Multiple Equilibrium Behaviors on Networks

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Models of traffic networks typically assume that the consumers of the transportation service either behave in a totally noncooperative (user equilibrium) or cooperative (system equilibrium) manner. This note presents a model in which each origin-destination pair can obey either behavioral principal. After discussing the issues of existence and uniqueness and the model's applicability to freight and urban transportation applications, it is shown how variational inequality methods can be employed to solve this new model. A small example is then used to illustrate the features of a multiple behavior traffic network equilibrium.

1. INTRODUCTION

In the modeling of competition on networks such as in the case of urban traffic flows, telecommunication flows, and intercity freight flows, one of two standard behavioral assumptions is typically employed: Wardropian User Equilibrium (UE) or Wardropian System Equilibrium (SE). In the case of UE, it is assumed that the system consists of many agents, each of whom is attempting to minimize his or her own travel costs; i.e., each agent plays noncooperatively against all others in the travel "game." In the case of SE, it is assumed that one agent controls the entire network and routes flows in such a way that the total system cost is minimized. The reader is referred to Friesz (1985) for an excellent review of these concepts.

When modeling the flow of passenger traffic on urban networks, the UE concept is always employed (FRIESZ[5]) and in the modeling of intercity freight movements by a single carrier, it is typically assumed that this carrier behaves via the SE concept on its own network (HARKER,[10] FRIESZ et al.,[3] FRIESZ, VITON and TOBIN[4]). However, there are several situations when neither concept is immediately applicable. Consider first the case of intercity freight flows. While it is reasonable to assume that the carriers (railroads, motor carriers, etc.) attempt to minimize the total cost of transportation (or transmission in the case of telecommunications) on their exclusive networks (HARKER[10]), what is the appropriate model for the shippers—those agents who are purchasing the spatially distributed outputs of the carriers? In Friesz[3] and Harker[10] it is assumed that the shippers behave via the UE principle when routing their freight over the sequence of carrier networks. Thus, one is assuming that each shipper is a price taker in the transportation market. This assumption is clearly unrealistic when one recognizes the presence of large freight forwarders, consolidators, and large individual shippers who routinely employ their substantial market power to negotiate better freight rates with carriers. However, not all shippers possess such market power. Thus, one must employ a mixed behavior model to capture the situation in which the smaller shippers behave according to the UE concept and the larger shippers each attempt to minimize their own costs in a way similar to SE.

In the case of urban mass transit, one typically uses network equilibrium models to analyze passenger flows. However, the advent of private market participation in the U.S. urban mass transit industry has led several analysts to consider network-based models of urban bus competition (e.g., VITON,[22] HARKER[12]). Typically, one characterizes this situation as a Cournot-Nash (MOULIN[18]) model in which the competing bus firms set the quantity and quality of service in a noncooperative fashion. However, it is also very reasonable to assume that the privatized urban mass transit market is composed of several large firms and many other smaller, price-taking firms. In this situation, one must employ a SE-like model for the large firms and a UE-like model for the smaller firms; again, mixed behaviors must be considered.

Finally, the telecommunications industry is fast approaching the type of situation described for intercity freight movements. Consumers now have the capability to choose a carrier for their calls on a route-by-route basis and furthermore, have the ability to choose a sequence of carriers to handle each call. As the competitive forces in this market increase, the type of mixed behavior model described for the intercity freight situation may prove very useful in analyzing the telecommunications market.

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In summary, there are many network equilibrium situations which do not fit neatly into either the UE or SE framework. The purpose of this note is to describe a simple variant of both concepts which can be employed in these situations. This work should be contrasted with the work by DEVARAJAN,[2] HAURIE and MARCOTTE[13] and others which deals with the issue of the convergence of a Nash equilibrium model of travel behavior to the set of User Equilibria. In these approaches, each player is an origin-destination (O-D) pair who acts in a Nash equilibrium manner. In the current approach, a player can own one or several O-D pairs and may either act in a purely competitive or UE manner, or in a SE-like manner. Furthermore, the emphasis of the papers listed above is on the convergence of this Nash model to UE; our emphasis will be on the construction and solution of a useable model of mixed behavior on traffic networks.

The remainder of this paper is structured as follows. The next section will describe the simple case of a mixed behavior traffic equilibrium problem in which all arc costs are separable and O-D demands are fixed, and Section 3 will describe the solution of this model. A small numerical example is then presented in Section 4 which provides some counterintuitive insights into the equilibria which result from a mixed behavior model. Extensions to the case of nonseparable arc costs and elastic O-D demands are considered in Section 4, and conclusions are drawn in Section 5.

2. MIXED BEHAVIOR NETWORK EQUILIBRIA

IN ORDER to begin our discussion, let us assume that each O-D pair is either controlled by a player with market power (i.e., a Cournot-Nash player) or a player acting as a price-taker in the market—the UE player. Note that a Cournot-Nash player can control more than one O-D pair but that each O-D pair can be controlled by only one player. In the case where several players compete over the same physical O-D pair, one simply makes copies of this O-D pair and treats it in the model as a set of O-D pairs (see, for example, AASHTIANI[1] for a discussion of the multicopy network concept on which this O-D pair splitting is based). Thus, the current modeling construct is more general than that of Devarajan[2] and Haurie and Marcotte[13] in that any ownership of the O-D pairs can be represented. Also note that when a single price-taking player controls all O-D pairs, one obtains the UE model and when all O-D pairs are controlled by a single Cournot-Nash player, the SE model results.

Let us define the following notation:

$G[M, A] \equiv$ the directed graph under consideration, M being the set of nodes and A the set of directed arcs,

$W \equiv$ the set of O-D pairs which are modeled as being purely competitive; i.e., obeying the UE principle,

$P_w \equiv$ the set of paths between $w \in W$,

$P \equiv \bigcup_{w\in W} P_w$,

$h_p \equiv$ the flow on path $p \in P$,

$h \equiv (\ldots, h_p, \ldots)^t$,

$T_w \equiv$ the O-D flow between $w \in W$,

$\delta_{ap} \equiv 1$ if $p \in P$ traverses $a \in A$, 0 otherwise,

$f_a{}^W =$ the flow of arc $a \in A$ arising out of the O-D flows from the set W,

$f^W \equiv (\ldots, f_a{}^W, \ldots)^t$,

$N \equiv$ the set of n Cournot-Nash firms on the graph,

$V^i \equiv$ the set of O-D pairs which are controlled by player $i \in N$,

$V \equiv \bigcup_{i\in N} V^i$,

$Q_v \equiv$ the set of paths between $v \in V$,

$Q^i \equiv \bigcup_{v\in V^i} Q_v$,

$Q \equiv \bigcup_{i\in N} Q^i$,

$g_q \equiv$ the flow on path $q \in Q$,

$g \equiv (\ldots, g_q, \ldots)^t$,

$S_v \equiv$ the O-D flow between $v \in V$,

$\gamma_{aq} \equiv 1$ if $q \in Q$ traverses $a \in A$, 0 otherwise,

$e_a{}^i \equiv$ the flow on arc $a \in A$ arising out of the O-D pairs controlled by player $i \in N$,

$e^i \equiv (\ldots, e_a{}^i, \ldots)^t$,

$e \equiv (\ldots, e^i, \ldots)^t$,

$e_a \equiv (\ldots, e_a{}^i, \ldots)^t$,

$$f_a \equiv \text{the total flow on arc } a \in A = f_a{}^W + \textstyle\sum_{i\in N} e_a{}^i, \tag{1}$$

$f \equiv (\ldots, f_a, \ldots)^t$,

$c_a(f_a) \equiv$ the average cost of transportation on arc $a \in A$, which is assumed to be separable in arc flows for the moment. Also assume that this function is twice continuously differentiable,

$c(f^W; e) \equiv (\ldots, c_a(f_a), \ldots)^t$.

Given the above notation, the feasible sets for player "W"—the UE player—and player $i \in N$ are given by:

$$\Omega^W = \{f^W: \textstyle\sum_{p\in P_w} h_p = T_w \qquad \forall w \in W \tag{2}$$

$$f_a{}^W = \textstyle\sum_{p\in P} \delta_{ap} h_p \quad \forall a \in A, \tag{3}$$

$$h_p \geqslant 0 \qquad \forall p \in P\} \quad ,$$

$$\Omega^i = \{e^i: \textstyle\sum_{q\in Q_v} g_q = S_v \qquad \forall v \in V^i \tag{4}$$

$$e_a{}^i = \textstyle\sum_{q\in Q^i} \gamma_{aq} g_q \quad \forall a \in A, \tag{5}$$

$$g_q \geqslant 0 \qquad \forall q \in Q^i\}.$$

Assuming that player W takes prices or costs (price simply denotes negative cost in this context) as given, this player's problem is:

$$\text{minimize}_{f^W\in\Omega^W}, \textstyle\sum_{a\in A} c_a(\bar{f}_a) f_a{}^W \tag{6}$$

where the bar denotes a fixed value. The first order conditions for this problem at the solution f^* can be written as:

$$\langle c(f^{W*}; e^*), f^W - f^{W*}\rangle \geqslant 0 \quad \forall f^W \in \Omega^W \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product operator in R^n.

Player $i \in N$ acts as a Cournot-Nash player and thus, treats all other players' flows on any arc $a \in A$ as fixed when solving his cost minimization problem. Defining e_a^{-i} to be the flow on arc $a \in A$ which is not due to player i, this minimization problem becomes:

$$\text{minimize}_{e^i \in \Omega^i} \sum_{a\in A} c_a(e_a^i + e_a^{-i})e_a^i \tag{8}$$

where e_a^{-i} is treated as fixed. If one assumes:

$$\text{(A-1)} \quad 2c_a'(f_a) + e_a^i c_a''(f_a) \geqslant 0 \quad \forall a \in A, i \in N,$$
$$\text{where} \quad c_a'(f_a) = \partial c_a(f_a)/\partial f_a$$
$$\text{and} \quad c_a''(f_a) = \partial^2 c_a(f_a)/\partial f_a^2,$$

then (8) is convex in e^i and the first-order conditions are both necessary and sufficient for a solution. Defining the marginal cost on arc $a \in A$ for player $i \in N$ as:

$$\mathrm{MC}_a^i(f^W; e) \equiv c_a(f_a) + e_a^i c_a'(f_a)$$

and

$$\begin{aligned} \mathrm{MC}^i(f^W; e) &\equiv (\ldots, \mathrm{MC}_a^i(f^W; e), \ldots)^t, \\ \mathrm{MC}(f^W; e) &\equiv (\ldots, \mathrm{MC}^i(f^W; e), \ldots)^t, \end{aligned} \tag{9}$$

the first-order necessary (and sufficient by assumption) conditions at the solution $(f^{W*}; e^*)$ can be written as:

$$\langle \mathrm{MC}^i(f^W; e^*), e^i - e^{i*}\rangle \geqslant 0 \quad \forall e^i \in \Omega^i. \tag{10}$$

Given the fact that the strategy sets Ω^i for each player are disjoint, it is well known (GABAY and MOULIN,[6] HARKER[7]) that conditions (7) and (10) can be written as a single *variational inequality (VI) problem* (KINDERLEHRER and STAMPACCHIA[14]), which is the problem of finding an $x^* \in K \subseteq R^n$ such that for the function $F: K \to R^n$ the following holds:

$$\langle F(x^*), x - x^*\rangle \geqslant 0 \quad \forall x \in K. \tag{11}$$

The following proposition follows directly from the disjoint nature of the strategy sets to state the VI formulation of our mixed behavior network equilibrium.

PROPOSITION 1. *Under assumption* (A-1) , *the vector* $(f^{W*}; e^*)$ *is a mixed behavior network equilibrium if and only if the following inequality holds*:

$$\begin{aligned} &\langle c(f^{W*}; e^*), f^W - f^{W*}\rangle \\ &\quad + \textstyle\sum_{i\in N} \langle \mathrm{MC}^i(f^{W*}; e^*), e^i - e^{i*}\rangle \geqslant 0 \\ &\qquad\qquad \forall (f^W; e) \in \Omega \quad \text{(12a)} \\ &= \textstyle\sum_{a\in A} F_a(f_a^{W*}; e_a^*)^t[(f_a^W; e_a) - (f_a^{W*}; e_a^*)] \geqslant 0 \\ &\qquad\qquad \forall (f^W; e) \in \Omega \quad \text{(12b)} \end{aligned}$$

where

$$\begin{aligned} F_a(f_a^W; e_a) &= [c_a(f_a), \mathrm{MC}_a^1(f^W; e), \ldots, \mathrm{MC}_a^n(f^W; e)]^t \\ &= [c_a(f_a), c_a(f_a) + c_a'(f_a)e_a^1, \ldots, \\ &\qquad c_a(f_a) + c_a'(f_a)e_a^n]^t \end{aligned}$$

and

$$\Omega = \Omega^W \times \textstyle\prod_{i\in N} \Omega^i. \tag{13}$$

The behavior which (12) exhibits is a mixture of user equilibrium on the O-D pairs W and "partial" systems equilibrium on the O-D pairs V^i. To see this mixture of behaviors more fully, let us rewrite the first-order conditions (7) and (10). Defining the average path cost on a path $p \in P$ by

$$C_p(h; g) = \textstyle\sum_{a\in A} \delta_{ap} c_a(f_a), \tag{14}$$

the first-order conditions (7) imply:

$$\begin{aligned} &[C_p(h; g) - u_w]h_p = 0 \quad \forall w \in W, \;\; p \in P_w \\ &C_p(h; g) - u_w \geqslant 0, \quad h_p \geqslant 0, \end{aligned} \tag{15}$$

where u_w is the minimum average cost between O-D pair $w \in W$; that is, the path cost equilization defining UE holds. Thus, the O-D pairs $w \in W$ exhibit user equilibrium at the solution to (12).

Defining the marginal cost to player $i \in N$ on path $q \in Q$ as

$$\mathrm{MC}_q^i(h; g) = \textstyle\sum_{i\in N} \gamma_{aq} \mathrm{MC}_a^i(f^W; e) \tag{16}$$

the first-order conditions (10) imply:

$$\begin{aligned} &[\mathrm{MC}_q^i(h; g) - y_v]g_q = 0 \quad \forall i \in N, v \in V^i, q \in Q_v \\ &\mathrm{MC}_q^i(h; g) - y_v \geqslant 0, \quad g_q \geqslant 0, \end{aligned} \tag{17}$$

where y_v is the minimum marginal cost between O-D pair $v \in V$. Conditions (17) state that player $i \in N$ will equate the "partial" marginal cost MC_q^i on all utilized paths. Note that (17) is different from the normal SE concept due to the use of a different concept of marginal costs. In SE, the marginal *system-wide* cost is equated, whereas in this mixed behavior model the marginal costs as defined by player i's actions alone—the "partial" marginal costs—are equated. Thus, each player exhibits a SE-like behavior, but the overall system is not in a system equilibrium.

One can employ several recent results in nonlinear complementarity theory in order to provide very general conditions insuring the existence of a solution to (12); e.g., the works by Aashtiani,[1], SMITH,[21] or HARKER.[9] However, let us employ the result of Kinderlehrer and Stampacchia[14] (Theorem 3.1) in order to provide a simple proof of existence. Assume that

(A-2) The flow on each arc $a \in A$ is bounded from above by a large positive scalar U_a.

This assumption rules out the possibility of infinite cycles at the solution. A simple bound would be for all arc flows to be less than or equal to the total flow in the system:

$$U = \sum_{w \in W} T_w + \sum_{v \in V} S_v. \tag{18}$$

Given this assumption, the feasible set Ω in (12) is compact and convex and due to the assumed continuity of all functions in (12), Theorem 3.1 of Kinderlehrer and Stampacchia[14] implies:

PROPOSITION 2. *Under assumptions* (A-1)–(A-2), *a solution to the VI formulation of the mixed behavior network equilibrium model* (12) *exists.*

In order to establish conditions for uniqueness, let us assume

(A-3) $F_a(f_a{}^W;\, e_a)$ is a strictly monotone function $\forall a \in A$.

Under this assumption we have:

PROPOSITION 3. *Under assumption* (A-3), *there exists at most one solution to the mixed behavior network equilibrium model* (12).

Proof. Follows directly from the well-known result in variational inequality theory that if $F(x)$ is a strictly monotone function, then the solution is unique. □

In general, assumption (A-3) will not hold for arbitrary arc cost functions $c_a(f_a)$ even if these functions are strictly monotone and convex. For example, consider the case where there exists one Cournot-Nash player with strategy e_a and one user equilibrium player controlling $f_a{}^W$, and let the arc cost function be given by $c_a = \exp(f_a)$ which is clearly convex and strictly monotone. Writing the definition of strict monotonicity for $F_a(f_a{}^W;\, e_a)$, one obtains after some manipulation:

$$\begin{aligned}[][F_a(f_a{}^W; e_a) - F_a(f_a{}^{W*}; e_a{}^*)]^t[(f_a{}^W; e_a) - (f_a{}^{W*}; e_a{}^*)] \\ = [\exp(f_a) - \exp(f_a{}^*)](f_a - f_a{}^*) \\ + [\exp(f_a)e_a - \exp(f_a{}^*)e_a{}^*](e_a - e_a{}^*).\end{aligned} \tag{19}$$

Defining two feasible points as $f_a = 5$, $e_a = 2$, $f_a{}^W = 3$, $f_a{}^* = 4$, $e_a{}^* = 3$, $f_a{}^{W*} = 1$ and substituting these values into (19) yields:

$$[e^5 - e^4](5 - 4) + [2e^5 - 3e^4](2 - 3) = e^4(2 - e) < 0.$$

Therefore, $c_a(f_a) = \exp(f_a)$ does not yield a strictly monotone F_a even though it is strictly monotone and convex.

In order to establish a *sufficient* set of conditions for uniqueness which are more meaningful in terms of the function $c_a(f_a)$ than what is given in Assumption (A-3), let us assume:

(A-4) $c_a(f_a)$ is a strictly monotone and linear function in f_a.

Under this assumption, the following result can be established:

PROPOSITION 4. *Under assumption* (A-4), *there exists at most one solution to the mixed behavior network equilibrium model* (12).

Proof. Given the arc-separability of the function F defining (12), one need only establish the strict monotonicity of $F_a(f_a{}^W;\, e_a)$ for all $a \in A$ to establish the strict monotonicity of F and hence, the uniqueness of a solution to (12). Writing out the definition of strict monotonicity for $F_a(f_a{}^W;\, e_a)$, one has:

$$\begin{aligned}[][F_a(f_a{}^W; e_a) - F_a(f_a{}^{W*}; e_a{}^*)]^t[(f_a{}^W; e_a) - (f_a{}^{W*}; e_a{}^*)] \\ = [c_a(f_a) - c_a(f_a{}^*)](f_a{}^W - f_a{}^{W*}) \\ + \textstyle\sum_{i \in N} [c_a(f_a) + c_a{}'(f_a)e_a{}^i \\ - c_a(f_a{}^*) - c_a{}'(f_a{}^*)e_a{}^{i*}](e_a{}^i - e_a{}^{i*}) \\ = [c_a(f_a) - c_a(f_a{}^*)](f_a - f_a{}^*) \\ + c_a{}' \textstyle\sum_{i \in N} (e_a{}^i - e_a{}^{i*})^2.\end{aligned} \tag{20}$$

where $c_a{}'$ is a constant due to the assumed linearity of $c_a(f_a)$. For $(f_a{}^W;\, e_a) \neq (f_a{}^{W*};\, e_a{}^*)$, the above function is clearly greater than zero due to the strict monotonicity of $c_a(f_a)$ and hence, $F_a(f_a{}^W;\, e_a)$ is strictly monotone. □

Therefore, this section has shown that a model of multiple equilibrium behaviors on a transportation network can be easily stated and shown to possess a unique equilibrium under fairly mild conditions. In order to achieve this inclusion of multiple behaviors, the dimensionality of the problem has increased from $|A|$, the number of arcs, in the case of pure UE or SE to $|A|(1 + n)$, where n is the number of Cournot-Nash players. Clearly, many Cournot-Nash players will drastically increase the dimensionality of the problem but for most realistic applications, n should be fairly small.

3. SOLUTION ALGORITHMS

IN ORDER to solve the mixed equilibrium model (12), one must first notice that even with separable arc costs one cannot obtain an equivalent optimization formulation due to the nonseparability of the cost functions with respect to their individual player's strategies. Thus, one must employ a variational inequality algorithm for the solution to (12). Due to the size of the typical application of this model, the two major choices of solution algorithms are the diagonalization algorithm and simplicial decomposition for asymmetric VI problems. In this section we shall briefly discuss the application of each algorithm to (12).

The diagonalization (or relaxation or nonlinear Jacobi) algorithm for VI problems such as (11) consists of forming a separable function $G_i(x)$ from $F_i(x)$ at each iteration; e.g., PANG and CHAN.[19] Harker[7] has shown how this approach, when applied to a general Nash equilibrium problem, leads to the solution of a sequence of mathematical programs, one for each player in the game. Specializing Harker's[7] result to (12), the diagonalization algorithm consists of solving a sequence of problems:

Player W

$$\underset{f^W \in \Omega^W}{\text{minimize}} \sum_{a \in A} \int_0^{f_a^W} c_a(s + f_a^{-W})\, ds \tag{21}$$

where f_a^{-W} denotes the flow not associated with O-D pairs W and is held fixed at the previous iteration's value in the solution of (21).

Player $i \in N$

$$\text{minimize}_{e^i \in \Omega^i} \textstyle\sum_{a \in A} c_a(e_a^i + e_a^{-i}) e_a^i \tag{22}$$

where e_a^{-i} (which includes f^W) is held at the previous iteration value. Note that (21) and (22) are standard UE and SE problems for which several efficient methods of solution exist; e.g., LEBLANC et al.[17] and LAWPHONGPANICH and HEARN.[15] Thus, the application of the diagonalization algorithm consists of the solution of a sequence of simultaneous UE and SE subproblems of the form (21) and (22), respectively.

The second approach is to apply the asymmetric version of the simplicial decomposition algorithm which was developed by LAWPHONGPANICH and HEARN[16] and PANG and YU[20] for the UE problem. This algorithm consists of the solution of a sequence of all-or-nothing assignments of O-D flows and a lower dimensional VI problem defined over the unit simplex. The only change to this algorithm from the case of UE is in the all-or-nothing assignments. Instead of using the same arc costs for each O-D pair, one must use c_a or MC_a^i depending on the behavior or "ownership" of each O-D pair. Thus, a minor modification to the existing simplicial decomposition algorithms will enable these approaches to solve (12).

The main issue in the solution of (12) is its dimensionality as discussed in the previous section. As the number of Cournot-Nash players increases, the dimensionality of the problem increases by the number of arcs in the network. For the type of large-scale networks encountered in practice, this fact may place some limitation on the size of the problem which can be addressed with this model. The increased dimensionality of the problem may also play a role in choosing which of the above two algorithms to employ. The simplicial decomposition algorithm is substantially faster than the standard diagonalization algorithm but one must generate and store a large number of arc flow vectors for this algorithm to be convergent in many situations. The diagonalization algorithm, on the other hand, only requires that the vector from the previous iteration be stored, and the UE or SE subproblems can be solved with either Frank-Wolfe or *restricted* simplicial decomposition (Lawphongpanich and Hearn[15]) which requires that only a few vectors be stored at each iteration. Furthermore, there are some modifications which one can make to the diagonalization algorithm to decrease its computational time in practice (HARKER[11]).

Therefore, one must carefully select which of the above two approaches to employ in solving (12) after considering the computer memory requirements of the application. In either case, the solution to (12) involves only minor modifications to existing algorithms for the traffic assignment problem and should be very easy to implement.

4. NUMERICAL EXAMPLE

IN ORDER to illustrate the mixed behavior network equilibrium and to point out the somewhat paradoxical behavior which this model can exhibit, consider the network depicted in Figure 1 and the following data:

$$c_1 = 20 + f_1 \quad c_2 = 20 + f_2 \quad c_3 = 5 + 2f_3$$

$$c_4 = 20 + f_4 \quad c_5 = 5 + 2f_5 \quad c_6 = 5 + 2f_6$$

$$c_7 = 20 + f_7$$

$$T_{(1,4)} = 10 \quad T_{(4,1)} = 10$$

Using these data, consider four cases: (a) user equilibrium, (b) one Cournot-Nash player controlling O-D pair (1, 4) and (4, 1) obeying UE, (c) (1, 4) being controlled by Cournot-Nash player 1 and (4, 1) by Cournot-Nash player 2, and (d) system equilibrium. Table I lists the results of solving these four cases analytically (i.e., no roundoff error is present except for calculator errors), where Cost_w denotes the total cost incurred by the movements between O-D pair $w \in W$ and Total Cost is the total system cost. As expected, the total system cost decreases with increas-

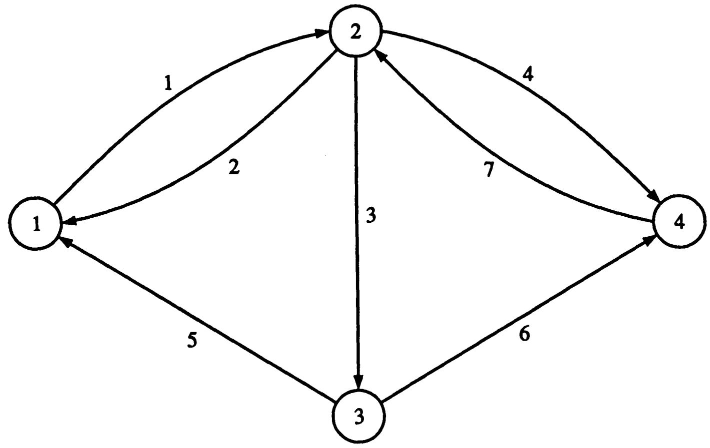

Fig. 1. Network for numerical example.

TABLE I
Results of Numerical Example

	CASE			
	(a) User equilibrium	(b) Nash player(1, 4)	(c) 2 Nash players	(d) System equilibrium
$\mathrm{Cost}_{(1,4)}$	571.4286	571.4083	568.7500	567.8571
$\mathrm{Cost}_{(4,1)}$	571.4286	569.5652	568.7500	567.8571
Total cost	1142.8571	1140.9735	1137.5000	1135.7143
f_1	10.0000	10.0000	10.0000	10.0000
f_2	7.1426	6.9565	7.5000	7.8571
f_3	5.7149	5.4348	5.0000	4.2857
f_4	7.1426	7.6087	7.5000	7.8571
f_5	2.8574	3.0435	2.5000	2.1429
f_6	2.8574	2.3913	2.5000	2.1429
f_7	10.0000	10.0000	10.0000	10.0000

ing market power and cooperation. However, the results of case (b) are somewhat paradoxical. In this situation, player (1, 4) is acting in a Cournot-Nash fashion while (4, 1) is passively taking the costs as given. One would expect in this situation that (1, 4) would have a lower cost than (4, 1). Clearly, (1, 4) does improve over the purely competitive costs in case (a) (571.4083 versus 571.4286) but player (4, 1) does significantly better when (1, 4) employs his market power (569.5652 versus 571.4286). The recognition and use of market power helps the other player more!

The seeming paradox illustrated by the above example can be easily understood when one recognizes that unlike the standard market oligopoly model in microeconomic theory, the mixed behavior network equilibrium model contains externalities due to the congestion phenomenon inherent in the arc cost functions $c_a(f_a)$. When these congestion externalities are present, one can easily get the type of situation depicted above.

Therefore, this small numerical example has shown that a mixed behavior network equilibrium model can be a very useful tool in recognizing the somewhat paradoxical outcomes which can occur in models of network-based competition; other such examples are easily generated. One cannot simply assume that those large firms with substantial market power will always fare better in the market than the smaller, purely competitive firms when externalities are present; one must carefully analyze the situation with the type of model described in this paper.

5. EXTENSIONS TO NONSEPARABLE COSTS AND ELASTIC DEMAND

IN THIS section the basic model which was developed in Section 3 will be extended to include the possibility of elastic rather than fixed O-D demands and nonseparable arc cost functions. In order to begin, define

$T \equiv (\ldots, T_w, \ldots)^t$,

$S^i \equiv (S_v \mid v \in V^i)^t$,

$S \equiv (\ldots, S_v, \ldots)^t$,

$\Theta_w(T; S) \equiv$ the inverse demand function for O-D pair $w \in W$ which is assumed to be twice-continuously differentiable,

$\Theta(T; S) \equiv (\ldots, \Theta_w(T; S), \ldots)^t$,

$\Psi_v(T; S) \equiv$ the inverse demand function for O-D pair $v \in V^i$, $i \in N$ which is assumed to be twice continuously differentiable,

$\Psi^i(T; S) \equiv (\Psi_v(T; S) \mid v \in V^i)^t$.

The inverse demand functions can arise from any standard transportation demand system; i.e., singly or doubly constrained gravity models, multinomial logits, etc. Given these definitions, the optimization problems (6) and (8) for players W and $i \in N$ become one of maximizing profits or:

Player W

$$\text{maximize}_{(f^W; T) \in \Omega^W} \langle \Theta(\bar{T}; \bar{S}), T \rangle - \langle c(\bar{f}^W; \bar{e}), f^W \rangle \quad (23)$$

Player $i \in N$

$$\text{maximize}_{(e^i; S^i) \in \Omega^i} \langle \Psi^i(\bar{T}; S^i, S^{-i}), S^i \rangle - \langle c(e^i; e^{-i}), e^i \rangle \quad (24)$$

where the bar denotes fixed values, S^{-i} and e^{-i} denote those variables not under player i's control which are held fixed during the solution to (24), and T and S^i are treated as variables in Ω^W and Ω^i respectively. The first-order conditions at the equilibrium point for (23) are:

$$\langle c(f^{W*}; e^*), f^W - f^{W*} \rangle - \langle \Theta(T^*; S^*), T - T^* \rangle \geq 0 \quad \forall (f^W; T) \in \Omega^W. \quad (25)$$

Denoting the marginal revenue and marginal cost functions for player $i \in N$ by:

$$\text{MR}^i(T; S) \equiv \Psi^i(T; S) + \nabla_S i \Psi^i(T; S) S^i \quad (26)$$

$$\text{MC}^i(f^W; e) \equiv c(f^W; e) + \nabla_f c(f^W; e) e^i \quad (27)$$

$$\text{MR}(T; S) \equiv (\ldots, \text{MR}^i(T; S), \ldots)^t,$$

$$\text{MC}(f^W; e) \equiv (\ldots, \text{MC}^i(f^W; e), \ldots)^t,$$

and assuming

(A-1′) $\text{MR}^i(T; S)$ is a monotone decreasing function with respect to S^i,
(A-2′) $\text{MC}^i(f^W; e)$ is a monotone increasing function with respect to e^i,

which imply that (24) is a concave function in $(e^i; S^i)$, the first-order necessary and sufficient conditions for (24) at the equlibrium point are:

$$\langle \text{MC}^i(f^{W*}; e^*), e^i - e^{i*} \rangle - \langle \text{MR}^i(T^*; S^*), S^i - S^{i*} \rangle \geq 0 \quad \forall (e^i; S^i) \in \Omega^i. \quad (28)$$

As in the case of the model in Section 3, conditions (25) and (28) can be combined to form a single variational inequality problem, again due to the disjoint nature of the feasible sets:

PROPOSITION 5. *Under assumptions* (A-1′) *and* (A-2′), *the vector* $(f^{W*}; e^*; T^*; S^*)$ *is a mixed behavior network equilibrium if and only if the following inequality holds*:

$$\langle c(f^{W*}; e^*), f^W - f^{W*} \rangle - \langle \Theta(T^*; S^*), T - T^* \rangle + \textstyle\sum_{i \in N} \langle \text{MC}^i(f^{W*}; e^*), e^i - e^{i*} \rangle - \langle \text{MR}^i(T^*; S^*), S^i - S^{i*} \rangle \geq 0 \quad (29)$$

for all $(f^W; e; T; S) \in \Omega$.

The existence of a solution to this generalized model can be established in much the same way as for the simplier model in Proposition 2. Let us assume:

(A-3′) The flow on each arc $a \in A$ is bounded by a positive scalar U_a and the O-D flows on $w \in W$ and $v \in V$ are bounded by positive scalars R_w and R_v respectively.

For example, if the inverse demand function is of the form of a doubly constrained gravity model where O_i and D_j represent the amount of flow out of and into nodes i and j respectively, then the bounds for the O-D flows could be derived from the relation $R_{ij} = \min(O_i, D_j)$. As in the case of Proposition 2, assumption (A-3′) and the continuity of the VI function leads to the following existence result:

PROPOSITION 6. *Under assumptions* (A-1′)–(A-3′), *a solution to the VI formulation of the mixed behavior network equilibrium model* (29) *exists.*

To establish the uniqueness of the solution, one must simply assume:

(A-4′) The composite function $[c(f), \text{MC}(f^W; e), \Theta(T; S), -\text{MR}(T; S)]$ is strictly monotone,

which implies:

PROPOSITION 7. *Under assumption* (A-4′), *there exists at most one solution to the mixed behavior network equilibrium model* (29).

As in the case of Proposition 1, the VI problem (29) will exhibit a mixture of user and "partial" system equilibrium. The first-order conditions (15) and (17) for the simple version of this model are changed in the current situation by replacing u_w and y_v with $\Theta_w(T; S)$ and $\text{MR}_v(T; S)$ respectively. Thus, average cost will equal average revenue, or price, on those O-D pairs obeying the UE principle, and marginal cost will equal marginal revenue on O-D pairs obeying SE.

Finally, the solution to (29) is no more involved than the solution to (12) except for the increased dimensionality of the problem due to the vectors

T and S. The same variational inequality algorithms as described in Section 4 can be employed, with the only difference coming in the LP solution phase. Instead of an all-or-nothing assignment of flow as in the solution to (12), one must solve a Koopmans-Hitchcock transportation problem in the case of a doubly constrained gravity model or some variant therein for other demand functions. Therefore, (29) should not prove to be much more difficult to solve than problem (12).

6. CONCLUSIONS

THIS PAPER has illustrated how a simple variant of the traffic assignment problem can be employed to model mixed user behaviors on the network. The resulting model is not a great deal more complicated to solve than either a UE or SE problem, and it can yield some surprising and counterintuitive results. Such results are very important in the type of freight network modeling described in HARKER[10] since both large and small shippers typically coexist on the same network and thus, unequal market power is typically observed. Extensions of this framework can be made to the spatial price equilibrium model (HARKER[8]) and to any other model of network-based competition, and such extensions should prove to be very useful in the various applied settings in which these types of models are used.

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