HIGHER-FORM SYMMETRIES AND TOPOLOGICAL PHASES

Sal Pace (MIT)

SP & X-G Wen, PRB 106, 045145 (2022)
SP & X-G Wen, PRB 107, 075112 (2023)
Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706
SP & X-G Wen, arXiv:2301.05261

THE MANY-BODY SAGA

Separation of scales

UV Scale → IR Scale

Short distance physics

Large distance, low-energy physics

Know UV degrees of freedom (electrons, neutrons, etc)

Emergent IR degrees of freedom (quasiparticles, strings, etc)

A GUIDING PRINCIPLE

UV Scale $\longrightarrow$ IR Scale

Too complicated: describes all but hides much

Reveals universal properties but hides their cause

It's desirable to have a set of guiding principles to help us understand their interplay

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$\boxed{\text{Symmetry}}$

THE SYMMETRY RENAISSANCE

Our understanding of symmetry has been recently revolutionized through modern generalizations

McGreevy, arXiv:2204.03045 (2022)
Córdova, Dumitrescu, Intriligator, & Shao, arXiv:2205.09545 (2022)

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Our understanding of symmetry has been recently revolutionized through modern generalizations

Higher-form symmetry

Non-invertible symmetry

Dipole symmetry

Loop group symmetry

Subsystem symmetry

Higher-group symmetry

Biform symmetry

Harmonic symmetry

Framed symmetry

Categorical symmetry

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WHY ARE THESE SYMMETRIES

- There's a symmetry operator that commutes with the Hamiltonian
- The objects carrying the symmetry charge can condense, causing spontaneous symmetry breaking
- Are associated with conservation laws that constrain dynamics
- Can have 't Hooft anomalies that constrain possible phases of matter

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Passes the duck test!

If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

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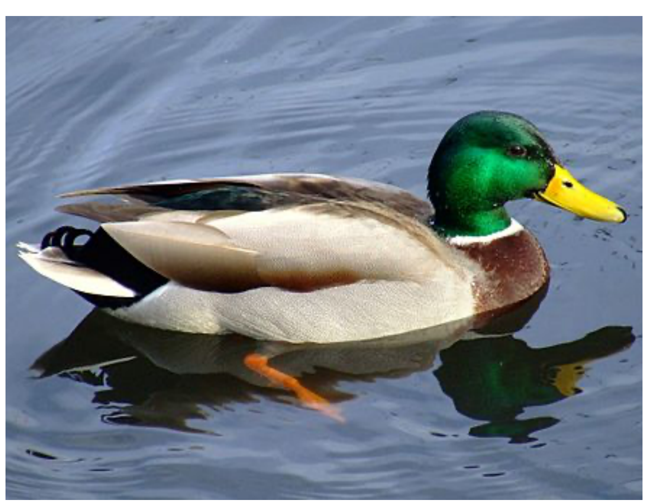

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GENERALIZED SYMMETRIES IN CMP

UV Scale $\longrightarrow$ IR Scale

G_{UV} $\quad$ $G_{\text{mid}-\text{IR}}$ $\quad$ G_{IR}

G_{UV} typically includes ordinary symmetries, but $G_{\text{mid}-\text{IR}}$ and G_{IR} can include emergent generalized symmetries

- Unifying perspective on different phases of matter
- New phases of matter characterized by generalized symmetries
- Classification scheme (generalized Landau paradigm)

THE PLAN FOR THIS TALK

Explore how higher-form symmetries arise in topological phases of quantum matter

1. How higher-form symmetries emerge & exact emergent symmetries
   [**SP** & X-G Wen, arXiv:2301.05261]
2. Symmetry Protected Trivial (SPT) phase protected by higher-form symmetries
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3. The rank-2 toric code, its symmetries, and UV/IR mixing
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HIGHER-FORM SYMMETRIES

Consider bosonic lattice model with Hilbert space $\mathscr{H} = \bigotimes \mathscr{H}_i$.

Ordinary symmetries act on entire lattice.

$$U = \prod_{i \in \text{lattice}} U_i \quad \text{and} \quad UH = HU$$

p-form symmetry acts on codimension p ($d - p$ dimensional) closed hypersurface Σ_{d-p}

$$U(\Sigma_{d-p}) = \prod_{i \in \Sigma_{d-p}} U_i \quad \text{and} \quad U(\Sigma_{d-p})H = HU(\Sigma_{d-p}) \; \forall \; \Sigma_{d-p}$$

- For a G p-form symmetry, $U : \; Z_{d-p}(\text{lattice}; G) \to \mathscr{U}(\mathscr{H})$

Zohar & Ortiz, PNAS 106.40 (2009): 16944-16949.
Gaiotto, Kapustin, Seiberg, & Willett, JHEP, 2015, 172 (2015)

HIGHER-FORM SYMMETRIES

Ordinary (0-form) symmetries transform operators acting on points

➤ Ex) $U(1)^{(0)}$ symmetry: $U_\alpha\, b_i\, U_\alpha^\dagger = \exp[i\alpha]\, b_i$

p-form symmetry transform operators acting on a p dimensional hypersurface C_p

$\#(\Sigma_{d-p}, C_p) \neq 0$ only if Σ_{d-p} and C_p are non-contractible

➤ Ex) $U(1)^{(p)}$ symmetry:

$$U_\alpha(\Sigma_{d-p})\; W(C_p)\; U_\alpha^\dagger(\Sigma_{d-p}) = \exp[i\alpha\; \#(\Sigma_{d-p}, C_p)\;]\; W(C_p)$$

Just as b_i creates a bosonic particle at site i, $W(C_p)$ creates a p-dimensional excitation at C_p

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➤ Ex) $U(1)^{(p)}$

$U_\alpha(\Sigma_{d-p}$ ... $]\; W(C_p)$

<u>$U(1)^{(p)}$ *Conservation laws:*</u>

$p = 0$: *Particle number conserved*

$p = 1$: *String flux conserved*

$p = 2$: *Membrane flux conserved*

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EXAMPLE: 2+1D $\mathbb{Z}_2$ TORIC CODE

Square lattice with one qubit on each edge with $\Delta_e, \Delta_m > 0$

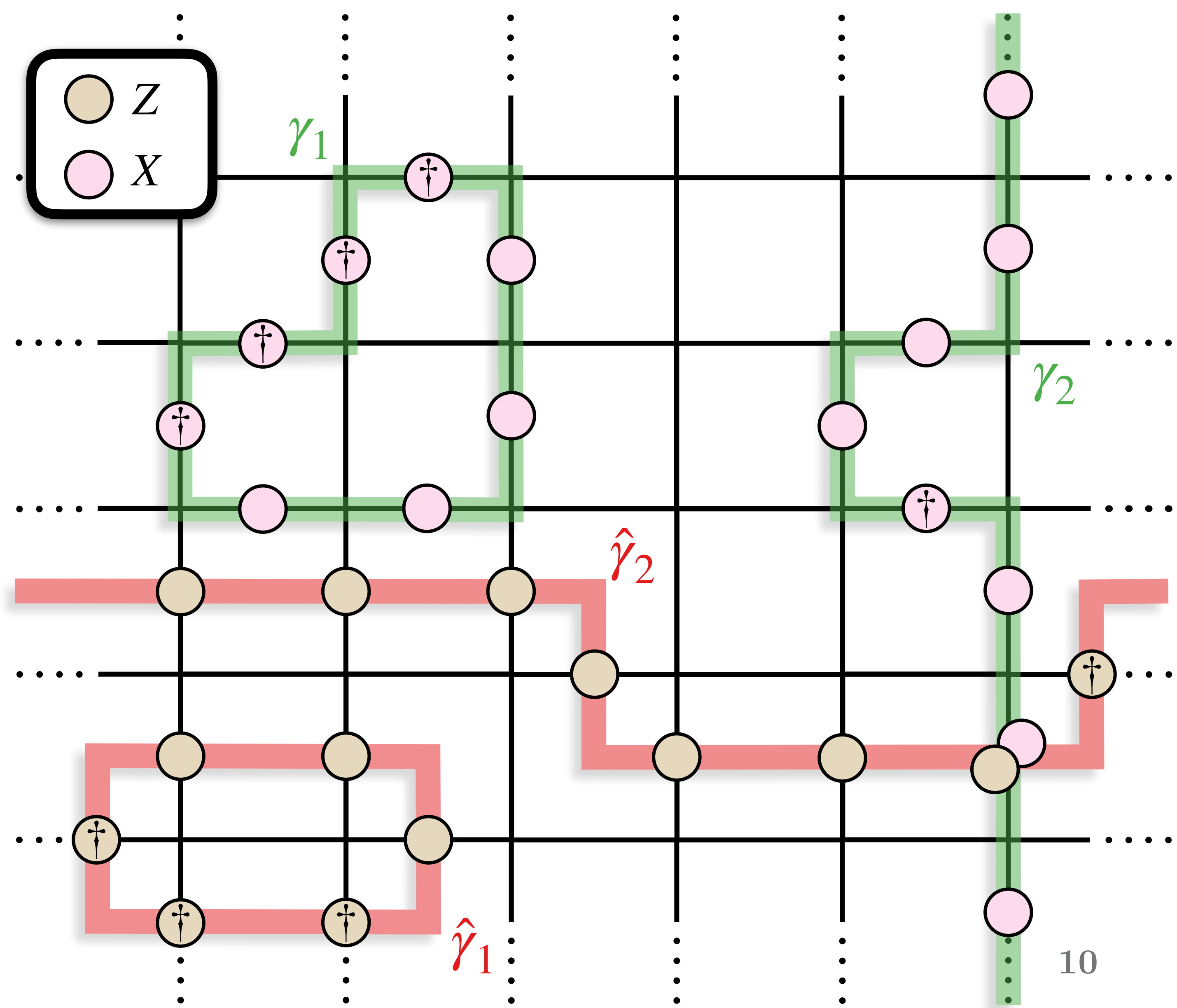
$$H = -\frac{\Delta_e}{2}\sum_s \prod_{e\in\delta s} Z_e - \frac{\Delta_m}{2}\sum_p \prod_{e\in\partial p} X_e$$

Exact $\mathbb{Z}_N^{(1)}\times\mathbb{Z}_N^{(1)}$ symmetry:

$$U_m(\gamma) = \prod_{e\in\gamma} X_e$$

$$U_e(\hat{\gamma}) = \prod_{e\perp\hat{\gamma}} Z_e$$

$U_m(\gamma)$ and $U_e(\hat{\gamma})$ do not commute: manifestation of mixed 't Hooft anomaly



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$E \geq \Delta_m$ $\quad \langle U_m(\gamma_1)\rangle = \langle U_m(\gamma_2)\rangle$ if $\gamma_1 = \gamma_2$

Faithful 1-form symmetry

$E < \Delta_m$ $\quad \langle U_m(\gamma_1)\rangle = \langle U_m(\gamma_2)\rangle$ iff $[\gamma_1] = [\gamma_2] \in H_1(\text{lattice}; \mathbb{Z}_N)$

Topological 1-form symmetry

$$U_m(\gamma_2 = \gamma_1 + \partial M) = U_m(\gamma_1)\prod_{p\in M}\prod_{e\in\partial p} X_e = U_m(\gamma_1)$$

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<u>Ground state subspace</u> ($E < \Delta_e, \Delta_m$)

$U_e(\hat{\gamma})$ ($U_m(\gamma)$) is the charged operator of $U_m(\gamma)$ ($U_e(\hat{\gamma})$)

➤ $W_m(\hat{\gamma}) = U_e(\hat{\gamma})$ and $W_e(\gamma) = U_m(\gamma)$

$$\langle W_m(\hat{\gamma} = \partial \hat{M})\rangle = \langle W_e(\gamma = \partial M)\rangle = 1$$

➤ $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$ SSB phase

$\mathbb{Z}_2$ TORIC CODE IN TRANSVERSE FIELD

$$H = -\frac{\Delta_e}{2}\sum_s \prod_{e\in\delta s} Z_e - \frac{\Delta_m}{2}\sum_p \prod_{e\in\partial p} X_e - h_x \sum_e X_e$$

$$HU_m(\gamma) = U_m(\gamma)H$$

$$HU_e(\hat{\gamma}) \neq U_e(\hat{\gamma})H$$

X_ℓ no longer excites e anyons on $\partial\ell$. The operator that does is $\tilde{X}_\ell = UX_\ell U^\dagger$, dressed due to quantum fluctuations

- U is a unitary such that $\tilde{X}_\ell$ is a local operator, acting on a neighborhood of edges near ℓ (fattened operator)

For $h_x \ll \Delta_e$, low-energy states satisfy $\prod_{e\in\delta s} \tilde{Z}_e = 1 \Longrightarrow$ emergent conservation law ($\mathbb{Z}_N$ electric flux)$\Longrightarrow$ emergent topological $\mathbb{Z}_N^{(1)}$ symmetry

$$U_e(\hat{\gamma}) = \prod_{e\perp\hat{\gamma}} \tilde{Z}_e$$

Hastings & Wen, PRB 72.4 (2005): 045141.

***SP** & Wen, arXiv:2301.05261*

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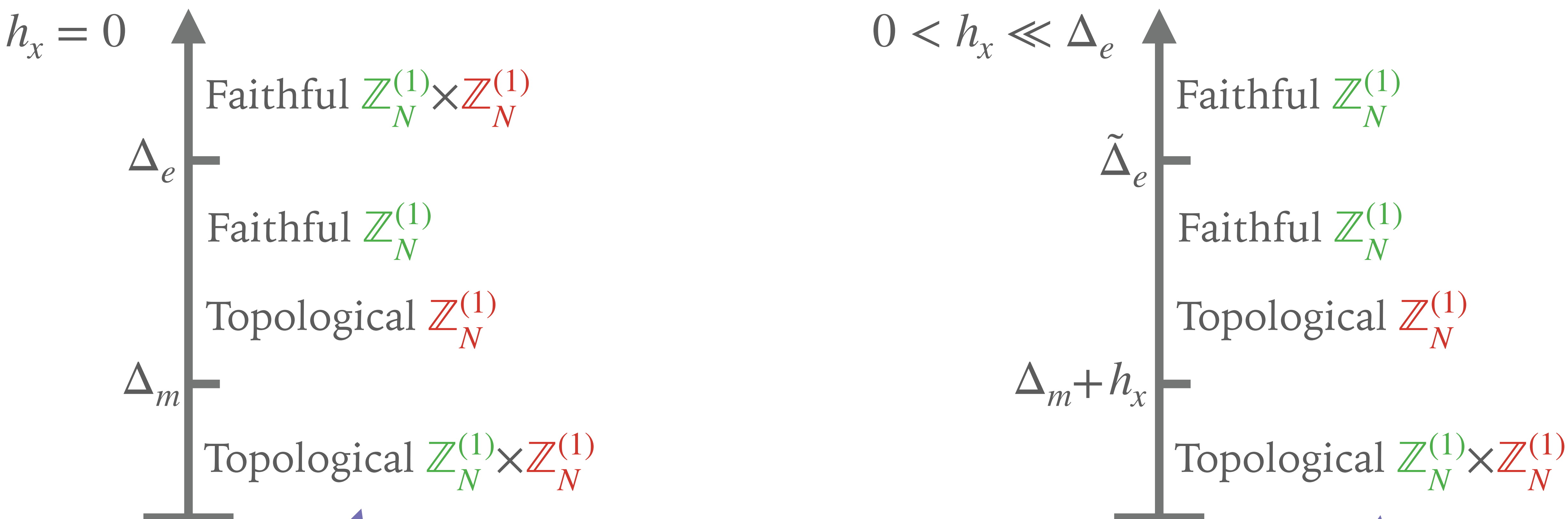

SSB ($\mathbb{Z}_2$ topological ordered phase)

Hastings & Wen, PRB 72.4 (2005): 045141.
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EFFECTIVE HAMILTONIAN

$$H = -\frac{\Delta_e}{2}\sum_s \prod_{e\in\delta s} Z_e - \frac{\Delta_m}{2}\sum_p \prod_{e\in\partial p} X_e - h_x \sum_e X_e \qquad 0 < h_x \ll \Delta_e$$

Effective Hamiltonian for $E < \tilde{\Delta}_e$ describes e anyon free sub-Hilbert space. Low-energy allowed operator: $W(\gamma) = \prod_{e\in\gamma} \tilde{X}_e$.

$$H_{e\text{ free}} = -\frac{\Delta_m}{2}\sum_p W(\partial p) - h_x \sum_{\gamma\in B_1(\text{lattice};\mathbb{Z}_N)} \epsilon_\gamma W(\gamma) - h_x \sum_{\gamma\in H_1(\text{lattice};\mathbb{Z}_N)} \epsilon_\gamma W(\gamma)$$

$$\epsilon_\gamma \sim (h_x/\Delta_e)^{|\gamma|-1}$$

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$$\mathcal{O}(\mathrm{e}^{-L})$$

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- Exact emergent topological $\mathbb{Z}_N^{(1)}$ symmetry: $\tilde{X}_e \to s_e \tilde{X}_e \qquad \prod_{e\in\partial p} s_e = 1 \, \forall\, p$
- $\langle W \rangle = 1$ for contractible loops. $\mathbb{Z}_N^{(1)}$ SSB phase

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Deconfined phase of $\mathbb{Z}_2$ gauge theory is SSB phase of exact emergent $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$ symmetry

$H_{e\text{ free}}$ … $W(\gamma)$

…ice;$\mathbb{Z}_N$)

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EVIDENCE FROM NUMERICS

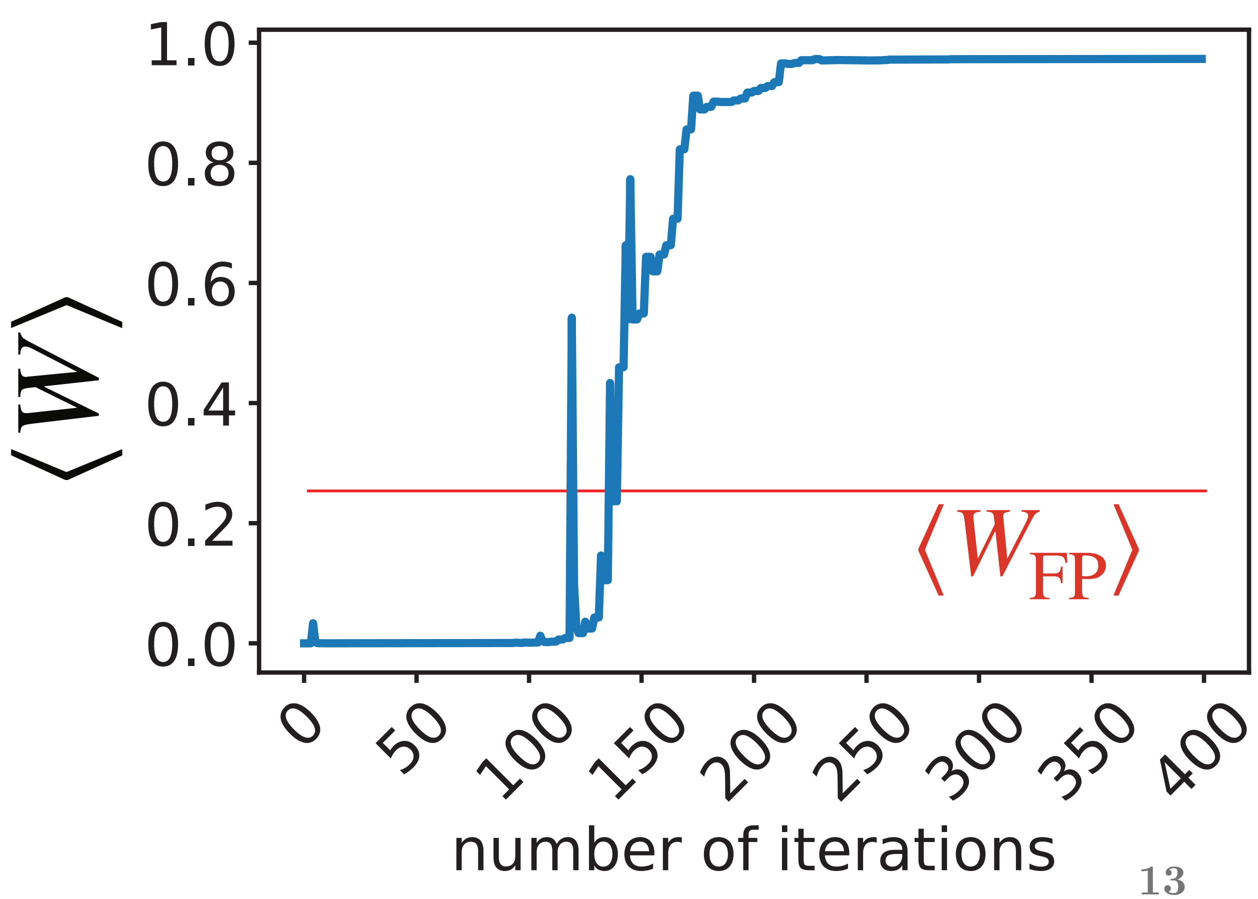
Precise form of dressed operators is not generally unique, depending on microscopic details

- Can be numerically constructed as a *matrix product operator* [Cian, Hafezi, & Barkeshli, arXiv:2209.14302]

Toric code with $h_x = 0.15$, $h_z = 0.05$ and $\Delta_e = \Delta_m = 2$

$$W_{\mathrm{FP}} = \prod_{e \in \gamma} X_e \implies \langle W_{\mathrm{FP}} \rangle \sim \mathrm{e}^{-|\gamma|/\xi}$$

$$W = \text{“} \prod_{e \in \gamma} \tilde{X}_e \text{”} \implies \langle W \rangle = 1$$



EMERGENCE WITHOUT SSB

$$H = -\frac{\Delta_e}{2}\sum_s \prod_{e\in\delta s} Z_e - \frac{\Delta_m}{2}\sum_p \prod_{e\in\partial p} X_e - h_x \sum_e X_e - h_z \sum_e Z_e$$

When $h_x \gg \Delta_e$, e anyons condense (Higgs phase)

- No e free low-energy subspace $\Longrightarrow$ no emergent conservation law ($\mathbb{Z}_N$ electric flux) $\Longrightarrow$ no emergent $\mathbb{Z}_N^{(1)}$ symmetry
- There can be an m free low-energy subspace $\Longrightarrow$ emergent conservation law ($\mathbb{Z}_N$ magnetic flux) $\Longrightarrow$ emergent $\mathbb{Z}_N^{(1)}$ symmetry

SP & X-G Wen, arXiv:2301.05261

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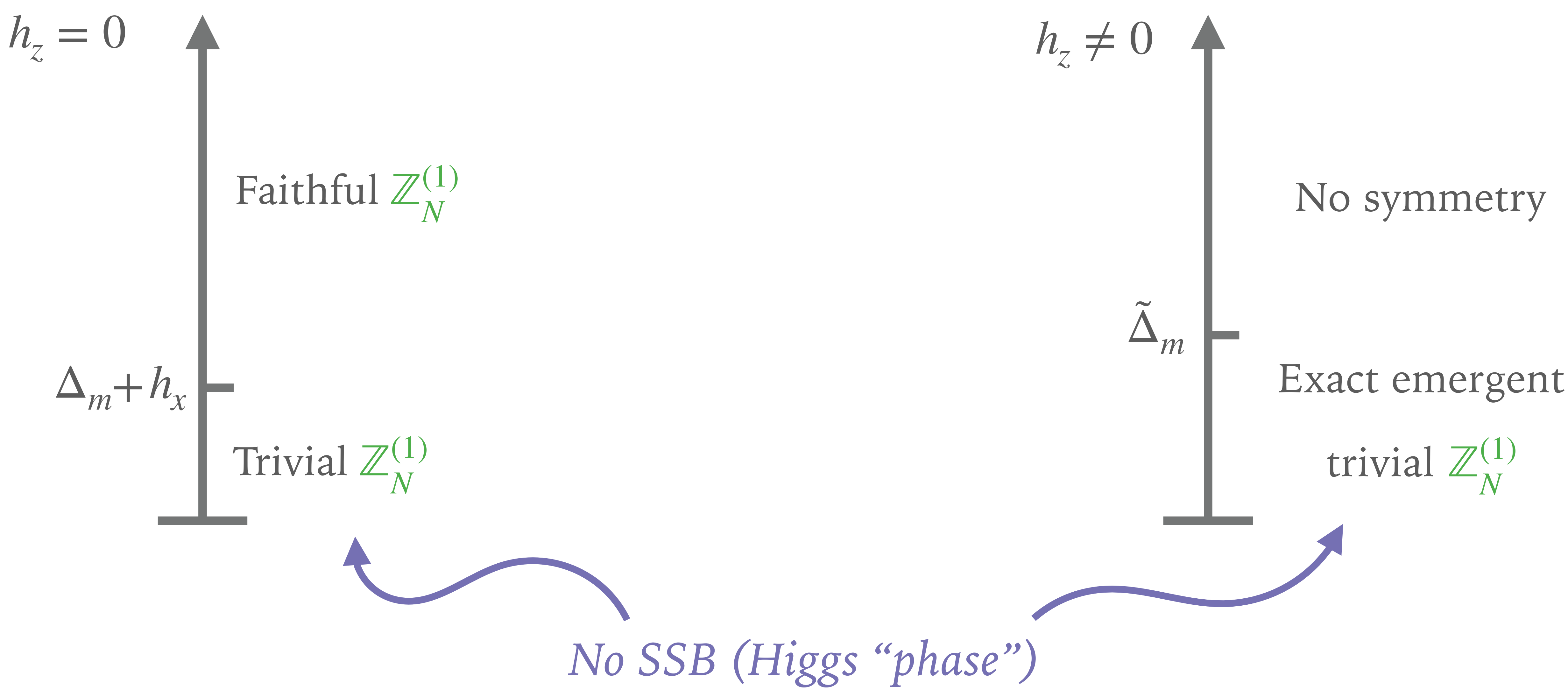

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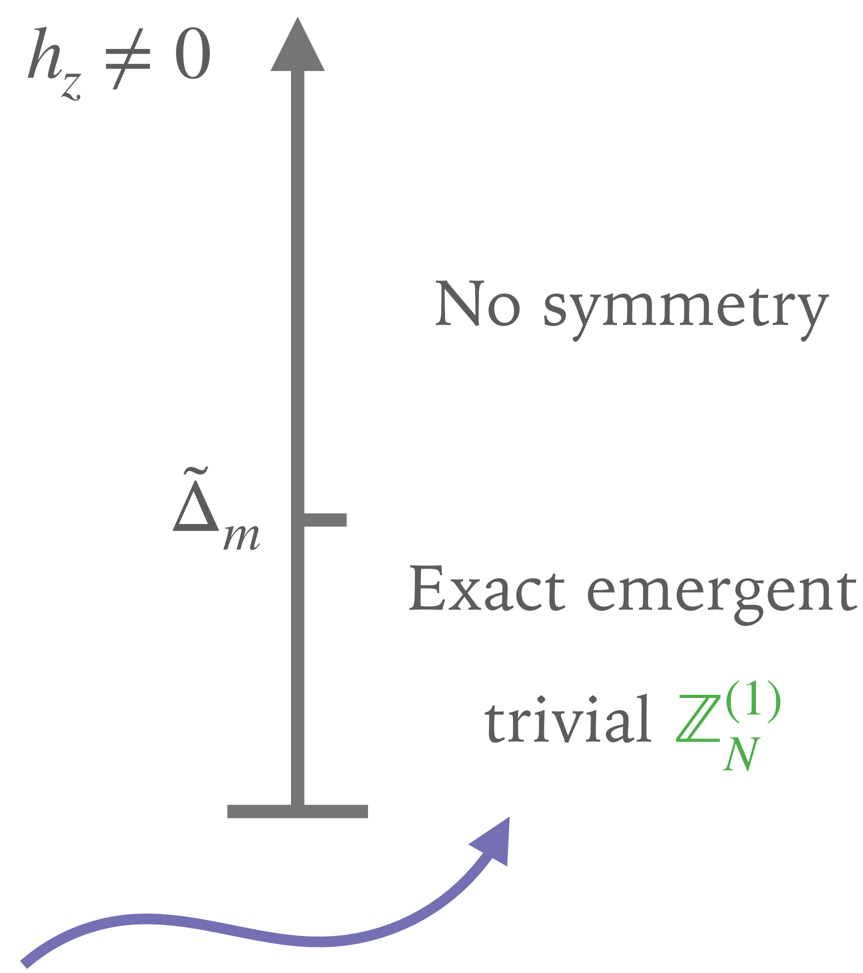
<u>Exact emergent *trivial* without SSB:</u>

- With PBC, unbroken exact emergent $\mathbb{Z}_N^{(1)}$ symmetry is trivial
- With a spatial boundary, unbroken exact emergent $\mathbb{Z}_N^{(1)}$ symmetry can be nontrivial (e.g., SPT phase with $\mathbb{Z}_N^{(1)}$ SSB on boundary)



No SSB (Higgs "phase")

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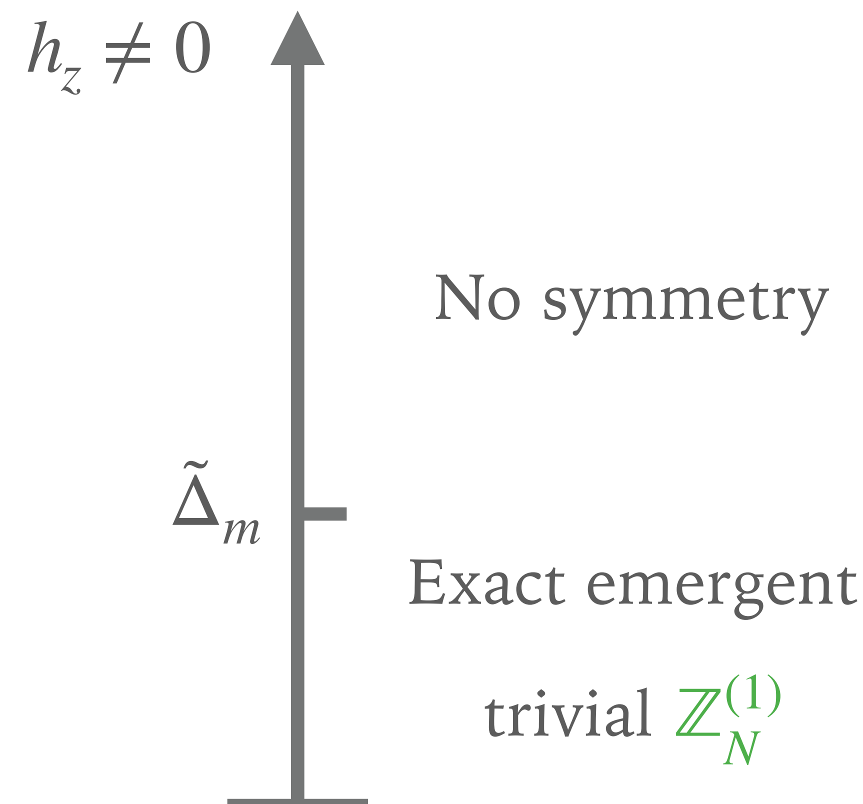
<u>Exact emergent *trivial* with SSB:</u>

- With PBC, spontaneously broken exact emergent $\mathbb{Z}_N^{(1)}$ symmetry is nontrivial
- On a disk, spontaneously broken exact emergent $\mathbb{Z}_N^{(1)}$ symmetry is trivial

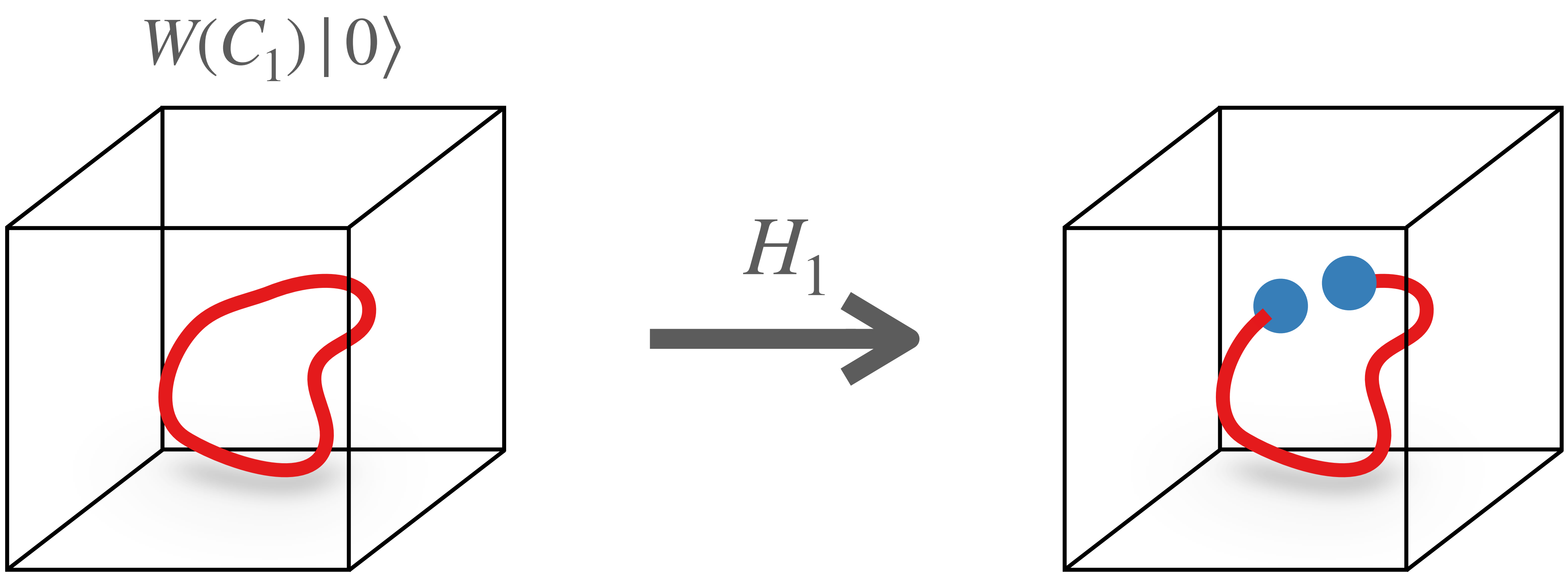


No SSB (Higgs "phase")

GENERAL DISCUSION

Typical Hamiltonians do not have exact higher-form symmetries.

- Consider $H = H_0 + H_1$ in $d = 3$. Suppose H_0 has a 1-form symmetry and H_1 is a generic local perturbation



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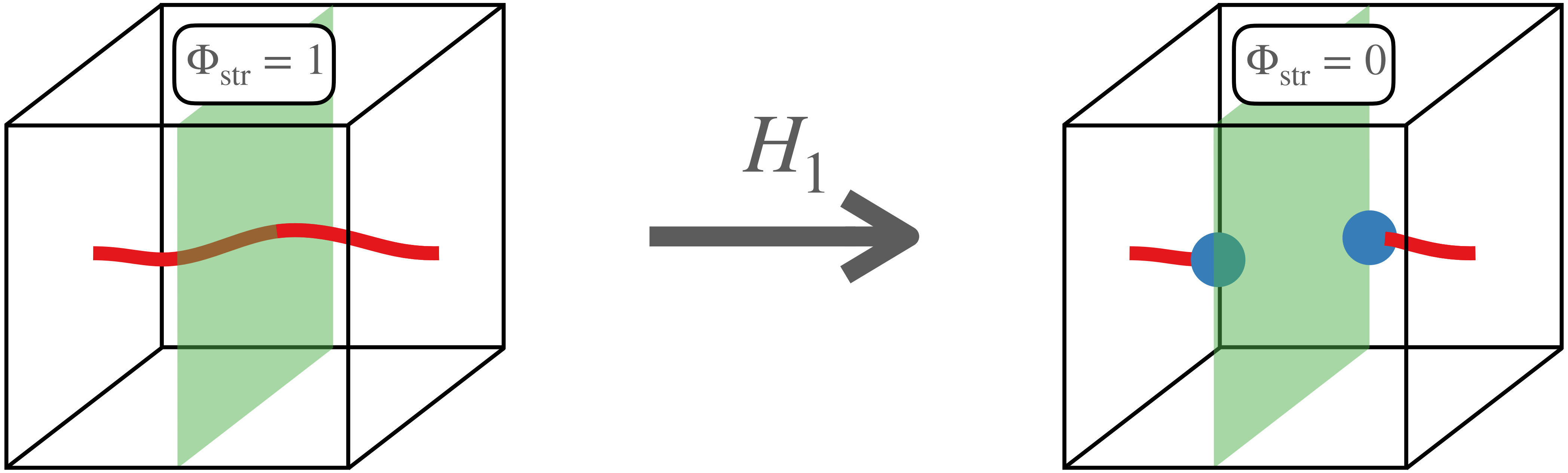

General H_1 violates the flux conservation law of the 1-form symmetry, hence explicitly breaking H_0's 1-form symmetry

EMERGENT HIGHER-FORM SYMMETRY

Ends of strings cost finite energy (Gauss law term in H_0)

- True gapped topological excitations are modified due to H_1. Correspond to ends of modified (fattened) strings
- Below gap, only closed modified loops $\Longrightarrow$ emergent string flux conservation $\Longrightarrow$ emergent 1-form symmetry

General p-form symmetry: gapped $(p-1)$ dimensional topological excitations. At energies below their gap, only modified (fattened) p-branes $\Longrightarrow$ emergent p-form symmetry.

- No 0-form symmetry analog ($p > 0$ only)

SP & Wen, arXiv:2301.05261

EXACT EMERGENT SYMMETRY

A conservation law implies a symmetry. This conservation law is violated by charged operators of the symmetry.

Emergent 0-form symmetries are approximate symmetries

- Emergent conservation laws *are* violated by local operators, which typically appear in a low-energy effective theory.

Emergent higher-form symmetries are exact symmetries

- Emergent conservation law *cannot* be violated by any local operators. The (local) low-energy effective theory will have the higher-form symmetry as an exact symmetry.

SP & Wen, arXiv:2301.05261

EXACT EMERGENT SYMMETRY

A conservation law implies a symmetry. This conservation law is vio ... y.

Emergent ... etries

- Emerg ... operato ... effective theory.

Emerge ... s

- Eme ... y local oper ... ll have the l

Emergent 0-form symmetry SSB phase:

- Discrete: no GSD
- Continuous: gapped goldstone modes

Emergent higher-form symmetry SSB phase:

- Discrete: exact GSD
- Continuous: gapless goldstone modes

SP & Wen, arXiv:2301.05261

EXACT EMERGENT HIGHER-FORM SYMMETRIES

Exact higher-form symmetries are rare, but exact emergent higher-form symmetry are common.

- Most ordinary SSB phases have exact emergent higher-form symmetries (To be posted on arXiv this spring… hopefully 🙏🙏🤞🤞)
- Known examples in $d = 3$: Ising ferromagnets ($\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(3)}$), bosonic superfluids ($U(1)^{(0)} \times U(1)^{(2)}$)

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2+1D ABELIAN BOSONIC TOPOLOGICAL ORDER

Deconfined phase of 2+1D $\mathbb{Z}_N$ gauge theory is an exact emergent anomalous $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$ SSB phase

- TQFT describing ground states: $S = \frac{\mathrm{i}N}{2\pi} \int \mathrm{d}t\mathrm{d}^2x \; \epsilon^{\mu\nu\rho} a_\mu^1 \partial_\nu a_\rho^2$

Ground states of a general abelian bosonic topological order in 2+1D described by

$$S = \frac{\mathrm{i}K_{IJ}}{4\pi} \int \mathrm{d}t\mathrm{d}^2x \; \epsilon^{\mu\nu\rho} a_\mu^I \partial_\nu a_\rho^J \qquad K_{IJ} \in \begin{cases} \mathbb{Z}, & I \neq J \\ 2\mathbb{Z} & I = J \end{cases}, \qquad K_{IJ} = K_{JI}$$

- Anomalous $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ symmetry

'T HOOFT ANOMALY AND SPT ORDER

The $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ symmetry is anomalous:

1. Theory has to be in a gapless, SSB, or T.O. phase.
2. The anomalous part of the symmetry cannot be gauged

$$Z[A] \to Z[A + \mathrm{d}\omega] = \mathrm{e}^{\mathrm{i}\theta(A,\omega)} Z[A]$$

3. Theory is perfectly well defined by itself (with $A = 0$)
4. Can be gauged if it resides on the boundary of an SPT.

Question: What is the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ SPT whose boundary has topological order described by the K matrix?

THE SPT INVARIANT

Turn on background fields $\mathscr{A}^I$: $S[\mathscr{A}] = \frac{\mathrm{i}K_{IJ}}{4\pi}\int_{X_3} \mathrm{d}t\mathrm{d}^2x\ \epsilon^{\mu\nu\rho} a^I_\mu \left(\partial_\nu a^J_\rho - \frac{1}{2}\mathscr{A}^J_{\nu\rho}\right)$

$$a^I_\mu \to a^I_\mu + \omega^I_\mu$$

$$\mathscr{A}^I_{\mu\nu} \to \mathscr{A}^I_{\mu\nu} + \partial_\mu \omega^I_\nu - \partial_\nu \omega^I_\mu$$

$$Z[X_3, \mathscr{A}] \to \mathrm{e}^{-\frac{\mathrm{i}K_{IJ}}{8\pi}\int_{X_3} \mathrm{d}t\mathrm{d}^2x\ \epsilon^{\mu\nu\rho}\omega^I_\mu \mathscr{A}^J_{\nu\rho}} Z[X_3, \mathscr{A}]$$

Extend $\mathscr{A}$ from 2+1D X_3 to 3+1D Y_4 such that $\partial Y_4 = X_3$

$$Z_{\mathrm{g-inv}}[Y_4, \mathscr{A}] = \mathrm{e}^{\frac{\mathrm{i}K_{IJ}}{16\pi}\int_{Y_4} \mathrm{d}t\mathrm{d}^3x\ \epsilon^{\mu\nu\rho\sigma}\mathscr{A}^I_{\mu\nu}\mathscr{A}^J_{\rho\sigma}} Z[X_3 = \partial Y_4, \mathscr{A}]$$

➤ Invertible theory characterizing the SPT order:

$$Z_{\mathrm{SPT}}[Y_4, \mathscr{A}] = \exp\left[\frac{\mathrm{i}K_{IJ}}{16\pi}\int_{Y_4} \mathrm{d}t\mathrm{d}^3x\ \epsilon^{\mu\nu\rho\sigma}\mathscr{A}^I_{\mu\nu}\mathscr{A}^J_{\rho\sigma}\right]$$

TWISTED $U(1)$ GAUGE THEORY

Z_{SPT} is the IR FP theory. What system is in this SPT phase?

➤ Trick: "ungauge" $\mathscr{A}^I$ by $\mathscr{A}^I_{\mu\nu} \to \partial_\mu a^I_\nu - \partial_\nu a^I_\mu \equiv f^I_{\mu\nu}$

$$S_{\text{SPT}} = \frac{\mathrm{i}K_{IJ}}{16\pi}\int_{Y_4} \mathrm{d}t\mathrm{d}^3x\ \epsilon^{\mu\nu\rho\sigma} f^I_{\mu\nu} f^J_{\rho\sigma} \equiv \frac{\mathrm{i}K_{IJ}}{32\pi}\int_{Y_4} \mathrm{d}t\mathrm{d}^3x\ \vec{e}^I \cdot \vec{b}^J$$

➤ General twisted $U(1)$ gauge theory

$$S = \frac{1}{4g^2}\int_{Y_4} f^I_{\mu\nu} f^{I\ \mu\nu} + \frac{\mathrm{i}K_{IJ}}{16\pi}\int_{Y_4} \mathrm{d}t\mathrm{d}^3x\ \epsilon^{\mu\nu\rho\sigma} f^I_{\mu\nu} f^J_{\rho\sigma}$$

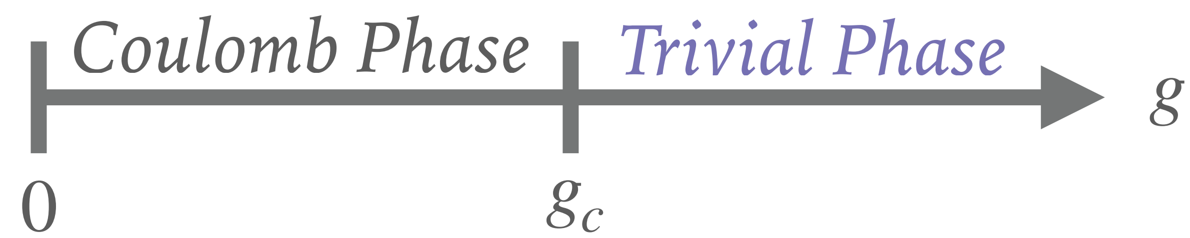
The confined phase ($g \to \infty$) of twisted $U(1)$ gauge theory is a $\mathbb{Z}^{(1)}_{k_1} \times \mathbb{Z}^{(1)}_{k_2} \times \cdots$ SPT

SP & X-G Wen, PRB 107, 075112 (2023)

TWISTED $U(1)$ GAUGE THEORY

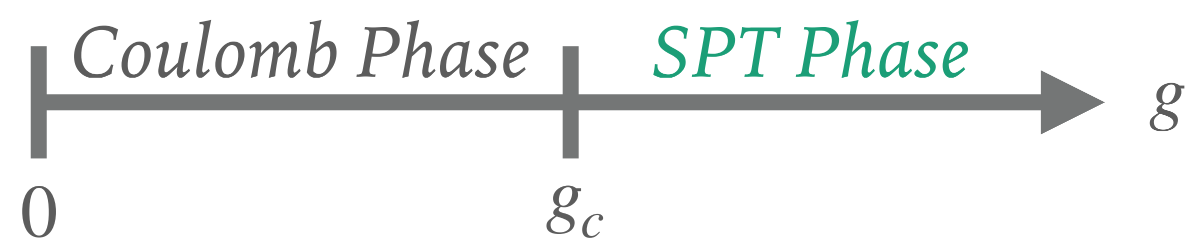
$U(1)$ gauge theory:

- $g > g_c$: monopoles condense



Twisted $U(1)$ gauge theory

- $g > g_c$: dyons condense



$(q_e, q_m) = (2{,}1)$ dyon $\to \mathbb{Z}_2^{(1)}$ SPT

Boundary: $\nu = 1/2$ bosonic FHQ

$(q_e^{1,2}, q_m^{2,1}) = (2{,}1)$ dyons $\to \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ SPT

Boundary: $\mathbb{Z}_2$ topological order

Need magnetic monopoles (dyons), why care?

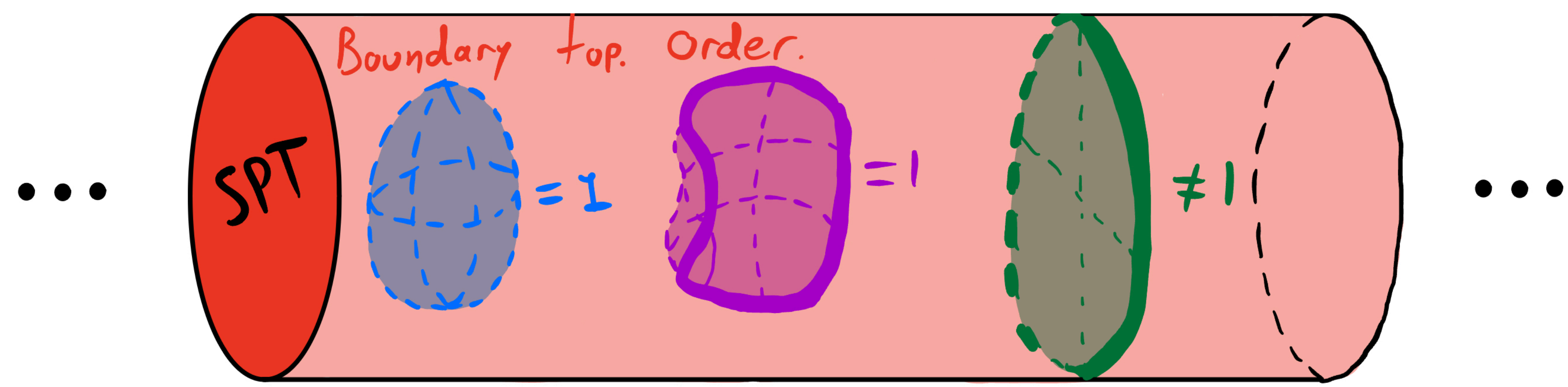
- Quantum spin ice on a breathing pyrochlore lattice

SP & Wen, PRB 107, 075112 (2023) *SP, Castelnovo, & Laumann, PRL 130, 076701 (2023)*

EMERGENT HIGHER-FORM SPT

$\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ symmetry operator (with $g \to \infty$)

$$U^{(I)}(\Sigma_2) = \exp\left[\frac{\mathrm{i}K^{IJ}}{2\pi}\int_{\Sigma_2} \mathrm{d}a^J\right] = \exp\left[\frac{\mathrm{i}K^{IJ}}{2\pi}\int_{\partial\Sigma_2} a^J\right]$$



- Nontrivial $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ exact emergent symmetry (exact conservation law in low-energy bulk+boundary theory)
- Robustness of exact emergent SSB boundary phase (topological order) reflects exact emergent SPT order

SP & Wen, PRB 107, 075112 (2023)

THE PLAN FOR THIS TALK

Explore how higher-form symmetries arise in topological phases of quantum matter

1. How higher-form symmetries emerge & exact emergent symmetries
   [**SP** & X-G Wen, arXiv:2301.05261]

2. Symmetry Protected Trivial (SPT) phase protected by higher-form symmetries
   [**SP** & X-G Wen, PRB 107, 075112 (2023)]

3. The rank-2 toric code, its symmetries, and UV/IR mixing
   [**SP** & X-G Wen, PRB 106, 045145 (2022)]
   [Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706]

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2+1D RANK-2 TORIC CODE

Square lattice with two $\mathbb{Z}_N$ quantum rotors on each site and one $\mathbb{Z}_N$ quantum rotors on each plaquette

- Toric code: exactly solvable point in deconfined phase of vector $\mathbb{Z}_N$ gauge theory
- Rank-2 Toric Code (R2TC): exactly solvable point in deconfined phase of symmetric rank-2 tensor $\mathbb{Z}_N$ gauge theory

Ma, Hermele, & Chen, PRB 98.3 (2018): 035111.
Bulmash & Barkeshli, PRB 97.23 (2018): 235112.
Oh et al., PRB 105.4 (2022): 045128.
***SP** & X-G Wen, PRB 106, 045145 (2022)*

2+1D RANK-2 TORIC CODE

Square lattice with two $\mathbb{Z}_N$ quantum rotors on each site and one $\mathbb{Z}_N$ quantum rotors on each plaquette

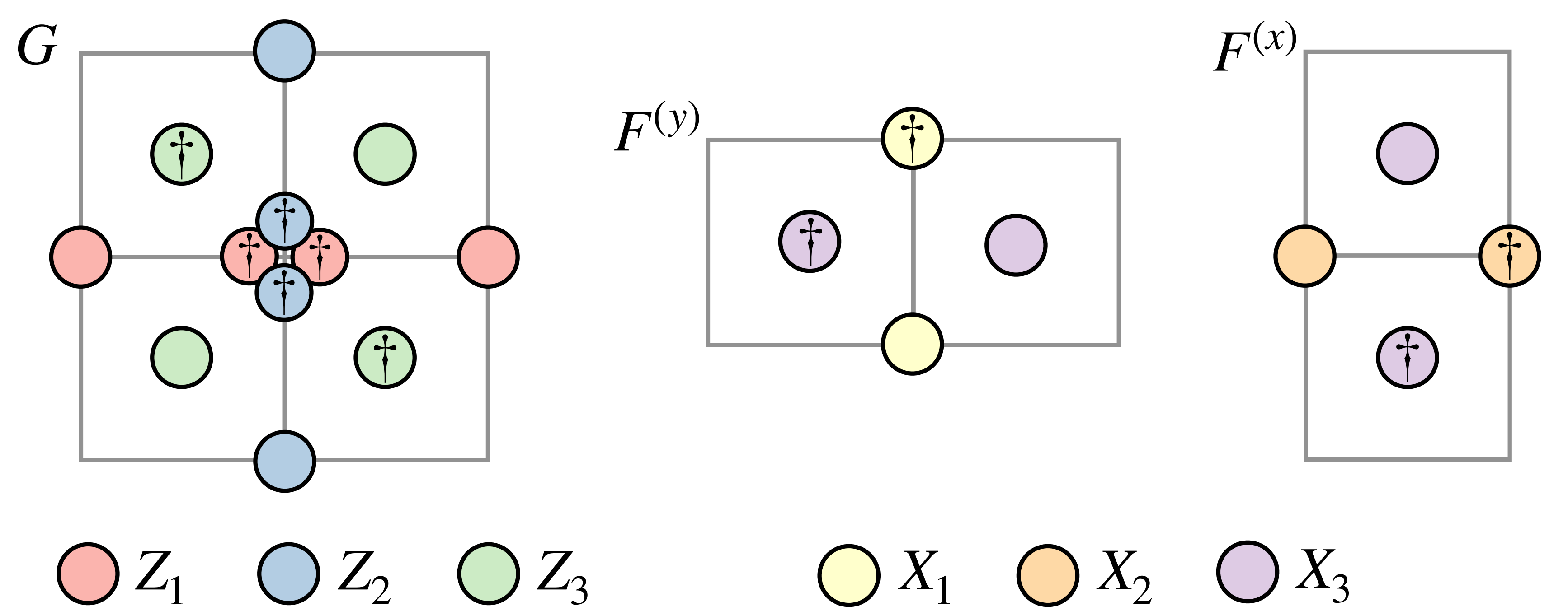
$$H_{\mathrm{R2TC}} = -\sum_{x,y}\left(G_{x,y} + F^{(x)}_{x,y} + F^{(y)}_{x,y} + \mathrm{h.c.}\right)$$

$\mathbb{Z}_N$ clock operators:

$$Z_j X_i = \omega^{\delta_{i,j}} X_i Z_j$$

$$\omega = \exp[2\pi \mathrm{i}/N]$$

$$X_i^N = Z_i^N = 1$$



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***SP** & X-G Wen, PRB 106, 045145 (2022)*

R2TC EXCITATIONS

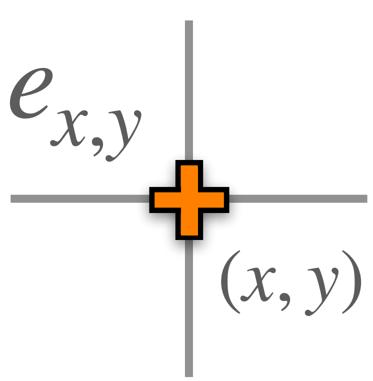
$$H_{\mathrm{R2TC}} = -\sum_{x,y} \left(G_{x,y} + F^{(x)}_{x,y} + F^{(y)}_{x,y} + \mathrm{h.c.} \right)$$

- Exactly solvable because: $[G_{x,y}, F^{(x)}_{x',y'}] = [G_{x,y}, F^{(y)}_{x',y'}] = 0$

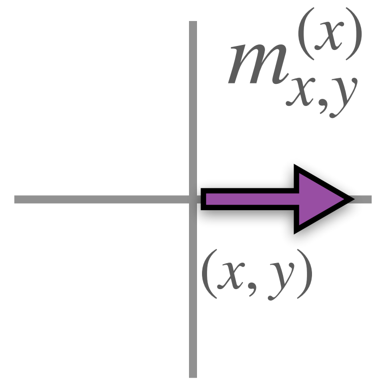
- A gapped excited states $|\psi\rangle$ satisfies:

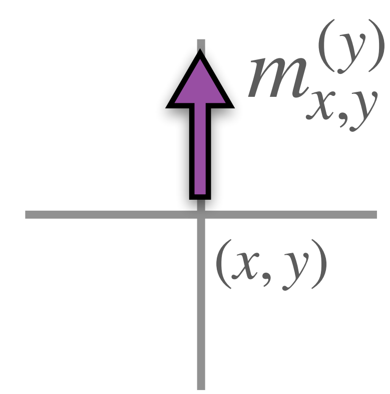
$$G_{x,y}|\psi\rangle = \omega^{\mathfrak{e}_{x,y}}|\psi\rangle \qquad F^{(x)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(x)}_{x,y}}|\psi\rangle \qquad F^{(y)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(y)}_{x,y}}|\psi\rangle$$

$\mathbb{Z}_N$ charge



$\mathbb{Z}_N$ fluxes





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***SP** & X-G Wen, PRB 106, 045145 (2022)*

SOME R2TC PROPERTIES

Restricted Mobility:

- e hops by N in all directions → pseudo-fracton
- $m^{(i)}$ hops by 1 in i-direction, by N in transverse direction → pseudo-lineon

Position-dependent braiding:

- Statistics of e_{x_e,y_e} and $m^{(i)}_{x_m,y_m}$ depend on $(x_e - x_m, y_e - y_m)$

UV/IR Mixing:

- Anyon counting on a $L_x \times L_y$ torus yields:

$$\mathrm{GSD} = N^3 \gcd(L_x, N) \gcd(L_y, N) \gcd(L_x, L_y, N)$$

Ma, Hermele, & Chen, PRB 98.3 (2018): 035111.
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***SP** & X-G Wen, PRB 106, 045145 (2022)*

R2TC GENERALIZED SYMMETRIES

Given exotic features, it is desirable to have a generalized symmetry understanding

Can use SET machinery, so why generalized symmetry?

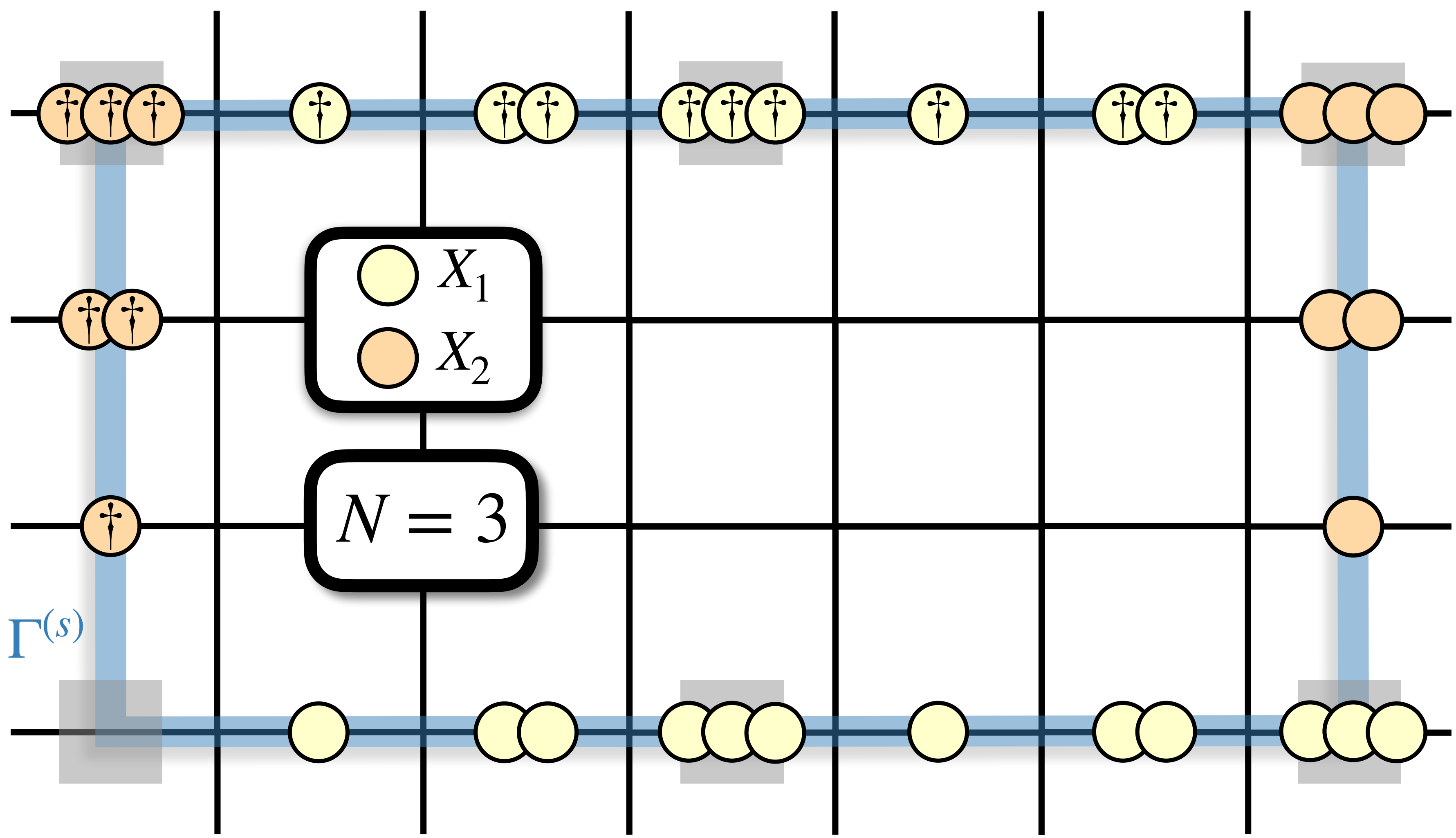
- Captures ground states' structures directly (naturalness)
- Easily generalizable to higher dimensions (practical utility)
- As we'll see, provides a natural unifying picture for intrinsic topological order and fracton topological order (conceptually useful)

*Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706*

R2TC GENERALIZED SYMMETRIES

There are six discrete 1-form symmetries, all spontaneously broken, and three types of mixed 't Hooft anomalies

➤ Three symmetries are topological only within a sublattice



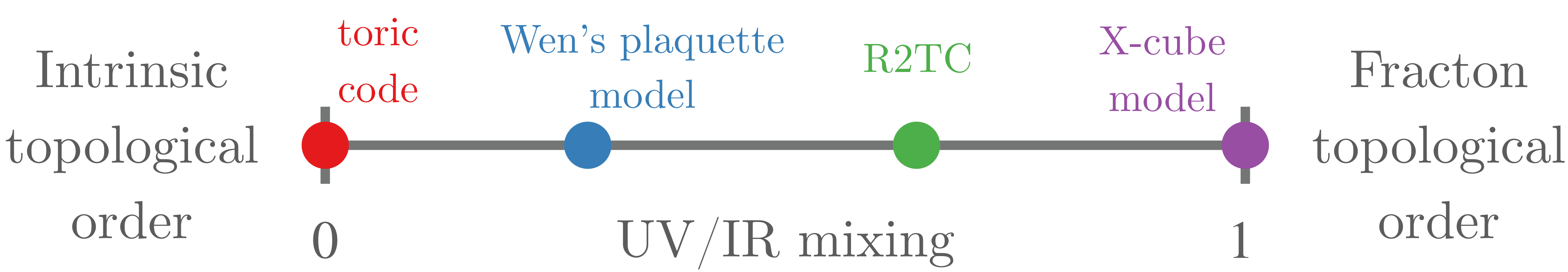
*Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706*

UV/IR MIXING AND SYMMETRY

The UV/IR mixing originates entirely from the sublattice topological 1-form symmetries:

- Low-energy emergent symmetries (IR) are defined in terms of the lattice regularization (UV)

Conjecture: All UV/IR mixing in "topological phases" arises from SSB of subsystem generalized symmetries



*Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706*

***SP** & X-G Wen, PRB 106, 045145 (2022)*

SUMMARY

1. Emergent higher-form symmetries are exact emergent symmetries and are common.
   [**SP** & X-G Wen, arXiv:2301.05261]
2. Emergent higher-form SPTs can appear in confined phases of twisted gauge theory
   [**SP** & X-G Wen, PRB 107, 075112 (2023)]
3. UV/IR mixing in gapped phases is naturally understood using generalized symmetries
   [**SP** & X-G Wen, PRB 106, 045145 (2022)]
   [Y-T Oh, **SP**, JH Han, Y You, H-Y Lee, arXiv:2301.04706]