Recovering the on-axis rotational transform from variational principles

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Contents

Outline

- Context: ι_a as a parameter of interest.
- Variational principles: the **magnetic field line action** as starting point.
- Suitable **near-axis expansion** combined with the **Floquet formalism** enables to recover Mercier's formula for ι_a.
- Second variation connected to **Hill's infinite determinant**.
- Second variation - **discrete approach** - ι_a as an reduced-dimensionality eigenvalue problem.

Context

- Rotational transform: $\iota\!\!\!-$ defined in toroidal coordinates as

$$\iota\!\!\!- := \frac{1}{2\pi}\frac{d\theta}{d\phi}\bigg|_{\text{along }\mathbf{B}} = \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\phi}. \tag{1}$$

- **On-axis** rotational transform $\iota\!\!\!-_a$ defined as the limit of Eq.(1) when the field line gets infinitely close to the magnetic axis.
- **Magnetic field line action**: Provided a closed curve $\mathcal{C}$ and $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\mathcal{S} := \oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{l}. \tag{2}$$

Magnetic field action yields on-axis rotational transform

Magnetic field line action

- **Magnetic field line action**: Provided a closed curve $\mathcal{C}$ and $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\mathcal{S} := \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}. \tag{3}$$

- First variation: change in the curve geometry $\delta\mathbf{x}$

$$\delta\mathcal{S} = \int_{\mathcal{C}} \mathbf{x}' \times \mathbf{B} \cdot \delta\mathbf{x} dl. \tag{4}$$

- Stationary integral curves are **tangential** to **B**: field-lines.
- To recover the on-axis $\iota\!\!\!-$, the **second** variation of the magnetic action $\delta^2\mathcal{S}$ is needed.

Second variation

- Second variation:

$$\delta^2\mathcal{S}[\delta\mathbf{x}] = \oint_{\mathcal{C}} \delta(\mathbf{x}' \times \mathbf{B} \cdot \delta\mathbf{x}) d\ell = \oint_{\mathcal{C}} d\ell\; \delta\mathbf{x} \cdot (\delta\mathbf{x}' \times \mathbf{B} + \mathbf{x}' \times \delta\mathbf{B})$$

- In tensor form:

$$\delta^2\mathcal{S} = \oint d\ell\; \delta\mathbf{x}^i \frac{\delta^2 S}{\delta\mathbf{x}^i \delta\mathbf{x}^j} \delta\mathbf{x}^j \tag{5}$$

where,

$$\frac{\delta^2\mathcal{S}}{\delta\mathbf{x}^i \delta\mathbf{x}^j} = \epsilon_{ijk}\mathbf{B}^k \frac{d}{d\ell} + \epsilon_{imk}\mathbf{x}'^m \partial_j \mathbf{B}^k \tag{6}$$

Second variation

- Defining the operator $\overline{\overline{\mathcal{M}}}$:

$$\overline{\overline{\mathcal{M}}} \equiv \frac{\delta^2 \mathcal{S}}{\delta \mathbf{x} \delta \mathbf{x}} = -\left(\mathbb{I} \times \mathbf{B}\right) \frac{d}{d\ell} + \mathbf{x}' \times \left(\nabla \mathbf{B}\right)^{\mathsf{T}}. \tag{7}$$

- Eigenspaces of operator $\overline{\overline{\mathcal{M}}}$ of importance to recover $\iota\!\!-$, in particular null eigenvectors:

$$\overline{\overline{\mathcal{M}}} \cdot \mathbf{v} = \mathbf{0}. \tag{8}$$

- Linked with (Floquet-)normal form of the Hamiltonian [DM21].

Floquet Theory

- Need to solve
$$\dot{\mathbf{x}} = A(t)\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{9}$$
with $A \in \mathcal{M}_{n\times n}$ periodic in t with period T.
- Let n solutions of Eq.(9), and group them to define $X(t,0) = (\mathbf{x}_1; \ldots; \mathbf{x}_n)$.
- **Theorem (Floquet-Lyapunov).**
$$X(t,0) = P(t)e^{tB}, \tag{10}$$
where P is symplectic, $T-$periodic, and B constant Hamiltonian. So the $\mathbf{x}_i$ are of the form
$$\mathbf{x}_i(t) = e^{\nu_i t}\mathbf{p}_i(t), \tag{11}$$
with $\mathbf{p}_i$ periodic of period T.
- ν_i: **Floquet exponents** and $e^{\nu_i T}$: **Floquet multipliers**.
- We will prove $\nu \leftrightarrow \boldsymbol{t}_a$.

Expansion of $\overline{\overline{\mathcal{M}}}$

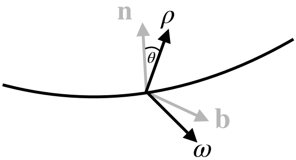

Figure: Mercier's basis

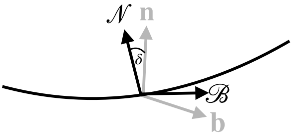

Figure: Solovev-Shafranov's basis

- Near axis exp:
  $\mathbf{x}(\ell) := \mathbf{x}_0(\ell) + \rho\boldsymbol{\rho}$

- Near axis exp:
  $\mathbf{r}(\ell) := \mathbf{r}_0(\ell) + x\boldsymbol{\mathcal{N}} + y\boldsymbol{\mathcal{B}}$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Mercier's coordinates I

- Mercier: $(\rho, \omega := \theta - \int \tau d\ell, \ell)$, τ the torsion, orthogonal coord system.
- Expand $\mathbf{B}$:

$$\mathbf{B} = B_0(\ell)\mathbf{t} + \rho\mathbf{B}_1, \quad \mathbf{B}_1 = \left(B_1^\rho \boldsymbol{\rho} + B_1^\omega \boldsymbol{\omega} + B_1^t \mathbf{t}\right). \tag{12}$$

- Compute $\mathbf{x}' \times \nabla\mathbf{B}$ and $\mathbb{I} \times \mathbf{B}$, and substitute in $\overline{\overline{\mathcal{M}}}$.
- Enforce MHD constraints $\nabla \cdot \mathbf{B} = \mathbb{I} : \nabla\mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mathbb{I} \overset{\times}{\cdot} \nabla\mathbf{B} = J_0\mathbf{t}$ for physical consistency.
- Enables to constrain B_1^ρ, B_1^ω and B_1^t:

$$B_1^t = \kappa B_0 \cos\theta, \quad B_1^\rho = -\frac{1}{2}(B_0' + \partial_\omega b_1), \quad B_1^\omega = \frac{1}{2}J_0 + b_1,$$

where κ denotes the curvature and b_1 satisfies

$$(\partial_\omega^2 + 4)b_1 = 0.$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Mercier's coordinates II

- Following [MEotEC87], the solution of Eq.(11) can be expressed as

$$b_1 = b_{c2}(\ell)\cos 2u + b_{s2}(\ell)\sin 2u, \tag{13}$$

$$u = \theta + \delta(\ell) = \omega - \int \tau d\ell + \delta(\ell) \tag{14}$$

$$\frac{b_{s2}}{B_0} = \frac{\eta'}{2}, \quad \frac{b_{c2}}{B_0} = \tanh\eta(\ell)\left(\delta' - \tau + \frac{J_0/2}{B_0}\right) \tag{15}$$

η, δ eccentricity and rotation of flux surfaces.

- Solve for $\overline{\overline{\mathcal{M}}}\cdot\mathbf{v} = 0$ with $\mathbf{v} = v^t\mathbf{t} + v^\rho\boldsymbol{\rho} + v^\omega\boldsymbol{\omega}$. Leads to system for v^ρ, $v^\omega(\omega,\ell)$:

$$\begin{cases} v_\omega' + \left(\frac{B_0'}{2B_0} - \frac{\partial_\omega b_1}{2B_0}\right)v_\omega - \left(\frac{J_0/2}{B_0} + \frac{b_1}{B_0}\right)v_\rho = 0 \\ v_\rho' + \left(\frac{B_0'}{2B_0} + \frac{\partial_\omega b_1}{2B_0}\right)v_\omega + \left(\frac{J_0/2}{B_0} - \frac{b_1}{B_0}\right)v_\omega = 0 \end{cases} \tag{16}$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Mercier's coordinates III

- In matrix form:

$$\frac{d\tilde{\mathbf{v}}}{d\ell} = A(\ell)\tilde{\mathbf{v}}, \quad \tilde{\mathbf{v}} = (v_\rho, v_\omega). \tag{17}$$

- Using Floquet theorems:

$$\tilde{\mathbf{v}} = U(\ell)e^{C\ell/L} \to \nu, \tag{18}$$

  with C constant matrix with exponents as eigenvalues.

- **Alternatively**: periodicity $\implies$ trigonometric expansion + change of variable (here for v_ω)

$$v_{\omega c} = \frac{1}{\sqrt{B_0'}} e^{+\eta/2} X(\ell), \quad v_{\omega s} = \frac{1}{\sqrt{B_0'}} e^{-\eta/2} Y(\ell) \tag{19}$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Mercier's coordinates IV

- Plugging the change of variable in the equation for v_ω:

$$X' + \Omega_0(\ell)Y = 0, \quad Y' - \Omega_0(\ell)X = 0, \tag{20}$$

and

$$\Omega_0(\ell) = \frac{\frac{J_0/2}{B_0} - \tau + \delta'}{2\cosh\eta} \tag{21}$$

- Combining in one single complex ODE $Z = X + iY$:

$$Z' - i\Omega_0 Z = 0 \quad \Rightarrow \quad Z(\ell) = Z_0 \exp\int_0^\ell i\Omega_0(s)ds \tag{22}$$

- Identifying periodic part:

$$\nu = \oint_{\mathcal{C}} \frac{\frac{J_0(\ell)/2}{B_0(\ell)} - \tau(\ell) + \delta'(\ell)}{2\cosh\eta(\ell)} ds \tag{23}$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Solovev-Shafranov's coordinates I

- Solovev Shafranov: expansion basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ with

$$\begin{aligned} \mathbf{e_1} &= \boldsymbol{\mathcal{N}} \\ \mathbf{e_2} &= \boldsymbol{\mathcal{B}} \\ \mathbf{e_3} &= h\mathbf{t} - (\delta' - \tau)\,(y\boldsymbol{\mathcal{B}} - x\boldsymbol{\mathcal{N}}) \end{aligned} \tag{24}$$

  and $h = 1 - \kappa\rho$.
- Expand $\mathbf{B}$:

$$\sqrt{g}\mathbf{B} = \sqrt{g}\sum_i B^i \mathbf{e}_i \tag{25}$$

  with the components

$$\sqrt{g}B^1 = a_1x + a_2y, \quad \sqrt{g}B^2 = b_1x + b_2y, \quad \sqrt{g}B^3 = B_0 + c_1x + c_2y.$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Solovev-Shafranov's coordinates II

- Compute $\nabla \mathbf{B}$, $\nabla \times \mathbf{B}$, $\nabla \cdot \mathbf{B}$ to constraint a_i, b_i, c_i.
- Substitute in $\overline{\overline{\mathcal{M}}}$ and solve $\overline{\overline{\mathcal{M}}} \cdot \mathbf{v} = 0$. Leads to

$$\frac{dv_{\mathcal{N}}}{a_1 v_{\mathcal{N}} + a_2 v_{\mathcal{B}}} = \frac{dv_{\mathcal{B}}}{b_1 v_{\mathcal{N}} + b_2 v_{\mathcal{B}}} = \frac{d\ell}{B_0}. \tag{26}$$

- So the components of $\mathbf{v}$ satisfy the general ODE:

$$\frac{d\mathcal{X}}{\sqrt{g} B^1(\mathcal{X}, \mathcal{Y})} = \frac{d\mathcal{Y}}{\sqrt{g} B^2(\mathcal{X}, \mathcal{Y})} = \frac{d\ell}{\sqrt{g} B^3(\mathcal{X}, \mathcal{Y})} \tag{27}$$

- With a change of variable:

$$v^{\mathcal{N}} = \mathcal{X} = \frac{1}{\sqrt{B_0'}} e^{+\eta/2} X(\ell), \quad v^{\mathcal{B}} = \mathcal{Y} = \frac{1}{\sqrt{B_0'}} e^{-\eta/2} Y(\ell) \tag{28}$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Solovev-Shafranov's coordinates III

- **Alternatively**: Use of **Hill's infinite determinant** to compute the exponents ν. Starting from Eq.(27), eliminating $v^{\mathcal{B}}$:

$$v^{\mathcal{N}''} + 2C_1 v^{\mathcal{N}} + C_2 = 0, \tag{29}$$

with C_1, C_2 periodic functions of ℓ. With the change of variable $\Psi = v^{\mathcal{N}} \exp \int C_1 d\ell$:

$$\Psi^{''} + \omega^2 \Psi = 0, \quad \omega^2 = C_2 - C_1^{'} - C_1^2 \tag{30}$$

- **Hill's equation** or Schrödinger with periodic potential.
- Expand solution with Laurent series:

$$\Psi = e^{i\nu\ell} \sum_n b_n e^{i\frac{2\pi n}{L}\ell} \tag{31}$$

Expansion of $\overline{\overline{\mathcal{M}}}$ - Solovev-Shafranov's coordinates IV

- Plugging the above expression in $(\partial_\ell^2 + \omega^2) = 0$ gives

$$\sum_m B_{nm} b_m = 0 \tag{32}$$

with

$$B_{nm} := \delta_{nm} \left(\omega_0^2 - \left(\nu + \frac{2\pi n}{L} \right)^2 \right) + \omega_0 \omega_n \tag{33}$$

and ω_n coming from expansion of periodic ω^2.

- Left with the infinite determinant equation

$$\det |B_{mn}|(\nu) = 0 \implies \nu. \tag{34}$$

Discrete formalism: towards ι

- **Infinite determinant**: hard to compute (approx. in [ZXW89]). **Discrete approach**: We follow MacKay and Meiss [MM83] for 1D Lagrangian.
- Discretize close curve $\mathcal{C}$ with $n-1$ curves $\mathcal{C}_i$

$$\begin{aligned} \mathcal{S} &= \sum_{i=1}^{n-1} \int_{\mathcal{C}_i} \mathbf{A} \cdot d\mathbf{l}, \\ &= \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \end{aligned} \tag{35}$$

- Notation:

$$\begin{aligned} \mathbf{S}_1^{[i,i+1]} &= \nabla_{\mathbf{x}_i} S^{[i,i+1]} := \nabla_{\mathbf{x}_i} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \\ \mathbf{S}_2^{[i,i+1]} &= \nabla_{\mathbf{x}_{i+1}} S^{[i,i+1]} := \nabla_{\mathbf{x}_{i+1}} S(\mathbf{x}_i, \mathbf{x}_{i+1}) \end{aligned} \tag{36}$$

Discrete formalism: towards ι

- Stationarity of the action for extremal curve

$$\begin{aligned} \delta S[\delta \mathbf{x}_i] &= \left[\nabla_{\mathbf{x}_i} S^{[i-1,i]} + \nabla_{\mathbf{x}_i} S^{[i,i+1]}\right] \cdot \delta \mathbf{x}_i = 0, \\ \iff \mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]} &= \mathbf{0}. \end{aligned} \tag{37}$$

- Total derivative of previous result (for tangent orbits - nearby trajectories)

$$\begin{aligned} \delta\left(\mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]}\right) =& \overline{\overline{S}}_{12}^{[i-1,i]} \cdot \delta \mathbf{x}_{i-1} + \overline{\overline{S}}_{21}^{[i,i+1]} \cdot \delta \mathbf{x}_{i+1} \\ & \left(\overline{\overline{S}}_{22}^{[i-1,i]} + \overline{\overline{S}}_{11}^{[i,i+1]}\right) \cdot \delta \mathbf{x}_i = \mathbf{0}, \\ & 1 \leq i \leq n-1. \end{aligned} \tag{38}$$

- Leads to tridiagonal-block matrix form.

Discrete formalism - ι as an eigenvalue problem

$$\begin{pmatrix} \left(\overline{\overline{S}}_{22}^{[01]} + \overline{\overline{S}}_{11}^{[12]}\right) & \overline{\overline{S}}_{12}^{[12]} & & & \lambda^{-1}\overline{\overline{S}}_{21}^{[01]} \\ \overline{\overline{S}}_{21}^{[12]} & \left(\overline{\overline{S}}_{22}^{[12]} + \overline{\overline{S}}_{11}^{[23]}\right) & \overline{\overline{S}}_{12}^{[23]} & & \\ & \overline{\overline{S}}_{21}^{[23]} & \ddots & \ddots & \\ & & \ddots & \ddots & \overline{\overline{S}}_{12}^{[n-1,n]} \\ \lambda\overline{\overline{S}}_{12}^{[n,n+1]} & & & \overline{\overline{S}}_{21}^{[n-1,n]} & \left(\overline{\overline{S}}_{22}^{[n-1,n]} + \overline{\overline{S}}_{11}^{[n,n+1]}\right) \end{pmatrix} \cdot \begin{pmatrix} \delta\mathbf{x}_1^T \\ \delta\mathbf{x}_2^T \\ \vdots \\ \\ \vdots \\ \delta\mathbf{x}_{n-1}^T \\ \delta\mathbf{x}_n^T \end{pmatrix} = \mathbf{0}$$

- Recover the same form (discrete this time) as previously for our operator equation:

$$\iff \overline{\overline{M}}(\lambda) \cdot \delta\mathbf{x} = \mathbf{0}, \tag{39}$$

Discrete formalism - ι as an eigenvalue problem

- For such a matrix form, exists formula for determinant [Mol08]

$$\det\Big(M(\lambda)\Big) = \frac{(-1)^{mn}}{(-\lambda)^m} \det\Big(T_S - \lambda \mathcal{I}_6\Big) \det\Big(\prod_{i=1}^{n} \overline{\overline{S}}_{12}[i, i+1]\Big)$$

- Requires to define the so-called transfer matrix $T_{\mathcal{S}}$

$$T_{\mathcal{S}} = \prod_{i=1}^{n} \begin{pmatrix} -\overline{\overline{S}}_{12}^{-1\,[i,i+1]} (\overline{\overline{S}}_{22}^{[i-1,i]} + \overline{\overline{S}}_{11}^{[i,i+1]}) & -\overline{\overline{S}}_{12}^{-1\,[i,i+1]} \overline{\overline{S}}_{12}^{[i-1,i]} \\ \mathcal{I}_3 & 0 \end{pmatrix}.$$

Discrete formalism - a necessary condition

- MacKay and Meiss use what they call a convexity condition on their Lagrangian: $-L_{12} > 0$, which is similar to the so-called *true angle-dynamics* from [HS14].
- We propose a **generalized criterion**: imposing that the upper diagonal matrices are negative definite

$$\mathbf{x}^T \overline{\overline{S}}_{12}^{[i,i+1]} \mathbf{x} < 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad 1 \leq i \leq n. \tag{40}$$

- Finally, $\det(M(\lambda)) = 0 \rightarrow$ eigenvalue equation:

$$(-\lambda)^{-3} \cdot \det\left(T_S - \lambda \mathcal{I}_6\right) = 0. \tag{41}$$

- Exponents λ are eigenvalues of transfer matrix.

Conclusion

- Second variation of action combined with Floquet theory: $\iota_a \leftrightarrow \nu$.
- Two near-axis expansions gave same result and proved to enable recovering Mercier's formula.
- Practical computation of exponent made difficult by continuous/infinite dimension character of the problem. Alternative: discretize trial-curve and apply variations.
- Next: investigate near X point.
- This work is incomplete and unpublished.

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