

An SPT-LSM theorem for weak SPTs with non-invertible symmetry

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Quantum phases and symmetry

A fundamental problem in quantum condensed matter physics is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

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Sometimes, phases are characterized by a symmetry

- Superfluids by U(1) boson number conservation
- Topological insulators by $U(1)_f$ and \mathbb{Z}_2^T symmetries

For such phases, symmetries provide answers to questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in **generalizing** the notion of **symmetries**

- Ordinary symmetries transform local operators in an invertible manner (e.g., $c_r^\dagger \rightarrow e^{i\theta} c_r^\dagger$)
- So-called **generalized symmetries** modify this definition

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Non-invertible symmetries have non-invertible transformations

[Bhardwaj, Tachikawa 2017; Chang, Lin, Shao, Wang, Yin 2018; ...]

- Can arise at **critical points** from Kramers-Wannier dualities
[... ; Choi, Córdova, Hsin, Lam, Shao 2021; ... ; Seiberg, Shao 2023; **SP**, Delfino, Lam, Aksoy 2024]
- Can emerge in **ordered phases** (are symmetries of nonlinear sigma models) [**SP** 2023; **SP**, Zhu, Beaudry, X-G Wen 2023]

Generalized symmetries

Q: Why should we consider these as **symmetries**?

The answer is that they have some nice properties.

For example, they are closed under composition.

That is, if σ and τ are symmetries, then $\sigma \circ \tau$ is also a symmetry.

Another nice property is that they form a group.

No, not exactly. They form a monoid.

That is, they have an identity element, but they are not necessarily closed under composition.

For example, if σ is a symmetry and τ is not, then $\sigma \circ \tau$ is not a symmetry.

Generalized symmetries

Q: Why should we consider these as symmetries?

A: They pass the **duck test**!



*If it looks like a duck, swims
like a duck, and quacks like a
duck, then it probably is a duck.*

- Have conservation laws
 - Can constrain phase diagrams (be anomalous)
 - Can characterize SSB and SPT phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by
generalized symmetries?

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Which quantum phases are characterized by
generalized symmetries?

Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$a_1 - a_2 - a_3 - \dots$ Symmetry

(2) Specify SSB and SPT pattern

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Ordered phases

Topological insulators

Topological order

Maxwell phases

Higgs phases

Fracton phases

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Phases we have yet to name!

Quantum phases + generalized symmetry

Which quantum phases are characterized by
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Why care?

1. Provides a **novel** and **unifying** perspective of **quantum phases**
2. Guides us towards new **quantum phases** and models
3. Further develops a classification of **quantum phases** based on **symmetries** (a “generalized Landau paradigm”)

TL;DR for this talk

In this talk, we explore Symmetry Protected Topological (SPT) phases characterized by non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries

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In this talk, we explore Symmetry Protected Topological (SPT) phases characterized by non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries

Outline:

1. Review SPTs (from a symmetry defect perspective)
2. Simple example of an entangled weak SPT characterized by a projective non-invertible symmetry
3. General discussion on (SPT-)LSM theorems from projective non-invertible symmetries

What are SPTs

An SPT phase is a gapped quantum phase described by a symmetry with a unique ground state on all closed spatial manifolds [Chen, Gu, Liu, Wen 2011; Wang, Senthil 2013; Else, Nayak 2014; ...]

- Interesting physics can arise on boundaries and interfaces between SPTs (e.g., topological order, gapless excitations)

SPTs are characterized in the bulk by their response to static probes

- Background gauge fields and symmetry defects

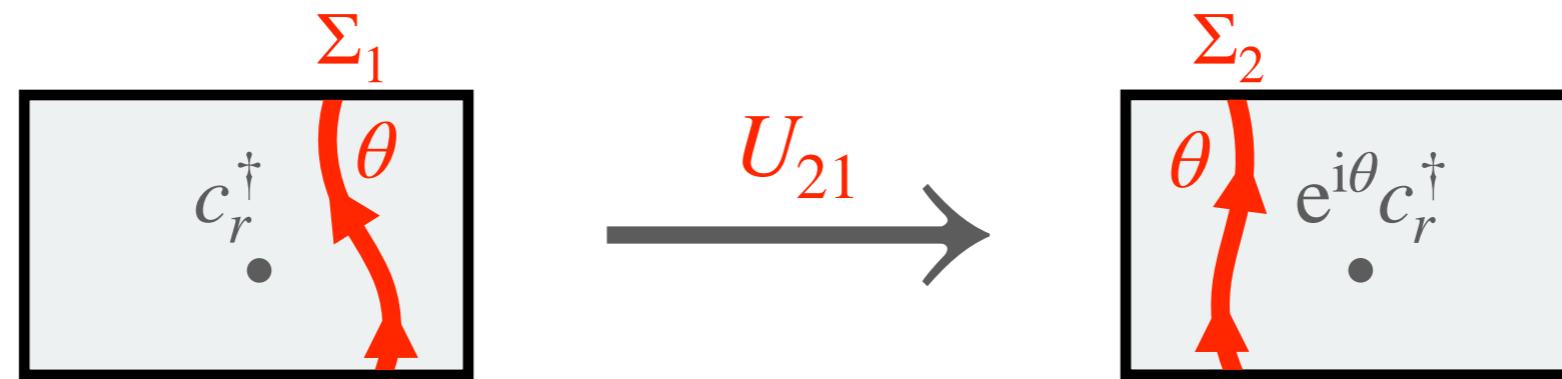
Symmetry defects

Symmetry defects (domain walls) are localized modifications to the Hamiltonian $H_{\text{defect}}^{(\Sigma)} = H + \delta H(\Sigma)$ and other operators

- Moved using unitary operators (are topological defects)

$$H_{\text{defect}}^{(\Sigma_2)} = U_{21} H_{\text{defect}}^{(\Sigma_1)} U_{21}^\dagger$$

- Implement the symmetry transformation across space



- Twisted boundary conditions $(T_\perp)^L = \text{Symmetry operator}$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d periodic lattice with **two qubits** on each site $j \sim j + L$ acted on by **Pauli operators** X_j, Z_j and \tilde{X}_j, \tilde{Z}_j .

$$H_p = - \sum_{j=1}^L (X_j + \tilde{X}_j)$$

$$H_c = - \sum_{j=1}^L (\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1})$$

$$|\text{GS}_p\rangle = |+++ \cdots +\rangle$$

$$|\text{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\text{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\text{GS}_c\rangle$$

- Both models have a **unique symmetric gapped ground state**
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\text{GS}_.\rangle = \tilde{U}|\text{GS}_.\rangle = |\text{GS}_.\rangle$

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Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a $\textcolor{teal}{U}$ symmetry defect at $\langle L, 1 \rangle$

- Gives rise to $\textcolor{brown}{U}$ -twisted boundary conditions: $Z_{j+L} = -Z_j$
- 1. H_p is unaffected, so its ground state still satisfies

$$U|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle \quad \tilde{U}|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle$$

- 2. H_c becomes $\textcolor{teal}{H}_c + 2Z_L \tilde{X}_L Z_1$, and its ground state satisfies

$$U|\text{GS}_{c;U}\rangle = +|\text{GS}_{c;U}\rangle \quad \tilde{U}|\text{GS}_{c;U}\rangle = -|\text{GS}_{c;U}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can distinguish them by looking at the domain wall decorations.

► G_{c} has a domain wall between two \mathbb{Z}_2 SPT phases, while G_p does not.

Different **domain wall** decorations imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

[Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

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Example: \mathbb{Z}_2 weak SPTs

1d periodic lattice with a **qubit** on each site $j \sim j + L$

$$H_+ = - \sum_j X_j \quad \text{vs.} \quad H_- = + \sum_j X_j$$

- Both have a unique gapped ground state $|\text{GS}_{\pm}\rangle = \otimes_j |\pm\rangle$
- **Symmetries:** $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \rightarrow j + 1$

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SPTs characterized by translations are called weak SPTs

H_+ and H_- are both in \mathbb{Z}_2 weak SPT phases

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

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	Even L	Even L , \mathbb{Z}_2 symmetry defect
$U \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$+ \text{GS}_{\pm} \rangle$
$T \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$\pm \text{GS}_{\pm} \rangle$

Different \mathbb{Z}_2 weak SPTs

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Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Translation defect carries \mathbb{Z}_2 symmetry charge in $|\text{GS}_-\rangle$

- Inserting a translation defect is done by

$$T^L = 1 \rightarrow T^L = T \implies L \rightarrow L - 1$$

- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

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A curious Hamiltonian

1d periodic lattice with a single **qubit** and \mathbb{Z}_4 qudit on each site $j \sim j + L$ [SP, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on **qubits**: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on \mathbb{Z}_4 qudits: $X^4 = Z^4 = 1$ and $ZX = iXZ$

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$$H = - \sum_j \sigma_j^x C_{j+1} \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger)$$

- C acts as $X \rightarrow X^\dagger$ and $Z \rightarrow Z^\dagger$
- Is a sum of commuting terms and has a **unique gapped ground state**

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- C
 - Is
 - gr
- $$|GS\rangle = \sum_{\substack{\{\varphi_j=0,1\} \\ \{\alpha_j=0,2\}}} i^{\sum_j \alpha_j(\varphi_j - \varphi_{j-1})} \bigotimes_j |\sigma_j^x = (-1)^{\varphi_j}, Z_j = i^{\alpha_j+1}\rangle$$

Some curious symmetries

$$H = - \sum_j \sigma_j^x C_{j+1} \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger)$$

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What are the **symmetries** of H ?

- \mathbb{Z}_L lattice **translations** $T: j \rightarrow j + 1$
- Three **\mathbb{Z}_2** symmetry operators

$$U = \prod_j X_j^2, \quad R_1 = \prod_j \sigma_j^z, \quad R_2 = \prod_j Z_j^2$$

Some curious symmetries

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- 🤔 symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

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What are the **symmetries** of H ?

- \mathbb{Z}_2 symmetry operator
 - R_E is a **non-invertible symmetry** operator
 - $R_E |\psi\rangle = 0$ if $R_1 |\psi\rangle = -|\psi\rangle$ or $R_2 |\psi\rangle = -|\psi\rangle$
 - R_E have zero-eigenvalues $\implies R_E$ is non-invertible
-  symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$UR_E = (-1)^L R_E U$$

- Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry

Dihedral group of order 8 $D_8 \simeq \langle r, s \mid r^2 = s^4 = 1, rsr = s^3 \rangle$

- Four 1d reps $1, P_1, P_2, P_3 = P_1 \otimes P_2$ and one 2d irrep E

$$P_i \otimes P_i = 1 \quad E \otimes E = 1 \oplus P_1 \oplus P_2 \oplus P_3 \quad E \otimes P_i = P_i \otimes E = E$$

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Ground state satisfies

$$T|GS\rangle = +|GS\rangle \quad U|GS\rangle = +|GS\rangle \quad R_1|GS\rangle = +|GS\rangle$$

$$R_2|GS\rangle = \begin{cases} +|GS\rangle, & L \text{ even} \\ -|GS\rangle, & L \text{ odd} \end{cases}$$

$$R_E|GS\rangle = \begin{cases} +2|GS\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1,$$

H is in a $\mathbb{Z}_2 \times \text{Rep}(D_8)$ weak SPT phase

- Translation defects carry $\text{Rep}(D_8)$ symmetry charge in $|\text{GS}\rangle$
- Form a (projective, $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetric)

$$R_i R_E = R_E$$

Ground state satisfies

$$T|\text{GS}\rangle = +|\text{GS}\rangle$$

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A curious projective algebra

This SPT is characterized by a projective symmetry:

$$UR_E = -R_E U \quad (\text{odd } L)$$

Projective unitary symmetries $U_1 U_2 = e^{i\theta} U_2 U_1$ forbid SPTs

► Assume non-degenerate symmetric ground state:

$$\left. \begin{array}{l} 1. \langle \psi | U_1 U_2 | \psi \rangle = \langle \psi | \psi \rangle = 1 \\ 2. \langle \psi | U_1 U_2 | \psi \rangle = e^{i\theta} \langle \psi | U_2 U_1 | \psi \rangle = e^{i\theta} \end{array} \right\} \begin{array}{l} \textit{Contradicts} \\ \textit{assumption} \end{array}$$

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Projective non-invertible symmetries are compatible with SPTs

- Loophole: symmetry operator has zero-eigenvalues
- $UR_E = (-1)^L R_E U$ enforces $R_E | \text{GS}_{\text{SPT}} \rangle = 0$ when L is odd

The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the **projective algebras**

U symmetry defect	R_E symmetry defect
$R_E T = -TR_E$	$TU = -UT$

For invertible symmetries, such **projective algebras** imply an 't Hooft anomaly (e.g., the type III anomaly $(-1)^{\int_{M_3} a \cup b \cup c}$)

[Matsui 2008; Yao, Oshikawa 2020; Seifnashri 2023; Kapustin, Sopenko 2024]

- This is not true for **non-invertible symmetries!**

The surprising lack of an 't Hooft anomaly

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Fails because of
 $R_E = 0$ loophole

[2023; Kapustin, Sopenko 2024]

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The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the projective algebras

U symmetry defect	R_E symmetry defect
$R_E T = -TR_E$	$TU = -UT$

For invertible symmetries, such as the type I
2023; Kapustin, Willett, and Zarembo

Fails because of the $R_E = 0$ loophole

Fails because the degeneracy is encoded in the defect's quantum dimension

Projective $Z(G) \times \text{Rep}(G)$ symmetry

The **projective** $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry is a **special case** of a more general **projective** $Z(G) \times \text{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

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- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

$Z(G)$ **symmetry** operator U_z , with $z \in Z(G)$, satisfies

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\text{Rep}(G)$ **symmetry** operator R_Γ , with Γ an irrep of G , satisfies

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\bigoplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- **Non-invertible symmetry** when G is non-Abelian

Projective $Z(G) \times \text{Rep}(G)$ symmetry

The **projectivity** arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma \quad \text{with } e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

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e.g., $e^{i\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ ($Z(\mathbb{Z}_2) = \mathbb{Z}_2$)

Γ	1	sign
$+1$	$+1$	$+1$
-1	$+1$	-1

Projective $Z(G) \times \text{Rep}(G)$ symmetry

The **projectivity** arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma \quad \text{with } e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

e.g., $e^{i\phi_\Gamma(z)}$ when $G = D_8$ ($Z(D_8) = \mathbb{Z}_2$)

Γ	1	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	E
z	+1	+1	+1	+1	+1
	-1	+1	+1	+1	-1

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	-1	+1	+1	+1	-1

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \dots, zg_L\rangle \langle g_1, \dots, g_L| \quad R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \dots, g_L\rangle \langle g_1, \dots, g_L|$$

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e.g., $e^{i\phi_\Gamma(z)}$

Projective algebras also arise from inserting
symmetry defects [SP, Lam, Aksoy arXiv:2409.18113]

$z \in Z(G)$ defect	$\Gamma \in \text{Rep}(G)$ defect
$R_\Gamma T_{\text{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\text{tw}}^{(z)} R_\Gamma$	$T_{\text{tw}}^{(\Gamma)} U_z = e^{i\phi_\Gamma(z)} U_z T_{\text{tw}}^{(\Gamma)}$

Explains

$\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \dots, zg_L\rangle \langle g_1, \dots, g_L| \quad R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \dots, g_L\rangle \langle g_1, \dots, g_L|$$

(SPT)-LSM theorems

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$$

There is an **Lieb-Schultz-Mattis (LSM) theorem** when $e^{i\phi_\Gamma(z)}$ is non-trivial for a unitary R_Γ

[…; Matsui 2008; Chen, Gu, Wen 2010;
Yao, Oshikawa 2020; Ogata, Tasaki 2021;
Seifnashri 2023; Kapustin, Sopenko 2024]

- The **LSM theorem** forbids SPT phases
- The ground state always has long-range entanglement

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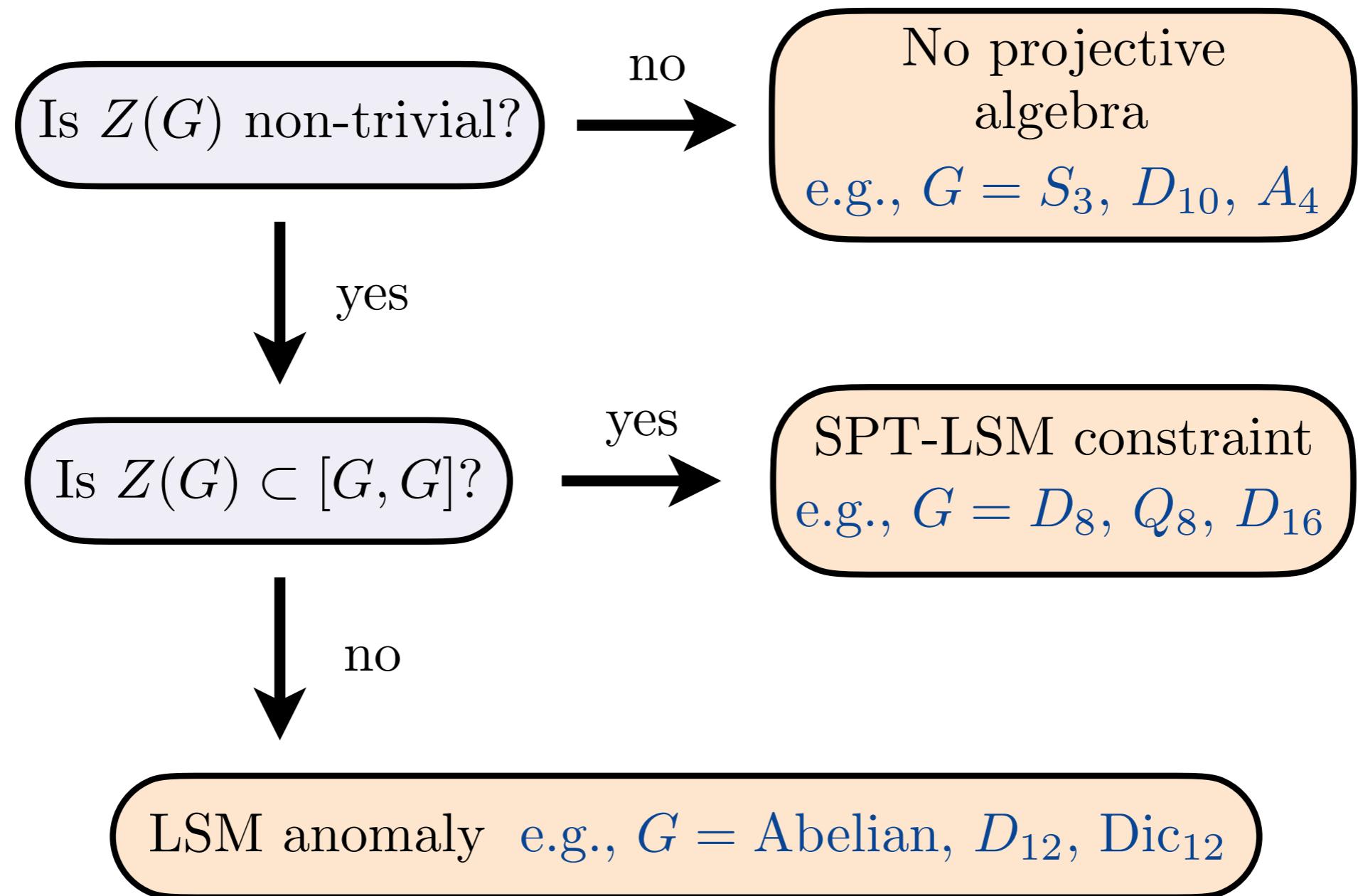
- The **LSM theorem** forbids SPT phases
- The ground state always has long-range entanglement

When there is no **LSM theorem**, the **projective algebra** gives rise to an **SPT-LSM theorem**

- Any **SPT state** must have non-zero entanglement
- [Lu 2017; Yang, Jiang, Vishwanath, Ran 2017; Lu, Ran, Oshikawa 2017; …]

(SPT)-LSM theorems

Whether there is an (SPT)-LSM theorem depends on G :



Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its ground state to satisfy $\textcolor{orange}{R_\Gamma | GS \rangle = 0}$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

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Two possibilities:

1. An SPT state satisfies $\textcolor{red}{R}_\Gamma |\text{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(e^{i\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $\textcolor{green}{R}_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$, but $\textcolor{red}{R}_\Gamma |\text{GS}\rangle = 0$ for $L \neq L^*$

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If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its ground state to satisfy $\textcolor{brown}{R}_\Gamma |\text{GS}\rangle = \mathbf{0}$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

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The first is incompatible with 1 + 1D TQFT, where $\langle R_\Gamma \rangle = d_\Gamma$

[Chang, Lin, Shao, Wang, Yin 2018]

- Reasonable to assume that this SPT state at some $L = L^*$ is described by a TQFT in the IR

Non-invertible weak SPT

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ground state to satisfy $R_\Gamma |GS\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

At $L = L^*$, SPTs satisfy $R_\Gamma |GS\rangle = \lambda_\Gamma |GS\rangle$

At $L = L^* + 1$, SPTs satisfy $R_\Gamma |GS\rangle = 0$

► All SPT states have translation defects dressed by
non-trivial $\text{Rep}(G)$ symmetry charge

► \exists a trivial SPT \implies SPT-LSM theorem

[Chang, Lin, Shao, Wang, Yin 2018]

► Reasonable to assume that this SPT state at some $L = L^*$
is described by a TQFT in the IR

SPT-LSM theorem

To prove this SPT-LSM theorem, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \text{ which satisfies } R_\Gamma U_j^{(z)} = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

2. Assume that any unique gapped ground state $|GS\rangle$ satisfies $R_\Gamma |GS\rangle \neq 0$ for some system size $L = L^*$

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2. Assume that any unique gapped ground state $|GS\rangle$ satisfies $R_\Gamma |GS\rangle \neq 0$ for some system size $L = L^*$

We prove this assumption for product states in $\otimes_j \mathbb{C}^{|G|}$, where

$$R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \dots, g_L\rangle \langle g_1, \dots, g_L|$$

but it is true as long as there is an IR TQFT description

SPT-LSM theorem

If there is a unique gapped $|GS\rangle$ that is a product state:

► $U_z|GS\rangle = |GS\rangle \implies U_j^{(z)}|GS\rangle = |GS\rangle$

Using the assumption, $R_\Gamma|GS\rangle = \lambda_\Gamma|GS\rangle$ at $L = L^*$:

$$\left. \begin{array}{l} 1. \quad R_\Gamma U_j^{(z)}|GS\rangle = R_\Gamma|GS\rangle = \lambda_\Gamma|GS\rangle \\ 2. \quad R_\Gamma U_j^{(z)}|GS\rangle = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma|GS\rangle = \lambda_\Gamma e^{i\phi_\Gamma(z)}|GS\rangle \end{array} \right\} \text{Contradiction}$$

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⇒ Cannot be an SPT state that is a product state at $L = L^*$

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⇒ Cannot be an SPT state that is a product state at $L = L^*$

⇒ By locality, there cannot be an SPT state that is a product state for any L

SPT-LSM theorem

If there is a unique gapped $|GS\rangle$ that is a product state:

$$\blacktriangleright U_z |GS\rangle = |GS\rangle \implies U_j^{(z)} |GS\rangle = |GS\rangle$$

Using $U_j^{(z)} = \sum_i z_i |GS\rangle \langle GS|$, we find

Therefore, the **projective non-invertible symmetry**

1. R_z prevents a product state SPT

2. R_j \blacktriangleright All SPTs must have **non-zero entanglement**

\implies Cannot be an SPT state that is a product state at $L = L^*$

\implies By locality, there cannot be an SPT state that is a product state for any L

Outlook

We found a new class of entangled weak SPTs characterized by a **projective** $Z(G) \times \text{Rep}(G)$ **non-invertible symmetry**

1. An exactly solvable model in a **weak SPT phase** characterized by a **projective** $\mathbb{Z}_2 \times \text{Rep}(D_8)$ **symmetry**
2. General discussion on decorated domain wall pattern of these $Z(G) \times \text{Rep}(G)$ weak SPTs \implies an **SPT-LSM theorem**

New **quantum phases** and models can be discovered using **generalized symmetries** as a guide!

Back-up slides

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^L J \sigma_j^x \sigma_{j+1}^x + K \sigma_j^y \sigma_{j+1}^y$$

- There is an **LSM anomaly** involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the **projective algebras** [Cheng, Seiberg 2023]

<i>Translation defects</i>	\mathbb{Z}_2^x defect	\mathbb{Z}_2^y defect
$U_x U_y = (-1)^L U_y U_x$	$U_y T = - T U_y$	$T U_x = - U_x T$

GROUP BASED QUDITS

A **G -qudit** is a $|G|$ -level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a **finite group**, e.g. $\mathbb{Z}_2, S_3, D_8, \text{SmallGroup}(32,49)$

Group based **Pauli operators** [Brell 2014]

- X operators labeled by group elements

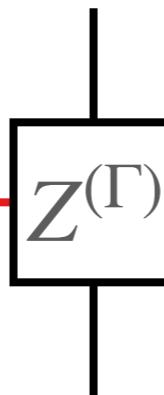
$$\vec{X}^{(g)} = \sum_h |gh\rangle\langle h|$$

$$\overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

$$\bar{g} \equiv g^{-1}$$

- Z operators are MPOs labeled by **irreps** $\Gamma: G \rightarrow \text{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv \alpha \xrightarrow{\hspace{-0.5cm}} Z^{(\Gamma)} \xrightarrow{\hspace{-0.5cm}} \beta \quad (\alpha, \beta = 1, 2, \dots, d_\Gamma)$$



GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{1, 1'\}$

$$\vec{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(1)}]_{11} = 1$$

$$\vec{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \quad [Z^{(1')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

1. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(gh)}$, $\overleftarrow{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\vec{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)} \vec{X}^{(g)}$
2. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(h)} \vec{X}^{(g)}$ iff g and h commute
3. $\vec{X}^{(g)} [Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma} [Z^{(\Gamma)}]_{\gamma\beta} \vec{X}^{(g)}$
4. **Unitarity:** $\vec{X}^{(g)\dagger} = \vec{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger} Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

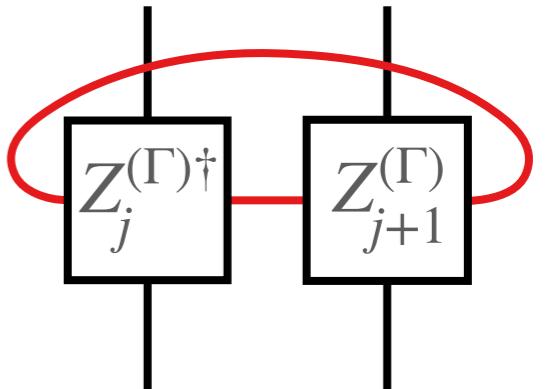
GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based XY model: Consider a periodic 1d lattice of L sites. On each site j resides a G -qudit and its Hamiltonian

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$$\text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_{\Gamma}(\bar{g}_j g_{j+1}) | \{g\} \rangle \langle \{g\} | \equiv$$



- For $G = \mathbb{Z}_2$, this is the ordinary quantum XY model

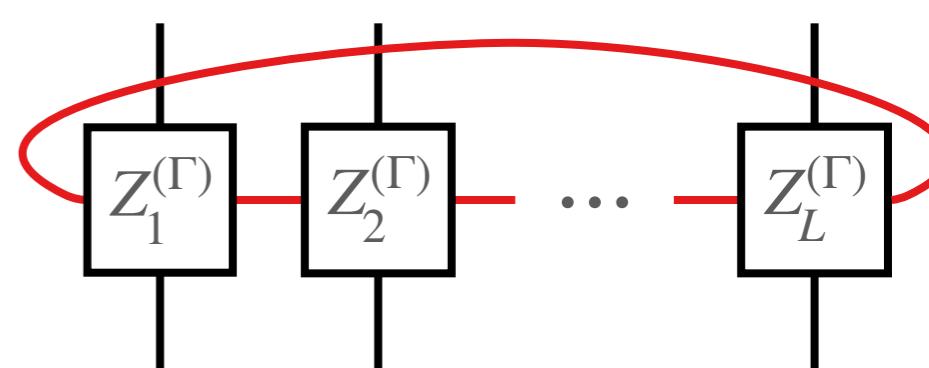
SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

\mathbb{Z}_L lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

► $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$

► $\text{Rep}(G)$ symmetry $R_{\Gamma} = \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\bigoplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \vec{X}_j^{(z)}$$

$$T_{\text{tw}}^{(z)} = \vec{X}_I^{(z)} T$$

$$R_\Gamma = \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right)$$

$$T_{\text{tw}}^{(\Gamma)} = \hat{Z}_I^{(\Gamma)} (T \otimes \mathbf{1})$$

Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

<i>Translation defects</i>	$z \in Z(G)$ defect	$\Gamma \in \text{Rep}(G)$ defect
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$	$R_\Gamma T_{\text{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\text{tw}}^{(z)} R_\Gamma$	$T_{\text{tw}}^{(\Gamma)} U_z = e^{i\phi_\Gamma(z)} U_z T_{\text{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ **projective algebra** of the ordinary quantum XY model

GAUGING WEB

