Results and Discussion: Optimization of Implicit Signal Weights

1. Experimental Setup and Objective Function

The core challenge addressed in this study is finding the optimal weight vector $\mathbf{w}^*$ for implicit user behavior events (e.g., product_view, add_to_cart, purchase) that maximizes a complex objective function, $f(\mathbf{w})$. This function is designed to enforce a crucial trade-off:

$$\text{Maximize } f(\mathbf{w}) = \underbrace{\mathbf{c}^T\mathbf{w}}_{\text{Engagement}} - \underbrace{1/2 \cdot \mathbf{w}^T\mathbf{Q}\mathbf{w}}_{\text{Penalty (Fairness + Regularization)}}$$

The quadratic term $\mathbf{w}^T\mathbf{Q}\mathbf{w}$ penalizes combinations of weights that lead to:

1. **High Covariance (λ_1):** Events that are highly correlated with each other, reducing the fairness of the resulting user-item scores.
2. **Overfitting (λ_2):** Excessive weight magnitude, promoting a smoother, more regularized solution.

We compared six distinct optimization methods, categorized into three groups, all constrained to the standard simplex ($\mathbf{w} \geq 0$ and $\sum w_i = 1$).

2. Summary of Methodologies and Performance Benchmarking

A. Category I: Baseline Methods

These methods require no iteration or complex matrix solving and serve as non-optimal benchmarks. They inherently fail to utilize the covariance structure ($\mathbf{Q}$) of the data.

Method	Description	Strength	Weakness
Heuristic	Arbitrary, fixed weights (e.g., 10:5:1 ratio for Purchase:Cart:View) normalized to sum to 1.	Simplicity and intuitive prioritization of high-value events.	Fails to incorporate the fairness penalty (λ_1) or data distribution, leading to sub-optimal $f(\mathbf{w})$.
Log-Sum (Uniform)	All event weights are set equally ($\mathbf{w}_i = 1/K$).	Maximizes portability across diverse schemas.	Dilutes the signal of high-impact events, yielding the lowest expected objective score.

B. Category II: Library-Based Solver

Method	Description	Precision & Speed	Complexity
QP Solver (CVXPY)	Solves the convex optimization problem directly using specialized, highly optimized numerical libraries (e.g., OSQP or SCS).	**Highest Precision.** Guarantees the global optimum for the convex objective. **Fastest Execution.**	**Lowest.** Simple API integration for defining the problem and constraints.

C. Category III: Custom Iterative Solvers

These methods rely on iterative updates and require specialized logic for handling the simplex constraints.

Method	Core Mechanism	Convergence Rate & Behavior	Diagnostic Tool
Projected Newton's Method (PNM)	Uses the **Hessian** (matrix $\mathbf{Q}$) to determine a second-order descent direction, then **projects** the result back onto the simplex.	**Quadratic Convergence** near the optimum. Highly effective, but step size is sensitive to projection.	ℓ_2 Norm of the step change ($\|\mathbf{w}_{k+1} - \mathbf{w}_k\|$).
Projected Conjugate Gradient (PCG)	Uses only **gradient** information to generate $\mathbf{Q}$-conjugate search directions, followed by a **projection**.	**Superlinear Convergence.** Slower than PNM but requires less memory (does not compute/store the Hessian).	ℓ_2 Norm of the step change.
Interior Point Method (IPM)	Solves the **KKT conditions** via a primal-dual path-following approach, iteratively reducing the **duality gap (μ)** towards zero.	**Very Fast.** Quadratic convergence. Provides both primal ($\mathbf{w}^*$) and dual solutions.	**Duality Gap (μ)** .

3. Interpretation of Results and Comparison

The ultimate measure of success is the maximized objective value $f(\mathbf{w}^*)$.

A. Objective Score Ranking

The experimental results consistently show the superiority of second-order optimization methods over simple baselines.

Rank	Method	Max Objective Value ($f(\mathbf{w}^*)$)	Key Takeaway
1 (Tie)	**QP Solver (Hierarchy)**	**Highest**	Confirmed as the global optimal solution.
1 (Tie)	**IPM Optimizer**	**Highest (Near-Identical)**	Validated the QP result through an independent, highly precise algorithm.
3	Projected Newton's Method	High (Slightly Lower)	Achieves near-optimal results quickly, limited mainly by the projection step complexity.
4	Projected CG Optimizer	Moderate	Significantly underperforms the second-order methods due to slower convergence.
5	Heuristic Baseline	Low/Moderate	**Crucially, it is sub-optimal.** The arbitrary weights incur an unnecessary fairness penalty.
6	Log-Sum (Uniform)	Lowest	Underscores that maximizing $f(\mathbf{w})$ requires non-uniform, data-driven weight distribution.

B. Analysis of Optimal Weights ($\mathbf{w}^*$)

The **optimal weights** derived by the QP and IPM solutions fundamentally differed from the **Heuristic Baseline** in how risk (covariance) was managed:

- **Heuristic Over-Prioritization:** The fixed Heuristic weights placed disproportionate emphasis on the highest-value event ('purchase'), often leading to a **higher quadratic penalty** due to the event being sparse but highly correlated with other downstream behavior.
- **Optimal Distribution:** The optimal weights distribute the magnitude across events more robustly. The optimization model learns to **de-emphasize** highly correlated events to mitigate the fairness penalty while still maximizing overall engagement. This demonstrates that the derived $\mathbf{w}^*$ represents a **risk-adjusted return** on engagement.

C. Conclusion on Methodological Choice

Requirement	Best Method	Rationale
Production Speed & Robustness	**QP Solver (CVXPY)**	Guaranteed global optimum, fast execution, and minimal code maintenance overhead.
Theoretical Rigor & Validation	**Interior Point Method (IPM)**	Provides comprehensive diagnostic information (primal and dual variables, duality gap convergence) that independently validates the QP solution.
Resource-Constrained Environments	**Projected Newton's Method**	A solid trade-off, achieving a fast convergence rate without requiring the overhead of dual variables like IPM.

Conclusion

This paper successfully defines and solves a formal optimization model for calculating risk-adjusted weights of implicit user-item interaction signals. We demonstrated that simple heuristic weighting schemes are fundamentally sub-optimal, failing to capture the complex trade-off between maximizing user engagement and minimizing covariance-induced fairness penalties.

The **Quadratic Programming Solver** and the **Primal-Dual Interior Point Method** both converge to the global optimal weight vector $\mathbf{w}^*$, proving that the mathematical formulation is both solvable and robust. The resulting optimal weights are not intuitively obvious, as they distribute magnitude to minimize the quadratic penalty, effectively providing a **data-driven, risk-aware weighting scheme** superior to any arbitrary fixed assignment.

This methodology can be generalized across diverse e-commerce schemas, providing a portable and mathematically grounded solution to a challenge traditionally addressed by unreliable heuristics.