An SPT-LSM theorem for weak SPTs with non-invertible symmetry

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Quantum phases and symmetry

A fundamental problem in quantum condensed matter physics is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

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Quantum phases are sometimes characterized by symmetry

- Superfluids from U(1) boson number conservation

- Topological insulators from $\mathrm{U}(1)_f$ and $\mathbb{Z}_2^{\mathrm{T}}$ symmetries

For such phases, symmetries provide answers to the above questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in generalizing the notion of symmetries

- Ordinary symmetries transform local operators in an invertible manner (e.g., $c_r^\dagger \to \mathrm{e}^{\mathrm{i}\theta} c_r^\dagger$)
- So-called generalized symmetries modify this definition

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Higher-form symmetries transform non-local operators that create extended excitations [Gaiotto, Kapustin, Seiberg, Willett 2014; ⋯]

- Can arise from anyons in 2d topological orders, vortex loops in 3d superfluids, *etc*

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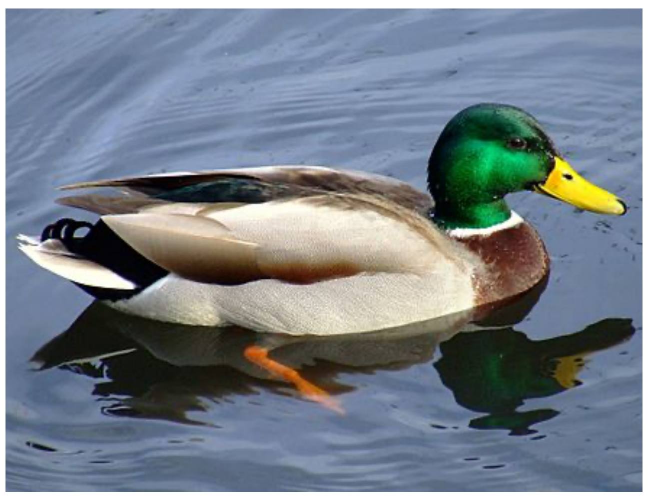
Non-invertible symmetries have non-invertible transformations
[Bhardwaj, Tachikawa 2017; Chang, Lin, Shao, Wang, Yin 2018; ⋯]

- Can arise from Kramers-Wannier self-dualities
  [⋯ ; Seiberg, Shao 2023; **SP**, Delfino, Lam, Aksoy 2024; Gorantla, Shao, Tantivasadakarn 2024]
- Generically emerge in ordered phases (are symmetries of nonlinear sigma models) [**SP** 2023; **SP**, Zhu, Beaudry, X-G Wen 2023]

Generalized symmetries

Q: Why should we consider these as symmetries?

A: They pass the duck test!



If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- Have conservation laws
- Can constrain phase diagrams (be anomalous)
- Can characterize SSB and SPT phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

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Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$$\mathrm{a}_1\text{–}\mathrm{a}_2\text{–}\mathrm{a}_3\text{–}\cdots \text{ Symmetry}$$

(2) Specify SSB and SPT pattern

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Ordered phases *Topological insulators*

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FQH phases *QSL phases* *Higgs phases* *Fracton phases*

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Ordered phases *Topological insulators*

FQH phases *QSL phases* *Higgs phases* *Fracton phases*

Phases we have yet to name!

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Why care?

1. Guides us towards new quantum phases
2. Provides a novel and unifying perspective of quantum phases
3. Further develops a classification of quantum phases based on symmetries (a "generalized Landau paradigm")

TL;DR for this talk

In this talk, we explore Symmetry Protected Topological (SPT) phases characterized by non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries

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In this talk, we explore Symmetry Protected Topological (SPT) phases characterized by non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries

Outline:

1. Review SPTs (from a symmetry defect perspective)
2. Simple example of an entangled weak SPT characterized by a non-invertible symmetry
3. (SPT-)LSM theorems from projective non-invertible symmetries

What are SPTs

An SPT phase is a gapped quantum phase described by a symmetry with [Chen, Gu, Liu, Wen 2011; ⋯]

1. A unique ground state on all closed spatial manifolds
2. Interesting physics arising on boundaries and interfaces between SPTs (e.g., topological order, gapless excitations)

SPTs are characterized by their response to static probes

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➤ Example: 3d insulators [Qi, Hughes, Zhang 2008; ⋯]

<u>Ordinary insulator</u>

$$\mathscr{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$$

<u>Topological insulator</u>

$$\mathscr{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{\pi}{4\pi^2}\mathbf{E}\cdot\mathbf{B}$$

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$$\boxed{Z_{\text{topological}}[A_\mu] = \mathrm{e}^{\frac{\mathrm{i}}{4\pi}\int \mathbf{E}\cdot\mathbf{B}\, \mathrm{d}^3x\, \mathrm{d}t}\, Z_{\text{ordinary}}[A_\mu]}$$

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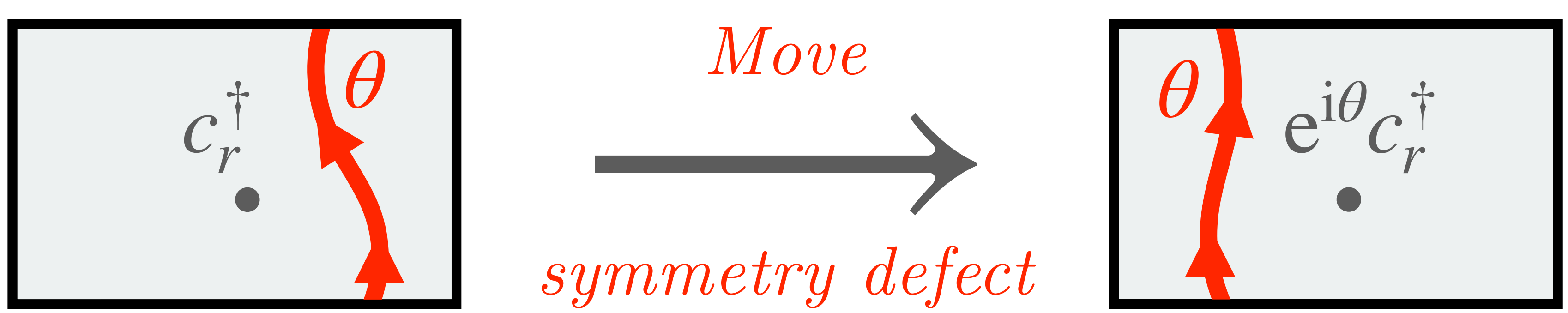
SPTs are characterized by their response to static probes

- Probes are generally background gauge fields and symmetry defects (i.e., twisted boundary conditions)

- e.g., $c_r^\dagger \to \mathrm{e}^{\mathrm{i}\theta} c_r^\dagger$



Example 1: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT

1d periodic lattice with two qubits on each site $j \sim j + L$ acted on by Pauli operators X_j, Z_j and $\tilde{X}_j$, $\tilde{Z}_j$.

$$H_p = -\sum_{j=1}^{L} (X_j + \tilde{X}_j)$$

- Unique gapped ground state $|\mathrm{GS}_p\rangle = |++\cdots+\rangle$

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- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ symmetry, where

$$U = \prod_j X_j \quad \text{and} \quad \tilde{U} = \prod_j \tilde{X}_j \quad \text{with} \quad U|\mathrm{GS}_p\rangle = \tilde{U}|\mathrm{GS}_p\rangle = |\mathrm{GS}_p\rangle$$

H_p is in a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phase

Example 2: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT

1d periodic lattice with two qubits on each site $j \sim j + L$

$$H_c = -\sum_{j=1}^{L} \left(\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1}\right)$$

➤ Sum of commuting terms and unique gapped ground state

$$\tilde{Z}_{j-1} X_j \tilde{Z}_j |\mathrm{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\mathrm{GS}_c\rangle = |\mathrm{GS}_c\rangle$$

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$\boxed{H_c \text{ is also in a } \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2 \text{ SPT phase}}$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Q: Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a probe $\mathbb{Z}_2$ symmetry defect

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We can check by inserting a probe $\mathbb{Z}_2$ symmetry defect

- Passing through $\langle L, 1 \rangle$ implements $U: Z_j \to -Z_j \implies$ twisted translation symmetry $T = X_1 T_{\text{defect-free}}$
- Gives rise to $\mathbb{Z}_2$-twisted boundary conditions $T^L = U$:

$$X_{j+L} = X_j \qquad Z_{j+L} = -Z_j \qquad \tilde{X}_{j+L} = \tilde{X}_j \qquad \tilde{Z}_{j+L} = \tilde{Z}_j$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Under $\mathbb{Z}_2$-twisted boundary conditions $Z_{j+L} = -Z_j$

1. H_p is unaffected, so its ground state still satisfies

$$U|\mathrm{GS}_{p;\mathbb{Z}_2}\rangle = +|\mathrm{GS}_{p;\mathbb{Z}_2}\rangle \qquad \tilde{U}|\mathrm{GS}_{p;\mathbb{Z}_2}\rangle = +|\mathrm{GS}_{p;\mathbb{Z}_2}\rangle$$

2. H_c is affected, and becomes

$$H_{\mathrm{c};\mathbb{Z}_2} = \sum_{j=1}^{L-1} \left(-\tilde{Z}_{j-1} X_j \tilde{Z}_j - Z_j \tilde{X}_j Z_{j+1}\right) - \tilde{Z}_{L-1} X_L \tilde{Z}_L + Z_L \tilde{X}_L Z_1$$

➤ Its ground state $|\mathrm{GS}_{c;\mathbb{Z}_2}\rangle$ now satisfies

$$U|\mathrm{GS}_{c;\mathbb{Z}_2}\rangle = +|\mathrm{GS}_{c;\mathbb{Z}_2}\rangle \qquad \tilde{U}|\mathrm{GS}_{c;\mathbb{Z}_2}\rangle = -|\mathrm{GS}_{c;\mathbb{Z}_2}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

$\mathbb{Z}_2$ domain walls dressed by:

- trivial $\tilde{\mathbb{Z}}_2$ symmetry charge in ground state of H_p
- Non-trivial $\tilde{\mathbb{Z}}_2$ symmetry charge in ground state of H_c

Different domain wall decorations imply that H_p and H_c are different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPTs [Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

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Low-energy effective field theories of H_p and H_c coupled to background $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ gauge fields:

$$Z_c[A, \tilde{A}] = (-1)^{\int A \cup \tilde{A}} Z_p[A, \tilde{A}]$$

Example 3: $\mathbb{Z}_2$ weak SPT

1d periodic lattice with a qubit on each site $j \sim j + L$

$$H_+ = +\sum_j X_j$$

- Unique gapped ground state $|\mathrm{GS}_+\rangle = \otimes_j |-\rangle$
- Symmetries: $\mathbb{Z}_2 \times \mathbb{Z}_L$, with $U = \prod_j X_j$ and $T: j \to j+1$

H_+ is in a $\mathbb{Z}_2 \times \mathbb{Z}_L$ SPT phase

SPTs characterized by translations are called weak SPTs

$\boxed{H_+ \text{ is in a } \mathbb{Z}_2 \text{ weak SPT phase}}$

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	Even L	Odd L	Even L, $\mathbb{Z}_2$ symmetry defect
$U|\mathrm{GS}_+\rangle =$	$+|\mathrm{GS}_+\rangle$	$-|\mathrm{GS}_+\rangle$	$+|\mathrm{GS}_+\rangle$
$T|\mathrm{GS}_+\rangle =$	$+|\mathrm{GS}_+\rangle$	$+|\mathrm{GS}_+\rangle$	$-|\mathrm{GS}_+\rangle$

Example 3: $\mathbb{Z}_2$ weak SPT

1d periodic lattice with a qubit on each site $j \sim j + L$

$\mathbb{Z}_2$ symmetry defect has $k = \pi$ crystal momentum in $|\mathrm{GS}_+\rangle$

Translation defect has $\mathbb{Z}_2$ symmetry charge in $|\mathrm{GS}_+\rangle$

➤ Inserting translation defect done by

$$T^L = 1 \to T^L = T \implies L \to L - 1$$

	Even L	Odd L	Even L, $\mathbb{Z}_2$ symmetry defect
$U\|\mathrm{GS}_+\rangle =$	$+\|\mathrm{GS}_+\rangle$	$-\|\mathrm{GS}_+\rangle$	$+\|\mathrm{GS}_+\rangle$
$T\|\mathrm{GS}_+\rangle =$	$+\|\mathrm{GS}_+\rangle$	$+\|\mathrm{GS}_+\rangle$	$-\|\mathrm{GS}_+\rangle$

A curious Hamiltonian

1d periodic lattice with a single qubit and $\mathbb{Z}_4$ qudit on each site $j \sim j + L$ [**SP**, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on qubits: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z\sigma^x = -\sigma^x\sigma^z$
- X, Z act on $\mathbb{Z}_4$ qudits: $X^4 = Z^4 = 1$ and $ZX = \mathrm{i}\, XZ$

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$$H = -\sum_j \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x + \frac{1}{4}\sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger)$$

- C acts as $X \to X^\dagger$ and $Z \to Z^\dagger$
- Is a sum of commuting terms and has a unique gapped ground state

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- C
- Is
gr

$$\boxed{|\mathrm{GS}\rangle = \sum_{\substack{\{\varphi_j = 0,1\} \\ \{\alpha_j = 0,2\}}} \mathrm{i}^{\sum_j \alpha_j(\varphi_j - \varphi_{j-1})} \bigotimes_j |\sigma^x_j = (-1)^{\varphi_j},\, Z_j = \mathrm{i}^{\alpha_j + 1}\rangle}$$

Some curious symmetries

$$H = -\sum_j \sigma_j^x \, C_{j+1} \, \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger) \, \sigma_j^z \, (Z_{j+1} - Z_{j+1}^\dagger)$$

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What are the symmetries of H?

- $\mathbb{Z}_L$ lattice translations T: $j \to j+1$
- Three $\mathbb{Z}_2$ symmetry operators

$$U = \prod_j X_j^2, \qquad R_1 = \prod_j \sigma_j^z, \qquad R_2 = \prod_j Z_j^2$$

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- 🤷‍♂️ symmetry operator

$$R_{\mathsf{E}} = \frac{1}{2}\left(1 + R_1\right)\left(1 + R_2\right)\prod_j Z_j^{\prod_{k=1}^{j-1}\sigma_k^z}$$

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What are the symmetries of H?

- $\mathbb{Z}$
- T

R_E is a non-invertible symmetry operator

- $R_\mathsf{E}|\psi\rangle = 0$ if $R_1|\psi\rangle = -|\psi\rangle$ or $R_2|\psi\rangle = -|\psi\rangle$
- R_E have zero-eigenvalues $\Longrightarrow$ R_E is non-invertible

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A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_\mathsf{E}^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_\mathsf{E} R_i = R_i R_\mathsf{E} = R_\mathsf{E}$$

$$U\, R_\mathsf{E} = (-1)^L R_\mathsf{E}\, U$$

- Form a (projective) $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

A curious SPT

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Ground state satisfies

$$T\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle \qquad U\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle \qquad R_1\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle$$

$$R_2\,|\mathrm{GS}\rangle = \begin{cases} +\,|\mathrm{GS}\rangle, & L \text{ even} \\ -\,|\mathrm{GS}\rangle, & L \text{ odd} \end{cases} \qquad R_{\mathsf{E}}\,|\mathrm{GS}\rangle = \begin{cases} +2\,|\mathrm{GS}\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

A curious SPT

These sym

$U^2 = 1,$ $\quad R_i R_{\mathsf{E}} = R_{\mathsf{E}}$

H is in a $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ weak SPT phase

- **Translation defects** carry $\mathsf{Rep}(D_8)$ symmetry charge in $|\mathrm{GS}\rangle$

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A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U\,R_{\mathsf{E}} = -R_{\mathsf{E}}\,U \qquad \text{(odd } L\text{)}$$

Projective unitary symmetries $U_1 U_2 = \mathrm{e}^{\mathrm{i}\theta} U_2 U_1$ forbid SPTs

- Assume non-degenerate symmetric ground state:

$$\left.\begin{array}{l} 1.\ \ \langle\psi|\,U_1U_2\,|\psi\rangle = \langle\psi|\psi\rangle = 1 \\[2ex] 2.\ \ \langle\psi|\,U_1U_2\,|\psi\rangle = \mathrm{e}^{\mathrm{i}\theta}\langle\psi|\,U_2U_1\,|\psi\rangle = \mathrm{e}^{\mathrm{i}\theta} \end{array}\right\}\ \textit{Contradicts assumption}$$

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Projective non-invertible symmetries <u>are compatible</u> with SPTs

- Loophole: symmetry operator has zero-eigenvalues
- $UR_{\mathsf{E}} = (-1)^L R_{\mathsf{E}} U$ enforces $R_{\mathsf{E}}|\mathrm{GS}_{\mathrm{SPT}}\rangle = 0$ when L is odd

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry is a special case of a more general projective symmetry

$$Z(G) \times \mathsf{Rep}(G)$$

- G is a discrete group with finitely many elements
- $Z(G) = \{z \in G \mid zg = gz, \forall g \in G\}$ is called the center of G
- Previous model realized the $G = D_8$ case
- Symmetry arises in the group-based XY model
  (see Ho Tat Lam's CMT Kid's Seminar)

This is a non-invertible symmetry when G is non-Abelian

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

$Z(G)$ symmetry operator U_z, with $z \in Z(G)$

$\mathsf{Rep}(G)$ symmetry operator R_Γ, with Γ an irrep of G, satisfies

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

Projectivity arises through $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$

- $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)]/d_\Gamma$ is a phase

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$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

Projectivity arises through $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$

- $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)] / d_\Gamma$ is a phase

e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ $(Z(\mathbb{Z}_2) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	sign
$+1$	$+1$	$+1$
-1	$+1$	-1

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = D_8$ $(Z(D_8) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	E
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1

Non-invertible weak SPT

If there is an SPT phase, the projective algebra $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its ground state to satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

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Two possibilities:

1. An SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$, but $R_\Gamma|\mathrm{GS}\rangle = 0$ for $L \neq L^*$

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The first is incompatible with 2D TQFT [Chang, Lin, Shao, Wang, Yin 2018]

- In a TQFT, $\langle R_\Gamma\rangle = d_\Gamma$, so all SPT states at some $L = L^*$ should satisfy $R_\Gamma|\mathrm{GS}\rangle = d_\Gamma|\mathrm{GS}\rangle$

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At $L = L^*$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$

At $L = L^* + 1$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$

- $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$ implies that any SPT state has translation defects dressed by non-trivial $Z(G) \times \mathsf{Rep}(G)$ symmetry charge

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(SPT)-LSM theorems

There is an Lieb-Schultz-Mattis (LSM) theorem when the projective algebra

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L \, U_z R_\Gamma$$

is non-trivial for a unitary R_Γ at some L

[⋯; Matsui 2008; Chen, Gu, Wen 2010; Yao, Oshikawa 2020; Ogata, Tasaki 2021; Seifnashri 2023; Kapustin, Sopenko 2024]

- The LSM theorem forbids SPT phases at any L
- The ground state always has long-range entanglement

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- The LSM theorem forbids SPT phases <u>at any</u> L
- The ground state always has long-range entanglement

When there is no LSM theorem, the projective algebra gives rise to an SPT-LSM theorem [Lu 2017; Yang, Jiang, Vishwanath, Ran 2017; Lu, Ran, Oshikawa 2017; ⋯]

- Any SPT state must have non-zero entanglement

SPT-LSM theorem proof

To prove the SPT-LSM theorem, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

2. Assume that any unique gapped ground state $|\mathrm{GS}\rangle$ satisfies $R_\Gamma|\mathrm{GS}\rangle \neq 0$ for some system size $L = L^*$

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We prove assumption 2 for product states $|\mathrm{GS}\rangle$ of G-qudits, but it is true as long as there is an IR TQFT description:

- In a TQFT, $\langle R_\Gamma \rangle = d_\Gamma$, so all SPT states at $L = L^*$ should satisfy $R_\Gamma |\mathrm{GS}\rangle = d_\Gamma |\mathrm{GS}\rangle$ [Chang, Lin, Shao, Wang, Yin 2018]

SPT-LSM theorem proof

If there is a unique gapped $|\mathrm{GS}\rangle$ that is a product state:

- $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \Longrightarrow U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

By assumption 2, $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$ at $L = L^*$:

$$\left.\begin{array}{l} 1.\quad R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle \\[2em] 2.\quad R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}U_j^{(z)}R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}|\mathrm{GS}\rangle \end{array}\right\} \mathit{Contradiction}$$

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By assumption 2, $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$ at $L = L^*$:

$$\left.\begin{aligned} &1.\quad R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle \\ &2.\quad R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}U_j^{(z)}R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}|\mathrm{GS}\rangle \end{aligned}\right\}\ \mathit{Contradiction}$$

$\Longrightarrow$ Cannot be a unique gapped SPT state that is a product state at $L = L^*$

$\Longrightarrow$ For any local Hamiltonian, there cannot be a unique gapped SPT state that is a product state for any L

SPT-LSM theorem proof

If there is a unique gapped $|\mathrm{GS}\rangle$ that is a product state:

➤ $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \implies U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

By a

1. R

diction

2. R

Therefore, the projective non-invertible symmetry prevents a product state SPT

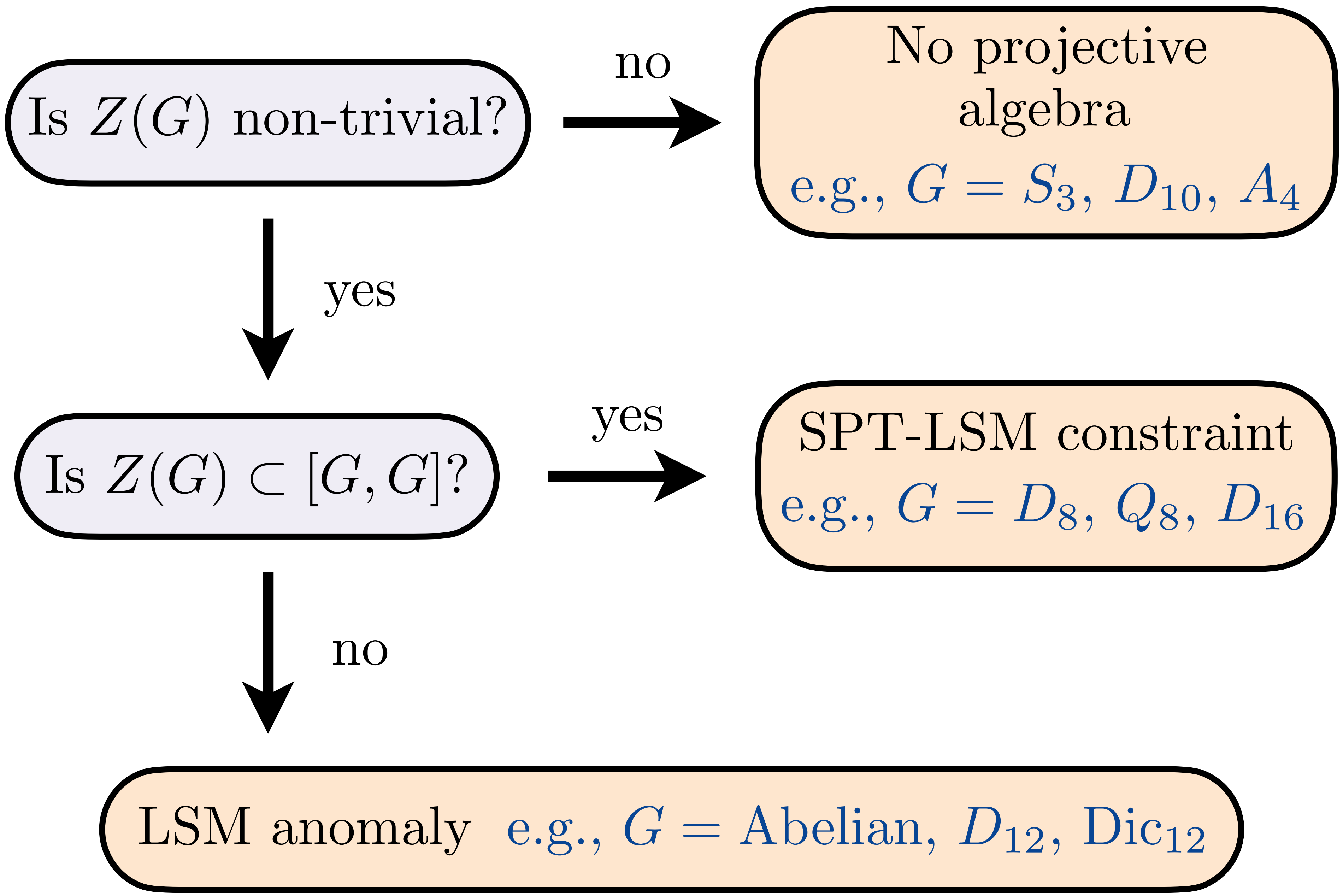
➤ All SPTs must have non-zero entanglement

$\implies$ Cannot be a unique gapped SPT state that is a product state at $L = L^*$

$\implies$ For any local Hamiltonian, there cannot be a unique gapped SPT state that is a product state for any L

(SPT)-LSM theorems

Whether there is a (SPT)-LSM theorem depends on G:



Outlook

We explored weak SPT phases characterized by non-invertible symmetries

1. An exactly solvable lattice model in an SPT phase characterized by $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ non-invertible symmetry and lattice translations

2. Projective non-invertible symmetries can give rise to SPT-LSM theorem that requires ground states to have nonzero-entanglement

3. Emphasized the effectiveness of using symmetry defects to characterize SPT states

arXiv:2409.18113

Back-up slides

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^{L} J\,\sigma_j^x\,\sigma_{j+1}^x + K\,\sigma_j^y\,\sigma_{j+1}^y$$

- There is an LSM anomaly involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the projective algebras [Cheng, Seiberg 2023]

Translation defects	$\mathbb{Z}_2^x$ *defect*	$\mathbb{Z}_2^y$ *defect*
$U_xU_y = (-1)^L U_yU_x$	$U_y\,T = -\,T\,U_y$	$T\,U_x = -\,U_x\,T$

GROUP BASED QUDITS

A G-qudit is a $|G|$-level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a finite group, e.g. $\mathbb{Z}_2$, S_3, D_8, SmallGroup(32,49)

Group based Pauli operators [Brell 2014]

$\bar{g} \equiv g^{-1}$

- X operators labeled by group elements

$$\overrightarrow{X}^{(g)} = \sum_h |gh\rangle\langle h| \qquad \overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

- Z operators are MPOs labeled by irreps $\Gamma : G \to \mathrm{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv \alpha \; \text{—} \boxed{Z^{(\Gamma)}} \text{—} \; \beta \qquad (\alpha, \beta = 1,2,\cdots,d_\Gamma)$$

GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{\mathbf{1}, \mathbf{1}'\}$

$$\overrightarrow{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(\mathbf{1})}]_{11} = 1$$

$$\overrightarrow{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \qquad [Z^{(\mathbf{1}')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

1. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(gh)}$, $\overleftarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\overrightarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$
2. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$ iff g and h commute
3. $\overrightarrow{X}^{(g)}\,[Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma}[Z^{(\Gamma)}]_{\gamma\beta}\,\overrightarrow{X}^{(g)}$
4. Unitarity: $\overrightarrow{X}^{(g)\dagger} = \overrightarrow{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger}\,Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

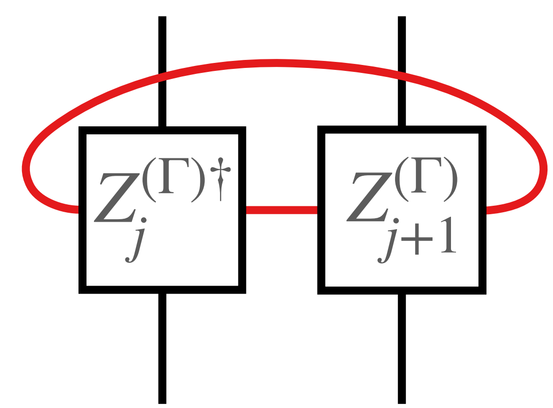
GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based XY model: Consider a periodic 1d lattice of L sites. On each site j resides a G-qudit and its Hamiltonian

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$$\operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_\Gamma(\bar{g}_j g_{j+1}) \, | \{g\} \rangle \langle \{g\} | \equiv$$



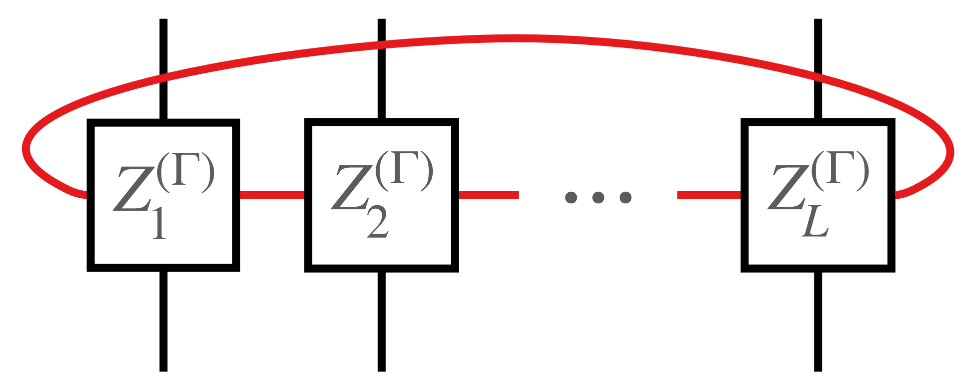
- For $G = \mathbb{Z}_2$, this is the ordinary quantum XY model

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_{g} K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$\mathbb{Z}_L$ lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

- $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$
- Rep(G) symmetry $R_\Gamma = \operatorname{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \overrightarrow{X}_j^{(z)} \qquad R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)}\right)$$

$$T_{\mathrm{tw}}^{(z)} = \overrightarrow{X}_I^{(z)}\, T \qquad T_{\mathrm{tw}}^{(\Gamma)} = \widehat{Z}_I^{(\Gamma)}\,(T \otimes \mathbf{1})$$

Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T_{\mathrm{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$	$T_{\mathrm{tw}}^{(\Gamma)}\, U_z = e^{i\phi_\Gamma(z)} U_z\, T_{\mathrm{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ projective algebra of the ordinary quantum XY model

GAUGING WEB

non-Abliean dipole symmetry

$$
\begin{tikzcd}
 & \mathsf{Rep}(G) \times Z(G) \times \mathbb{Z}_L \text{ with projective algebra} \arrow[dl, "\mathsf{Rep}(G)"'] \arrow[dr, "Z(G)"] & \\
(G \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dr, "Z(G)"'] & & (\mathsf{Rep}(G) \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dl, "\mathsf{Rep}(G)"] \\
 & G \times Z(G) \times \text{non-invertible translations} &
\end{tikzcd}
$$

Non-invertible dipole symmetry