INTERPLAYS OF GENERALIZED AND CRYSTALLINE SYMMETRIES IN G-QUDIT MODELS

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Applications of Generalized Symmetries and Topological Defects to Quantum Matter

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A TALE OF TWO SYMMETRIES

There are two types of symmetries of quantum systems

- Internal symmetries: preserve spacetime coordinates

$$\phi(t, r) \to \phi'(t, r)$$

- Spacetime symmetries: transform spacetime coordinates

$$\phi(t, r) \to \tilde{\phi}(\,\tilde{t}(t, r), \tilde{r}(t, r)\,)$$

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Can have non-trivial interplays [Nati's, Maissams's, Weicheng's, and Ömer's talks]

- For *ordinary* symmetries:

  Supersymmetry, Lieb-Schultz-Mattis (LSM) anomalies,
  $1 \to G_{\text{int}} \to G \to G_{\text{st}} \to 1$, symmetry fractionalization, $\cdots$

A GENERALIZED TALE

How can generalized symmetries and crystalline symmetries interplay in quantum lattice models?

Why care?

1. Searching for new interplays provides guidance towards novel phenomena in quantum matter
2. Exploring examples helps motivate the mathematical structure of symmetries in quantum lattice models

TL;DR FOR THIS TALK

In a group-based XY model, we find a projective algebra involving a $\mathsf{Rep}(G) \times Z(G)$ symmetry and lattice translations that constrains the allowed phases

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- Gauging internal sub-symmetries of $\mathsf{Rep}(G) \times Z(G)$ leads to lattice models with non-invertible dipole symmetries and non-invertible translation symmetries

TL;DR FOR THIS TALK

In a group-based XY model, we find a projective algebra involving a $\mathsf{Rep}(G)\times Z(G)$ symmetry and lattice translations that constrains the allowed phases

- Gauging internal sub-symmetries of $\mathsf{Rep}(G)\times Z(G)$ leads to lattice models with non-invertible dipole symmetries and non-invertible translation symmetries
- The SymTFT is a non-Abelian topological order enriched by lattice translations. It is a foliated field theory, not a topological field theory

  [see Ho Tat's Symmetries 2024 talk]

LSM ANOMALY IN THE XY MODEL

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^{L} J\,\sigma_j^x\,\sigma_{j+1}^x + K\,\sigma_j^y\,\sigma_{j+1}^y$$

- There is an LSM anomaly involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the projective algebras [Cheng, Seiberg 2023]

Translation defects	$\mathbb{Z}_2^x$ *defect*	$\mathbb{Z}_2^y$ *defect*
$U_x U_y = (-1)^L U_y U_x$	$U_y T = -\,T\,U_y$	$T\,U_x = -\,U_x T$

GROUP BASED QUDITS

A G-qudit is a $|G|$-level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a finite group, e.g. $\mathbb{Z}_2$, S_3, D_8, SmallGroup(32,49)

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Group based Pauli operators [Brell 2014]

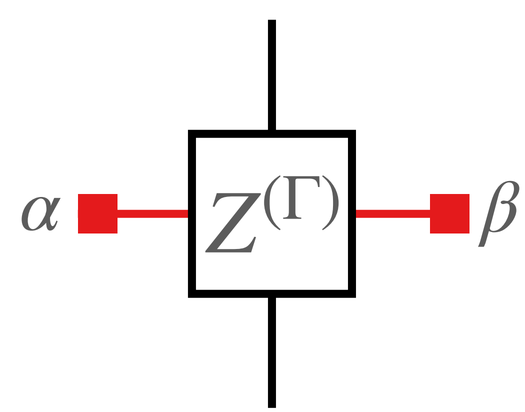
- X operators labeled by group elements

$\bar{g} \equiv g^{-1}$

$$\overrightarrow{X}^{(g)} = \sum_h |gh\rangle\langle h| \qquad \overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

- Z operators are MPOs labeled by irreps $\Gamma: G \to \mathrm{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv$$



$$(\alpha, \beta = 1,2,\cdots, d_\Gamma)$$

GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{\mathbf{1}, \mathbf{1}'\}$

$$\overrightarrow{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(\mathbf{1})}]_{11} = 1$$

$$\overrightarrow{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \qquad [Z^{(\mathbf{1}')}]_{11} = \sigma^z$$

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Group based Pauli operators satisfy

1. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(gh)}$, $\overleftarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\overrightarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$
2. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$ iff g and h commute
3. $\overrightarrow{X}^{(g)}\,[Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma}[Z^{(\Gamma)}]_{\gamma\beta}\,\overrightarrow{X}^{(g)}$
4. Unitarity: $\overrightarrow{X}^{(g)\dagger} = \overrightarrow{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger}\,Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

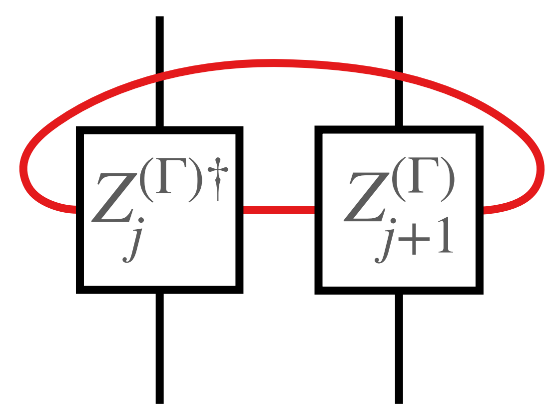
GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based XY model: Consider a periodic 1d lattice of L sites. On each site j resides a G-qudit and its Hamiltonian

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$$\operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_\Gamma(\bar{g}_j g_{j+1}) \,|\{g\}\rangle\langle\{g\}| \equiv$$



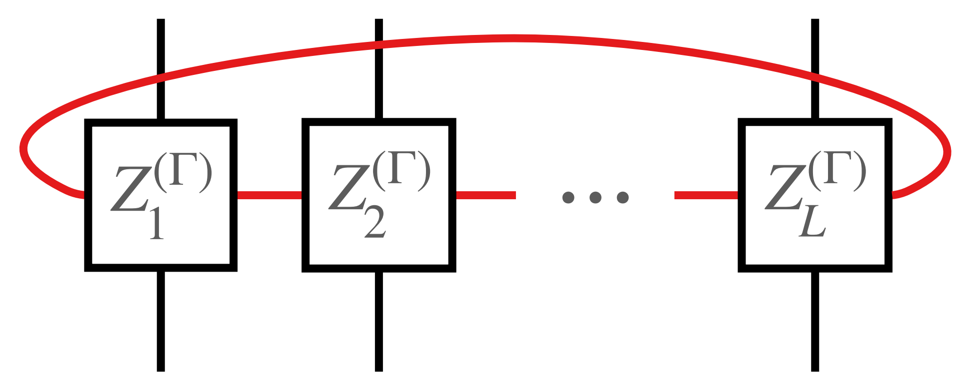
➤ For $G = \mathbb{Z}_2$, this is the ordinary quantum XY model

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_{g} K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$\mathbb{Z}_L$ lattice translations: $T\mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

- $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$
- Rep(G) symmetry $R_\Gamma = \operatorname{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)} \right) \equiv$ 

$$R_\Gamma = \sum_{\{g\}} \chi_\Gamma(g_1 g_2 \cdots g_{L-1} g_L) \,|\, \{g\} \rangle \langle \{g\} \,|$$

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_{\Gamma} \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_{g} K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

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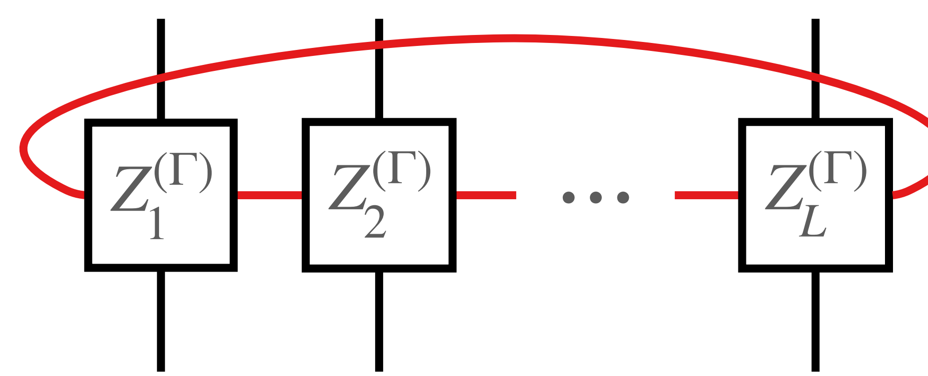


$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

SYMMETRY OPERATORS

$$H_{XY} = \sum^{L} \left(\sum J_\Gamma \,\mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)}\right) + \sum K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

When $G = A$ is Abelian, R_Γ is an A symmetry operator

$$\mathsf{Rep}(G) \times Z(G) \times \mathbb{Z}_L \to A \times A \times \mathbb{Z}_L$$

When G is non-Abelian, R_Γ is a non-invertible symmetry

$$R_\Gamma = \sum_{\{g\}} \chi_\Gamma(g_1 g_2 \cdots g_{L-1} g_L) \,|\{g\}\rangle\langle\{g\}|$$

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

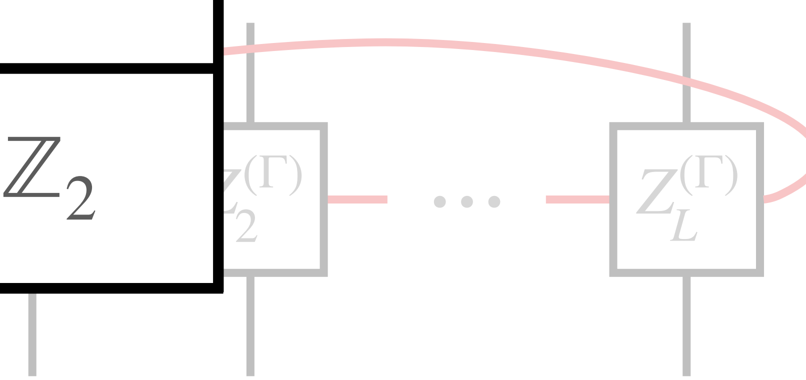
$\mathbb{Z}_L$ lattice tra

Various inter

➤ $Z(G)$ symm

➤ Rep(G) sy

G	Rep(G) $\times Z(G)$
$\mathbb{Z}_2$	$\mathbb{Z}_2^x \times \mathbb{Z}_2^y$
S_3	Rep(S_3)
D_8	Rep(D_8) $\times \mathbb{Z}_2$



$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

$Z(G)$ SYMMETRY DEFECTS

On an infinite chain, a $z \in Z(G)$ symmetry defect can be created at link $\langle I-1, I\rangle$ using

$$U_z(I) = \prod_{j \geq I} \overrightarrow{X}_j^{(z)}$$

- $\overrightarrow{X}_I^{(z)\dagger}$ moves this defect from $\langle I-1, I\rangle$ to $\langle I, I+1\rangle$
- Twisted translation operator $T_{\mathrm{tw}}^{(z)} = \overrightarrow{X}_I^{(z)}\, T$

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Defect Hamiltonian on a ring found using the twisted boundary conditions $(T_{\mathrm{tw}}^{(z)})^L = U_z$

$$H_{XY;z}^{\langle L,1\rangle} = H_{XY} + \sum_{\Gamma} \left(\frac{\chi_\Gamma(\bar{z})}{d_\Gamma} - 1 \right) J_\Gamma \operatorname{Tr}\left(Z_L^{(\Gamma)\dagger} Z_1^{(\Gamma)} \right) + \mathrm{hc}$$

Rep(G) SYMMETRY DEFECTS

A Rep(G) symmetry defect Γ has quantum dimension d_Γ

➤ $R_\Gamma \,|\psi\rangle = d_\Gamma \,|\psi\rangle$ on symmetric product state $|\psi\rangle = \bigotimes_{j=1}^{L} |1\rangle$

➤ To insert a Γ symmetry defect, must enlarge Hilbert space:

$$\mathscr{H}_\Gamma = \mathscr{H} \otimes \mathbb{C}^{d_\Gamma}$$

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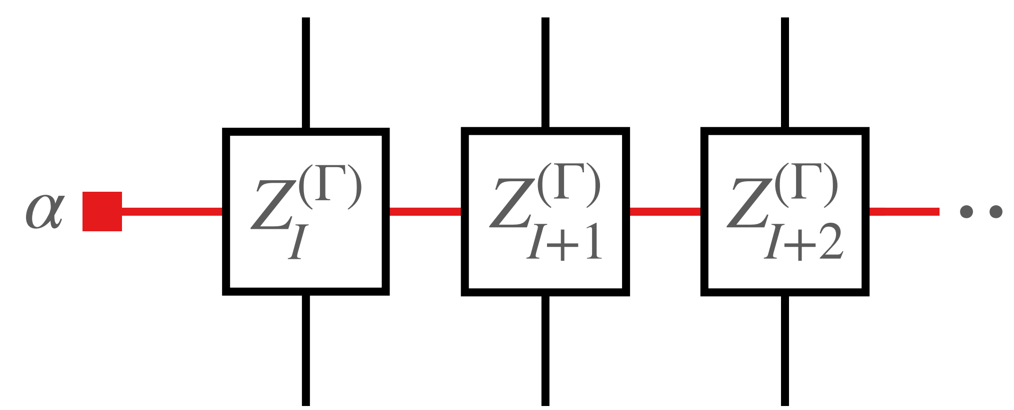
Create $\Gamma \in$ Rep(G) defect at $\langle I-1, I\rangle$ on infinite chain using truncated symmetry operator $R_\Gamma(I) = \sum_{\alpha,\beta} R_\Gamma(I;\alpha) \otimes |\alpha\rangle\langle\beta|$

- $R_\Gamma(I;\alpha) = [Z_I^{(\Gamma)}]_{\alpha,\alpha_I} \prod_{j>I} [Z_j^{(\Gamma)}]_{\alpha_{j-1}\alpha_j} \equiv$



Rep(G) SYMMETRY DEFECTS

Γ symmetry defect moved from $\langle I-1, I\rangle$ to $\langle I, I+1\rangle$ using

$$\widehat{Z}_I^{(\Gamma)\dagger} = \sum_{\alpha,\beta} [Z_I^{(\Gamma)\dagger}]_{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$

because $R_\Gamma(I+1) = \widehat{Z}_I^{(\Gamma)\dagger} R_\Gamma(I)$

- Twisted translation operator $T_{\text{tw}}^{(\Gamma)} = \widehat{Z}_I^{(\Gamma)}(T \otimes \mathbf{1})$

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Defect Hamiltonian on a ring found using the twisted boundary conditions $\left(T_{\mathrm{tw}}^{(\Gamma)}\right)^L = \prod_{j=1}^{L} \widehat{Z}_j^{(\Gamma)}$

$$\widehat{\Gamma}(g) = \sum_{\alpha,\beta} [\Gamma(g)]_{\alpha\beta} |\alpha\rangle\langle\beta|$$

$$H_{XY;\Gamma}^{\langle L,1\rangle} = H_{XY} \otimes \mathbf{1} + \sum_g K_g \overleftarrow{X}_L^{(g)} \overrightarrow{X}_1^{(g)} \otimes \left(\widehat{\Gamma}(g) - \mathbf{1}\right) + \mathrm{hc}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \overrightarrow{X}_j^{(z)} \qquad R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)}\right)$$

$$T_{\mathrm{tw}}^{(z)} = \overrightarrow{X}_I^{(z)}\, T \qquad T_{\mathrm{tw}}^{(\Gamma)} = \widehat{Z}_I^{(\Gamma)}\,(T \otimes \mathbf{1})$$

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Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T_{\mathrm{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$	$T_{\mathrm{tw}}^{(\Gamma)}\, U_z = e^{i\phi_\Gamma(z)} U_z\, T_{\mathrm{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ projective algebra of the ordinary quantum XY model

PROJECTIVE ALGEBRA FROM DEFECTS

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$	$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}} U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z T^{(\Gamma)}_{\mathrm{tw}}$

Example 1: $G = S_3 \implies \mathsf{Rep}(S_3) \times \mathbb{Z}_1 \times \mathbb{Z}_L$

$$\exp[\mathrm{i}\phi_\Gamma(z)] = 1$$

$$\boxed{\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma}$$

PROJECTIVE ALGEBRA FROM DEFECTS

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
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$$\boxed{\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma}$$

Example 2: $G = D_8 \implies \mathsf{Rep}(D_8) \times \mathbb{Z}_2 \times \mathbb{Z}_L$

$$\exp[\mathrm{i}\phi_{\mathbf{2}}(-1)] = -1$$

PROJECTIVE ALGEBRA FROM DEFECTS

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$	$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}} U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z T^{(\Gamma)}_{\mathrm{tw}}$

$$\boxed{\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma}$$

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$$\exp[\mathrm{i}\phi_\Gamma(z)] = 1$$

Example 2: $G = D_8 \implies \mathsf{Rep}(D_8) \times \mathbb{Z}_2 \times \mathbb{Z}_L$

$$\exp[\mathrm{i}\phi_{\mathbf{2}}(-1)] = -1$$

Example 3: $G = D_{12} \implies \mathsf{Rep}(D_{12}) \times \mathbb{Z}_2 \times \mathbb{Z}_L$

$$\exp[\mathrm{i}\phi_{\mathbf{1}_3}(-1)] = \exp[\mathrm{i}\phi_{\mathbf{1}_4}(-1)] = \exp[\mathrm{i}\phi_{\mathbf{2}_6}(-1)] = -1$$

PROJECTIVE ALGEBRA FROM DEFECTS

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T_{\text{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\text{tw}}^{(z)} R_\Gamma$	$T_{\text{tw}}^{(\Gamma)}\, U_z = e^{i\phi_\Gamma(z)} U_z\, T_{\text{tw}}^{(\Gamma)}$

The projective algebras are nontrivial for any G with a nontrivial center $Z(G)$

- Will assume $Z(G)$ is nontrivial from here on

Example 3: $G = D_{12} \implies \mathsf{Rep}(D_{12}) \times \mathbb{Z}_2 \times \mathbb{Z}_L$

$$\exp[i\phi_{1_3}(-1)] = \exp[i\phi_{1_4}(-1)] = \exp[i\phi_{2_6}(-1)] = -1$$

```
SmallGroup(8,3): Structure = D8; Center = C2
SmallGroup(8,4): Structure = Q8; Center = C2
SmallGroup(12,1): Structure = C3 : C4; Center = C2
SmallGroup(12,4): Structure = D12; Center = C2
SmallGroup(16,3): Structure = (C4 x C2) : C2; Center = C2 x C2
SmallGroup(16,4): Structure = C4 : C4; Center = C2 x C2
SmallGroup(16,6): Structure = C8 : C2; Center = C4
SmallGroup(16,7): Structure = D16; Center = C2
SmallGroup(16,8): Structure = QD16; Center = C2
SmallGroup(16,9): Structure = Q16; Center = C2
SmallGroup(16,11): Structure = C2 x D8; Center = C2 x C2
SmallGroup(16,12): Structure = C2 x Q8; Center = C2 x C2
SmallGroup(16,13): Structure = (C4 x C2) : C2; Center = C4
SmallGroup(18,3): Structure = C3 x S3; Center = C3
SmallGroup(20,1): Structure = C5 : C4; Center = C2
SmallGroup(20,4): Structure = D20; Center = C2
SmallGroup(24,1): Structure = C3 : C8; Center = C4
SmallGroup(24,3): Structure = SL(2,3); Center = C2
SmallGroup(24,4): Structure = C3 : Q8; Center = C2
SmallGroup(24,5): Structure = C4 x S3; Center = C4
SmallGroup(24,6): Structure = D24; Center = C2
SmallGroup(24,7): Structure = C2 x (C3 : C4); Center = C2 x C2
SmallGroup(24,8): Structure = (C6 x C2) : C2; Center = C2
SmallGroup(24,10): Structure = C3 x D8; Center = C6
SmallGroup(24,11): Structure = C3 x Q8; Center = C6
SmallGroup(24,13): Structure = C2 x A4; Center = C2
SmallGroup(24,14): Structure = C2 x C2 x S3; Center = C2 x C2
SmallGroup(27,3): Structure = (C3 x C3) : C3; Center = C3
SmallGroup(27,4): Structure = C9 : C3; Center = C3
SmallGroup(28,1): Structure = C7 : C4; Center = C2
SmallGroup(28,3): Structure = D28; Center = C2
SmallGroup(30,1): Structure = C5 x S3; Center = C5
SmallGroup(30,2): Structure = C3 x D10; Center = C3
SmallGroup(32,2): Structure = (C4 x C2) : C4; Center = C2 x C2 x C2
SmallGroup(32,4): Structure = C8 : C4; Center = C4 x C2
SmallGroup(32,5): Structure = (C8 x C2) : C2; Center = C4 x C2
SmallGroup(32,6): Structure = (C2 x C2 x C2) : C4; Center = C2
SmallGroup(32,7): Structure = (C8 : C2) : C2; Center = C2
SmallGroup(32,8): Structure = C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2); Center = C2
```

IS THERE AN LSM THEOREM?

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}}\, U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z\, T^{(\Gamma)}_{\mathrm{tw}}$

A projective algebra for invertible symmetry operators implies an obstruction to SPT states (an 't Hooft anomaly):

- A projective algebra arising from inserting an *invertible* defect also obstructs SPTs states in the defect-free model
  [Matsui 2008; Yao, Oshikawa 2020; Seifnashri 2023; Kapustin, Sopenko 2024]

LSM theorem for G with $Z(G)$ nontrivial in a 1d irrep

- e.g., D_{2n} with $n \in 4\mathbb{Z}_{\geq 0} + 2$

IS THERE AN LSM THEOREM?

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$	$R_\Gamma T_{\mathrm{tw}}^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$	$T_{\mathrm{tw}}^{(\Gamma)} U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z T_{\mathrm{tw}}^{(\Gamma)}$

A projective algebra with non-invertible symmetry operators does *not* imply an 't Hooft anomaly

- i.e., $R_\Gamma T_{\mathrm{tw}}^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$ supports SPT state with $R_\Gamma \,|\, \psi\rangle = 0$

IS THERE AN LSM THEOREM?

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}}\, U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z\, T^{(\Gamma)}_{\mathrm{tw}}$

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- i.e., $R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$ supports SPT state with $R_\Gamma |\psi\rangle = 0$

A projective algebra of invertible symmetry operators from a non-invertible defect does *not* imply an 't Hooft anomaly

- Degeneracy can reflect the defects' quantum dimension

IS THERE AN LSM THEOREM?

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}}\, U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z\, T^{(\Gamma)}_{\mathrm{tw}}$

A projective algebra with non-invertible symmetry operators

does

➤ i. $= 0$

A pr n a

non-

No LSM theorem for G with $Z(G)$ trivial in all 1d irreps (i.e., $Z(G) \subset [G, G]$)

➤ Example $G = D_8$: using the SymTFT, there are ≥ 6 allowed $\mathsf{Rep}(D_8) \times \mathbb{Z}_2$ weak SPT states

➤ Degeneracy can reflect the defects' quantum dimension

NON-INVERTIBLE WEAK SPTs

For L such that the projective algebra $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$ is nontrivial, SPT ground states must satisfy $|\langle R_\Gamma \rangle| = 0$

Two possibilities:

1. An SPT state satisfies $|\langle U_z \rangle| = 1$ and $|\langle R_\Gamma \rangle| = 0$ for all system sizes L
2. For $L = L^*$ where all $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $|\langle U_z \rangle| = 1$ and $|\langle R_\Gamma \rangle| = d_\Gamma$, but $|\langle R_\Gamma \rangle| = 0$ for $L \neq L^*$

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The first possibility is incompatible with TQFT

- In a TQFT, $\langle$contractible TDL$\rangle$ = quantum dimension, so all SPT states at $L = L^*$ should satisfy $|\langle R_\Gamma \rangle| = d_\Gamma$

NON-INVERTIBLE WEAK SPTs

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T

1

2

At $L = L^*$, SPTs satisfy $|\langle R_\Gamma \rangle| = d_\Gamma$

At $L = L^* + 1$, SPTs satisfy $|\langle R_\Gamma \rangle| = 0$

➤ $R_\Gamma U_z = (\mathrm{e}^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ implies that any SPT is a non-invertible weak SPT with translation defects dressed by non-invertible symmetry charge

The first possibility is incompatible with TQFT

➤ In a TQFT, $\langle$contractible TDL$\rangle$ = quantum dimension, so all SPT states at $L = L^*$ should satisfy $|\langle R_\Gamma \rangle| = d_\Gamma$

GAUGING WEB [Nat's talk]

Projective algebras arising from inserting symmetry defects affect the symmetries in the gauging web

- Gauging web = duality web = orbifold groupoid
- e.g., gauging the anomaly-free $\mathbb{Z}_2^a$ sub-symmetry of an anomalous $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symmetry in $1+1$D leads to a dual $\mathbb{Z}_4$ symmetry [Bhardwaj, Tachikawa 2017; Chatterjee, Wen 2022; Zhang, Levin 2022]

The nontrivial projective algebras affect the symmetries in the gauging web of $\mathrm{Rep}(G) \times Z(G) \times \mathbb{Z}_L$

GAUGING WEB

non-Abliean dipole symmetry

$$\begin{tikzcd} & \mathsf{Rep}(G)\times Z(G)\times \mathbb{Z}_L \text{ with projective algebra} \arrow[dl, "\mathsf{Rep}(G)"'] \arrow[dr, "Z(G)"] & \\ (G\times Z(G))\rtimes\mathbb{Z}_L \arrow[dr, "Z(G)"'] & & (\mathsf{Rep}(G)\times Z(G))\rtimes \mathbb{Z}_L \arrow[dl, "\mathsf{Rep}(G)"] \\ & G\times Z(G)\times \text{non-invertible translations} & \end{tikzcd}$$

Non-invertible dipole symmetry

- Generalizes and unifies $G = \mathbb{Z}_2$ results from Aksoy, Mudry, Furusaki, Tiwari 2023 and Seifnashri 2023

GAUGING WEB

$$\begin{tikzcd}[column sep=small]
 & \text{Rep}(G) \times Z(G) \times \mathbb{Z}_L \text{ with projective algebra} \arrow[dl, bend right, "\text{Rep}(G)"'] \arrow[dr, bend left, "Z(G)"] & \\
(G \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dr, bend right, "Z(G)"'] & & (\text{Rep}(G) \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dl, bend left, "\text{Rep}(G)"] \\
 & G \times Z(G) \times \text{non-invertible translations} &
\end{tikzcd}$$

GAUGING UNIFORM $Z(G)$

To gauge $Z(G)$, we add $Z(G)$-qudits on links and enforce the Gauss laws

$$G_j^{(z)} = \overleftarrow{\mathscr{X}}^{(z)}_{j-1,j} \overrightarrow{X}^{(z)}_j \overrightarrow{\mathscr{X}}^{(z)}_{j,j+1} = 1$$

- Trivializes the $Z(G)$ symmetry operator $U_z = \prod \overrightarrow{X}^{(z)}_j$

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$Z(G)$-gauged G-based XY model is $(\rho_\Gamma(z) = \chi_\Gamma(z)/d_\Gamma)$

$$H_{XY/Z(G)} = \sum_{j=1}^{L}\left(\sum_\Gamma J_\Gamma\,\mathcal{Z}^{(\rho_\Gamma)}_{j,j+1}\,\mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)}\right) + \sum_g K_g\,\overleftarrow{X}^{(g)}_j\,\overrightarrow{X}^{(g)}_{j+1}\right) + \mathrm{hc}$$

Dual $Z(G)$ symmetry

$$U^\vee_\rho = \prod_j \mathcal{Z}^{(\rho)}_{j,j+1}$$

Rep(G) symmetry becomes

$$R_\Gamma = \mathrm{Tr}\Big(\prod_{j=1}^{L} Z_j^{(\Gamma)}\,[\mathcal{Z}^{(\rho_\Gamma)}_{j,j+1}]^{-j}\Big)$$

GAUGING UNIFORM $Z(G)$

To gauge $Z(G)$, we add $Z(G)$-qudits on links and enforce the Gauss law

Rep(G) is a modulated symmetry

$$T R_\Gamma T^\dagger = U^\vee_{\rho_\Gamma} R_\Gamma$$

- $\mathbb{Z}_L$ extended by $Z(G) \times \mathrm{Rep}(G)$

- Trivializes … $\overrightarrow{X}^{(z)}_j$

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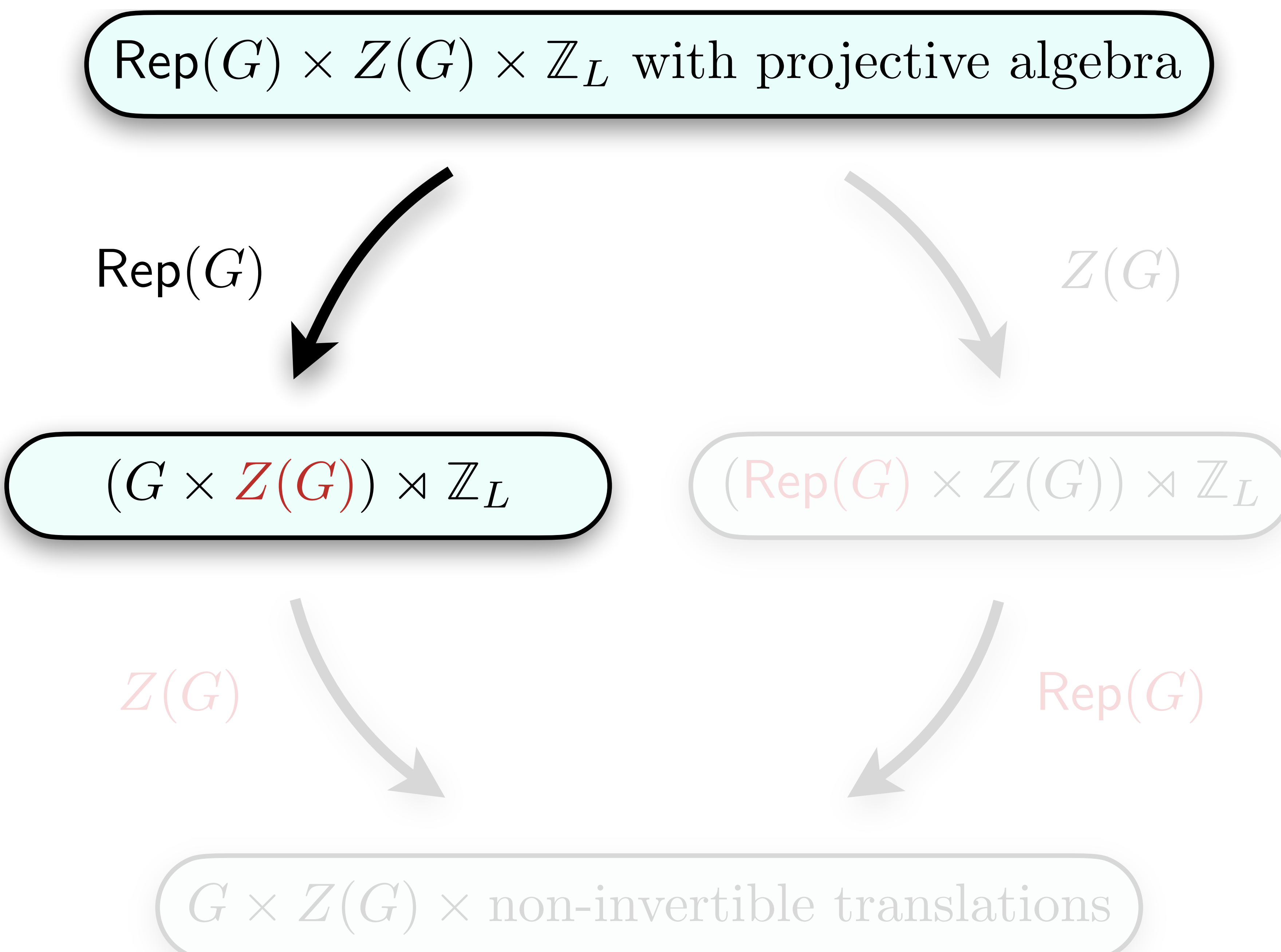
GAUGING UNIFORM Rep(G)



GAUGING UNIFORM Rep(G)

To gauge Rep(G), we add G-qudits on links and enforce the matrix product operator Gauss laws

$$[G_j^{(\Gamma)}]_{\alpha\beta} = [\mathscr{Z}_{j-1,j}^{(\Gamma)}\, Z_j^{(\Gamma)}\, \mathscr{Z}_{j,j+1}^{(\Gamma)\dagger}]_{\alpha\beta} = \delta_{\alpha,\beta}$$

- Equivalent to requiring $g_j = \bar{g}_{j-1,j}\, g_{j,j+1}$
- Trivializes the Rep(G) symmetry operator $R_\Gamma = \mathrm{Tr}\Big(\prod_{j=1}^{L} Z_j^{(\Gamma)}\Big)$

Minimal coupling leads to the Rep(G)-gauged model

$$H_{XY/\mathsf{Rep}(G)} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma\, \mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_{g} K_g\, \overleftarrow{X}_j^{(g)}\, \overleftarrow{\mathscr{X}}_{j,j+1}^{(g)}\, \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

GAUGING UNIFORM Rep(G)

To find the dual symmetry, it is useful to perform the unitary transformation

$$Z_j^{(\Gamma)} \to \mathcal{Z}_{j-1,j}^{(\Gamma)\dagger} Z_j^{(\Gamma)} \mathcal{Z}_{j,j+1}^{(\Gamma)} \qquad \overleftarrow{X}_j^{(g)} \overleftarrow{\mathcal{X}}_{j,j+1}^{(g)} \overrightarrow{X}_{j+1}^{(g)} \to \overleftarrow{\mathcal{X}}_{j,j+1}^{(g)}$$

- Gauss's laws $[Z_j^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$ decouple original G qudits
- $H_{XY/\mathrm{Rep}(G)} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \mathrm{Tr} \left(\mathcal{Z}_{j,j+1}^{(\Gamma)\dagger} \mathcal{Z}_{j-1,j}^{(\Gamma)} \mathcal{Z}_{j,j+1}^{(\Gamma)\dagger} \mathcal{Z}_{j+1,j+2}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{\mathcal{X}}_{j,j+1}^{(g)} \right) + \mathrm{hc}$

<u>$Z(G)$ symmetry becomes</u>

$$U_z = \prod_j [\overrightarrow{\mathcal{X}}_{j,j+1}^{(z)}]^j$$

<u>Dual G symmetry</u>

$$R_g^\vee = \prod_j \overrightarrow{\mathcal{X}}_{j,j+1}^{(g)}$$

GAUGING UNIFORM Rep(G)

To find the dual symmetry, it is useful to perform the unitary transformati

$Z(G)$ is a modulated symmetry

$$T\, U_z\, T^\dagger = [R_z^\vee]^\dagger\, U_z$$

➤ $\mathbb{Z}_L$ extended by $Z(G) \times G$

$Z_j^{(\Gamma)} \to$ $\to \overleftarrow{\mathcal{X}}^{(g)}_{j,j+1}$

➤ Gauss's la udits

➤ $H_{XY/\mathrm{Rep}(G)} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(\mathcal{Z}^{(\Gamma)\dagger}_{j,j+1} \mathcal{Z}^{(\Gamma)}_{j-1,j} \mathcal{Z}^{(\Gamma)\dagger}_{j,j+1} \mathcal{Z}^{(\Gamma)}_{j+1,j+2} \right) + \sum_g K_g \overleftarrow{\mathcal{X}}^{(g)}_{j,j+1} \right) + \mathrm{hc}$

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Dual G symmetry

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GAUGING MODULATED $Z(G)$

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\end{tikzcd}
$$

GAUGING MODULATED $Z(G)$

We can gauge the modulated $Z(G)$ symmetry $U_z = \prod_j [\overrightarrow{\mathcal{X}}^{(z)}_{j,j+1}]^j$ using $Z(G)$-qudits and the Gauss's laws

$$G^{(z)}_j = \overleftarrow{X}^{(z)}_j [\overrightarrow{\mathcal{X}}^{(z)}_{j,j+1}]^j \overrightarrow{X}^{(z)}_{j+1} = 1$$

- Dual $G \times Z(G)$ symmetry $R^\vee_g = \prod_j \overrightarrow{\mathcal{X}}^{(g)}_{j,j+1}$ and $U^\vee_\rho = \prod_j Z^{(\rho)}_j$

GAUGING MODULATED $Z(G)$

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This gauging explicitly breaks translations: $T\, G^{(z)}_j \, T^\dagger \neq G^{(z)}_{j+1}$

- There is a new non-invertible translation symmetry

$$\mathsf{D}_T = \mathsf{D}\, T$$

where, for instance, $\mathsf{D}\colon \overleftarrow{X}^{(z)}_j \, \overrightarrow{X}^{(z)}_{j+1} \to \overleftarrow{X}^{(z)}_j \, \overrightarrow{\mathscr{X}}^{(z)}_{j,j+1} \, \overrightarrow{X}^{(z)}_{j+1}$

THE SYMMETRY TFT

A discrete gauging web in 1+1D can be formulated through a 2+1D topological theory called the SymTFT [Sakura's, Paul's, Tian's talks]

[⋯ ; Gaiotto, Kapustin, Seiberg, Willet (2014); Kong, Wen, Zheng (2015), Freed, Teleman (2018); Ji, Wen (2019); Lichtman, Thorngren, Lindner, Stern, Berg (2020); Kong, Lan, Wen, Zhang, Zheng (2020); Gaiotto, Kulp (2020); Aasen, Fendley, Mong (2020); Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki (2021); Chatterjee, Wen (2022); ⋯]

THE SYMMETRY TFT

⋮

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Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki (2021);

Chatterjee, Wen (2022); Apruzzi (2022); Chatterjee, Wen (2022); Moradi, Moosavian, Tiwari (2022); Freed, Moore, Teleman (2022); Kaidi, Ohmori, Zheng (2022); Chatterjee, Ji, Wen (2022);

Kaidi, Nardoni, Zafrir, Zheng (2023); Zhang, Córdova (2023); Lan, Zhou (2023); Bhardwaj, Schafer-Nameki (2023); Chen, Cui, Haghighat, Wang (2023); Apruzzi, Bonetti, Gould, Schafer-Nameki (2023); Bah, Leung, Waddleton (2023); Córdova, Hsin, Zhang (2023); Cao, Jia (2023); ***SP*** *(2023); Baume, Heckman, Hübner, Torres, Turner, Yu (2023); Huang, Cheng (2023); Wen, Potter (2023); Inamura, Wen (2023); Schuster, Tantivasadakarn, Vishwanath, Yao (2023); Bhardwaj, Bottini, Pajer, Schafer-Nameki (2023);* ***SP****, Zhu, Beaudry, Wen (2023); Motamarri, McLauchlan, Béri (2023);*

Brennan, Sun (2024); Antinucci, Benini (2024); Bonetti, Del Zotto, Minasian (2024); Apruzzi, Bedogna, Dondi (2024); Del Zotto, Nadir Meynet, Moscrop (2024); Bhardaj, Pajer, Schafer-Nameki, Warman (2024); Argurio, Benini, Bertolini, Galati, Niro (2024); Wen, Ye, Potter (2024); Franco, Yu (2024); Putrov, Radhakrishnan (2024); Chatterjee, Aksoy, Wen (2024); Bhardwaj, Bottini, Schafer-Nameki, Tiwari (2024); Arbalestrier, Arguio, Tizzano (2024); Huang (2024); Bhardwaj, Inamura, Tiwari (2024); Hasan, Meynet, Migliorati (2024); Nardoni, Sacchi, Sela, Zafrir, Zheng (2024); Heckman, Hübner (2024); Ji, Chen (2024); Antinucci, Benini, Rizi (2024); Copetti (2024); Bhardaj, Pajer, Schafer-Nameki, Tiwari, Warman, Wu (2024)

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THE SYMMETRY TFT

A discrete gauging web in 1+1D can be formulated through a 2+1D topological theory called the SymTFT [Sakura's, Paul's, Tian's talks]

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Can construct the SymTFT by extending the $(G \times Z(G)) \rtimes \mathbb{Z}_L$ symmetry to 2+1D and gauging the internal sub-symmetry

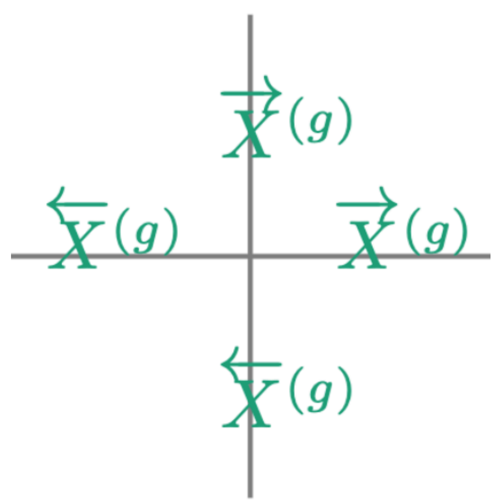
- SymTFT is $G \times Z(G)$ gauge theory enriched by $\mathbb{Z}_L$ lattice translations in only one direction (a spacetime SET)
- Can formulate as a quantum code made up $G \times Z(G)$ qudits on edges of a square lattice

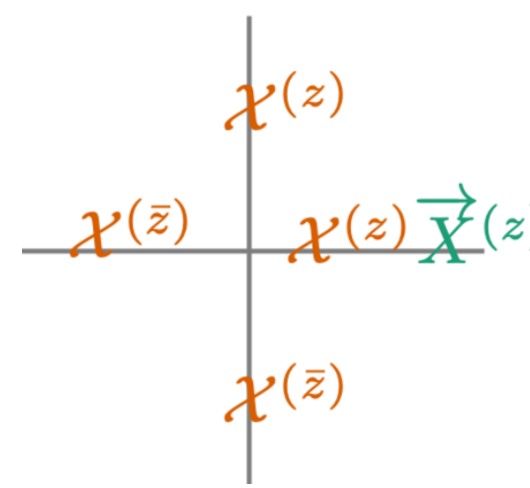
SYMTFT AS A QUANTUM CODE

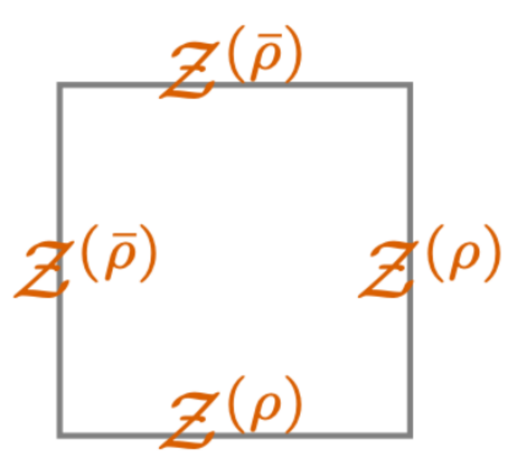
Code space is $\mathscr{V} = \mathrm{Span}_{\mathbb{C}} \left\{ |\psi\rangle \in \bigotimes_e \mathbb{C}^{|G \times Z(G)|} \mid \mathbb{A}_r = \mathbb{G}_r = \mathbb{B}_\square = \mathbb{F}_\square = 1 \right\}$

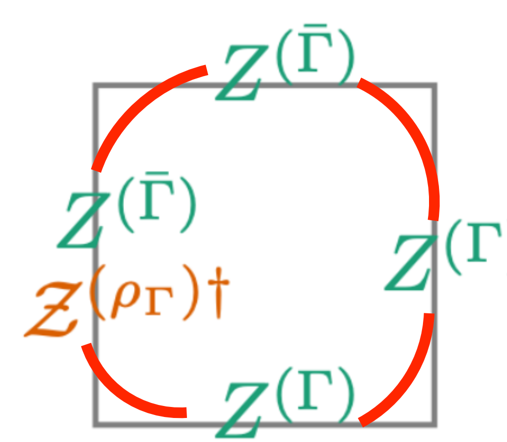
with $\mathbb{A}_r = \dfrac{1}{|G|} \displaystyle\sum_g A_r^{(g)}$, $\mathbb{G}_r = \dfrac{1}{|Z(G)|} \displaystyle\sum_z \mathscr{G}_r^{(z)}$, $\mathbb{B}_\square = \dfrac{1}{|Z(G)|} \displaystyle\sum_\rho B_\square^{(\rho)}$ and

$$\mathbb{F}_\square = \frac{1}{|G|} \sum_\Gamma d_\Gamma F_\square^{(\Gamma)}$$

$A_r^{(g)} =$ 

$\mathscr{G}_r^{(z)} =$ 

$B_\square^{(\rho)} =$ 

$F_\square^{(\Gamma)} =$ 

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with ...

This code space corresponds to a foliated field theory, not a TFT

- Has discrete translation symmetry that acts as an anyon automorphism
- $G = \mathbb{Z}_N$: $S[e_x^{(1)}] = \frac{\mathrm{i}N}{2\pi}\int A^{(1)}\mathrm{d}a^{(1)} + B^{(1)}\mathrm{d}b^{(1)} + A^{(1)}B^{(1)}e_x^{(1)}$

$(e_x^{(1)} = \Lambda\,\mathrm{d}x)$

[see Ho Tat's Symmetries 2024 talk]

GAUGING WEB IN THE SYMTFT

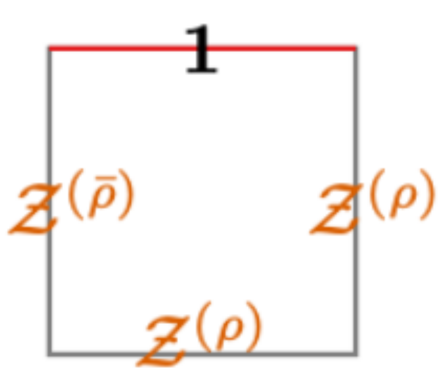
Different symmetries in the gauging web correspond to different gapped boundaries of the SymTFT

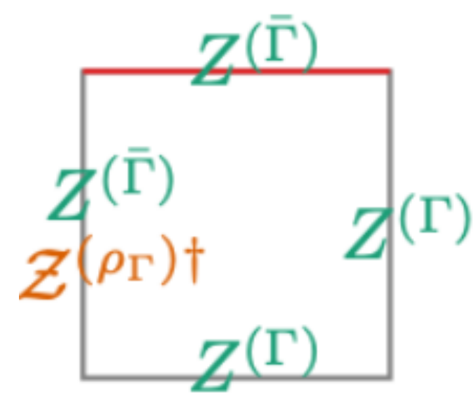
GAUGING WEB IN THE SYMTFT

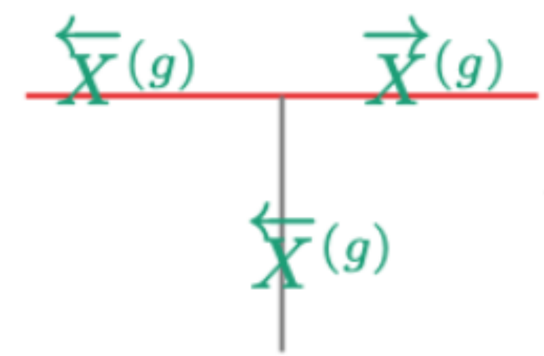
Different symmetries in the gauging web correspond to different gapped boundaries of the SymTFT

$\mathsf{Rep}(G) \times Z(G) \times \mathbb{Z}_L$ with projective algebra

- A smooth (rough) boundary for G ($Z(G)$) qudits







- Boundary symmetry operators

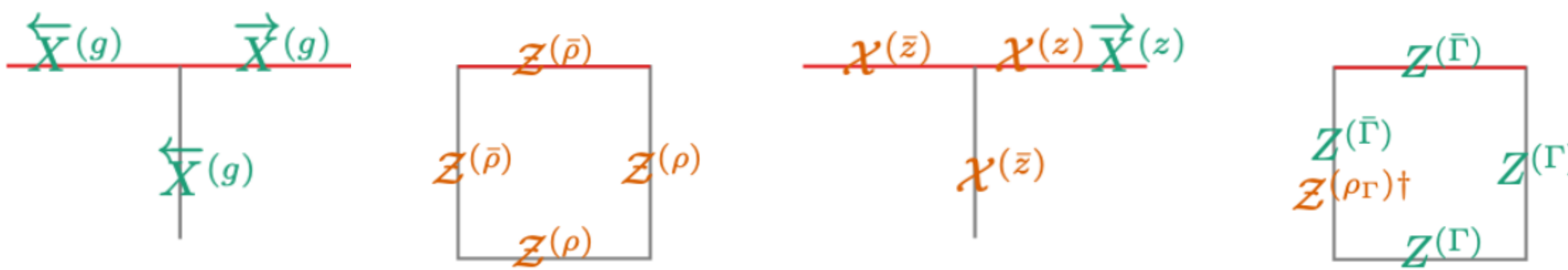
$$R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L_x} Z^{(\Gamma)}_{(j,L_y),x}\right) \qquad U_z = \prod_{j=1}^{L_x} \overrightarrow{X}^{(z)}_{(j,L_y),x} \overleftarrow{\mathcal{X}}^{(z)}_{(j,L_y-1),y}$$

GAUGING WEB IN THE SYMTFT

Different symmetries in the gauging web correspond to different gapped boundaries of the SymTFT

$$\boxed{(\mathsf{Rep}(G) \times Z(G)) \rtimes \mathbb{Z}_L}$$

- A smooth (smooth) boundary for G ($Z(G)$) qudits



- Boundary symmetry operators

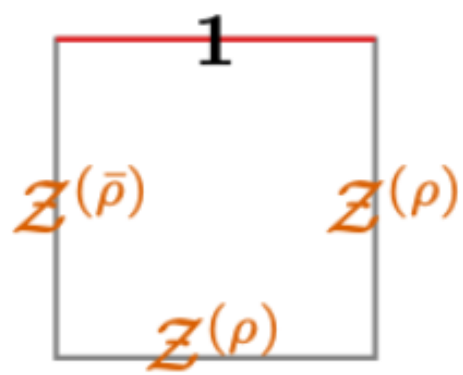
$$R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L_x} Z^{(\Gamma)}_{(j,L_y),x}\left[\mathcal{Z}^{(\rho_\Gamma)}_{(j,L_y),x}\right]^{-j}\right) \qquad U^\vee_\rho = \prod_{j=1}^{L_x} \mathcal{Z}^{(\rho_\Gamma)\dagger}_{(j,L_y),x}$$

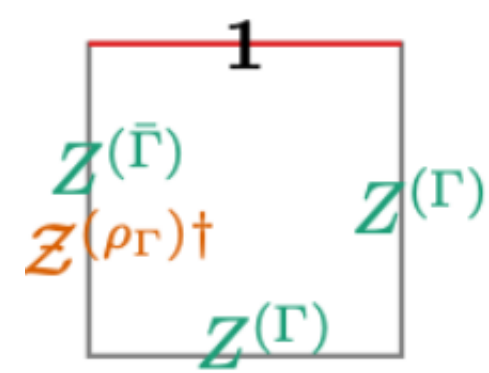
GAUGING WEB IN THE SYMTFT

Different symmetries in the gauging web correspond to different gapped boundaries of the SymTFT

$$\boxed{(G \times Z(G)) \rtimes \mathbb{Z}_L}$$

- A rough (rough) boundary for G ($Z(G)$) qudits





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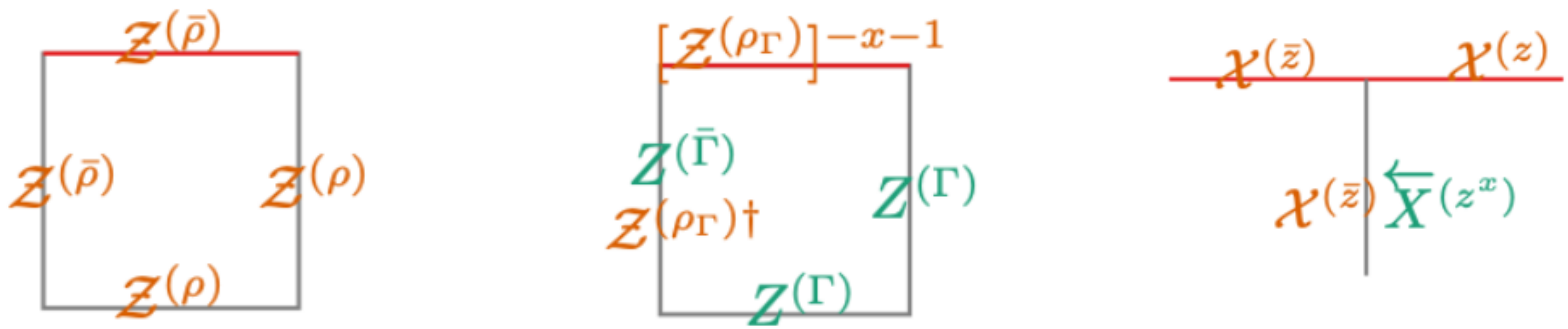
GAUGING WEB IN THE SYMTFT

Different symmetries in the gauging web correspond to different gapped boundaries of the SymTFT

$$\boxed{G \times Z(G) \times \text{non-invertible translations}}$$

- A rough (smooth) boundary for G ($Z(G)$) qudits



- Boundary symmetry operators

$$R_g^\vee = \prod_{j=1}^{L_x} \overleftarrow{X}^{(g)}_{(j,L_y-1),y} \qquad U_\rho^\vee = \prod_{j=1}^{L_x} \mathcal{Z}^{(\rho)}_{(j,L_y),x}$$

Non-invertible translations

OUTLOOK

We explored how generalized symmetries and crystalline symmetries interplay in quantum lattice models of G-qudits

1. Generalized and crystalline symmetries with projective algebras
2. Non-invertible weak SPTs
3. Non-invertible dipole and translation symmetries

This is just the tip of the iceberg!