Lagrangian techniques and on-axis rotational transform
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Vacuum field energy for coil design

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Contents

Outline

- Second variation of magnetic field action enables to recover Mercier's formula.
- Second variation - discrete approach - ι as an eigenvalue problem.
- Coil design - introducing a new penalty: Vacuum field energy $\mathcal{E}$.
- Issues encountered in the coil-design problem.

1. Magnetic field action yields on-axis rotational transform

Magnetic field line action

- **Magnetic field line action**: Provided a closed curve $\mathcal{C}$ and $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\mathcal{S} := \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}. \tag{1}$$

- The first variation with respect to a change in the curve geometry $\delta \mathbf{x}$, is

$$\delta \mathcal{S} = \int_{\mathcal{C}} \mathbf{x}' \times \mathbf{B} \cdot \delta \mathbf{x} dl. \tag{2}$$

- Stationary integral curves are tangential to the magnetic field.
- To recover the on-axis ι, the **second** variation of the magnetic action $\delta^2 \mathcal{S}$ is needed.

Second variation yields on-axis ι

- Second variation:

$$\delta^2\mathcal{S}[\delta\mathbf{x}] = \oint_{\mathcal{C}} \delta(\mathbf{x}' \times \mathbf{B} \cdot \delta\mathbf{x}) d\ell = \oint_{\mathcal{C}} d\ell \; \delta\mathbf{x} \cdot (\delta\mathbf{x}' \times \mathbf{B} + \mathbf{x}' \times \delta\mathbf{B})$$

- In tensor form:

$$\delta^2\mathcal{S} = \oint d\ell \; \delta\mathbf{x}^i \frac{\delta^2 S}{\delta\mathbf{x}^i \delta\mathbf{x}^j} \delta\mathbf{x}^j \tag{3}$$

where,

$$\frac{\delta^2\mathcal{S}}{\delta\mathbf{x}^i \delta\mathbf{x}^j} = \epsilon_{ijk}\mathbf{B}^k + \epsilon_{imk}\mathbf{x}'^m \partial_j \mathbf{B}^k \tag{4}$$

Second variation yields on-axis ι

- Defining the operator $\overline{\overline{\mathcal{M}}}$:

$$\overline{\overline{\mathcal{M}}} \equiv \frac{\delta^2 \mathcal{S}}{\delta \mathbf{x} \delta \mathbf{x}} = -\left(\mathbb{I} \times \mathbf{B}\right) \frac{d}{d\ell} + \mathbf{x} \times \left(\nabla \mathbf{B}\right)^{\mathsf{T}}. \tag{5}$$

- Null eigenvectors are of importance to recover ι:

$$\overline{\overline{\mathcal{M}}} \cdot \mathbf{v} = \mathbf{0}. \tag{6}$$

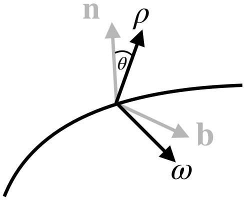
- For $\mathbf{v}$ to be nontrivial we must have

$$\det\left(\overline{\overline{\mathcal{M}}}\right) = 0. \tag{7}$$

The second variation in Mercier coordinates

- Define Mercier coordinates:

  - Start from Frenet - Serret frame$(\mathbf{t}, \mathbf{n}, \mathbf{b})$
  - Rotation of θ: $(\mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\omega})$
  - $(\rho, \omega = \theta + \int \tau d\ell, \ell)$ is an orthogonal coordinate system



- Useful identities:

$$\mathbb{I} = \boldsymbol{\rho}\boldsymbol{\rho} + \boldsymbol{\omega}\boldsymbol{\omega} + \mathbf{t}\mathbf{t}, \quad \nabla = \boldsymbol{\rho}\partial_\rho + \frac{\boldsymbol{\omega}}{\rho}\partial_\omega + \frac{\mathbf{t}}{h}\partial_\ell$$

$$\boldsymbol{\rho}_{,\omega} = \boldsymbol{\omega}, \quad \boldsymbol{\rho}_{,\ell} = -\mathbf{t}\ \kappa\cos\theta, \quad \boldsymbol{\omega}_{,\omega} = -\boldsymbol{\rho}, \quad \boldsymbol{\omega}_{,\ell} = \mathbf{t}\ \kappa\sin\theta$$

$$\nabla\mathbf{t} = \mathbf{t}\mathbf{n}\frac{\kappa}{h}, \quad \nabla\boldsymbol{\rho} = \boldsymbol{\omega}\boldsymbol{\omega}\frac{1}{\rho} - \mathbf{t}\mathbf{t}\frac{\kappa\cos\theta}{h}, \quad \nabla\boldsymbol{\omega} = -\boldsymbol{\omega}\boldsymbol{\rho}\frac{1}{\rho} + \mathbf{t}\mathbf{t}\frac{\kappa\sin\theta}{h}$$

The second variation in Mercier coordinates

- Mercier coordinate system has the metric

$$ds^2 = d\rho^2 + \rho^2 d\omega^2 + h^2 d\ell^2, \quad h = 1 - \kappa\rho\cos\theta, \tag{8}$$

  with κ the curvature.
- For the behavior near the magnetic axis: expansion in the parameter $\kappa\rho \ll 1$. To evaluate $\overline{\overline{\mathcal{M}}}$ to lowest order in $\kappa\rho$, we need to expand the magnetic field up to first order.
- We assume

$$\mathbf{B} = B_0(\ell)\mathbf{t} + \rho\mathbf{B}_1, \quad \mathbf{B}_1 = \left(B_1^\rho \boldsymbol{\rho} + B_1^\omega \boldsymbol{\omega} + B_1^t \mathbf{t}\right). \tag{9}$$

The second variation in Mercier coordinates

- Finally, $\overline{\overline{\mathcal{M}}}$ as given by Eq.(5) simplifies to

$$\begin{aligned}
\overline{\overline{\mathcal{M}}} &\equiv \overline{\overline{\mathcal{M}_1}}\frac{d}{d\ell} + \overline{\overline{\mathcal{M}_2}} \\
\overline{\overline{\mathcal{M}_1}} &= (\boldsymbol{\rho}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\rho})B_0 \\
\overline{\overline{\mathcal{M}_2}} &= \kappa B_0 \mathbf{b}\mathbf{t} + \left(B_1^\rho \boldsymbol{\omega} - B_1^\omega \boldsymbol{\rho}\right)\boldsymbol{\rho} \\
&\quad + \boldsymbol{\omega}\boldsymbol{\omega}\left(\partial_\omega B_1^\rho - B_1^\omega\right) - \boldsymbol{\omega}\boldsymbol{\rho}\left(\partial_\omega B_1^\omega + B_1^\rho\right)
\end{aligned} \tag{10}$$

- Remains to enforce constraints from MHD:

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \mathbf{J} = \nabla \times \mathbf{B} = J_0(\ell)\mathbf{t}.$$

MHD constraints

- Using dyiadic algebra, we can express $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{B}$ from $\nabla \mathbf{B}$.
- Leads to the following components for $\mathbf{B}_1$:

$$B_1^t = \kappa B_0 \cos\theta, \quad B_1^\rho = -\frac{1}{2}(B_0' + \partial_\omega b_1), \quad B_1^\omega = \frac{1}{2}J_0 + b_1,$$

- b_1 satisfies Laplace $(\partial_\omega^2 + 4)b_1 = 0$, and from [MEotEC87] it can be expressed as

$$\begin{aligned} b_1 &= b_{c2}(\ell)\cos(2u) + b_{s2}(\ell)\sin(2u), \\ u &= \theta + \delta(\ell) = \omega - \int \tau d\ell + \delta(\ell) \end{aligned} \tag{11}$$

- We get conditions on b_{c2} and b_{s2}, depending on $\delta(\ell)$ and $\eta(\ell)$ (eccentricity and rotation of flux surfaces).

MHD constraints

- Now let us solve for null eigenvector $\mathbf{v} = v^t \mathbf{t} + v^\rho \boldsymbol{\rho} + v^\omega \boldsymbol{\omega}$ of $\overline{\overline{\mathcal{M}}}$ such that $\overline{\overline{\mathcal{M}}} \cdot \mathbf{v} = 0$
- Developing $\overline{\overline{\mathcal{M}}} \cdot \mathbf{v} = 0$ using Eq.(10): only $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ components with derivatives of v_ρ, v_ω w.r.t. ℓ.
  $\implies$ absorb v_t by redefining v_ρ and v_ω
- We set $v_t = 0$.
- Leads to the system:

$$\begin{cases} v_\omega' + \left(\dfrac{B_0'}{2B_0} - \dfrac{\partial_\omega b_1}{2B_0}\right) v_\omega - \left(\dfrac{J_0/2}{B_0} + \dfrac{b_1}{B_0}\right) v_\rho = 0 \\ v_\rho' + \left(\dfrac{B_0'}{2B_0} + \dfrac{\partial_\omega b_1}{2B_0}\right) v_\omega + \left(\dfrac{J_0/2}{B_0} - \dfrac{b_1}{B_0}\right) v_\omega = 0 \end{cases} \tag{12}$$

MHD constraints

- Therefore Eq.(12) can be written in matrix form:

$$\frac{d\tilde{\mathbf{v}}}{d\ell} = A(\ell)\tilde{\mathbf{v}}, \quad \tilde{\mathbf{v}} = (v_\rho, v_\omega). \tag{13}$$

- Floquet theory:

$$\tilde{\mathbf{v}} = U(\ell)e^{C\ell/L}, \tag{14}$$

- C is constant Hamiltonian matrix whose eigenvalues are the Floquet exponents, and are of the form $\nu = \pm i\nu$, $\nu \in \mathbb{R}$ near elliptic axis.

Other approach: use u-harmonics order

- Another approach: b_1 has only 2nd order harmonics in u: seek for solution with 1st order harmonics for $\tilde{\mathbf{v}}$
- We show that $\exists$ solution with

$$\begin{pmatrix} v_\rho(\ell) \\ v_\omega(\ell) \end{pmatrix} = \begin{pmatrix} v_{\rho c}(\ell) \\ v_{\omega c}(\ell) \end{pmatrix} \cos u + \begin{pmatrix} v_{\rho s}(\ell) \\ v_{\omega s}(\ell) \end{pmatrix} \sin u$$
$$u = \omega - \int \tau d\ell + \delta(\ell). \tag{15}$$

- Equating $\sin(u)$ and $\cos(u)$ terms, the v_ω equation reads

$$v'_{\omega c} + \left(\frac{B'_0}{2B_0} - \frac{\eta'}{2}\right) v_{\omega c} + \frac{e^{+\eta}}{\cosh(\eta)} \left(\frac{J_0/2}{B_0} - \tau + \delta'\right) v_{\omega s} = 0$$
$$v'_{\omega s} + \left(\frac{B'_0}{2B_0} + \frac{\eta'}{2}\right) v_{\omega s} - \frac{e^{-\eta}}{\cosh(\eta)} \left(\frac{J_0/2}{B_0} - \tau + \delta'\right) v_{\omega c} = 0.$$

Other approach: use u-harmonics order

- Further reduce the system introducing new variables (X, Y):

$$v_{\omega c} = \frac{1}{\sqrt{B_0'}} e^{+\eta/2} X(\ell), \quad v_{\omega s} = \frac{1}{\sqrt{B_0'}} e^{-\eta/2} Y(\ell) \tag{16}$$

- It becomes:

$$X' + \Omega_0(\ell) Y = 0, \quad Y' - \Omega_0(\ell) X = 0, \quad \Omega_0(\ell) = \frac{\frac{J_0/2}{B_0} - \tau + \delta'}{2\cosh\eta}$$

- Using complex variable $Z = X + iY$: system reduces to complex ODE:

$$Z' - i\Omega_0 Z = 0 \quad \Rightarrow \quad Z(\ell) = Z_0 \exp \int_0^\ell i\Omega_0(s) ds \tag{17}$$

Recover ι

- Comparing Eq.(17) and Eq.(14), we identify the Floquet exponent:

$$\nu = \int_0^L \Omega_0(s)ds = \oint \frac{\frac{J_0(s)/2}{B_0(s)} - \tau(s) + \delta'(s)}{2\cosh\eta(s)} ds \tag{18}$$

- Matches Mercier's formula!

Floquet Theory I

- Suppose one needs to solve the following linear system,

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{19}$$

  where $A \in \mathcal{M}_{n\times n}$ is periodic in t with period T.
- Considering n solutions of Eq.(19) $\{\mathbf{x}_1, \dots \mathbf{x}_n\}$: rewrite the system in the form of a matrix equation, defining the so-called fundamental matrix X by grouping the $\mathbf{x}_i$:

$$X(t, t_0) := \Big(\mathbf{x}_1; \mathbf{x}_2; \dots ; \mathbf{x}_n\Big) \tag{20}$$

$$\frac{d}{dt}X(t, t_0) = A(t)X(t, t_0), \quad X(t_0, t_0) = \mathbb{I}. \tag{21}$$

Floquet Theory II

- Fundamental matrix:

$$\det\left(X(t,t_0)\right) = \prod_{i=1}^{n} e^{\lambda_i T}, \tag{22}$$

  where the λ_i are the Floquet exponents, which will be shown to be linked with the rotational transform. (Proof of thm. in [Mei07]). Note: $\exp(\lambda_i T)$ are called Floquet multipliers.
- Therefore Floquet multipliers are the eigenvalues of the fundamental matrix of the linear system Eq.(19).
- Main result: **Theorem (Floquet-Lyapunov).** The fundamental matrix X solution of the system Eq.(21) is of the form

$$X(t,0) = P(t)e^{tB} \tag{23}$$

  where the matrix P is symplectic and T-periodic, and B is a constant hamiltonian matrix.

Discrete formalism: towards ι

- We follow MacKay and Meiss [MM83] for 1D Lagrangian.
- Discretize close curve $\mathcal{C}$ with $n-1$ curves $\mathcal{C}_i$

$$\begin{aligned} \mathcal{S} &= \sum_{i=1}^{n-1} \int_{\mathcal{C}_i} \mathbf{A} \cdot d\mathbf{l}, \\ &= \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \end{aligned} \tag{24}$$

- Notation:

$$\begin{aligned} \mathbf{S}_1^{[i,i+1]} &= \nabla_{\mathbf{x}_i} S^{[i,i+1]} := \nabla_{\mathbf{x}_i} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \\ \mathbf{S}_2^{[i,i+1]} &= \nabla_{\mathbf{x}_{i+1}} S^{[i,i+1]} := \nabla_{\mathbf{x}_{i+1}} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \end{aligned} \tag{25}$$

Discrete formalism: towards ι

- Stationarity of the action for extremal curve

$$\begin{aligned} \delta S[\delta \mathbf{x}_i] = \left[\nabla_{\mathbf{x}_i} S^{[i-1,i]} + \nabla_{\mathbf{x}_i} S^{[i,i+1]}\right] \cdot \delta \mathbf{x}_i = 0, \\ \iff \mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]} = \mathbf{0}. \end{aligned} \tag{26}$$

- Total derivative of previous result

$$\begin{aligned} \delta\left(\mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]}\right) = & \overline{\overline{S}}_{12}^{[i-1,i]} \cdot \delta \mathbf{x}_{i-1} + \overline{\overline{S}}_{21}^{[i,i+1]} \cdot \delta \mathbf{x}_{i+1} \\ & \left(\overline{\overline{S}}_{22}^{[i-1,i]} + \overline{\overline{S}}_{11}^{[i,i+1]}\right) \cdot \delta \mathbf{x}_i = \mathbf{0}, \\ & 1 \le i \le n-1. \end{aligned} \tag{27}$$

- Leads to tridiagonal-block matrix form.

Discrete formalism - ι as an eigenvalue problem

$$\begin{pmatrix} \left(\overline{\overline{S}}_{22}^{[01]} + \overline{\overline{S}}_{11}^{[12]}\right) & \overline{\overline{S}}_{12}^{[12]} & & & \lambda^{-1}\overline{\overline{S}}_{21}^{[01]} \\ \overline{\overline{S}}_{21}^{[12]} & \left(\overline{\overline{S}}_{22}^{[12]} + \overline{\overline{S}}_{11}^{[23]}\right) & \overline{\overline{S}}_{12}^{[23]} & & \\ & \overline{\overline{S}}_{21}^{[23]} & \ddots & \ddots & \\ & & \ddots & \ddots & \overline{\overline{S}}_{12}^{[n-1,n]} \\ \lambda\overline{\overline{S}}_{12}^{[n,n+1]} & & & \overline{\overline{S}}_{21}^{[n-1,n]} & \left(\overline{\overline{S}}_{22}^{[n-1,n]} + \overline{\overline{S}}_{11}^{[n,n+1]}\right) \end{pmatrix} \cdot \begin{pmatrix} \delta\mathbf{x}_1^T \\ \delta\mathbf{x}_2^T \\ \vdots \\ \\ \vdots \\ \delta\mathbf{x}_{n-1}^T \\ \delta\mathbf{x}_n^T \end{pmatrix} = \mathbf{0}$$

- Recover the same form (discrete this time) as previously for our operator equation:

$$\iff \overline{\overline{M}}(\lambda) \cdot \delta\mathbf{x} = \mathbf{0}, \tag{28}$$

Discrete formalism - ι as an eigenvalue problem

- For such a matrix form, exists formula for determinant [Mol08]

$$\det\Big(M(\lambda)\Big) = \frac{(-1)^{mn}}{(-\lambda)^m}\det\Big(T_S - \lambda\mathcal{I}_6\Big)\det\Big(\prod_{i=1}^{n}\overline{\overline{S}}_{12}[i,i+1]\Big)$$

- Requires to define the so-called transfer matrix $T_{\mathcal{S}}$

$$T_{\mathcal{S}} = \prod_{i=1}^{n}\begin{pmatrix} -\overline{\overline{S}}_{12}^{-1\,[i,i+1]}(\overline{\overline{S}}_{22}^{[i-1,i]} + \overline{\overline{S}}_{11}^{[i,i+1]}) & -\overline{\overline{S}}_{12}^{-1\,[i,i+1]}\overline{\overline{S}}_{12}^{[i-1,i]} \\ \mathcal{I}_3 & 0 \end{pmatrix}.$$

Discrete formalism - a necessary condition

- MacKay and Meiss use what they call a convexity condition on their Lagrangian: $-L_{12} > 0$, which is similar to the so-called *true angle-dynamics* from [HS14].
- We propose a generalized criterion: imposing that the upper diagonal matrices are negative definite

$$\mathbf{x}^T \overline{\overline{S}}_{12}^{[i,i+1]} \mathbf{x} < 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad 1 \leq i \leq n. \tag{29}$$

- Finally, $\det\left(M(\lambda)\right) = 0$ yields the following eigenvalue equation:

$$(-\lambda)^{-m} \cdot \det\left(T_S - \lambda \mathcal{I}\right) = 0. \tag{30}$$

- Exponents λ are eigenvalues of transfer matrix.

Discrete formalism - an example: piecewise linears

- One way to discretize: use **segments**

$$\begin{aligned} \mathcal{S} &= \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}), \\ S(\mathbf{x}_i, \mathbf{x}_{i+1}) &= \int_0^1 \mathbf{A}\Big(\zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i\Big) \cdot (\mathbf{x}_{i+1} - \mathbf{x}_i) d\zeta \\ &= \int_0^1 \mathbf{A}\big(\mathbf{v}(\mathbf{x}(\zeta))\big) \cdot \mathbf{u} d\zeta, \end{aligned} \tag{31}$$

with $\mathbf{x}(0) = \mathbf{x}_i$ and $\mathbf{x}(1) = \mathbf{x}_{i+1}$, and the vector field $\mathbf{v}$ defined as follows:

$$\mathbf{v}(\zeta, \mathbf{x}_i, \mathbf{x}_{i+1}) = \zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i \tag{32}$$

Discrete formalism - an example: piecewise linears

- For the transfer matrix $T_{\mathcal{S}}$ to be implemented, need the second derivatives of the action $\overline{\overline{S}}_{ij}$:

$$\overline{\overline{S}}_{12}^{[i-1,i]} = \int_0^1 d\zeta\zeta\Big[(1-\zeta)\nabla_{\mathbf{v}} J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\cdot(\mathbf{x}_i - \mathbf{x}_{i-1}) - J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\Big] + (1-\zeta) J_{\mathbf{A}}(\mathbf{v})$$

$$\overline{\overline{S}}_{22}^{[i-1,i]} = \int_0^1 d\zeta\zeta\Big[\zeta\nabla_{\mathbf{v}} J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\cdot(\mathbf{x}_i - \mathbf{x}_{i-1}) + \big(J_{\mathbf{A}}(\mathbf{v}) + J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\big)\Big]$$

$$\overline{\overline{S}}_{21}^{[i,i+1]} = \int_0^1 d\zeta(1-\zeta)\Big[\zeta\nabla_{\mathbf{v}} J_{\mathbf{A}}^{T}(\mathbf{v})\cdot(\mathbf{x}_{i+1} - \mathbf{x}_{i}) + J_{\mathbf{A}}^{T}(\mathbf{v})\Big] + \zeta J_{\mathbf{A}}(\mathbf{v})$$

$$\overline{\overline{S}}_{11}^{[i,i+1]} = \int_0^1 d\zeta(1-\zeta)\Big[(1-\zeta)\nabla_{\mathbf{v}} J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\cdot(\mathbf{x}_{i+1} - \mathbf{x}_{i}) - \big(J_{\mathbf{A}}(\mathbf{v}) + J_{\mathbf{A}}^{\mathsf{T}}(\mathbf{v})\big)\Big]$$

Discrete formalism - an example: piecewise linears

- With vector potential $\mathbf{A}$ provided, $T_{\mathcal{S}}$ can be easily implemented numerically.
- Model can be verified.
- Strength of this approach: In the limit $n \uparrow \infty$, even if the dimension of M is infinite, finding the exponents λ still reduces to finding the eigenvalues of a 6×6 matrix.

2. Vacuum field energy for coil design

The filamentary coil description

- Coils are represented as filaments: **0 thickness**
- Coil: curve in 3D space $\mathcal{C} \subset \mathbb{R}^3$ + current $I_C \in \mathbb{R}_+$.
- **Degrees of freedom**: targeted by the algorithm to optimize for particular penalty involving the coils: Fourier modes (e.g. cartesian or cylindrical coordinates) + I_{C_i}.

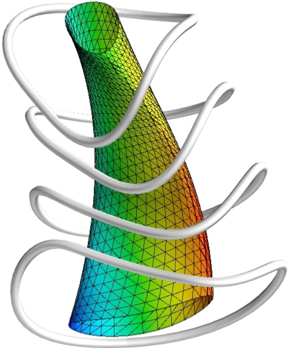


Figure: Filamentary coils for precise QA [LP22] configuration - penalty on $\mathcal{L}$.

Introduction of a new penalty

- $\mathbf{j} \times \mathbf{B}$ force is gradient of energy.
- Our guess: instead of minimizing force: minimize the "potential" that provokes it.
- Introduce the vacuum energy:

$$\mathcal{E} = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} d\mathbf{x}\; B^2 = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} d\mathbf{x}\; \mathbf{j} \cdot \mathbf{A}. \tag{33}$$

- Current constrained on set of coils $\{C_i\}$:

$$\mathcal{E} = \frac{1}{2\mu_0} \sum_{i=1}^{N_C} \oint_{C_i} I_i d\mathbf{l}_i \cdot \mathbf{A}(\mathbf{x}), \tag{34}$$

Introduction of a new penalty

- We can write $\delta\mathcal{E}$ in a shape gradient form:

$$\delta\mathcal{E}[\{\delta\mathbf{x}_i\}] = \sum_i I_i \oint_{C_i} dl \ (\delta\mathbf{x}_i \times \mathbf{t}) \cdot \mathbf{B}. \tag{35}$$

- Enables to express parameters derivatives to be implemented in Simsopt (w.r.t. Fourier modes for example) [LMW+21]

$$\begin{aligned} \frac{\partial\mathcal{E}}{\partial\Omega_k} &= \sum_i I_i \oint_{C_i} dl \ (\mathbf{t} \times \mathbf{B}) \cdot \frac{\partial\mathbf{x}_i}{\partial\Omega_k} \\ &= \sum_i \oint_{C_i} (\mathbf{j}_i \times \mathbf{B}) \cdot \frac{\partial\mathbf{x}_i}{\partial\Omega_k}, \end{aligned} \tag{36}$$

- $\mathbf{B}$ can be expressed from Biot-Savart law

Limitations due to filamentary coils

- **B** is **singular** on the coils. Field generated by a filament and evaluated on the filament itself.
- Biot-Savart:

$$\mathbf{B}(\mathbf{x}) = \sum_{i=1}^{N_C} I_i \frac{\mu_0}{4\pi} \oint_{C_i} \frac{d\mathbf{l} \times (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3}. \tag{37}$$

- The 0-thickness approx. works for objective functions such as integrated curvature, coil to coil separation etc. How can we overcome this singularity problem?

Overcome singularity limitations

- Introduce thickness: probably most relevant solution (see Matt's work)
- Define a *ghost*-curve: displace the filament in an arbitrary direction (e.g. normal) by a distance ϵ to evaluate the field here.
- Obvious limitations to the ghost-curve:
  1. Assumes smooth behavior of the field around the curve.
  2. Depends on behavior of $\mathbf{n}$, $\mathbf{b}$ along the filament.
  3. Need for additional numerical parameter: ϵ, adds arbitrariness to the model.

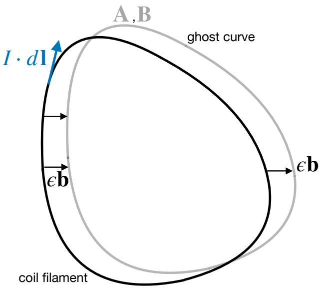


Figure: Filamentary coil curve (black) and ghost curve defined by binormal displacement.

Ghost-curves

- One issue with ghost curves: behavior of $\mathbf{n}$ might be not smooth with abrupt changes in direction
- Solution (?): $\mathcal{E}$ as an average of integral over 4 ghost curves at $\pm\epsilon\mathbf{n}$ and $\pm\epsilon\mathbf{b}$

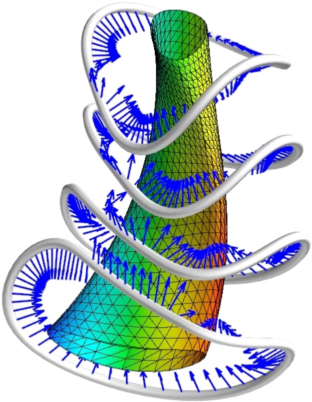


Figure: Same coils as in Fig.(28) with $\mathbf{n}$ plotted in blue.

Results I

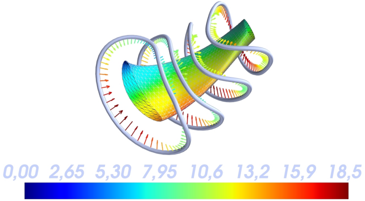
- Goal 1: recover configuration (here precise QA from Landreman-Paul [LP22]) minimizing quadratic flux Φ_2.
- Need penalty to prevent coils length going to ∞.
- Can we replace length penalty by vacuum-energy penalty?

$$\mathcal{F} = \int_S (\mathbf{B} \cdot \mathbf{n})^2 dS + \cancel{\rho \mathcal{L}} + \omega \mathcal{E} \tag{38}$$

- Goal 2: see if coils resulting from penalizing the energy have lower (inter-coil) forces.

Results II

- Penalty on energy not sufficient to constraint the coils lengths: need additional length penalty to prevent the coils from linking.
- The coils shapes are not regular.
- $\max_{\{C_i\}} |\mathbf{j} \times \mathbf{B}|$ is higher.



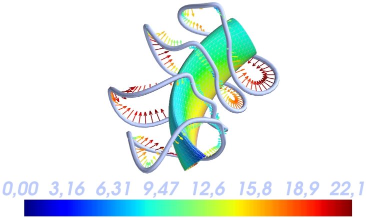


Figure: Up: Φ_2 and $\mathcal{L}$ - Down: same $+\mathcal{E}$

What could cause these "errors"?

- Our intuition could be wrong.
- The implementation approach might not be correct:
  - Ghost curve not a good-enough tool.
  - Simsopt needs the objective function $\mathcal{E}.J()$ which is technically infinite for filamentary coils. Here it scales with ϵ^{-1}, and so do the derivatives. Hence the weight has to be $\propto \epsilon \rightarrow$ huge gap between energy weight and other weights.
  - In general: such discrepancies between variables that are combined are not good.
- The whole implementation of the energy in itself might contain errors.

Conclusion

- The vacuum-field energy still remains a promising quantity to minimize in order to reduce inter-coil forces, as long as we haven't found the proper way to implement it in the more general framework of non-linear optimization for coil-design.
- Probably need to give non-zero thickness to the coils?
- To sum things up: we need to think more!

References I

S. R. Hudson and Y. Suzuki.
Chaotic coordinates for the Large Helical Device.
Physics of Plasmas, 21(10):102505, 10 2014.

Matt Landreman, Bharat Medasani, Florian Wechsung, Andrew Giuliani, Rogerio Jorge, and Caoxiang Zhu.
Simsopt: A flexible framework for stellarator optimization.
Journal of Open Source Software, 6(65):3525, 2021.

Matt Landreman and Elizabeth Paul.
Magnetic fields with precise quasisymmetry for plasma confinement.
Physical Review Letters, 128(3), jan 2022.

J.D. Meiss.
Differential Dynamical Systems.
SIAM, 2007.

References II

C. Mercier, Euratom, and Commission of the European Communities.
Lectures in Plasma Physics: The Magnetohydrodynamic Approach to Plasma Confinement in Closed Magnetic Configurations.
EUR 5127 EN/1987. Commission of the European Communities, 1987.

R.S. Mackay and J.D. Meiss.
Linear stability of periodic orbits in lagrangian systems.
Physics Letters A, 98(3):92–94, 1983.

Luca Guido Molinari.
Determinants of block tridiagonal matrices.
Linear Algebra and its Applications, 429(8-9):2221–2226, oct 2008.