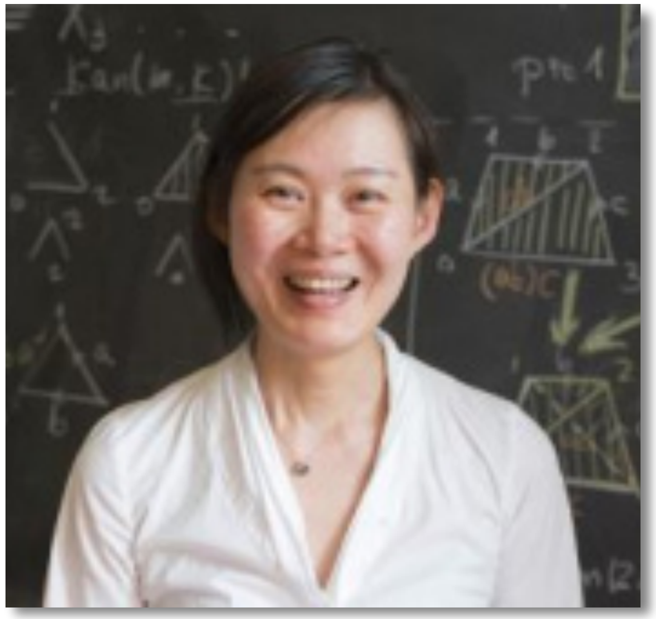
GENERALIZED SYMMETRIES IN ORDERED PHASES

Salvatore Pace

MIT

Based on:

1. **SP**, arXiv:2308.05730 [SciPost Phys. 17, 080 (2024)]
2. **SP**, C Zhu, A Beaudry, and X-G Wen, arXiv:2310.08554



Chenchang Zhu

Agnès Beaudry

Xiao-Gang Wen

The power of symmetry

Symmetries give sacred non-perturbative insights into quantum many-body systems

1. They constrain systems

Goldstone's theorem *Bloch's Theorem* *Noether's theorem*

Anomalies

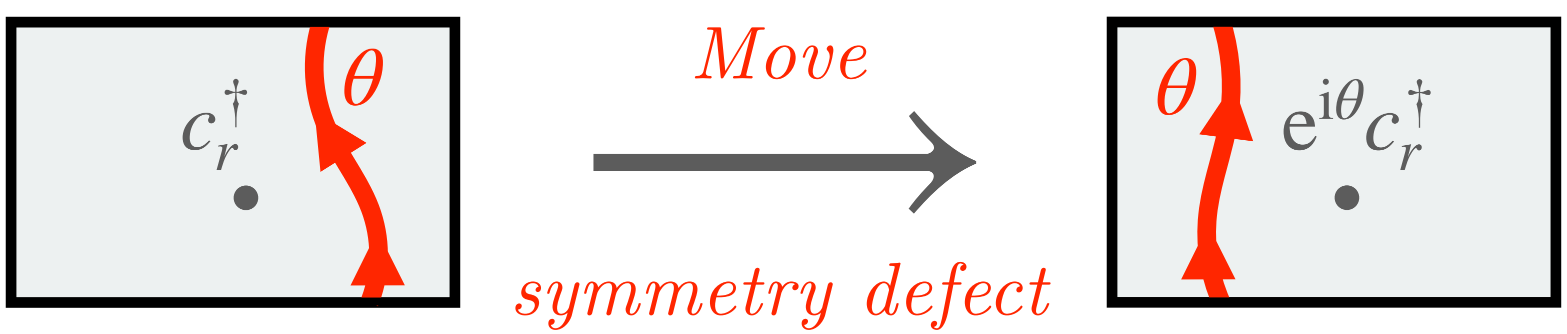
2. They can spontaneously break and protect topological phases

Classify phases of matter

Symmetry defects

Symmetries can be used to probe a theory via their symmetry defects

- A localized modification to the theory that implements the symmetry when moved in space
- e.g., Symmetry defect for $c_r^\dagger \to \mathrm{e}^{\mathrm{i}\theta} c_r^\dagger$



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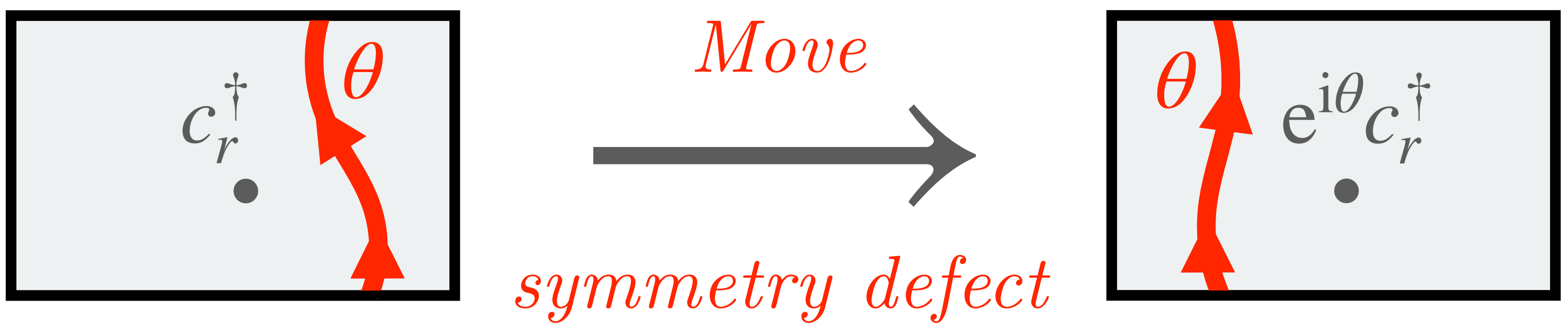

Symmetry defects are topological defects

- Correlation functions are invariant under infinitesimal deformations of them (moved using unitary operators)

Generalized symmetries

- Ordinary symmetries are generated by codimension 1 invertible topological defects (domain walls)

$$T^{(g_1)}_{D-1} \times T^{(g_2)}_{D-1} = T^{(g_1 g_2)}_{D-1} \quad g_1, g_2 \in G$$

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[*Reviews: McGreevy (2022), Schäfer-Nameki (2023), Shao (2024)*]

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[*Reviews: McGreevy (2022), Schäfer-Nameki (2023), Shao (2024)*]

- p-form symmetry: codimension $p+1$ topological defect
- Noninvertible symmetry: non-invertible topological defect

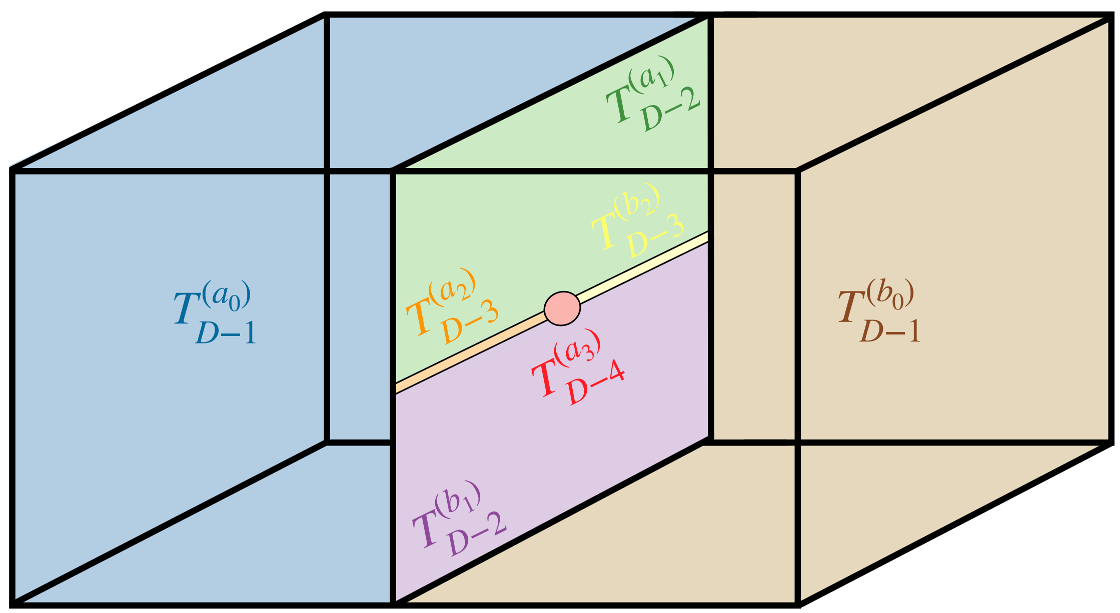
$$T^{(a)}_{D-p-1} \times T^{(b)}_{D-p-1} = \sum_c N^{ab}_c T^{(c)}_{D-p-1}$$

Higher categories and symmetry

Symmetry defects are described by a monoidal $(D-1)$-category

$D=$ spacetime dim.

- p-morphisms → p-form symmetry defects



- Monoidal product → symmetry defects fusion ring

$$T^{(a)}_{D-p-1} \otimes T^{(b)}_{D-p-1} = \bigoplus_c N_c^{ab} T^{(c)}_{D-p-1}$$

Higher categories and symmetry

Symmetry defects are described by a monoidal $(D-1)$-category

➤ p-morphisms $\rightarrow$ p-form symmetry defects

$D =$ spacetime dim.

Some common symmetry categories

➤ Finite $G \implies (D-1)\text{-}\mathsf{Vec}_G$

➤ $(D-1)\text{-}\mathsf{Rep}(G) = [BG, (D-1)\text{-}\mathsf{Vec}]$

➤ Tambara-Yamagami TY(G)

➤ Monoid

$$T^{(a)}_{D-p-1} \otimes T^{(b)}_{D-p-1} = \bigoplus_c N^{ab}_c T^{(c)}_{D-p-1}$$

IMPORTANT QUESTIONS

(1) What physical systems enjoy higher categorical symmetries?

(2) What do they teach us about such physical systems?

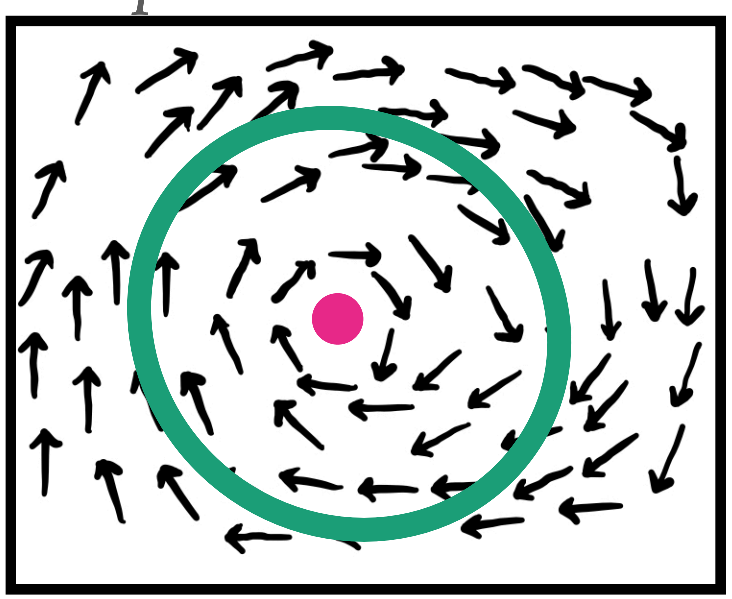
1-form symmetry in superfluids

Superfluid modes in $D = 3$ Euclidean spacetime X_3 described by

$$U(1) \xrightarrow{\text{ssb}} 1 \qquad\qquad \theta : X_3 \to \mathscr{M} = \mathbb{R}/\mathbb{Z}$$

➤ 1-dimensional vortices (defects) detected by $\Sigma_1 \subset X_3$:

2d space



$$Q(\Sigma_1) = \int_{\Sigma_1} \mathrm{d}\theta \in \pi_1(\mathscr{M}) \simeq \mathbb{Z}$$

Vortex is a singularity in the order parameter field $\theta(x)$

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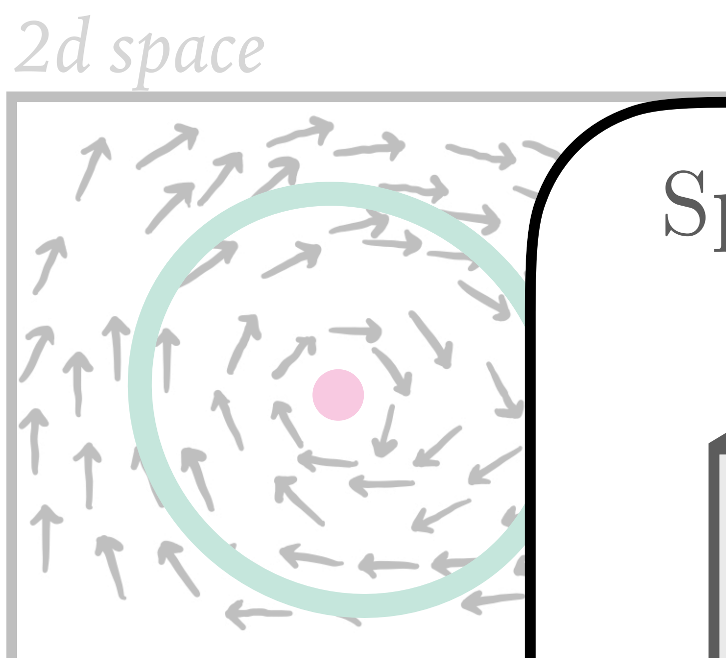
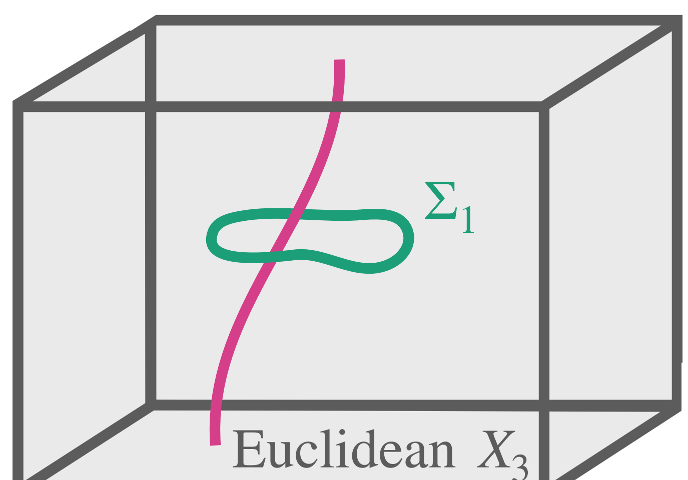


$\pi_1(\mathscr{M}) \simeq \mathbb{Z}$

Vortex is a singu… ter field $\theta(x)$

Space → spacetime



1-form symmetry in superfluids

There's a $U(1)$ 1-form symmetry generated by the topological defect

$$T^{(\alpha)}(\Sigma_1) = \exp(\mathrm{i}\alpha Q(\Sigma_1)) \qquad Q(\Sigma_1) = \int_{\Sigma_1} \mathrm{d}\theta \in \mathbb{Z}$$

$T^{(\alpha)}(\Sigma_1)$ is a topological defect since

1. (Topological) $Q(\Sigma_1 + \partial D) = Q(\Sigma_1)$
2. (Defect) Inserting it into a path integral "$Z = \int D\theta e^{iS}$" gives rise to a local modification of the action

$$S \to S + \mathrm{i}\alpha Q(\Sigma_1)$$

➤ The vortex transforms under this $U(1)$ 1-form symmetry

In this talk

Explore emergent generalized symmetry $\mathcal{S}_\pi$ in *generic* ordered phases and its spontaneous breaking

Why should you care?

- Physics: ordered phases are common and $\mathcal{S}_\pi$ can predict/classify exotic disordered phases
- Math: $\mathcal{S}_\pi$ is a physical realization of higher representations of a higher group related to a Postnikov tower

In this talk

Explore emergent generalized symmetry $\mathcal{S}_\pi$ in *generic* ordered phases and its spontaneous breaking

1. General features of ordered phases and homotopy defects
2. Generalized symmetries and their symmetry categories
3. Spontaneous symmetry breaking and nontrivial disordered phases

Ordered phases

A phase where an ordinary internal symmetry G is spontaneously broken

- Universal features determined by the SSB pattern

$$G \xrightarrow{\text{ssb}} H \subset G$$

- Ground states labeled by order parameter manifold

$$\mathscr{M} = G/H = \{gH : g \in G\}$$

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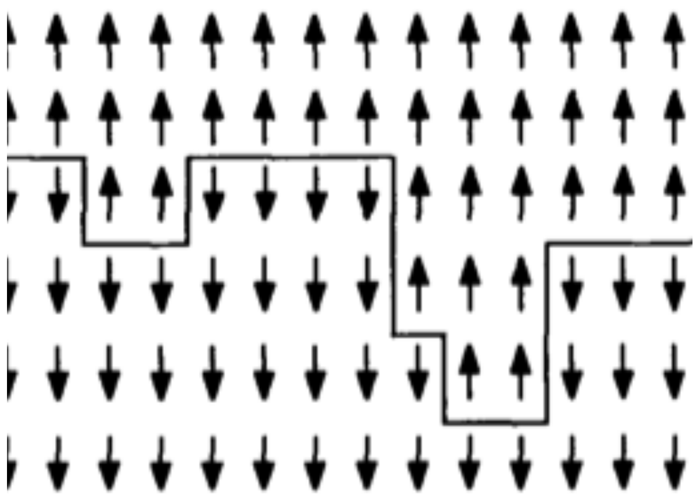
- There are (generally non-topological) defects called Homotopy defects, characterized by the homotopy of $\mathscr{M}$

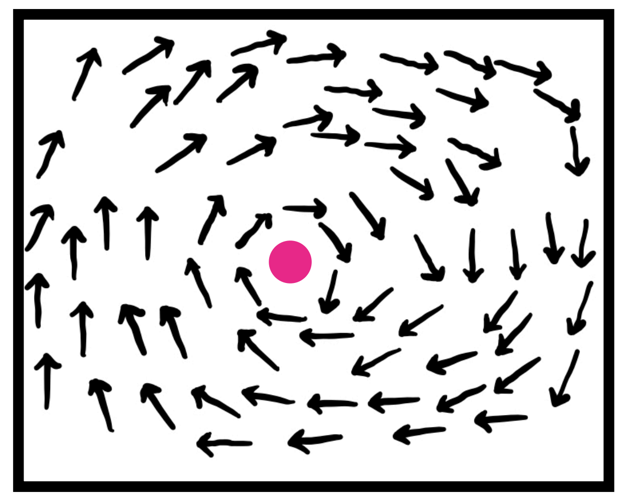
Ordered phases

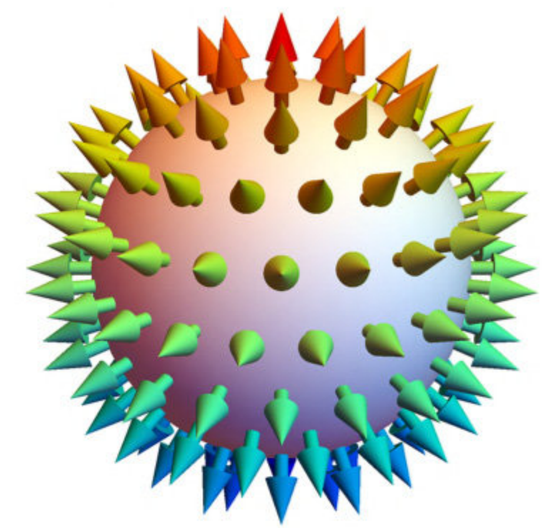
A phase where an ordinary internal symmetry G is

sp

e.g., domain walls, vortices, hedgehogs, etc







$$\mathscr{M} = G/H = \{gH : g \in G\}$$

➤ There are (generally non-topological) defects called Homotopy defects, characterized by the homotopy of $\mathscr{M}$

Homotopy defects in the IR

Continuous, connected G: Effective field theory is a nonlinear σ model with target space $\mathscr{M} = G/H$

- Path integral over order parameter field $U : X \to \mathscr{M}$

$$\text{Example:}\ \mathscr{L} = (-iU^{-1}\partial_\mu U)^2$$

Callan, Coleman, Wess, Zumino (1969)
Watanabe, Murayama (2014)

- Homotopy defects are singularities of U

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- Homotopy defects are singularities of U

Homotopy defects are not topological defects

- Deforming them creates Goldstone modes (e.g., sound waves)
- They are confined in ordered phases (cost energy to separate)

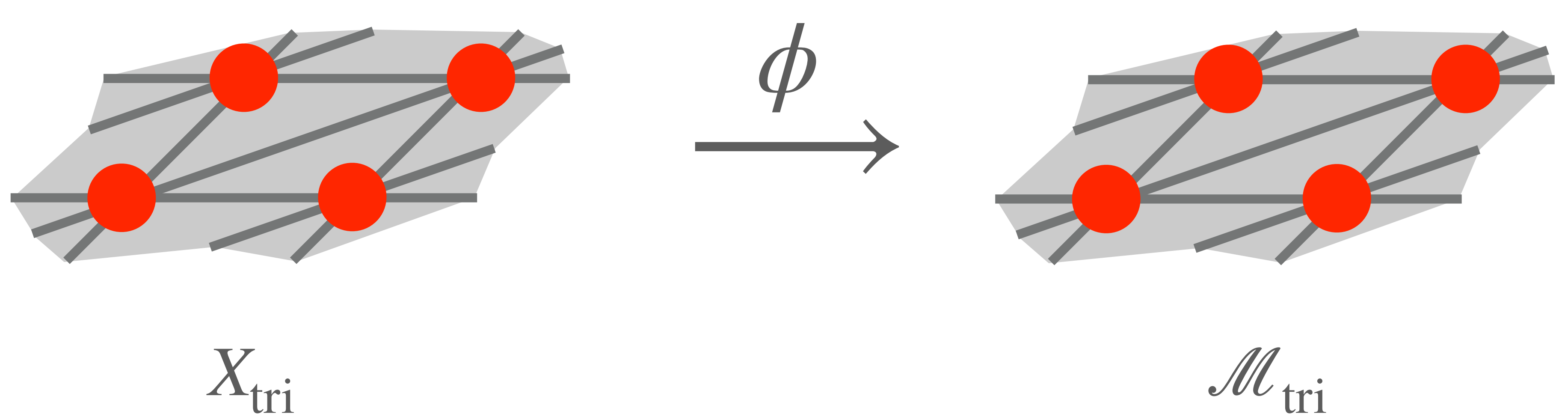
Lattice nonlinear σ model

Most of this talk will avoid explicit models (i.e., field theories, Hamiltonians, etc.)

Discussion applies generally to a spacetime lattice nonlinear σ model described by the partition function

$$Z\left(X_{\text{tri}}; \mathscr{M}_{\text{tri}}, \mathscr{L}\right) = \sum_{\phi} \mathrm{e}^{-\int_{X_{\text{tri}}} \mathscr{L}(\phi)}$$

- X_{tri} and $\mathscr{M}_{\text{tri}}$ are triangulations of spacetime X and $\mathscr{M} = G/H$
- ϕ is a simplicial homomorphism



Homotopy defect classification

Homotopy defects detected by the k-submanifold Σ_k are classified by

$$\mathrm{Maps}(\Sigma_k, \mathscr{M} = G/H)/\mathrm{homotopy}$$

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- $\Sigma_k \simeq S^k$: defects are detected via linking, are codimension $k+1$, and classification is based on homotopy groups

$$\pi_k(\mathscr{M})/\alpha_k, \qquad k = 1,\, 2, \cdots,\, D-2,\, D-1$$

where $\alpha_k : \pi_1(\mathscr{M}) \to \text{Aut}(\pi_k(\mathscr{M}))$ [*Mermin (1979)*]

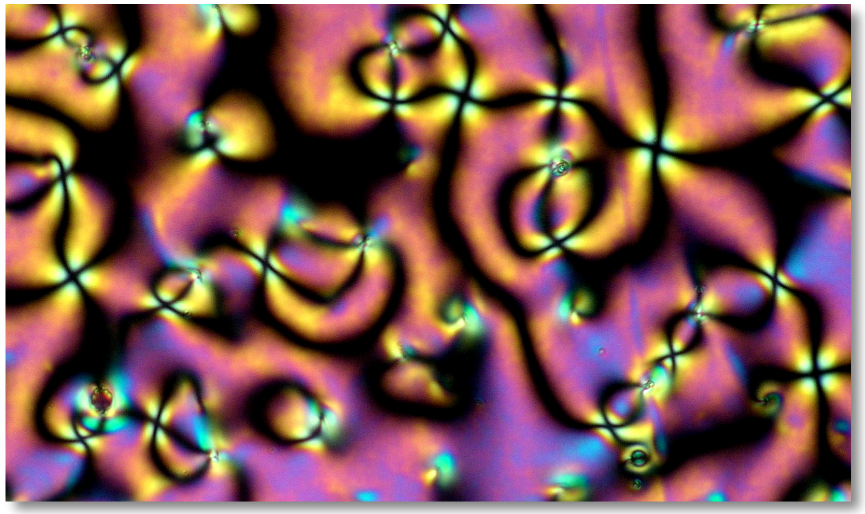
- e.g., α_1 is the inner automorphism, so codimension 2 homotopy defects classified by conjugacy classes $\text{Cl}(\pi_1(\mathscr{M}))$

Nematic liquid crystal

Ordered phase with SSB pattern

$$SO(3) \xrightarrow{\text{ssb}} O(2)$$

$$\mathscr{M} = SO(3)/O(2) \simeq \mathbb{RP}^2$$



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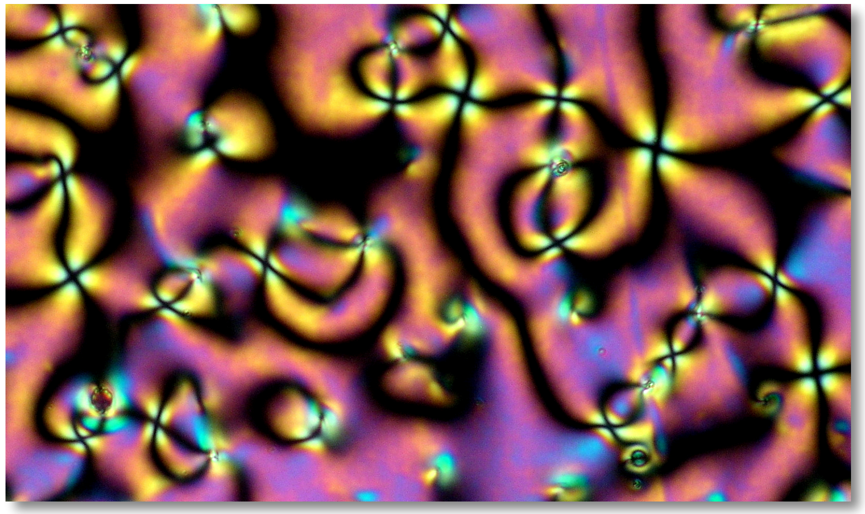

$$SO(3) \xrightarrow{\text{ssb}} O(2)$$

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- In $D = 3$ dimensional spacetime: [*Volovik, Mineev (1977)*]

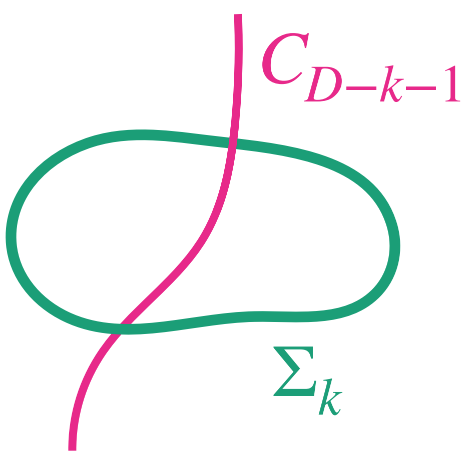
$$\pi_0(\mathbb{RP}^2) = 0 \qquad \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2 \qquad \pi_2(\mathbb{RP}^2) = \mathbb{Z}$$

$\alpha_2 : \pi_1(\mathbb{RP}^2) \to \mathrm{Aut}(\pi_2(\mathbb{RP}^2))$ flips sign of $\pi_2(\mathbb{RP}^2)$

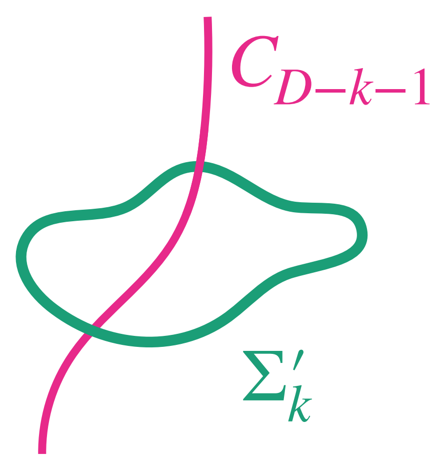
$\mathbb{Z}_2$ strings and $\mathbb{Z}_{\geq 0}$ particles

Topological defects

Since homotopy defects are classified by $\mathrm{Maps}(\Sigma_k, \mathscr{M})$ modulo homotopy, the defects detecting homotopy defects have topological properties.



$$\xrightarrow[\Sigma_k]{\text{deform}}$$



$\mathrm{Maps}(\Sigma_k, \mathscr{M})/\mathrm{homotopy}$ depend only on $\mathrm{Link}(\Sigma_k, C_{D-k-1})$ when $\partial C_{D-k-1} = \varnothing$

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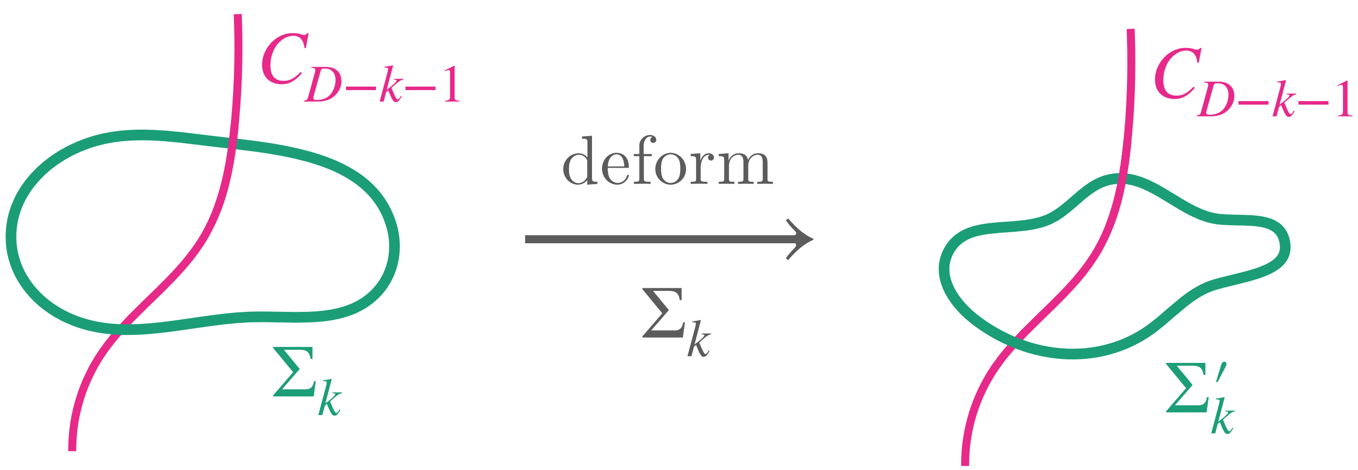

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When homotopy defects cannot end

They are detected by topological defects $\Longrightarrow$ They carry symmetry charge of a symmetry $\mathscr{S}_\pi$

Topological currents

For homotopy defects classified by $\pi_k(\mathcal{M} = G/H) = \mathbb{Z}$, the number of homotopy defects detected by $\Sigma_k \subset X$ is

[*D'Hoker, Weinberg (1994)*]

$$Q(\Sigma_k) = \int_{\Sigma_k} \Omega^{(k)} \in \mathbb{Z}$$

- $\Omega^{(k)}$ is generator of $H^k_{\mathrm{dR}}(\mathcal{M})$ pulled back to X (i.e., $\Omega^{(1)} = \mathrm{d}\theta$)
- $\mathrm{d}\Omega^{(k)} = \hat{C}$, the Poincaré dual of the homotopy defect's location
- $\mathcal{S}_\pi$: topological defect $T^{(\alpha)}(\Sigma_k) = \exp(\mathrm{i}\alpha Q(\Sigma_k))$ generates a $U(1)^{(D-k-1)}$ symmetry

Gaiotto, Kapustin, Seiberg Willet (2015)
Grozdanov, Poovuttikul (2018)
Delacrétaz, Hofman, Mathys (2020)
Armas, Jain (2020)
Brauner (2021)

The symmetry $\mathcal{S}_\pi$

What is this generalized symmetry for general connected G?

➤ We will now arrive at the answer:

$$\mathcal{S}_\pi = [\mathcal{M}_{\leq D-1}, (D-1)\text{-}\mathsf{Vec}] = (D-1)\text{-Rep}(\mathbb{G}_\pi^{(D-1)})$$

Examples with $G = SO(3)$:

D	SSB Pattern	$\mathcal{S}_\pi$
3	$SO(3) \xrightarrow{\text{ssb}} 1$	$\mathbb{Z}_2^{(1)} \quad (\mathcal{S}_\pi = 2\text{-Rep}(\mathbb{Z}_2))$
3	$SO(3) \xrightarrow{\text{ssb}} \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{S}_\pi = 2\text{-Rep}(Q_8)$
4	$SO(3) \xrightarrow{\text{ssb}} SO(2)$	$0 \to B^2\mathbb{Z} \to \mathbb{G}_\pi^{(3)} \to B\mathbb{Z} \to 0$

Abelian homotopy defects

Consider general abelian homotopy defects

- Defects of different dimension are independent from one another
- Trivial $\alpha_k : \pi_1(\mathscr{M}) \to \mathrm{Aut}(\pi_k(\mathscr{M})) \implies$ classified by $\pi_k(\mathscr{M})$.

Each $\pi_k(\mathscr{M})$ describes symmetry charges of a $(D-k-1)$-form symmetry.

- $T^{(\alpha)}(\Sigma_k) = \exp(\mathrm{i}\alpha Q(\Sigma_k)) \implies (D-k-1)$-form symmetry group is the Pontryagin dual of $\pi_k(\mathscr{M})$

$$G^{(D-k-1)} = \mathrm{Hom}(\pi_k(\mathscr{M}), U(1))$$

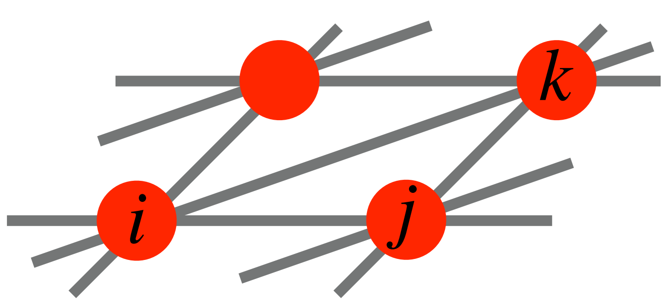
Codimension 2 homotopy defects

Since we only care about $\pi_1(\mathscr{M})$, let's truncate $\mathscr{M}$ to $\mathscr{M}_{\leq 1}$:

$$\pi_k(\mathscr{M}_{\leq 1}) = \begin{cases} \pi_k(\mathscr{M}) & k = 1 \\ 0 & \text{else} \end{cases} \quad \Longrightarrow \quad \mathscr{M}_{\leq 1} = B\pi_1(\mathscr{M})$$

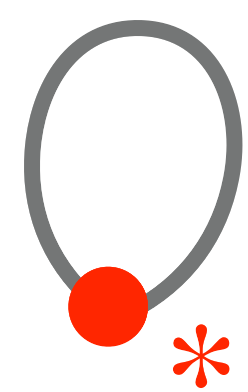
$\mathcal{S}_\pi$ from codimension 2 homotopy defects of $\mathscr{M}$ is the same as $\mathcal{S}_\pi$ from $\mathscr{M}_{\leq 1} = B\pi_1(\mathscr{M})$

- Maps $X \to BG$ describe discrete G gauge theory



Spacetime lattice

$\longrightarrow$



$g \in G$

$g_{ij} g_{jk} g_{ik}^{-1} = 1$

Realization of BG

Codimension 2 homotopy defects

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- These homotopy defects are $\pi_1(\mathscr{M})$ magnetic defects (a.k.a.,'t Hooft lines, magnetic fluxes)
- $\mathscr{S}_\pi =$ the magnetic symmetry of $\pi_1(\mathscr{M})$ gauge theory: the symmetry formed by electric defects lines (e.g, Wilson lines $= e$ particle worldlines)

Codimension 2 homotopy defects

Since we only care about $\pi_1(\mathscr{M})$, let's truncate $\mathscr{M}$ to $\mathscr{M}_{\leq 1}$:

Finite $\pi_1(\mathscr{M})$: electric defect lines described by irreps

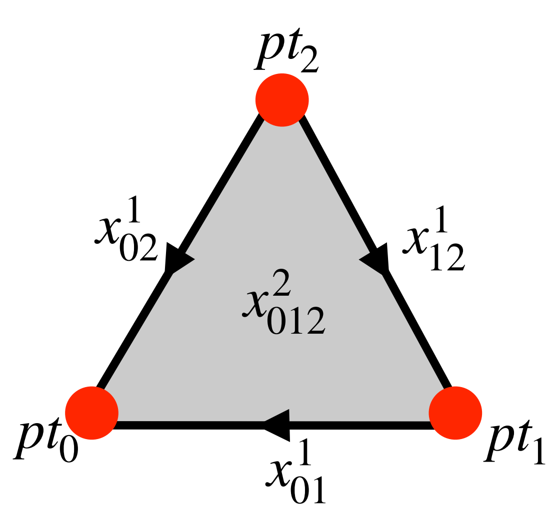
- Finite abelian $\pi_1(\mathscr{M})$: $\mathscr{S}_\pi$ includes a $\pi_1(\mathscr{M})$ $(D-2)$-form symmetry
- Finite non-abelian $\pi_1(\mathscr{M})$: $\mathscr{S}_\pi$ includes a non-invertible $\mathsf{Rep}(\pi_1(\mathscr{M}))$ $(D-2)$-form symmetry
- $\mathscr{S}_\pi = (D-1)\text{-Rep}(\pi_1(\mathscr{M})) = [\mathscr{M}_{\leq 1}, (D-1)\text{-}\mathsf{Vec}]$

General connected $\mathscr{M}$

Since $\pi_k(\mathscr{M})$ homotopy defects for $k > D-1$ are absent in D dimensions, we truncate $\mathscr{M}$ to $\mathscr{M}_{\leq D-1}$:

$$\pi_k(\mathscr{M}_{\leq n}) = \begin{cases} \pi_k(\mathscr{M}) & 1 \leq k \leq n \\ 0 & \text{else} \end{cases}$$

- $\mathscr{M}_{\leq n}$ is the nth Postnikov stage of $\mathscr{M}$



Realization of $\mathscr{M}_{\leq D-1}$

$pt_i \sim pt_j$

$x^k \in \pi_k(\mathscr{M}) \qquad k = 1,2,\cdots,n$

General connected $\mathscr{M}$

Postnikov stages obey the fibrations ($2 \le n \le D-1$)

$$B^n\pi_n(\mathscr{M}) \to \mathscr{M}_{\le n} \to \mathscr{M}_{\le n-1}$$

Classified by the twisted $(n+1)$-cocycle [Baez, Shulman (2009)]

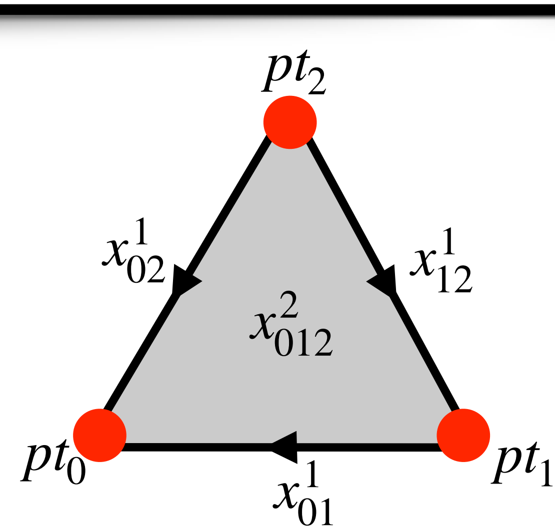
$$[\beta^{n+1}] \in H^{n+1}_{\alpha_n}(\mathscr{M}_{\le n-1}, \pi_n(\mathscr{M})) \qquad \alpha_n : \pi_1(\mathscr{M}) \to \mathrm{Aut}(\pi_n(\mathscr{M}))$$

- $\mathscr{M}_{\le n}$ is the classifying space of an n-group: $\mathscr{M}_{\le n} = B\mathbb{G}^{(n)}_{\pi}$,

$$\mathbb{G}^{(n)}_{\pi} = (\pi_1(\mathscr{M})\ ;\ \pi_2(\mathscr{M}), \alpha_2, \beta^3\ ;\ \cdots\ ;\ \pi_n(\mathscr{M}), \alpha_n, \beta^{n+1})$$

$$\mathrm{d}x^1 = 1$$

$$\mathrm{d}x^k = \beta_{k+1}$$



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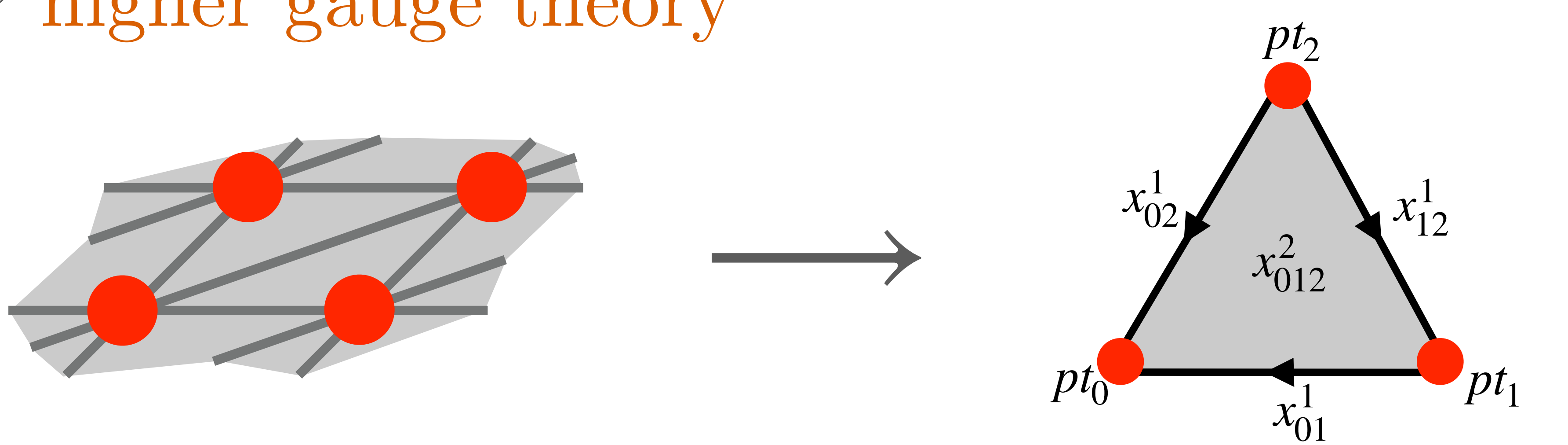
General connected $\mathscr{M}$

$\mathcal{S}_\pi$ from $\mathscr{M}$ is the same as $\mathcal{S}_\pi$ from $\mathscr{M}_{\leq D-1} = B\mathbb{G}_\pi^{(D-1)}$

- Relationship between maps $X \to \mathscr{M}_{\leq D-1} = B\mathbb{G}_\pi^{(D-1)}$ and $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory



Spacetime lattice   *Realization of* $\mathscr{M}_{\leq D-1}$

- Homotopy defects are $\mathbb{G}_\pi^{(D-1)}$ magnetic defects

- $\mathcal{S}_\pi =$ magnetic symmetry of $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory (formed by electric defect lines, surfaces, volumes, etc)

General connected $\mathscr{M}$

$\mathcal{S}_\pi$ from $\mathscr{M}$ is the same as $\mathcal{S}_\pi$ from $\mathscr{M}_{\leq D-1} = B\mathbb{G}_\pi^{(D-1)}$

- Relationship between maps $X \to \mathscr{M}_{\leq D-1} = B\mathbb{G}_\pi^{(D-1)}$ and

Finite $\pi_k(\mathscr{M})$:

- $\mathcal{S}_\pi = (D-1)\text{-Rep}(\mathbb{G}_\pi^{(D-1)}) \equiv [\mathscr{M}_{\leq D-1}, (D-1)\text{-}\mathsf{Vec}]$ is complicated
- Case for $D > 3$ seems to not be understood

Abstract nonsense appears in generic ordered phases!

- $\mathcal{S}_\pi =$ magnetic symmetry of $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory (formed by electric defect lines, surfaces, volumes, etc)

Check in

Homotopy defects carry symmetry charge of a generalized symmetry $\mathscr{S}_\pi$

- $\mathscr{S}_\pi$ = magnetic symmetry of $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory
- $\mathscr{S}_\pi$ includes invertible and non-invertible, 0-form and higher-form symmetries.

What are some physical applications of $\mathscr{S}_\pi$?

1. Spontaneous symmetry breaking *(we'll discuss here)*
2. Mixed 't Hooft anomaly with G *(we won't discuss here)*

Spontaneously breaking $\mathcal{S}_\pi$

$\mathcal{S}_\pi$ is not spontaneously broken in the ordered phase

- If it were, ordered phases would have GSD dependent on space's topology and exotic gapless modes
- Homotopy defects are not topological and are confined in the IR (*i.e., area law*)

Spontaneously breaking $\mathcal{S}_\pi$

$\mathcal{S}_\pi$ is not spontaneously broken in the ordered phase

- If it were, ordered phases would have GSD dependent on space's topology and exotic gapless modes
- Homotopy defects are not topological and are confined in the IR (*i.e., area law*)

$\mathcal{S}_\pi$ can spontaneously break, driving a transition out of the ordered phase

- Typically leads to an exotic phase of matter
- Homotopy defects will deconfine (*i.e., perimeter law*)

Spontaneously breaking $\mathcal{S}_\pi$

What happens to the microscopic G symmetry when spontaneously breaking $\mathcal{S}_\pi$?

Spontaneously breaking $\mathcal{S}_\pi$

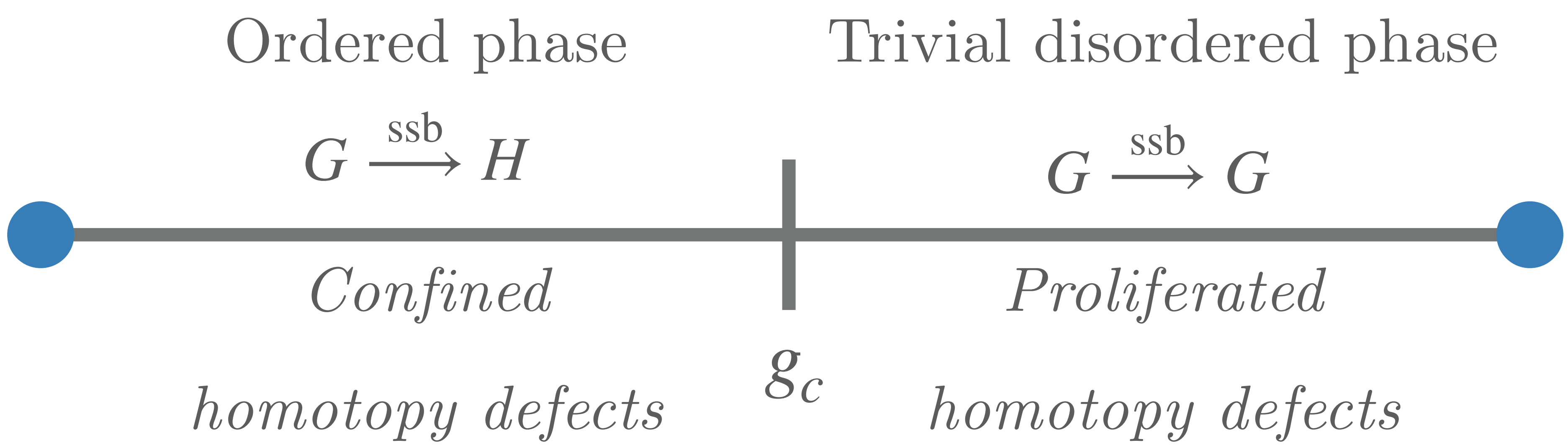
What happens to the microscopic G symmetry when spontaneously breaking $\mathcal{S}_\pi$?

A physical argument

- Ordered vacua ($G \xrightarrow{\text{ssb}} H$) want to confine homotopy defects
- $\mathcal{S}_\pi$ SSB vacua have a $\mathcal{S}_\pi$ symmetry charge condensate that wants to deconfine homotopy defects
- The latter contradicts the former, so spontaneously breaking $\mathcal{S}_\pi$ must restore G

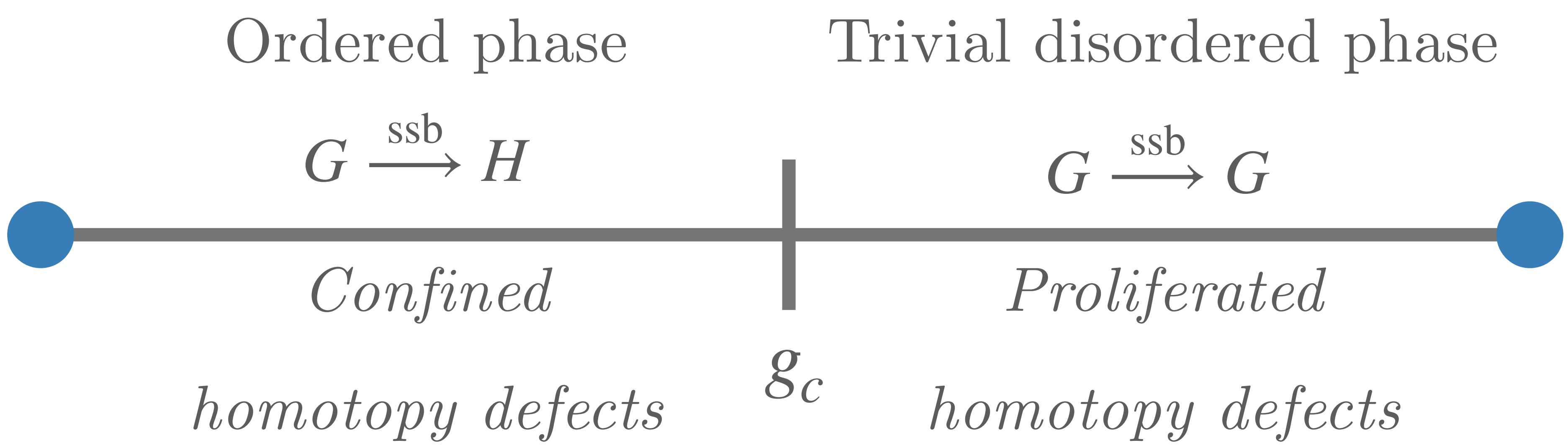
Two types of Disordered phases

Without the generalized symmetry $\mathcal{S}_\pi$



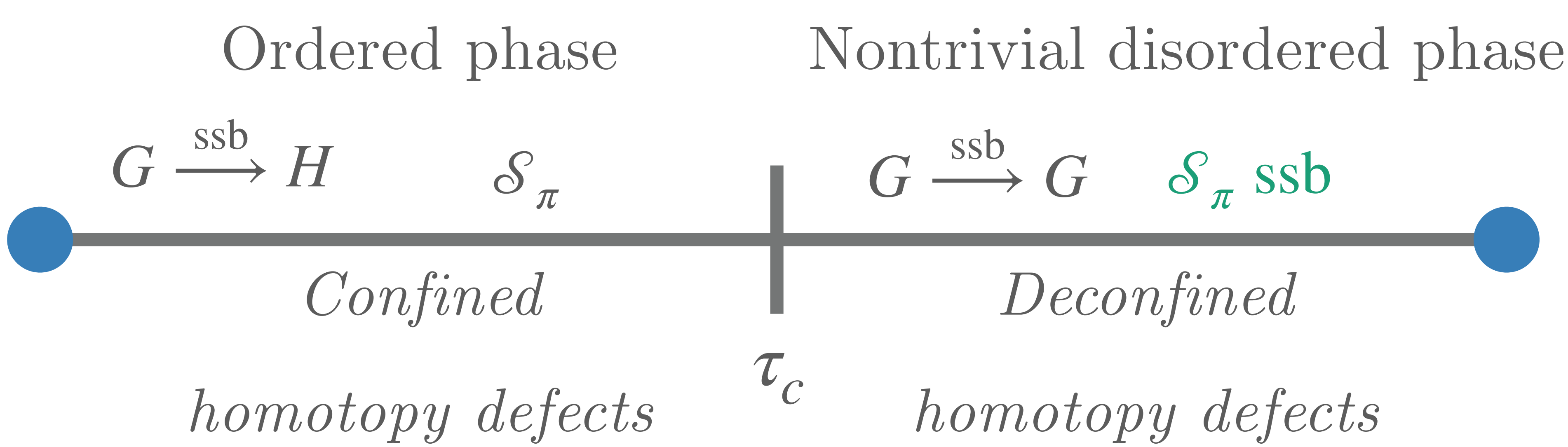
Two types of Disordered phases

Without the generalized symmetry $\mathcal{S}_\pi$



With the generalized symmetry $\mathcal{S}_\pi$:

Chubukov, Senthil, Sachdev (1994)
Lammert, Rokhsar, Toner (1995)
Motrunich, Vishwanath (2004)
Grover, Senthil (2011)
Xu, Ludwig (2012)
Zhu, Lan, Wen (2019)



The power of symmetry

$\mathcal{S}_\pi$ is a non-perturbative tool to identify/classify exotic phases neighboring ordered phases

D	Ordered phase $G \xrightarrow{\text{ssb}} H$	Nontrivial disordered phase $\mathcal{S}_\pi$ ssb
4	$U(1) \xrightarrow{\text{ssb}} 1$	none
4	$U(1) \times U(1) \xrightarrow{\text{ssb}} 1$	$U(1)^{(1)} \xrightarrow{\text{ssb}} 1$
3	$SO(3) \xrightarrow{\text{ssb}} 1$	$\mathbb{Z}_2^{(1)} \xrightarrow{\text{ssb}} 1$
3	$SO(3) \xrightarrow{\text{ssb}} \mathbb{Z}_2 \times \mathbb{Z}_2$	$\text{Rep}(Q_8)^{(1)} \xrightarrow{\text{ssb}} 1$

- For finite $\mathbb{G}_\pi^{(D-1)}$: Nontrivial disordered phase is the deconfined phase of untwisted $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory

Example with $G = SO(3)$

Consider SSB pattern $SO(3) \xrightarrow{\text{ssb}} H$ with finite H in $D = 3$

$$\mathscr{M} = SO(3)/H$$

$$\pi_0(\mathscr{M}) = 0 \qquad \pi_1(\mathscr{M}) = \tilde{H} \qquad \pi_2(\mathscr{M}) = 0$$

where $\tilde{H}$ is the cover of H that lifts it to a subgroup of $SU(2)$.

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- e.g., $H = \mathbb{Z}_N = \tilde{H}$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \implies \tilde{H} = Q_8$
- $1D$ homotopy defects classified by conjugacy classes $\mathrm{Cl}(\tilde{H})$
- Symmetry $\mathscr{S}_\pi = 2\text{-Rep}(\tilde{H})$

Let's build a Euclidean lattice model with the $\mathscr{S}_\pi$ SSB phase

Example with $G = SO(3)$

Consider the order parameter presentation

- On lattice sites i, $\tilde{G} = SU(2)$ rotors $z_i \in \mathbb{C}^2$ with $z_i^\dagger z_i = 1$.
  $SO(3)$ realized as $SU(2)$ transforming z_i in $\square$ of $SU(2)$
- On lattice edges (ij), $\tilde{H}$ gauge fields $h_{ij} \in \tilde{H}$

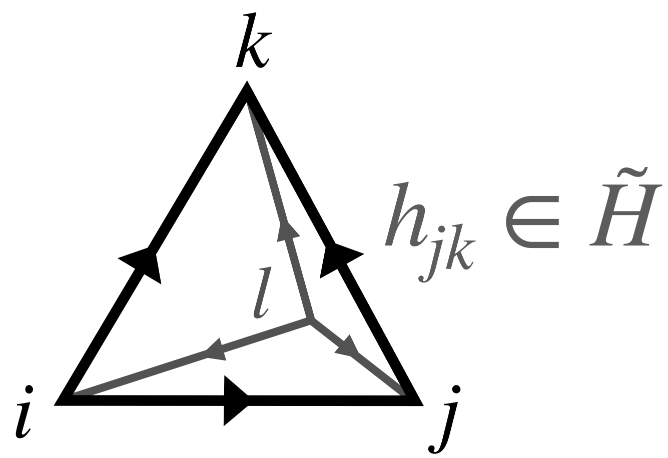
Why?

$\square$ of $SU(2)$

- Gauge redundancy $z_i \sim a(\tilde{h}_i) z_i$, $h_{ij} \sim \tilde{h}_i h_{ij} \tilde{h}_j^{-1}$ restricts physical z_i values in $SU(2)/\tilde{H} = SO(3)/H \equiv \mathscr{M}$
- 1D homotopy defects realized as $\tilde{H}$ magnetic fluxes

$$\pi_1(\mathscr{M}) = \tilde{H}$$

Example with $G = SO(3)$

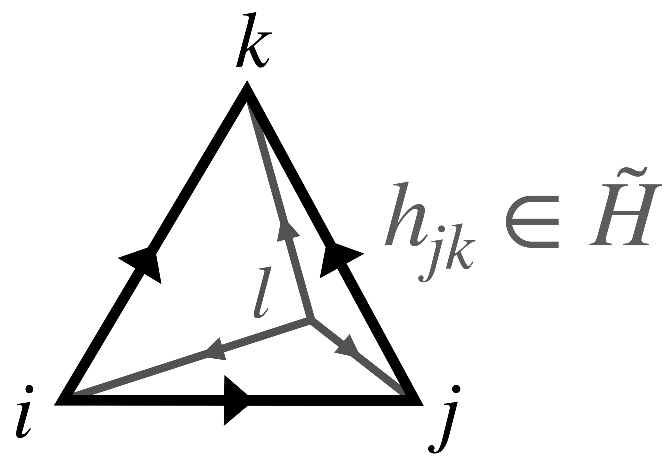


$$S = -J \sum_{(ij)} z_i^\dagger a(h_{ij}) z_j + K \sum_{(ijk)} [1 - \delta_{(\delta h)_{ijk}, 1}]$$

$$(\delta h)_{ijk} = h_{ij} h_{jk} h_{ik}^{-1}$$

- J term wants z_i (gauge charges) to condense
- K term penalizes $\pi_1(\mathscr{M})$ homotopy defects ($\tilde{H}$ gauge fluxes)

Example with $G = SO(3)$



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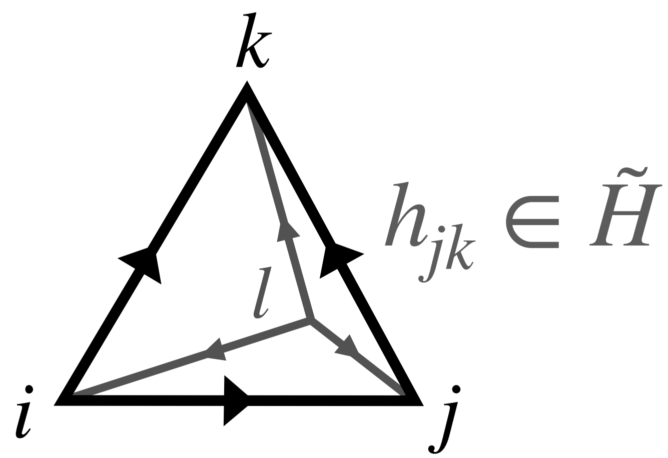
When $K \to \infty$, there is the topologial defect line

$$T_\Gamma(\gamma) = \mathrm{Tr} \prod_{(ij)\in\gamma} \Gamma(h_{ij}) \equiv \chi_\Gamma\left(\prod_{(ij)\in\gamma} h_{ij} \right)$$

- $T_{\Gamma_a} T_{\Gamma_b} = T_{\Gamma_a \otimes \Gamma_b} = T_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c T_{\Gamma_c} \implies$ Rep$(\tilde{H})$ 1-form symmetry

Example with $G = SO(3)$



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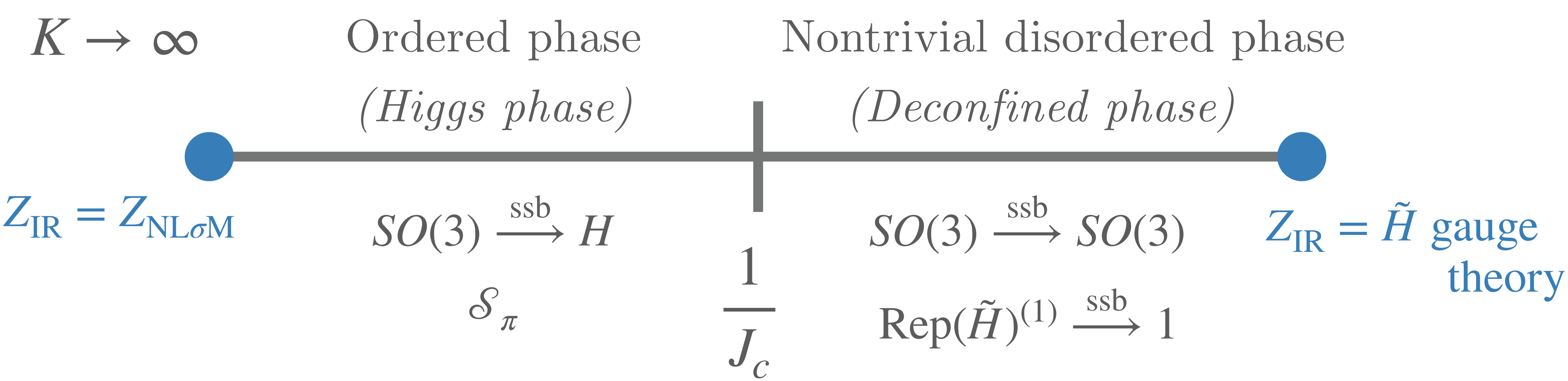

Summary

SP, arXiv:2308.05730 [SciPost Phys. 17, 080 (2024)]
SP, C Zhu, A Beaudry, and X-G Wen, arXiv:2310.08554

Generalized symmetries exist in ordered phases

- Symmetry charge carried by homotopy defects
- Symmetry defects described by $(D-1)$-representations of $\mathbb{G}_{\pi}^{(D-1)} = (\pi_1(\mathscr{M})\ ;\ \pi_2(\mathscr{M}), \alpha_2, \beta^3\ ;\ \cdots\ ;\ \pi_{D-1}(\mathscr{M}), \alpha_{D-1}, \beta^D)$

Their spontaneous breaking gives rise to nontrivial disordered phases

- $\mathcal{S}_\pi$ is a non-perturbative tool to identify exotic phases neighboring ordered phases

Back Up Slides

Emergent Symmetries

In generic UV models, homotopy defects are dynamical

- High energy processes cut open homotopy defects
- $\mathcal{S}_\pi$ is not a symmetry in the UV

In the IR, homotopy defects are not dynamical

- $\mathcal{S}_\pi$ is a symmetry in the IR

Generic ordered phases have an emergent $\mathcal{S}_\pi$ symmetry

- We will always implicitly refer to $\mathcal{S}_\pi$ at the lowest energy scale (the deep IR)

Generalized Symmetries in Practice

UV Scale → IR Scale

$\mathcal{S}_{\mathrm{UV}}$ $\mathcal{S}_{\mathrm{mid-IR}}$ $\mathcal{S}_{\mathrm{IR}}$

$\mathcal{S}_{\mathrm{UV}}$ includes ordinary/no symmetries, but $\mathcal{S}_{\mathrm{mid-IR}}$ and $\mathcal{S}_{\mathrm{IR}}$ can include emergent generalized symmetries

- Emergent higher-form symmetries are exact symmetries, not approximate symmetries *Iqbal, McGreevy (2022)* *McGreevy (2022)* *Cheng, Seiberg (2023)* ***SP**, Wen (2023)* *Cherman, Jacobson (2023)*
- The generalized Landau paradigm is really a classification scheme using $F\colon \mathcal{S}_{\mathrm{UV}} \to \mathcal{S}_{\mathrm{IR}}$

Example with $G = SO(3)$: part II

Consider SSB pattern $SO(3) \xrightarrow{\text{ssb}} SO(2)$ [$\mathscr{M} = S^2$] in $D = 4$

$\pi_0(S^2) = 0$ $\quad$ $\pi_1(S^2) = 0$ $\quad$ $\pi_2(S^2) = \mathbb{Z}$ $\quad$ $\pi_3(S^2) = \mathbb{Z}$

- Because $\pi_1(S^2)$ is trivial

$$\mathbb{G}_\pi^{(3)} = (\pi_2(S^2)\ ;\ \pi_3(S^2),\, \beta^4) \qquad [\beta^4] \in H^4(B^2\mathbb{Z}, \mathbb{Z})$$

- Consider 2 & 3 cochains $x^{(2)}$ & $x^{(3)}$ on $B\mathbb{G}_\pi^{(3)} = S^2_{\tau\leq 3}$

$$\mathrm{d}x^{(2)} = 0 \qquad \mathrm{d}x^{(3)} = x^{(2)} \cup x^{(2)} \equiv \beta^4(x^{(2)})$$

$\mathcal{S}_\pi$ = magnetic symmetry of $\mathbb{G}_\pi^{(3)}$ gauge theory

- Non-invertible symmetry since $\mathbb{G}_\pi^{(3)}$ does not have a Pontryagin dual 3-group [*Chen, Tanizaki (2023)*]

Example with $G = SO(3)$: part II

To construct effective theory for the nontrivial disordered phase, consider the $\mathbb{CP}^1$ presentation of the S^2 NLσM

- $U(1)$ 1-form gauge field $A^{(1)}$

$$\int_{S^2} \frac{1}{2\pi} \mathrm{d}A^{(1)} \in \pi_2(S^2) \qquad \int_{S^3} \frac{1}{4\pi^2} A^{(1)} \wedge \mathrm{d}A^{(1)} \in \pi_3(S^2)$$

Motivates us to introduce the $U(1)$ 2-form gauge field $B^{(2)}$ and gauge invariant field strengths

$$F^{(2)} = \mathrm{d}A^{(1)} \qquad H^{(3)} = \frac{1}{2\pi} A^{(1)} \wedge \mathrm{d}A^{(1)} + \mathrm{d}B^{(2)}$$

$$A^{(1)} \sim A^{(1)} + \mathrm{d}f_1^{(0)} \qquad B^{(2)} \sim B^{(2)} + \mathrm{d}f_2^{(1)} - \frac{1}{2\pi} f_1^{(0)} \mathrm{d}A^{(1)}$$

Example with $G = SO(3)$: part II

Effective field theory of the nontrivial disordered phase

$$S_{\rm IR} = \int_{M_4} \frac{1}{2e^2} F^{(2)} \wedge \star F^{(2)} + \frac{1}{4\pi v^2} H^{(3)} \wedge \star H^{(3)}$$

Dualizing $B^{(2)}$ to the compact boson $\phi^{(0)}$

$$S_{\rm IR} = \int_{M_4} \frac{1}{2e^2} F^{(2)} \wedge \star F^{(2)} + \frac{v^2}{2} \mathrm{d}\phi^{(0)} \wedge \star \mathrm{d}\phi^{(0)} + \frac{1}{4\pi^2} \phi^{(0)} F^{(2)} \wedge F^{(2)}$$

➤ Massless axion electrodynamics enriched by $SO(3)$!