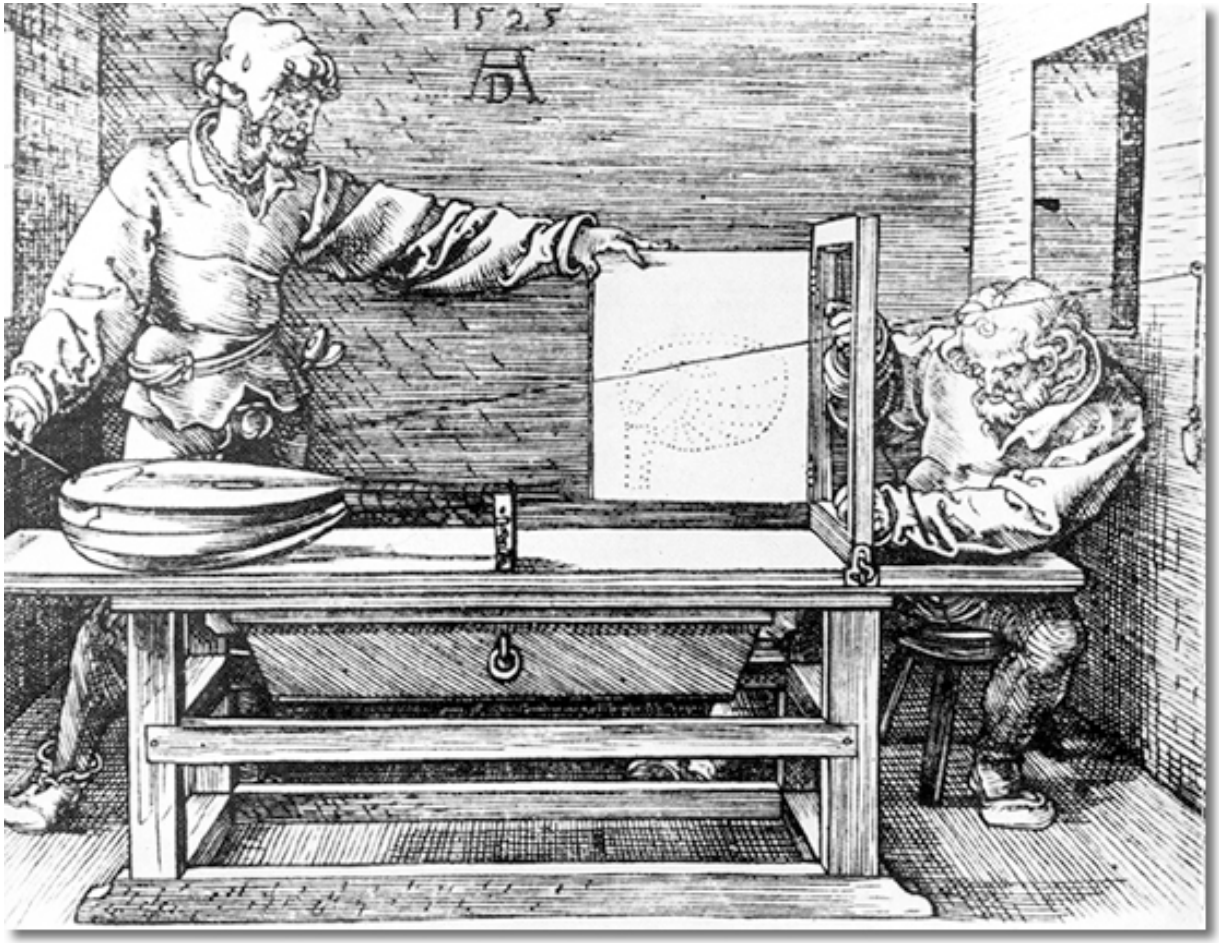


Ancient Greeks - Emission Theory



Albrecht Dürer (1525)

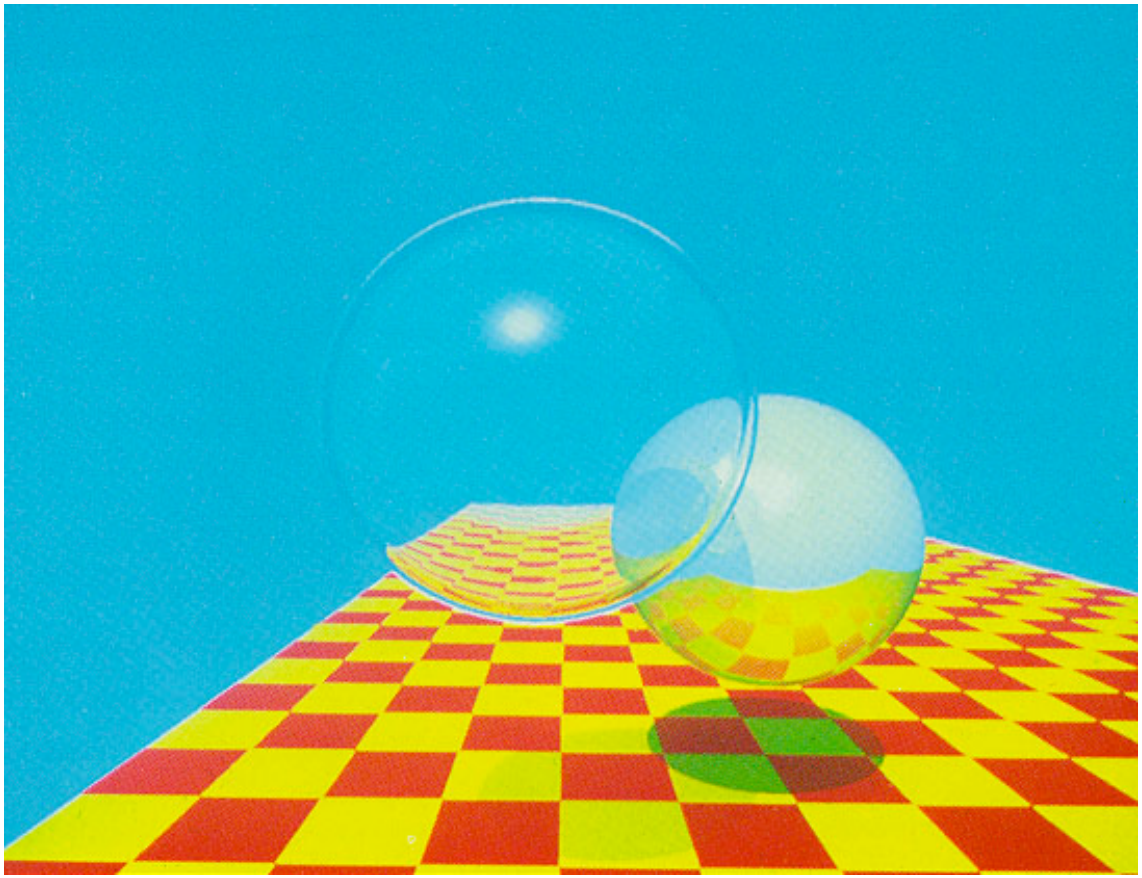


Raycasting Machine
Images from Wikipedia

Albrecht Dürer (1525)

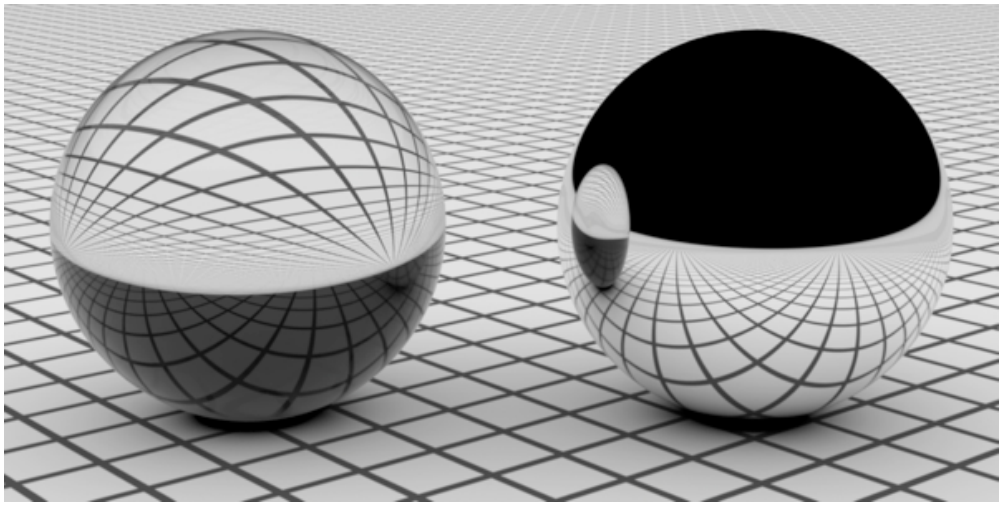
Rasterization
Images from Wikipedia

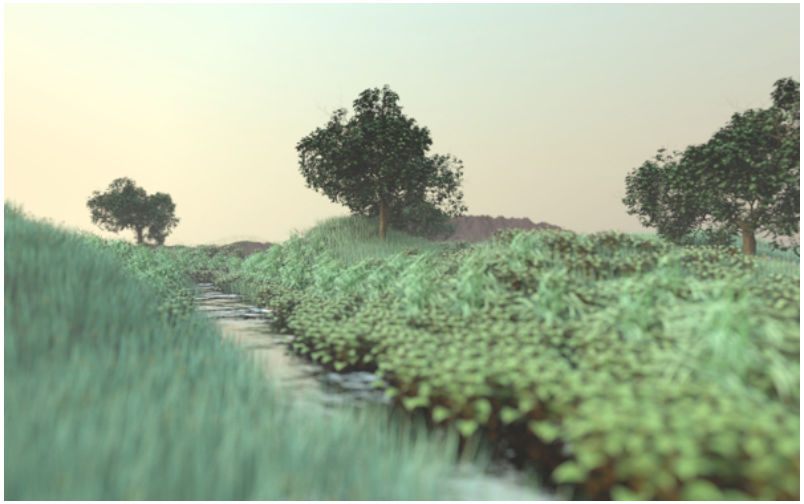
Turner Whitted (1979)



ACM SIGGRAPH 1979 - An improved illumination model for shaded display

Examples from PBRT

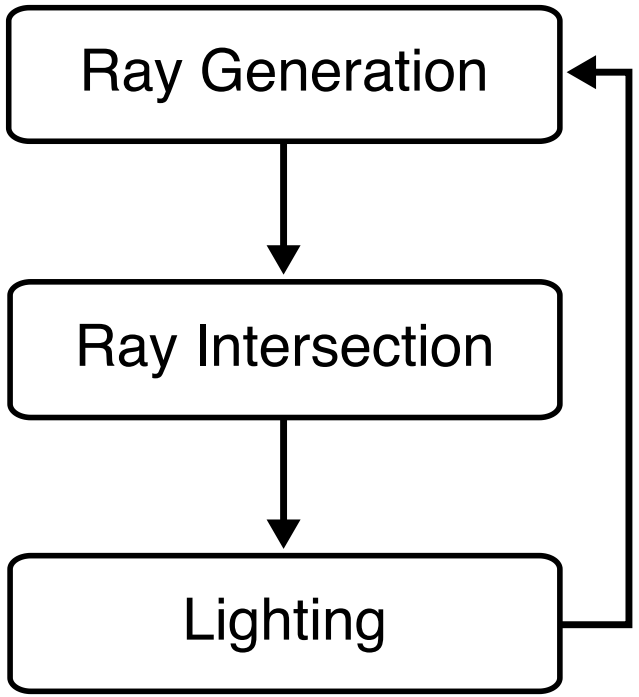


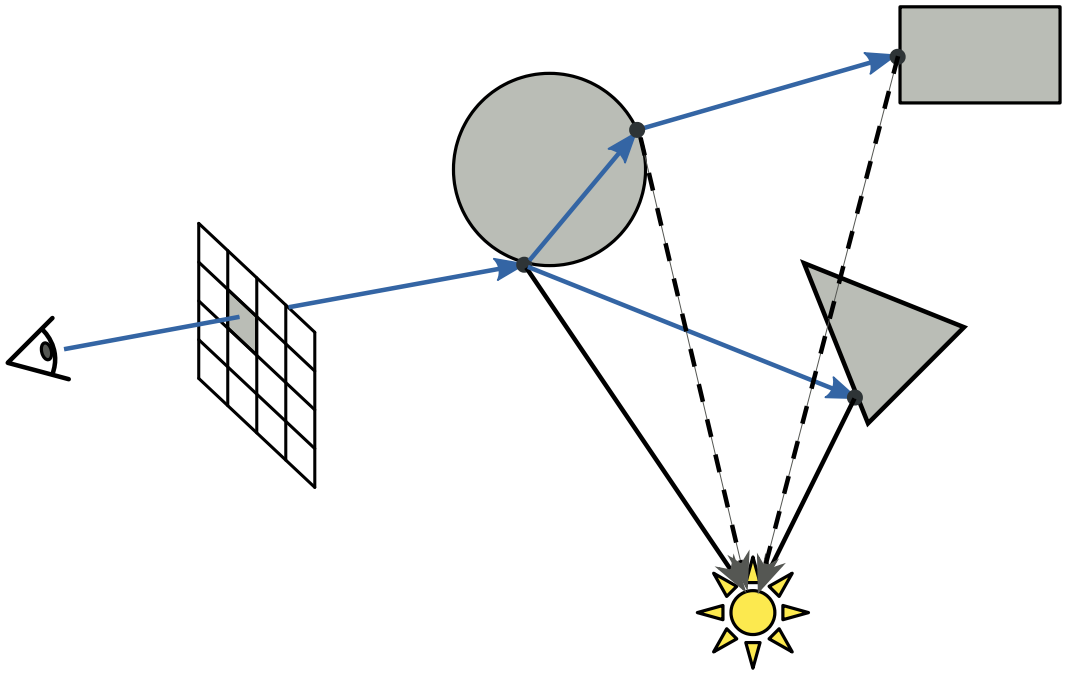


http://www.pbrt.org/gallery.html

Basic Ray Tracing Pipeline

Implementing a ray tracer basically comes down to three operators that we have to understand and implement.





Ray Tracing Pseudo Code

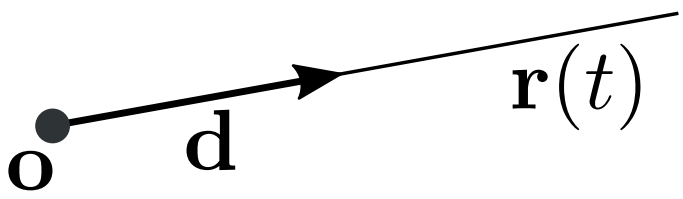
```
void raytrace()
{
    for (int x=0; x<width; ++x)
    {
        for (int y=0; y<height; ++y)
        {
            // generate primary ray through pixel (x,y)
            ray = primary_ray(x,y);

            // determine color through (recursive!) trace function
            color[x,y] = trace(ray);
        }
    }
}
```

Ray Generation

Ray Equation

- Explicit equation for ray $\mathbf{r}(t)$, starting at origin $\mathbf{o}$ and going in (normalized) direction $\mathbf{d}$:

$$\mathbf{r}(t) = \mathbf{o} + t\,\mathbf{d}$$



Vector Space $\mathbb{R}^n$

- Scalars $\alpha, \beta, \gamma \in \mathbb{R}$ are written in italic (mostly Greek) font.
- Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are written in bold font, their components are denoted as $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$.
- Linear combinations $\alpha\,\mathbf{x} + \beta\,\mathbf{y}$ stay in vector space and are performed component-wise.

Euclidean Vector Space $\mathbb{R}^n$

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- Inner product (or scalar product, or dot product)

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

  is symmetric and linear in both arguments

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  is symmetric and linear in both arguments
- The induced metric measures geometric length

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

Euclidean Vector Space $\mathbb{R}^n$

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- The induced metric measures geometric length

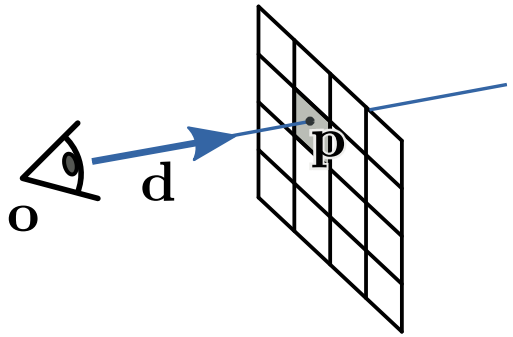
$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

- Scalar product can measure angle θ between two vectors

Primary Rays

Shoot a primary ray through each pixel (x, y) of the image plane:

1. use eye point as origin $\mathbf{o}$
2. compute 3D position $\mathbf{p}$ of pixel (x, y)
3. compute direction $\mathbf{d} = \frac{\mathbf{p}-\mathbf{o}}{\|\mathbf{p}-\mathbf{o}\|}$



Ray Intersection

Ray-Object Intersections

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- Which surface types or surface primitives do we want to support?

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- Which properties should surface primitives have?

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- Spheres, cylinders, planes, boxes, triangles…

Ray-Object Intersections

- Which surface types or surface primitives do we want to support?
- Which properties should surface primitives have?
- Spheres, cylinders, planes, boxes, triangles…

We will focus on spheres and planes first.

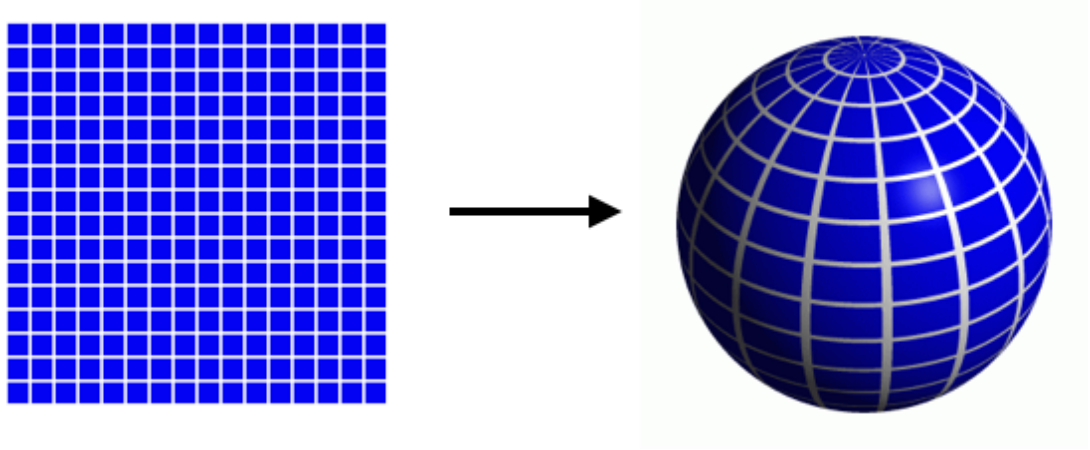
Ray-Sphere Intersection

- How to represent a sphere?
- How to intersect a ray and a sphere?

Unit Sphere

- Unit sphere parameterized by polar coordinates:

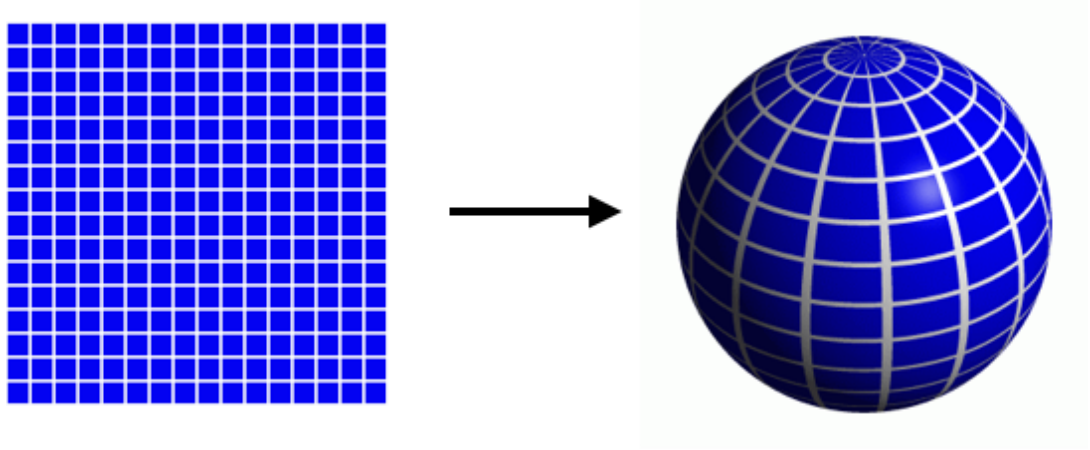
$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \cos\phi\ \cos\theta \\ \sin\phi\ \cos\theta \\ \sin\theta \end{pmatrix}$$



Arbitrary Sphere

- Sphere with center $\mathbf{c}$ and radius r:

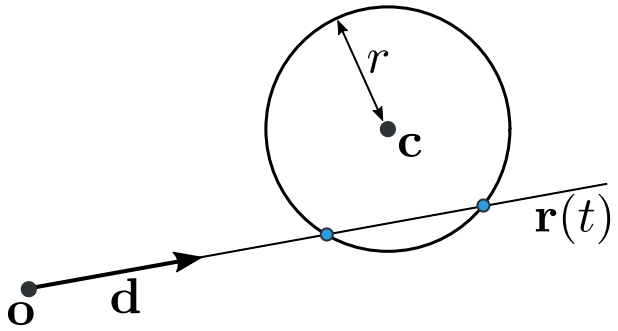
$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} \mapsto \mathbf{c} + r \begin{pmatrix} \cos\phi\ \cos\theta \\ \sin\phi\ \cos\theta \\ \sin\theta \end{pmatrix}$$



Ray-Sphere Intersection

- Ray and sphere have to coincide, leading to this equation

$$\mathbf{o} + t\,\mathbf{d} = \mathbf{c} + r \begin{pmatrix} \cos\phi\ \cos\theta \\ \sin\phi\ \cos\theta \\ \sin\theta \end{pmatrix}$$

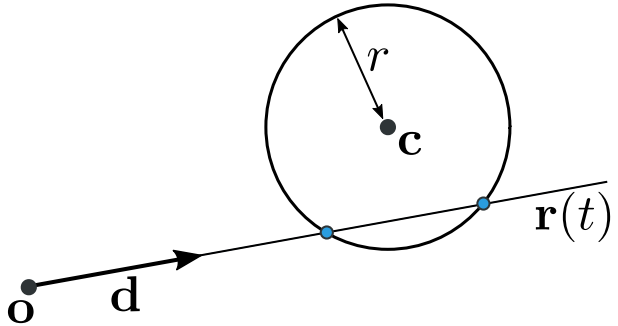


which we have to solve for t, ϕ, and θ.

Ray-Sphere Intersection

- Ray and sphere have to coincide, leading to this equation

$$\mathbf{o} + \mathrm{t}\,\mathbf{d} = \mathbf{c} + \mathrm{r} \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$



which we have to solve for t, φ and θ.

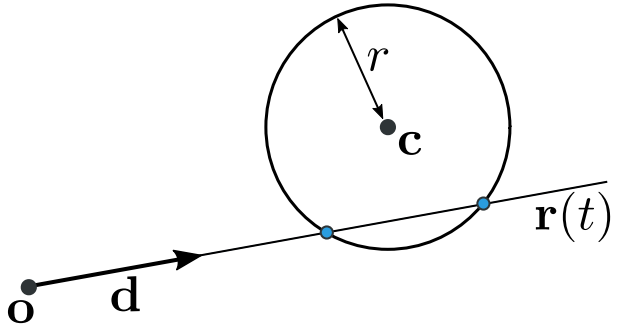
Trigonometric functions make this equation nonlinear and complicated!

Ray-Sphere Intersection, 2nd try

- Let us use the *implicit* representation of a sphere with center $\mathbf{c}$ and radius r:

$$\|\mathbf{x} - \mathbf{c}\| - r = 0$$

This equation is satisfied by all points $\mathbf{x} \in \mathbb{R}^3$ that lie on the sphere.

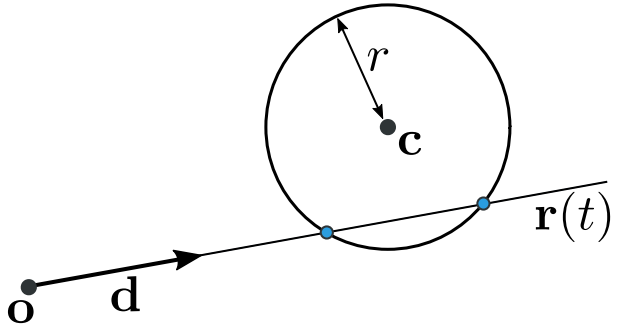


Ray-Sphere Intersection, 2nd try

- Let us use the *implicit* representation of a sphere with center $\mathbf{c}$ and radius r:

$$\|\mathbf{x} - \mathbf{c}\| - \mathrm{r} = 0$$

This equation is satisfied by all points $\mathbf{x} \in \mathbb{R}^3$ that lie on the sphere.



$$\|o + td - c\| = r$$

$$\Leftrightarrow \|o + td - c\|^2 = r^2$$

$$\Leftrightarrow \|td + (o-c)\|^2 = r^2$$

$$\Leftrightarrow (td + (o-c))^T (td + (o-c)) = r^2$$

$$\Leftrightarrow \underbrace{t^2 d^T d}_{a} + \underbrace{2t\, d^T (o-c)}_{t\,2\,d^T(o-c)} + \underbrace{(o-c)^T (o-c) = r^2}_{(o-c)^T(o-c) - r^2}$$

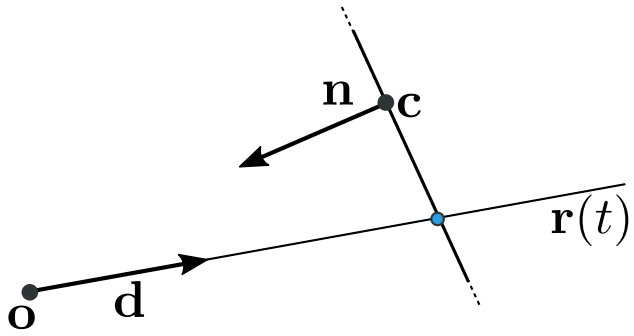
$$\Leftrightarrow at^2 + bt + c = 0$$

Ray-Plane Intersection

- Implicit equation for a plane with normal $\mathbf{n}$ centered at point $\mathbf{c}$ (or with distance d from $(0,0,0)$):

$$\mathbf{n}^\mathsf{T}(\mathbf{x}-\mathbf{c}) = 0$$

$$\mathbf{n}^\mathsf{T}\mathbf{x} - d = 0$$



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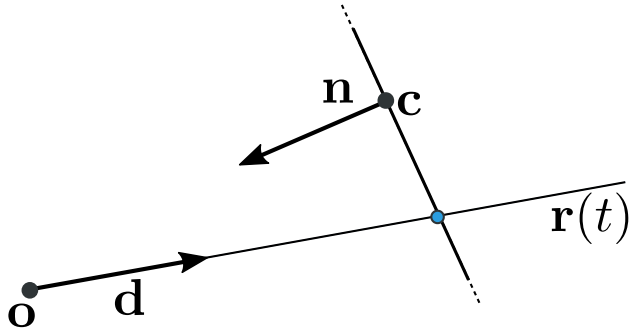


- Insert *explicit* ray equation into *implicit* plane equation

$$\mathbf{n}^{\mathrm{T}}(\mathbf{o} + \mathrm{t}\,\mathbf{d}) - \mathrm{d} = 0$$

and solve for t.

Ray-Triangle Intersection

Ray-Triangle Intersection

- How can we represent a triangle?

Ray-Triangle Intersection

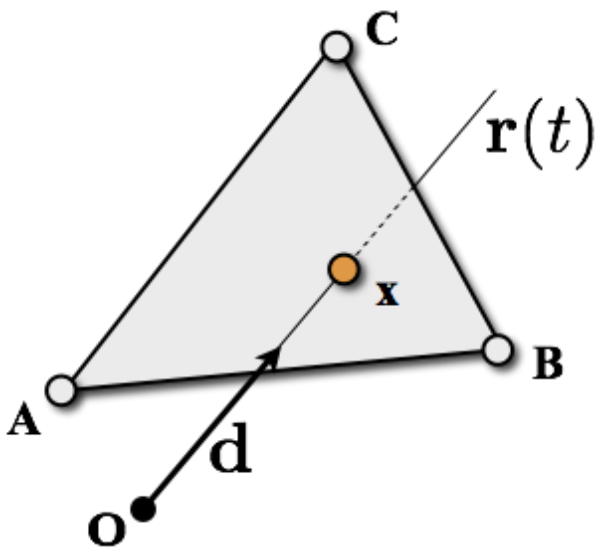
- How can we represent a triangle?
- Should we rather use an *implicit* or *explicit* formulation?

Ray-Triangle Intersection

- How can we represent a triangle?
- Should we rather use an *implicit* or *explicit* formulation?
- Which one can be intersected with a ray more efficiently?

Ray-Triangle Intersection (*implicit version*)

1. Construct triangle's supporting plane

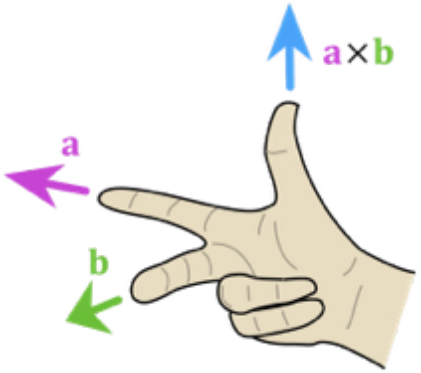


3D Cross Product

Images from Wikipedia

3D Cross Product

- Given two 3D vectors $\mathbf{a}$ and $\mathbf{b}$, finds a vector $\mathbf{c}$ that is orthogonal to them



$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

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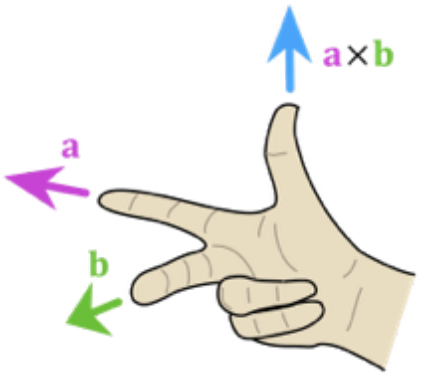

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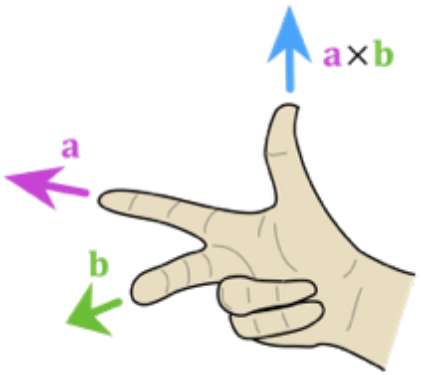

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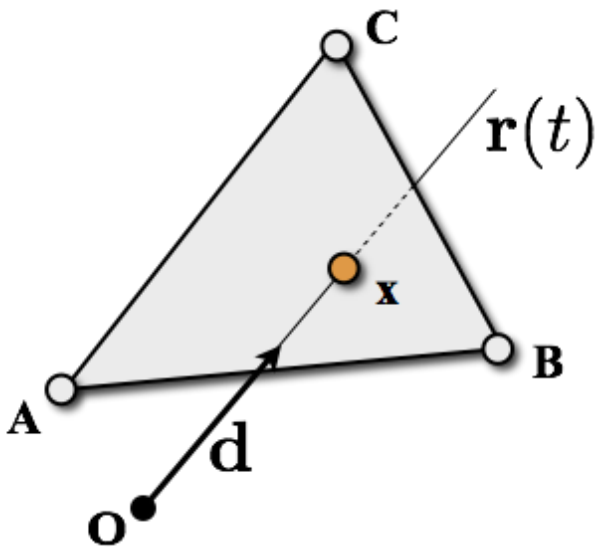
Ray-Triangle Intersection (*implicit version*)

1. Construct triangle's supporting plane
   - Normal $\mathbf{n} = \frac{(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})}{\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\|}$
   - Distance from origin $d = \mathbf{n}^\mathsf{T}\mathbf{A} = \mathbf{n}^\mathsf{T}\mathbf{B} = \mathbf{n}^\mathsf{T}\mathbf{C}$
2. Intersect ray with triangle's plane
   - Solve $(\mathbf{o} + t\mathbf{d})^\mathsf{T}\mathbf{n} - d = 0$
3. Is intersection point inside/outside of triangle?
   - Project to 2D coordinates
   - Point-in-polygon test



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   - Distance from origin
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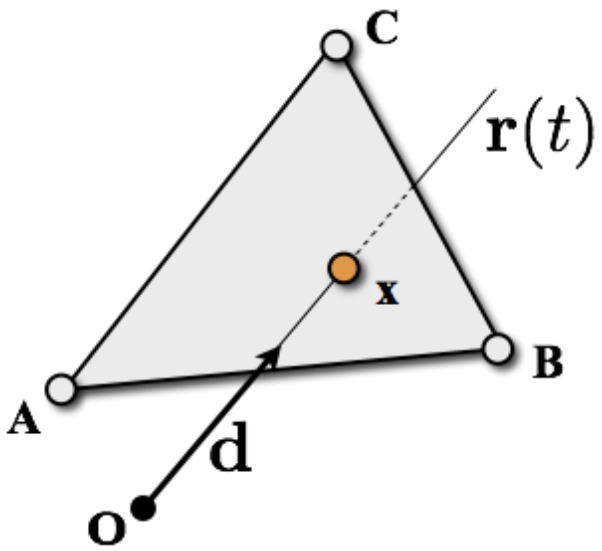

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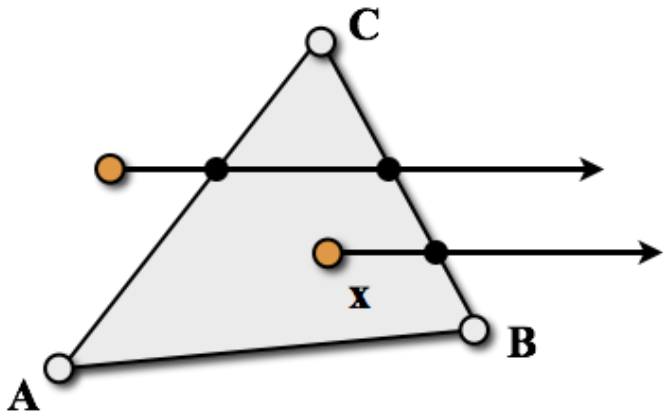
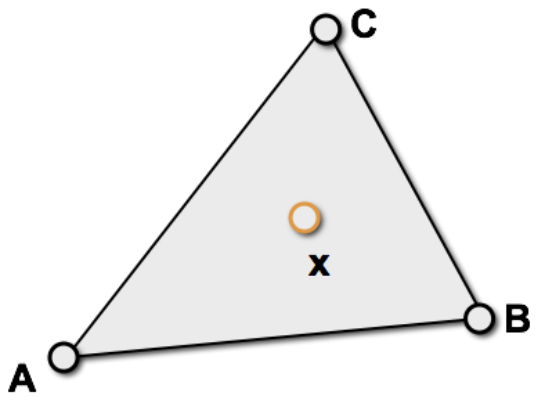


Barycentric Coordinates

Barycentric Coordinates

- Explicit triangle parameterization (first guess)
  - Origin $\mathbf{A}$, edges $\mathbf{U} = \mathbf{B} - \mathbf{A}$ and $\mathbf{V} = \mathbf{C} - \mathbf{A}$
  - Triangle points $\mathbf{x} = \mathbf{A} + u\,\mathbf{U} + v\,\mathbf{V}$ with $u, v \geq 0$ and $u + v \leq 1$



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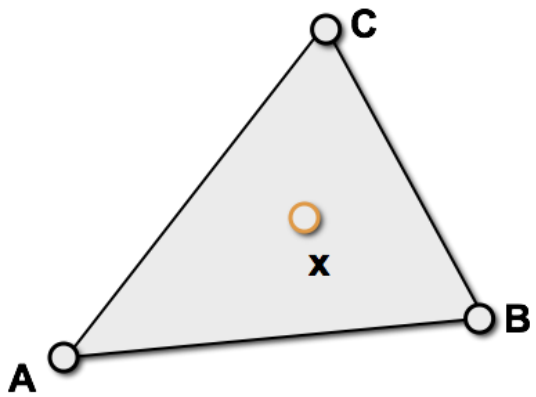
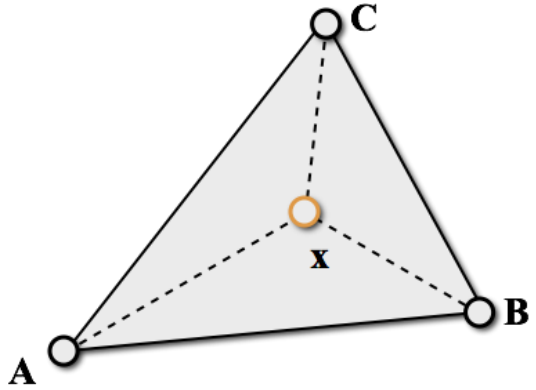


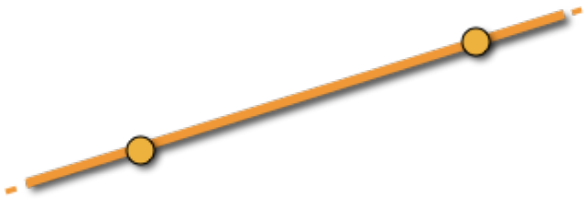
- *Barycentric coordinates* (more useful later)
  - $\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$ with $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$.
  - Equivalent to above (why?)



Barycentric Coordinates

- *Affine combination* of two points

$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B}$$



with $\alpha + \beta = 1$.

Barycentric Coordinates

- *Affine combination* of two points

$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B}$$

with $\alpha + \beta = 1$.

- *Convex combination* of two points

$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B}$$

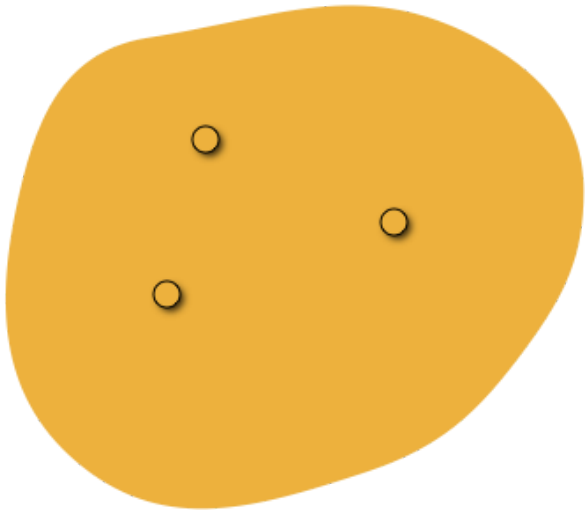
with $\alpha + \beta = 1$ **and** $\alpha, \beta \geq 0$.

Barycentric Coordinates

- *Affine combination* of three points

$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

with $\alpha + \beta + \gamma = 1$.



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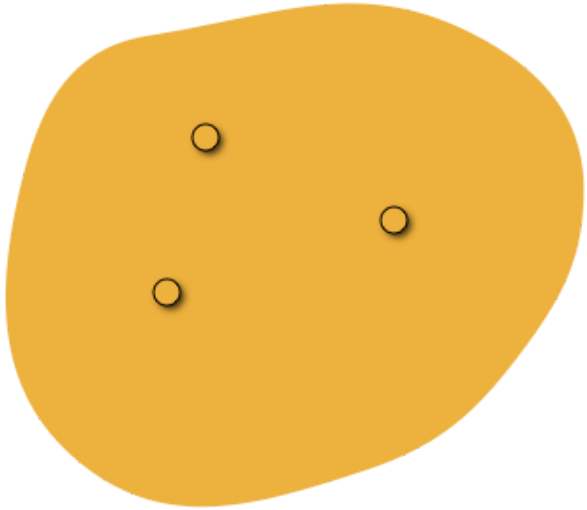
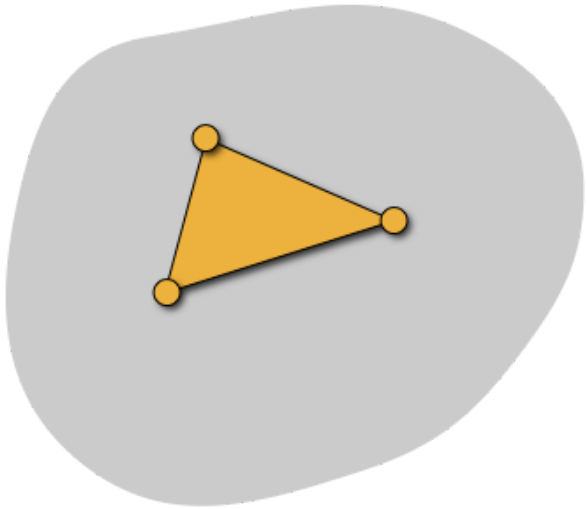


- *Convex combination* of three points

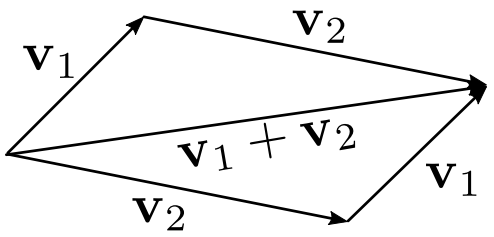
$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

with $\alpha + \beta + \gamma = 1$ **and** $\alpha, \beta, \gamma \geq 0$.

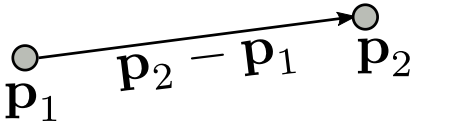


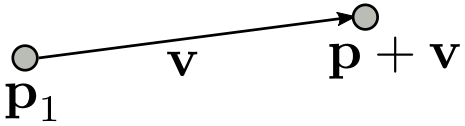
Points vs. Vectors

- Subtle distinction
  - Points denote positions in $\mathbb{R}^3$
  - Vectors denote differences of points



- Meaningful operations
  - vector + vector = vector
  - point - point = vector
  - point + vector = point
  - point + point = ???



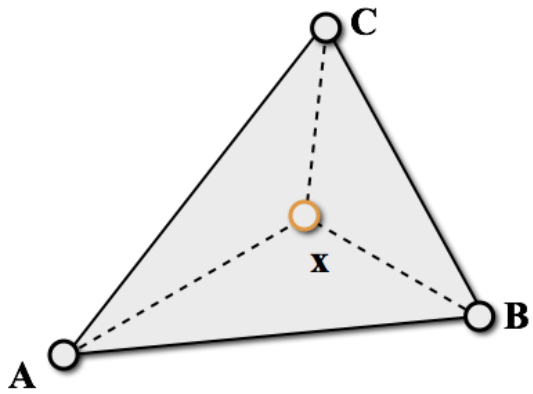


Barycentric Coordinates

Barycentric Coordinates

- Ratio of (signed!) areas (volumes in 3D)

$$\alpha(\mathbf{x}) = \text{area}(\mathbf{x}, \mathbf{B}, \mathbf{C}) / \text{area}(\mathbf{A}, \mathbf{B}, \mathbf{C})$$
$$\beta(\mathbf{x}) = \text{area}(\mathbf{A}, \mathbf{x}, \mathbf{C}) / \text{area}(\mathbf{A}, \mathbf{B}, \mathbf{C})$$
$$\gamma(\mathbf{x}) = \text{area}(\mathbf{A}, \mathbf{B}, \mathbf{x}) / \text{area}(\mathbf{A}, \mathbf{B}, \mathbf{C})$$



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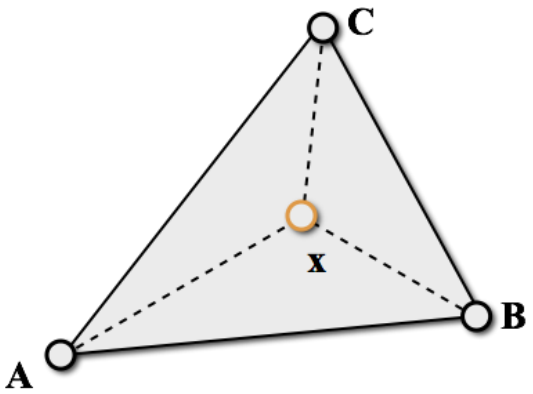
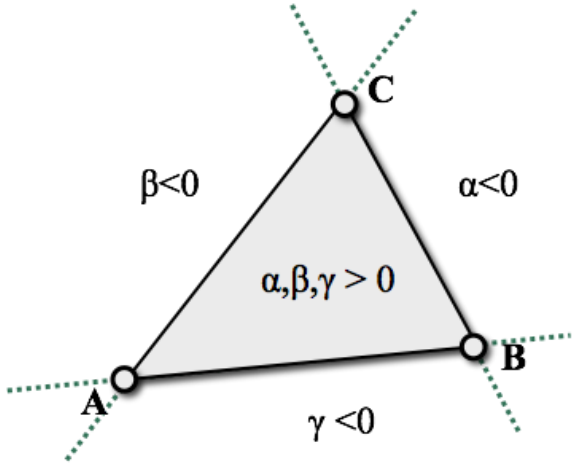


- Gives the required inside/outside information!

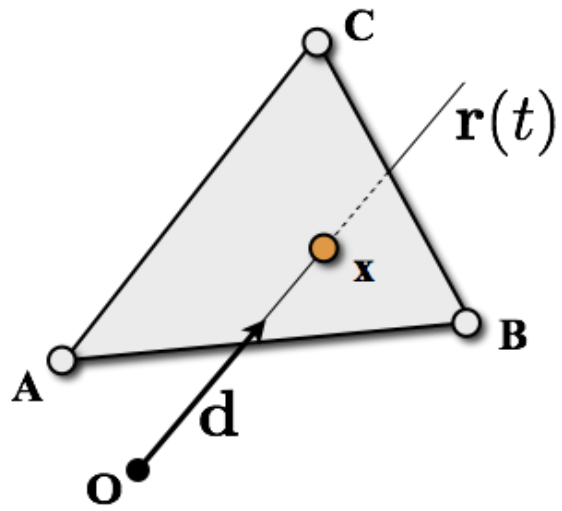


Ray-Triangle Intersection (explicit)

- Ray and triangle have to coincide

$$\mathbf{o} + t\,\mathbf{d} \;=\; \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

Four unknowns, but only three equations!



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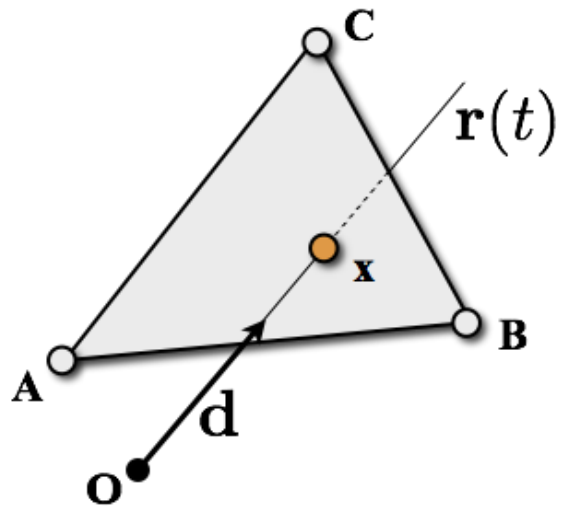

- Exploit condition $\alpha + \beta + \gamma = 1$ to eleminate α

$$\mathbf{o} + t\,\mathbf{d} \;=\; (1 - \beta - \gamma)\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

Solve 3×3 linear system (Cramer's rule)

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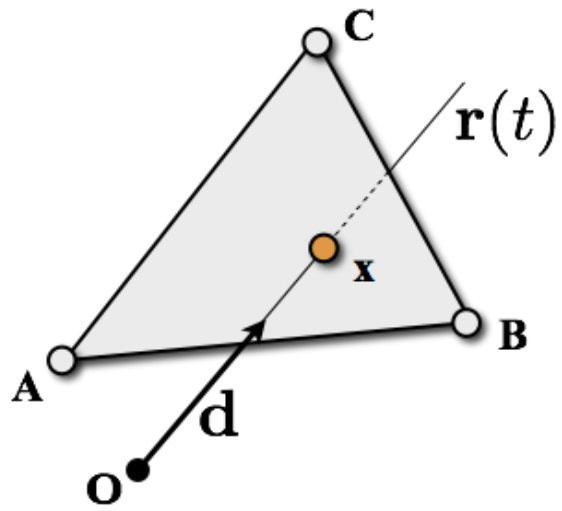

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Other Primitives

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- Cylinder (exercise)

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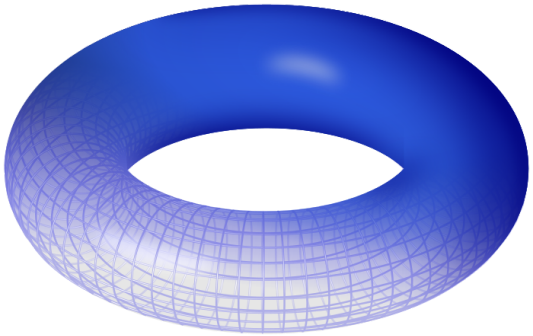
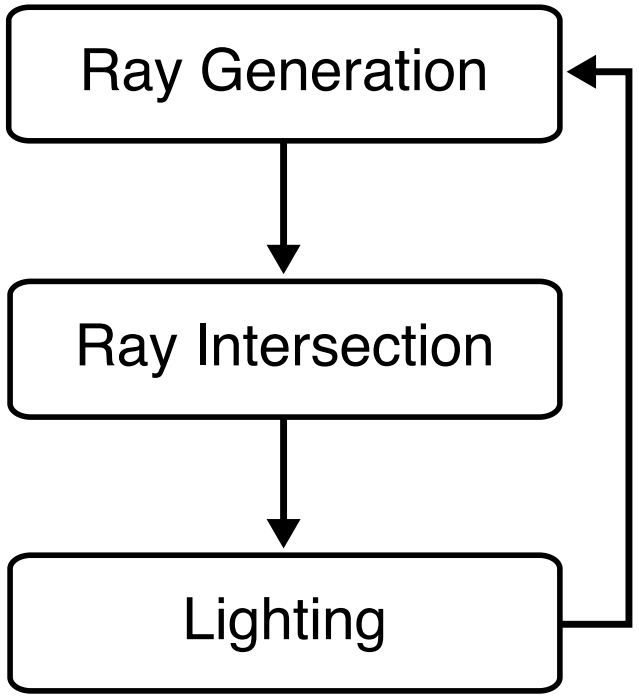
$$(\mathrm{R} - \sqrt{\mathbf{x}^2 + \mathbf{y}^2})^2 + \mathbf{z}^2 = \mathrm{r}^2$$

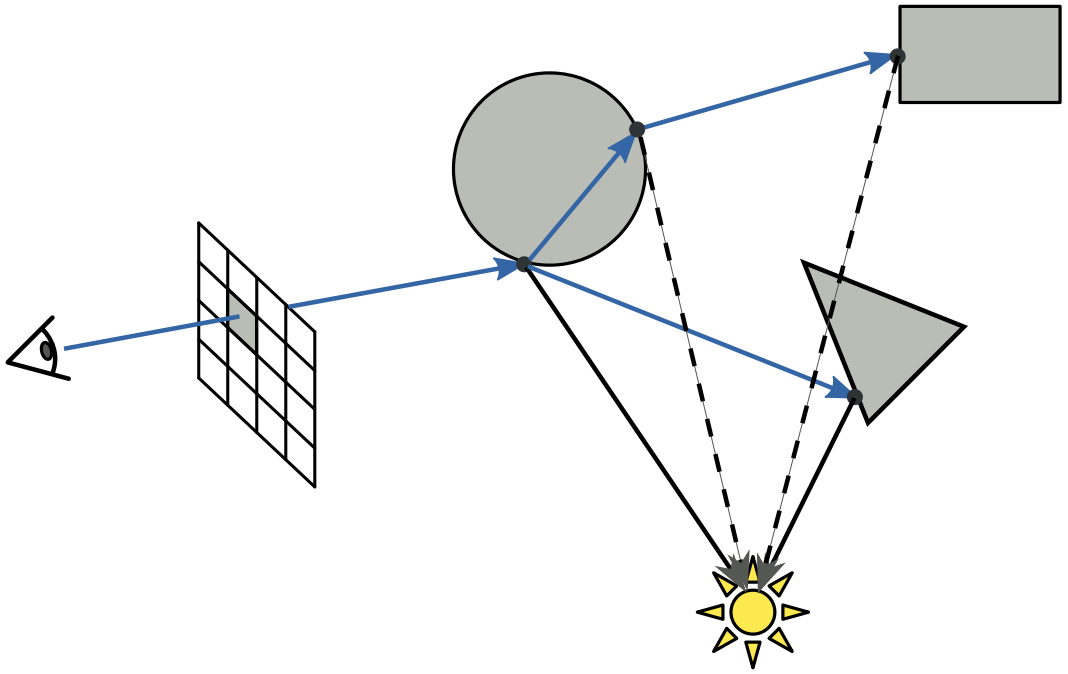


Wikipedia

Ray Tracing Pipeline

- We understood **ray generation** and **ray intersections**. Next week we'll talk about **lighting**.





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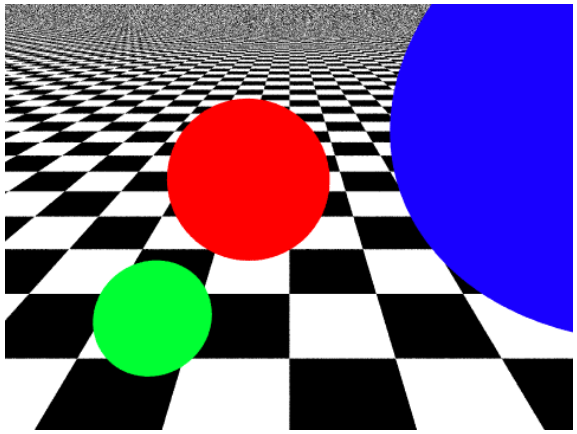
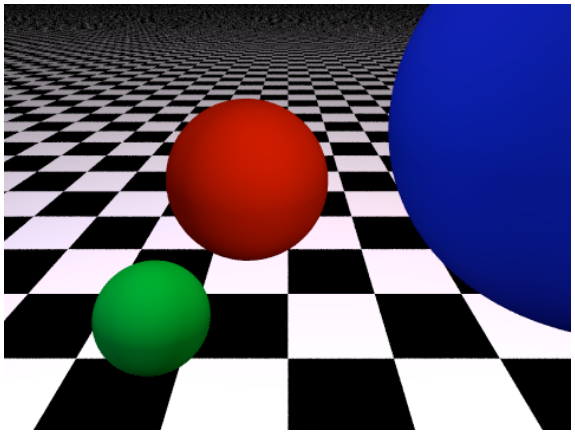
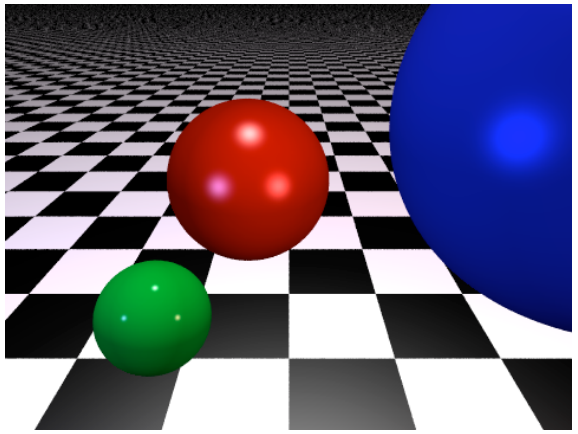


color only



+diffuse

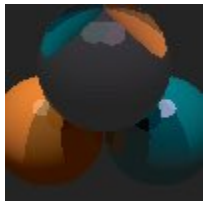


+specular

Is it difficult to code?

Is it difficult to code?

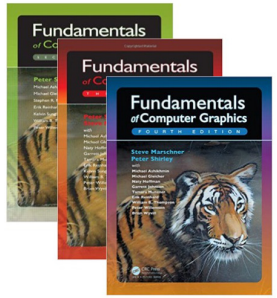
- Paul Heckbert's Minimal Ray Tracer
  (was printed on the back of his business card)



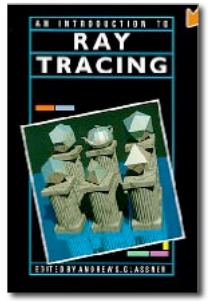
```
typedef struct{double x,y,z}vec;vec U,black,amb={.02,.02,.02};struct sphere{
vec cen,color;double rad,kd,ks,kt,kl,ir}*s,*best,sph[]={0.,6.,.5,1.,1.,1.,.9,
.05,.2,.85,0.,1.7,-1.,8.,-.5,1.,.5,.2,1.,.7,.3,0.,.05,1.2,1.,8.,-.5,.1,.8,.8,
1.,.3,.7,0.,0.,1.2,3.,-6.,15.,1.,.8,1.,7.,0.,0.,0.,.6,1.5,-3.,-3.,12.,.8,1.,
1.,5.,0.,0.,0.,.5,1.5,};yx;double u,b,tmin,sqrt(),tan();double vdot(A,B)vec A
,B;{return A.x*B.x+A.y*B.y+A.z*B.z;}vec vcomb(a,A,B)double a;vec A,B;{B.x+=a*
A.x;B.y+=a*A.y;B.z+=a*A.z;return B;}vec vunit(A)vec A;{return vcomb(1./sqrt(
vdot(A,A)),A,black);}struct sphere*intersect(P,D)vec P,D;{best=0;tmin=1e30;s=
sph+5;while(s-->sph)b=vdot(D,U=vcomb(-1.,P,s->cen)),u=b*b-vdot(U,U)+s->rad*s
->rad,u=u>0?sqrt(u):1e31,u=b-u>1e-7?b-u:b+u,tmin=u>=1e-7&&u<tmin?best=s,u:
tmin;return best;}vec trace(level,P,D)vec P,D;{double d,eta,e;vec N,color;
struct sphere*s,*l;if(!level--)return black;if(s=intersect(P,D));else return
amb;color=amb;eta=s->ir;d= -vdot(D,N=vunit(vcomb(-1.,P=vcomb(tmin,D,P),s->cen
)));if(d<0)N=vcomb(-1.,N,black),eta=1/eta,d= -d;l=sph+5;while(l-->sph)if((e=l
->kl*vdot(N,U=vunit(vcomb(-1.,P,l->cen))))>0&&intersect(P,U)==l)color=vcomb(e
,l->color,color);U=s->color;color.x*=U.x;color.y*=U.y;color.z*=U.z;e=1-eta*
eta*(1-d*d);return vcomb(s->kt,e>0?trace(level,P,vcomb(eta,D,vcomb(eta*d-sqrt
(e),N,black))):black,vcomb(s->ks,trace(level,P,vcomb(2*d,N,D)),vcomb(s->kd,
color,vcomb(s->kl,U,black))));}main(){printf("%d %d\n",32,32);while(yx<32*32)
U.x=yx%32-32/2,U.z=32/2-yx++/32,U.y=32/2/tan(25/114.5915590261),U=vcomb(255.,
trace(3,black,vunit(U)),black),printf("%.0f %.0f %.0f\n",U);}/*minray!*/
```

Literature

- Shirley et al.: *Fundamentals of Computer Graphics*, 3rd Edition, AK Peters, 2009.
  - Chapter 4



- Glassner: *An Introduction to Ray Tracing*, Academic Press, 1989.
  - Chapters 2 & 4



- Pharr, Humphreys: *Physically Based Rendering*, Morgan Kaufmann, 2004.
  - Chapters 1-3