Computational Tractability

Plan: Explore the space of computationally hard problems to arrive at a mathematical characterization of a large class of them.

Technique: Compare relative difficulty of different problems.

Loose definition: If problem X is at least as hard as problem Y, it means that if we could solve X, we could also solve Y.

Formal definition:

$Y \leq_p X$ (Y is polynomial time reducible to X)

if Y can be solved using a polynomial number of standard computational steps plus a polynomial number of calls to a blackbox that solves X.

Suppose $Y \leq_p X$, if X can be solved in polynomial time, then Y can be solved in polynomial time.

Suppose $Y \leq_p X$, if Y cannot be solved in polynomial time, then

Independent Set

Def. In a graph $G=(V,E)$, we say that a set of nodes $S \subseteq V$ is "independent" if no two nodes in S are joined by an edge.

Independent set problem

- Find the largest independent set in graph G.

Vertex Cover

Def. Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a vertex cover if every edge in E has at least one end in S.

Vertex Cover problem

- Find the smallest vertex cover set in G.

FACT: Let $G=(V,E)$ be a graph, then S is an independent set if and only if its complement $V-S$ is a vertex cover set.

Proof: A) First suppose that S is an independent set

Claim: Indep. set $\leq_p$ vertex cover

Proof: If we have a blackbox to solve vertex cover, we can decide if G has an independent set of size at least $\underline{k}$, by asking the blackbox

Claim: Vertex Cover $\leq_p$ Indep set

Proof: If we have a blackbox to solve independent set, we can decide if G has a vertex cover set of size at most $\underline{k}$, by asking the blackbox

Set Cover Problem

Given a set U of $\underline{n}$ elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, and a number $\underline{k}$, does there exist a collection of at most $\underline{k}$ of these sets whose union is equal to all of U.

Claim: Vertex Cover $\leq_p$ Set Cover

Need to show that G has a vertex cover of size k, iff the corresponding Set Cover instance has <u>k</u> sets whose union ~~is~~ contains ~~equal to~~ all edges in G.

Proof:

A) If I have a vertex cover set of size $\underline{k}$ in G, I can find a collection of $\underline{k}$ sets whose union contains all edges in G.

B) If I have k sets whose union contains all edges in G, I can find a vertex cover set of size k in G.

Reduction Using Gadgets

- Given n Boolean variables $x_1 \ldots x_n$, a **clause** is a disjunction of terms $t_1 \vee t_2 \vee \ldots t_\ell$ where $t_i \in \{x_1 \ldots x_n, \bar{x}_1 \ldots \bar{x}_n\}$

- A **truth assignment** for X is an assignment of values 0 or 1 to each x_i.

- An assignment **satisfies** a clause C if it causes C to evaluate to 1.

- An assignment satisfies a collection of clauses if

$$C_1 \wedge C_2 \wedge \ldots C_k$$

evaluates to 1.

Problem Statement: Given a set of clauses $C_1 \ldots C_k$ over a set of variables $X = \{x_1 \ldots x_n\}$ does there exist a satisfying truth assignment?

Problem Statement: Given a set of clauses $C_1 \ldots C_k$ each of length $\underline{3}$ over a set of variables $X = \{x_1 \ldots x_n\}$ does there exist a satisfying truth assignment?

Claim: 3SAT $\leq_p$ Independent Set

Plan: Given an instance of 3SAT with <u>k</u> clauses, build a graph G that has an indep. set of size <u>k</u> iff the 3SAT instance is satisfiable.

Claim: The 3-SAT instance is satisfiable iff the graph G has an independent set of size $\underline{k}$.

Proof: A) If the 3-SAT instance is satisfiable, then there is at least one node label per triangle that evaluates to 1.

Let S be a set containing one such node from each triangle

B) Suppose G has an independent set S of size at least k.

if x_i appears as a label in S
then

if $\bar{x}_i$ appears as a label in S
Then

if neither x_i nor $\bar{x}_i$ appear as a label in S, then

Efficient Certification

To show efficient certification:

1. Polynomial length certificate

2. Polynomial time certifier

Efficient certification

3-SAT Certificate $\underline{t}$ is an assignment of truth values to variables (x_i)

Certifier: evaluate the clauses. If all of them evaluate to $\underline{1}$ then it answers yes.

Indep set Certificate $\underline{t}$ is a set of nodes of size at least $\underline{k}$ in G.

certifier: check each edge to make sure no edges have both ends in the set ✓
check size of the set $\geq k$ ✓
no repeating nodes ✓

Class NP is the set of all problems for which there exists an efficient certifier

Discussion 10

1. Given the SAT problem from lecture for a Boolean expression in Conjunctive Normal Form with any number of clauses and any number of literals in each clause. For example,

$$(X_1 \vee \neg X_3) \wedge (X_1 \vee \neg X_2 \vee X_4 \vee X_5) \wedge \ldots$$

Prove that SAT is polynomial time reducible to the 3-SAT problem (in which each clause contains at most 3 literals.)

2. The *Set Packing* problem is as follows. We are given m sets $S_1, S_2, \ldots S_m$ and an integer k. Our goal is to select k of the m sets such that no selected pair have any elements in common. Prove that this problem is **NP**-complete.

3. The *Steiner Tree* problem is as follows. Given an undirected graph $G=(V,E)$ with nonnegative edge costs and whose vertices are partitioned into two sets, R and S, find a tree $T \subseteq G$ such that for every v in R, v is in T with total cost at most C. That is, the tree that contains every vertex in R (and possibly some in S) with a total edge cost of at most C.
Prove that this problem is **NP**-complete.