Non-invertible reflections in quantum spin chains

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SCAN ME

tl;dr

1. We gauge finite abelian modulated symmetries in $\mathbb{Z}_N$ spin chains

$$G_{\text{total}} = A \rtimes_{\varphi} G_{\text{space}} \xrightarrow{\text{Gauge } A} G^{\vee}_{\text{total}} = A \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

➤ Dual symmetry $G^{\vee}_{\text{total}}$ can have different modulation (i.e., $\varphi^{\vee} \neq \varphi$)

2. We establish sufficient conditions for the existence of an isomorphism between G_{total} and $G^{\vee}_{\text{total}}$, naturally implemented by lattice reflections.

➤ Gives rise to non-invertible reflection symmetries, and for ordinary-reflection symmetric models, KW symmetries

Quantum lattice models of $\mathbb{Z}_N$ qudits

We consider 1+1D Hamiltonian lattice models of $\mathbb{Z}_N$ qudits

➤ Degrees of freedom labeled by $g \in \mathbb{Z}_N$ ($N = 2$ is qubit): $\mathscr{H} = \bigotimes_{j=1}^{L} \mathbb{C}^N$

➤ Acted on by generalized Pauli operators

$$X|g\rangle = |g+1\rangle, \qquad Z|g\rangle = (e^{2\pi i/N})^g |g\rangle, \qquad X^N = Z^N = 1, \qquad ZX = e^{2\pi i/N} XZ$$

➤ Enforce periodic boundary conditions $X_j = X_{j+L}$, $Z_j = Z_{j+L}$.

Example: $\mathbb{Z}_N$ clock model, $H_\lambda = -\sum_{j=1}^{L} (Z_j Z^{\dagger}_{j+1} + \lambda X_j) + \text{hc}$

➤ $\mathbb{Z}_N$ symmetry $U = \prod_{j=1}^{L} X_j$, $\mathbb{Z}_L$ translations, and $\mathbb{Z}_2$ reflections.

➤ At $\lambda = 1$, non-invertible symmetry (Kramers-Wannier symmetry)

$$\mathsf{D}_{\text{KW}}: \quad Z_j Z^{\dagger}_{j+1} \to X_j, \qquad X_j \to Z_{j-1} Z^{\dagger}_j$$

A curious example with $\mathbb{Z}_5$ qudits

Consider $\mathbb{Z}_5$ qudits and $H_\lambda = -\sum_{j=1}^{L} (Z_j Z^{\dagger\,2}_{j+1} + \lambda X_j) + \text{hc}$ * *we assume $L \in 4\mathbb{Z}_{>0}$*

➤ $\mathbb{Z}_5$ symmetry* $U = \prod_{j=1}^{L} (X_j)^{3^j}$

➤ $\mathbb{Z}_L$ translation symmetry T

$$TUT^{-1} = U^3 \implies G_{\text{sym}} = \mathbb{Z}_5 \rtimes \mathbb{Z}_L$$

Kramers-Wannier (KW) transformation D_{KW}: $Z_j Z^{\dagger\,2}_{j+1} \to X_j$, $\quad X_j \to Z^2_{j-1} Z^{\dagger}_j$

➤ Can find by gauging the $\mathbb{Z}_5$ symmetry using $G_j = X_j \, \mathcal{X}^{\dagger\,2}_{j-1,j} \mathcal{X}_{j,j+1} = 1$

Under KW transformation D_{KW}: $H_{\lambda=1} \to M H_{\lambda=1} M$, with $M: j \to -j$

➤ D_{KW} does *not* commute with $H_{\lambda=1}$ because $H_\lambda \neq M H_\lambda M$

➤ $\mathsf{D}_{\text{M}} := M\,\mathsf{D}_{\text{KW}}$ does commute with $H_{\lambda=1} \implies$ non-invertible reflection

$$\mathsf{D}_{\text{M}}: \quad Z_j Z^{\dagger\,2}_{j+1} \to X_{-j}, \qquad X_j \to (Z_{-j} Z^{\dagger\,2}_{-j+1})^{\dagger}$$

D_{M} operator algebra (where C: $X_j, Z_j \to X^{\dagger}_j, Z^{\dagger}_j$)

$$\mathsf{D}_{\text{M}}\mathsf{D}_{\text{M}} = \mathsf{C}\left(1 + U + U^2 + U^3 + U^4\right), \qquad \mathsf{D}^{\dagger}_{\text{M}} = \mathsf{D}_{\text{M}}\mathsf{C}$$

$$\mathsf{D}_{\text{M}} U = U \mathsf{D}_{\text{M}} = \mathsf{D}_{\text{M}} \qquad \mathsf{D}_{\text{M}} T = T^{-1} \mathsf{D}_{\text{M}}$$

Modulated symmetries

$\mathscr{S}$ is a modulated symmetry if its symmetry operators transform nontrivially under spatial symmetries.

$$\mathscr{S}_{\text{total}} = \text{“}\mathscr{S} \rtimes_{\varphi} \mathscr{S}_{\text{space}}\text{”} \qquad \varphi: \mathscr{S}_{\text{space}} \to \text{Aut}(\mathscr{S})$$

➤ $\mathscr{S}$ can be a generalized symmetry [Oh, **SP**, Han, You, Lee (2023)]

➤ Symmetry defects are still topological defects

Gauging and Non-invertible reflections

How are finite abelian modulated symmetries gauged on the lattice?

➤ Consider modulated symmetry operators $U_q = \prod_j (X_j)^{f^{(q)}_j}$ with n independent lattice functions $f^{(q)}_j \in \mathbb{Z}_N$ closed under translations

➤ The bond algebra is an algebra of local symmetric operators

$$\mathscr{B} = \langle X_j, \prod_l Z_l^{\Delta^{(a)}_{j,l}} \rangle \quad \text{where} \quad \sum_k \Delta^{(a)}_{j,k} f^{(q)}_k = 0 \mod N$$

Despite the symmetry group being $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$, when there is a single $\Delta_{j,l}$, it can sometimes be gauged using only $\mathbb{Z}_N$ qudits and Gauss's law

$$G_j = X_j \prod_l \mathcal{X}^{\Delta^{\mathsf{T}}_{j,l}}_{l,l+1} = 1$$

➤ We prove this is always true when N is a prime integer

➤ For general N, we prove a sufficient condition based on $\mathscr{F}^{(r)}_{ij} = f^{(i)}_j$, where $i = 1,\cdots,n$ and $j = 1,\cdots,r$, in terms of a generalized inverse $\mathscr{F}^{(r)} \cdot \mathscr{G}^{(r)} \cdot \mathscr{F}^{(r)} = \mathscr{F}^{(r)}$: determinantal rank $\rho(1 - \mathscr{G}^{(r)} \cdot \mathscr{F}^{(r)}) = r - n$

Gauging using G_j and then gauge fixing yields the dual bond algebra

$$\mathscr{B}^{\vee} = \langle \mathcal{Z}_{j,j+1}, \prod_l \mathcal{X}^{\Delta^{\mathsf{T}}_{j,l}}_{l,l+1} \rangle$$

➤ With lattice translation symmetry, $\Delta_{j,l} = \Delta_{0,l-j}$, and

$$\mathscr{B}^{\vee} = \langle \mathcal{Z}_{-j,-j+1}, \prod_l \mathcal{X}^{\Delta_{j,l}}_{-l,-l+1} \rangle \simeq M\,\mathscr{B}\,M$$

➤ Non-invertible reflections = Reflection × KW transformation

Non-invertible reflections in spin chains

Consider $\mathbb{Z}_N$ qudits and $H_\lambda = -\sum_j \left(\prod_k Z_k^{\Delta_{j,k}} + \lambda X_j\right) + \text{hc}$ with $\Delta_{j,k} \in \mathbb{Z}_N$

1. Translation invariance: $\Delta_{j,k} = \Delta_{0,k-j}$

2. Locality, finite range r: $\Delta_{0,k-j} = 0$ if $k - j < 0$ or $k - j > r - 1$
$\Delta_{0,0}, \Delta_{0,r-1} \neq 0 \mod N$

3. Coprime condition: $\gcd(\Delta_{0,0}, N) = \gcd(\Delta_{0,r-1}, N) = 1$

Under these assumptions, we prove that:

➤ H_λ has a modulated $\mathbb{Z}_N^{\times(r-1)}$ symmetry

$$U_f = \prod_j (X_j)^{f_j}, \quad \text{where} \quad f_j \in \mathbb{Z}_N \quad \text{and} \quad \sum_k \Delta_{j,k} f_k = 0 \mod N$$

➤ At $\lambda = 1$, H has the non-invertible reflection symmetry

$$\mathsf{D}_{\text{M}}: \quad \prod_k Z_k^{\Delta_{j,k}} \to X_{-j} \qquad X_j \to \prod_k Z_k^{\dagger\,\Delta_{-j,k}}$$

➤ Satisfies operator algebra:

$$\mathsf{D}_{\text{M}}\mathsf{D}_{\text{M}} = \mathsf{C}\sum_f U_f, \quad \mathsf{D}^{\dagger}_{\text{M}} = \mathsf{D}_{\text{M}}\mathsf{C}, \quad \mathsf{D}_{\text{M}} U_f = U_f \mathsf{D}_{\text{M}} = \mathsf{D}_{\text{M}}, \quad \mathsf{D}_{\text{M}} T = T^{-1}\mathsf{D}_{\text{M}}$$

If H_λ commutes with reflections M, then $\Delta_{j,-k} = \sigma\,\Delta_{-(j+r-1),k}$ $\quad (\sigma = \pm 1)$

➤ At $\lambda = 1$, H has the KW symmetry

$$\mathsf{D}_{\text{KW}} := T^{-\lfloor r/2 \rfloor} M\,\mathsf{D}_{\text{M}}$$

➤ KW transformation related to gauging $\mathbb{Z}_N^{\times r-1}$

➤ Satisfies operator algebra

$$\mathsf{D}_{\text{KW}}\mathsf{D}_{\text{KW}} = \mathsf{C}^{(1+\sigma)/2}\, T^{(r-1 \bmod 2)} \sum_f U_f,$$