Lattice T-duality from non-invertible symmetries in quantum spin chains

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TSVP Symposium: Aspects of Generalized Symmetries

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arXiv:2409.12220 [PRL 134, 021601 (2025)]
arXiv:2412.18606 [SciPost Phys. 18, 121 (2025)]

The lattice and the continuum

The interplay between lattice and continuum theories has a long and fruitful history in physics

- Lattice $\longrightarrow$ continuum: extract universal properties
- Continuum $\longrightarrow$ lattice: a laboratory for QFTs to do numerics and have an intrinsic UV cutoff

The two continually push each other forward.

$$\begin{tikzcd} \text{Lattice} \arrow[rr, bend left, leftrightarrow, "\text{Knowledge}"] & & \text{Continuum} \end{tikzcd}$$

- This is a mutualistic relationship!

The lattice and the continuum

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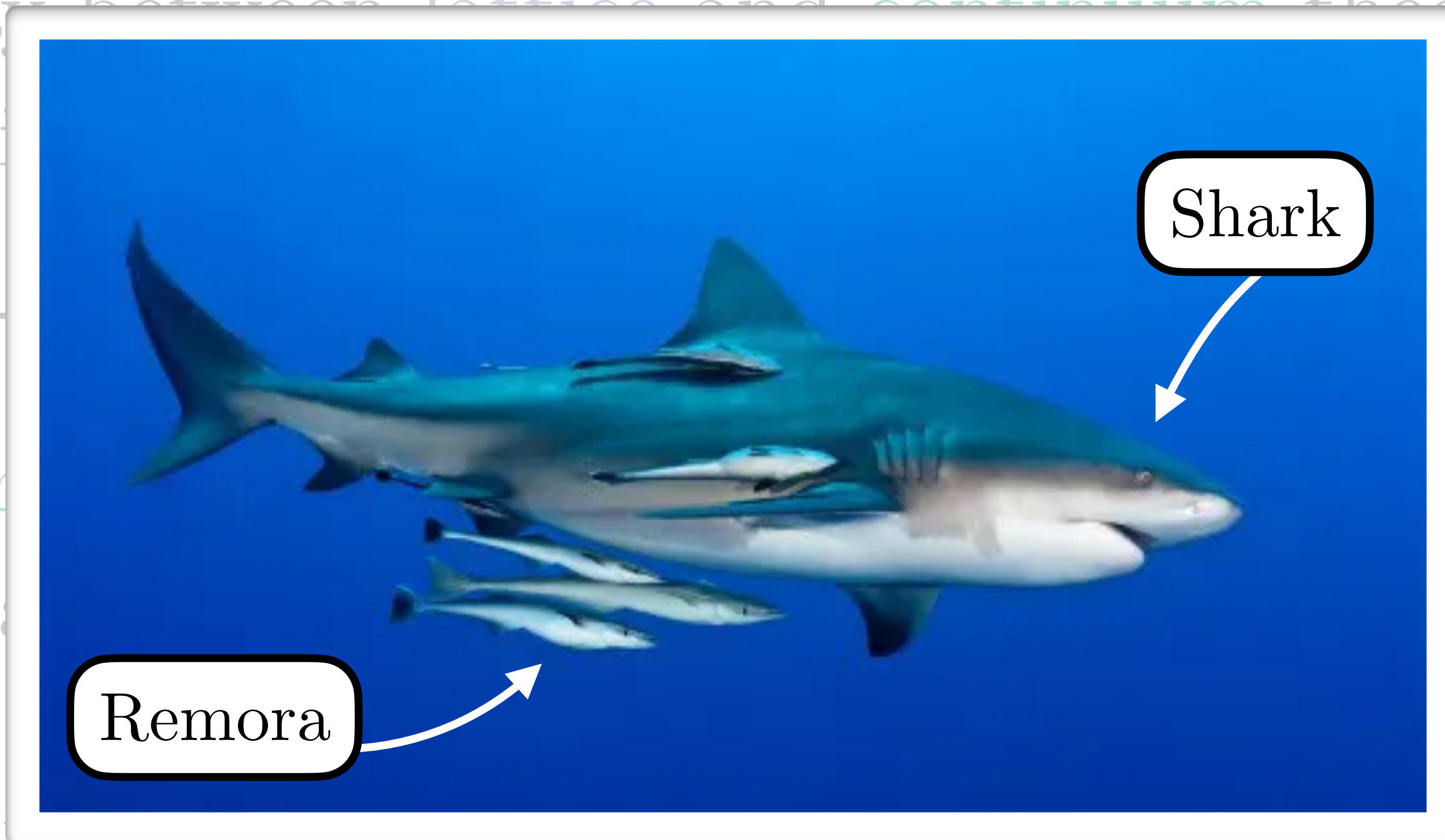
- Lattice — ... rties
- Continuum ... to do numerics ...



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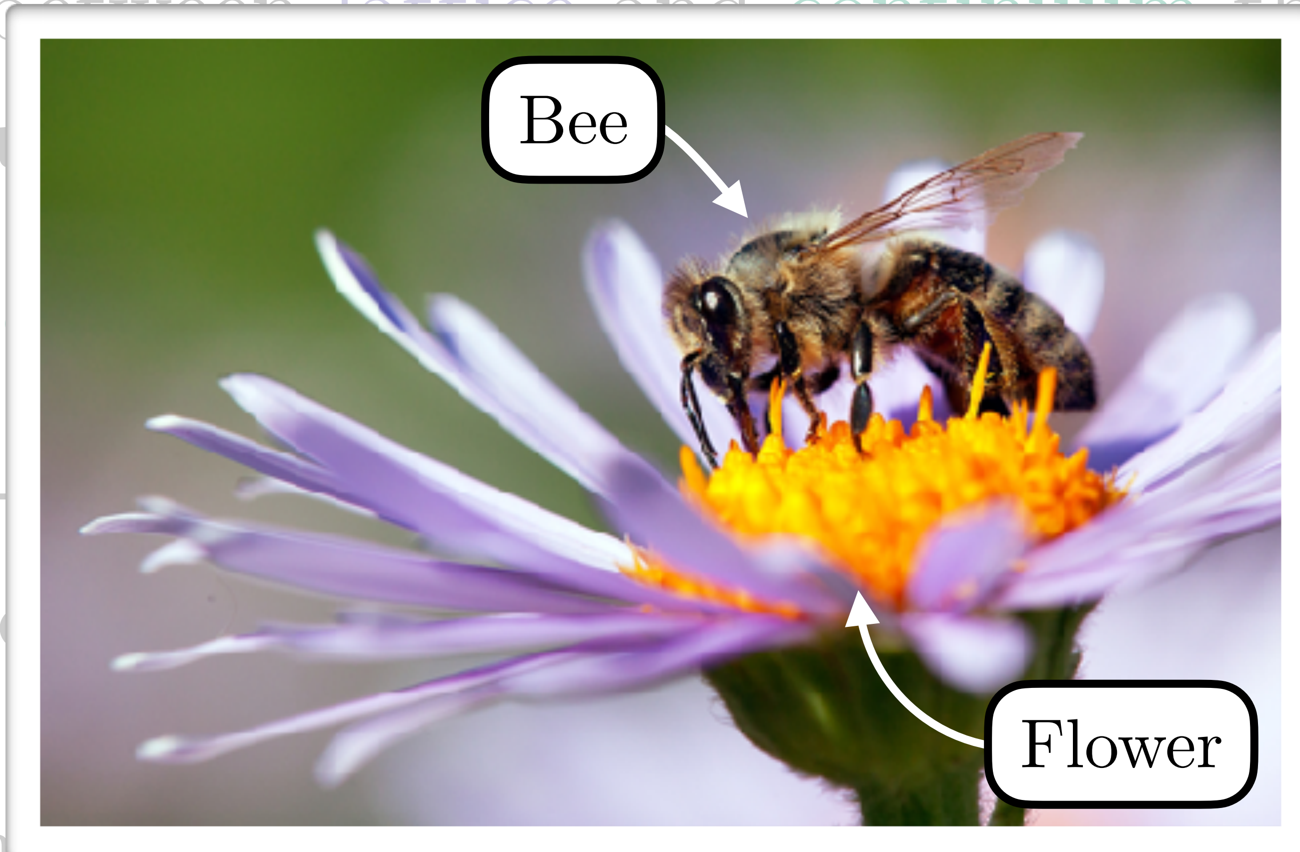

Knowledge

Lattice ⟷ Continuum

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Dualities in quantum systems

A theme in both lattice and continuum models is duality

- Dualities are maps between two seemingly distinct theories that are "secretly the same."
- Both conceptually and practically useful

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Ask people on the street their favorite duality and hear:

T-duality, Particle-Vortex duality,

Kramers-Wannier transformation

- These are not all the same notion of duality!
- Need to be more precise with "secretly the same."

Three* types of dualities

1. Exact duality: relates two different presentations of the same quantum system (is an isomorphism).
   - ➤ Electromagnetic duality, T-duality, Level-rank duality

* *there are certainly more than just three*

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2. IR duality: relates two quantum systems that are distinct in the UV but the same in the IR.
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3. Discrete gauging: relates two distinct quantum systems by gauging a discrete symmetry.
   - Kennedy-Tasaki (gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$), Kramers-Wannier (gauging $\mathbb{Z}_2$), bosonization (gauging $(-1)^F$)

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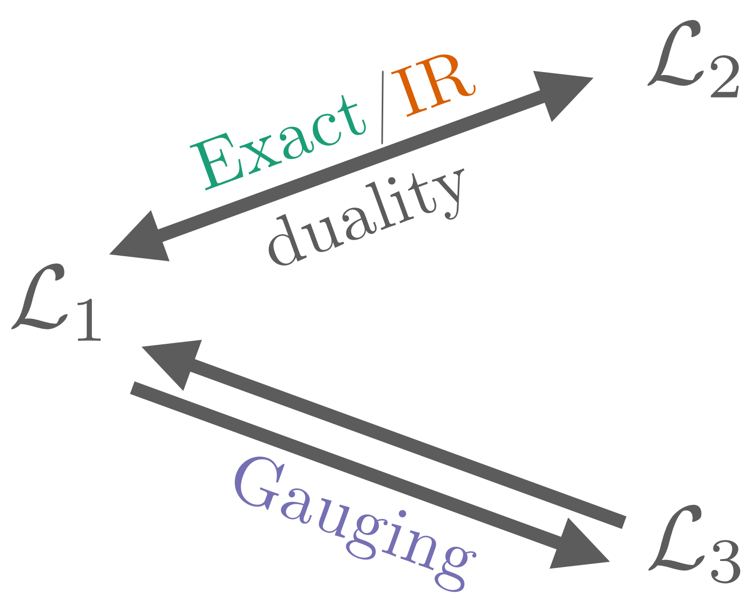
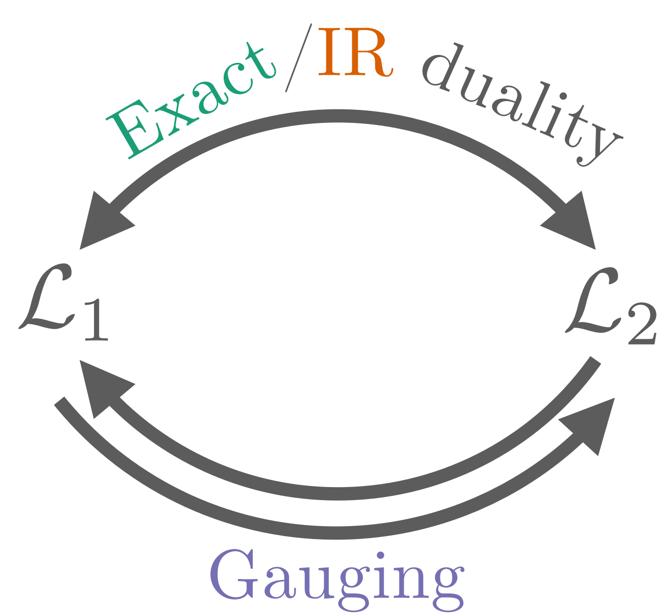
Three* types of dualities

Discrete gauging is generally unrelated to exact duality and IR duality!

Typical Scenario



Special Scenario



➤ Exact/IR dualities and discrete gauging always implement different maps

* *there are certainly more than just three*

T-duality in the compact boson CFT

The compact boson CFT at radius R is a $1+1$D CFT with

$$\mathcal{L}_R = \frac{R^2}{4\pi}\partial_\mu\Phi\partial^\mu\Phi, \qquad\qquad \Phi \sim \Phi + 2\pi$$

- T-duality is an isomorphism of *all* operators & *all* states of $\mathcal{L}_R$ and $\mathcal{L}_{1/R}$ — it is an exact duality.

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- The CFT has $\mathrm{U}(1)^{\mathcal{M}}$ momentum and $\mathrm{U}(1)^{\mathcal{W}}$ winding symmetries

$$J_\mu^{\mathcal{M}} = \frac{R^2}{2\pi}\partial_\mu \Phi$$

$$J_\mu^{\mathcal{W}} = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu \Phi$$

$$\begin{array}{ccc} \mathcal{L}_R & & \mathcal{L}_{1/R} \\ J^{\mathcal{M}} & \xrightarrow[\text{map}]{\text{T-duality}} & J^{\mathcal{W}} \\ J^{\mathcal{W}} & & J^{\mathcal{M}} \\ \vdots & & \vdots \end{array}$$

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$J^{\mathcal{M}}_\mu =$ $\qquad$ $\mathcal{L}_{1/R}$

$J^{\mathcal{W}}$

$J^{\mathcal{W}}$ $\qquad$ map $\qquad$ $J^{\mathcal{M}}$

$$J^{\mathcal{W}}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Phi$$

$\vdots$ $\qquad$ $\vdots$

Exchange of momentum and winding symmetries is a signature of T-duality

Gauging in the compact boson CFT

The $\mathrm{U}(1)^{\mathcal{M}}$ symmetry transformation $\Phi \to \Phi + C$

- Gauging $\mathbb{Z}_N^{\mathcal{M}} \subset \mathrm{U}(1)^{\mathcal{M}}$ causes $\Phi \sim \Phi + 2\pi/N$, which is equivalent to preserving $\Phi \sim \Phi + 2\pi$ while $R \to R/N$

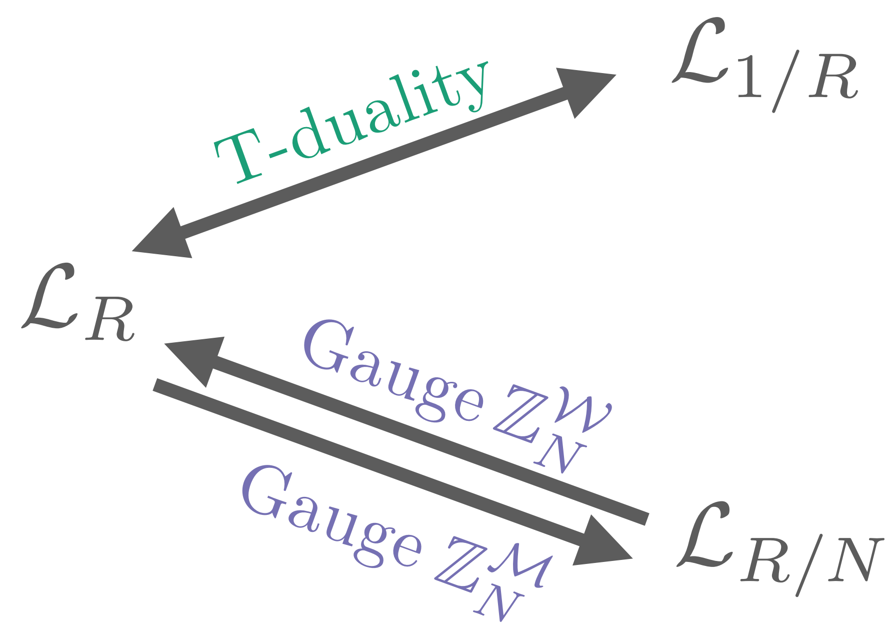
$$\begin{array}{ccc} \mathcal{L}_R & & \mathcal{L}_{R/N} \\ J^{\mathcal{M}} & \xrightarrow[\mathbb{Z}_N^{\mathcal{M}}]{\text{Gauge}} & N J^{\mathcal{M}} \\ J^{\mathcal{W}} & & J^{\mathcal{W}}/N \\ \vdots & & \vdots \end{array}$$

- Has a nontrivial kernel spanned by $Q^{\mathcal{M}} \notin N\mathbb{Z}$ states

$$\boxed{\text{Gauging maps are non-invertible (not bijective)!}}$$

T-duality versus gauging

When $\underline{R \neq \sqrt{N}}$, the image of $\mathcal{L}_R$ under T-duality and Gauging $\mathbb{Z}_N^{\mathcal{M}}$ is different.



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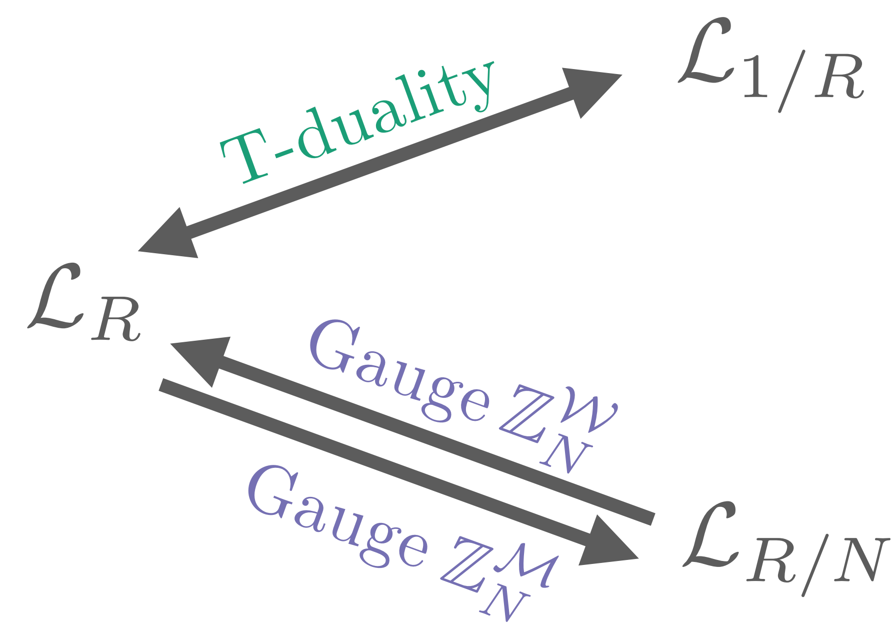
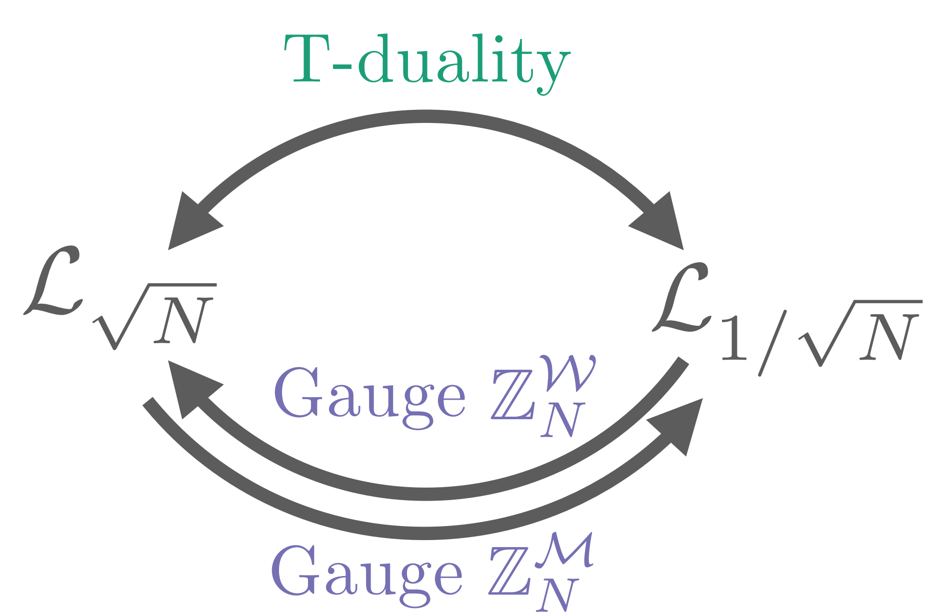


When $\underline{R = \sqrt{N}}$, the image of $\mathcal{L}_R$ under T-duality and Gauging $\mathbb{Z}_N^{\mathcal{M}}$ is <u>the same</u>.

- T-duality: Isomorphism of $\mathcal{L}_{\sqrt{N}}$ and its $\mathbb{Z}_N^{\mathcal{M}}$-gauged theory



Non-invertible symmetry at $R = \sqrt{N}$

- The Isomorphism between $\mathcal{L}_R$ and its $\mathbb{Z}_N^{\mathcal{M}}$-gauged theory at $R = \sqrt{N}$ indicates a non-invertible symmetry

[Thorngren, Wang '21; Choi, Córdova, Hsin, Lam, Shao '21]

$$
\begin{array}{ccccc}
\mathcal{L}_{\sqrt{N}} & & \mathcal{L}_{1/\sqrt{N}} & & \mathcal{L}_{\sqrt{N}} \\
J^{\mathcal{M}} & \xrightarrow[\mathbb{Z}_N^{\mathcal{M}}]{\text{Gauge}} & N J^{\mathcal{M}} & \xrightarrow[\text{map}]{\text{T-duality}} & N J^{\mathcal{W}} \\
J^{\mathcal{W}} & & J^{\mathcal{W}}/N & & J^{\mathcal{M}}/N \\
\vdots & & \vdots & & \vdots
\end{array}
$$

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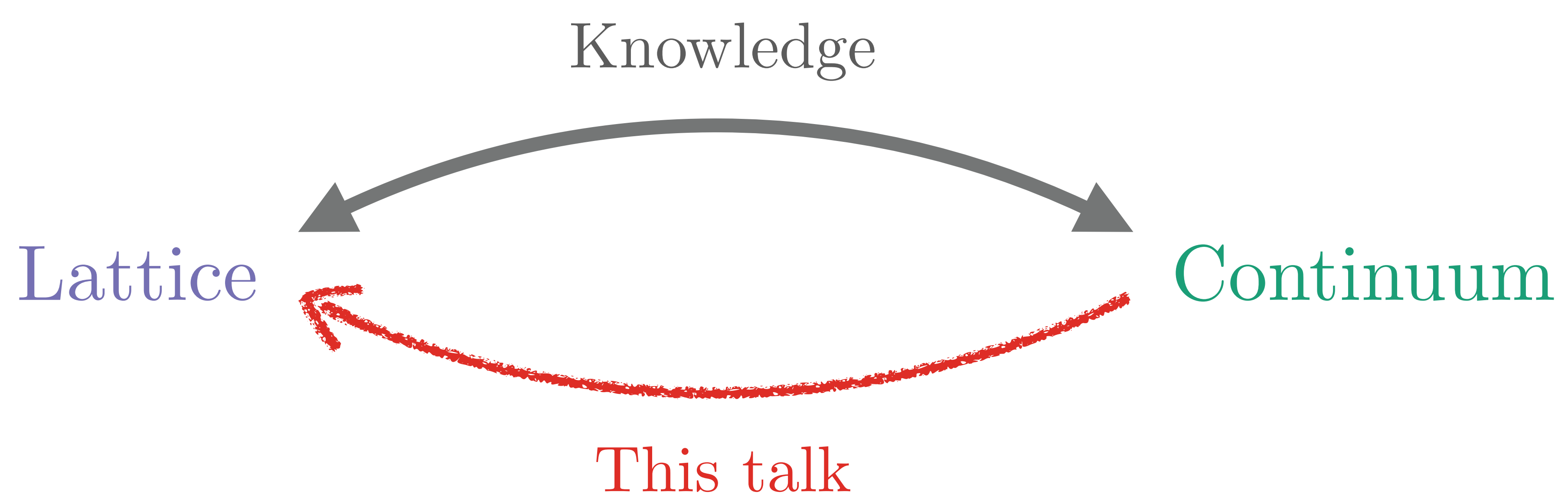
The existence of U(1) momentum, U(1) winding, & this non-invertible symmetry provides an invariant *definition* of T-duality.

T-duality and qubits?

Can T-duality exist in lattice models that flow to the compact boson in the IR? How about in qubit models?



T-duality and qubits?

Can T-duality exist in lattice models that flow to the compact boson in the IR? <u>How about in qubit models?</u>

<u>tl;dr</u>

1. In the XX spin chain, there is a non-invertible symmetry and corresponding lattice T-duality
2. Encounter conserved charges forming the Onsager algebra that match the CFT's 't Hooft anomalies on the lattice: enforce a gaplessness constraint
3. Explore phase diagram from symmetric deformations of the XX model

The XX model

Consider $1+1$D quantum lattice model on a finite ring with a qubit residing on each site j

- The number of sites L is even
- Pauli operators satisfy $X_{j+L} = X_j$ and $Z_{j+L} = Z_j$

XX model Hamiltonian [Lieb, Schultz, Mattis '61; Baxter '71; ⋯]

$$H_{\mathrm{XX}} = \sum_{j=1}^{L} (X_j X_{j+1} + Y_j Y_{j+1})$$

- Spin rotation $\mathrm{U}(1)^{\mathrm{M}}$ symmetry

$$Q^{\mathrm{M}} = \frac{1}{2} \sum_{j=1}^{L} Z_j$$

The XX model

$$H_{\mathrm{XX}} = 2\sum_{j=1}^{L}\left(\sigma_j^{+}\sigma_{j+1}^{-} + \sigma_j^{-}\sigma_{j+1}^{+}\right) \qquad \sigma_j^{\pm} = \frac{1}{2}\left(X_j \pm \mathrm{i}Y_j\right)$$

- $\mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}$ transforms $\sigma_j^{\pm} \to \mathrm{e}^{\pm\mathrm{i}\phi}\sigma_j^{\pm}$

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IR limit of the XX model

The IR of the XX model is described by the compact boson CFT at $R = \sqrt{2}$ *

[Alcaraz, Barber, Batchelor '87; Baake, Christe, Rittenberg '88]

- The IR limit: focus on low-energy states within an $\mathcal{O}(L^0)$ energy window above the ground state and take $L \to \infty$

$$\begin{array}{ccc} \sigma_j^+ & & \exp[\mathrm{i}\,\Phi] \\ Q^{\mathrm{M}} & \xrightarrow{\text{IR limit}} & \mathcal{Q}^{\mathcal{M}} = \int J_0^{\mathcal{M}} \\ \vdots & & \vdots \end{array}$$

- Q^{M} generates a U(1) momentum symmetry on the lattice

* *Reminder: in our convention, $R = 1$ is the self-dual radius*

Gauging $\mathbb{Z}_2^{\mathrm{M}}$ in the XX model

Does the XX model have a lattice T-duality?

➤ In the IR: implements an isomorphism between the $R = \sqrt{2}$ compact boson CFT and its $\mathbb{Z}_2^{\mathcal{M}}$ gauged theory

Let's gauge the $\mathbb{Z}_2^{\mathrm{M}}$ symmetry $\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \prod_{j=1}^{L} (-1)^j Z_j$ in the XX model

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$$\begin{pmatrix} (-1)^j Z_j \\ X_j X_{j+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} Z_j Z_{j+1} \\ X_{j+1} \end{pmatrix}$$

- $\mathbb{Z}_2^{\mathrm{M}}$ gauged Hamiltonian

$$H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} = \sum_{j=1}^{L} (X_j + Z_{j-1} X_j Z_{j+1})$$

Lattice T-duality

Hamiltonians are unitarily equivalent: $H_{\mathrm{XX}} = U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1}$

$$U_{\mathrm{T}} = \prod_{n=1}^{L/2} \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4} Z_{2n+1}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4} X_{2n+1}} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4} X_{2n}} \mathrm{CZ}_{2n,2n+1} \right)$$

- U_{T} implements an isomorphism between the XX model and its $\mathbb{Z}_2^{\mathrm{M}}$ gauged theory

T-duality from the isomorphism pov!

What about the symmetry pov?

Non-invertible symmetry of the XX model

Unitary equivalence $H_{\mathrm{XX}} = U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1}$ implies there is a non-invertible symmetry D

$$\mathsf{D}\, H_{\mathrm{XX}} = (U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1})\mathsf{D} = H_{\mathrm{XX}}\, \mathsf{D}$$

- Non-invertible because $\mathsf{D}\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \mathsf{D}$

  $\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}}|\psi\rangle = -|\psi\rangle \implies \mathsf{D}|\psi\rangle = 0 \implies \mathsf{D}^{-1}$ does not exist

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- Symmetry transformation:*

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_{\mathrm{T}}} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

* *this is a non-invertible <u>modulated</u> symmetry*

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In the continuum limit, this becomes the non-invertible symmetry of the compact boson CFT

- Non-in

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Lattice winding symmetry

What about the winding symmetry?

Lattice winding symmetry

What about the winding symmetry?

- Acting D on Q^{M}

$$\mathsf{D}Q^{\mathrm{M}} = 2Q^{\mathrm{W}}\mathsf{D}$$

where $Q^{\mathrm{W}} = \frac{1}{4}\sum_{n=1}^{L/2}(X_{2n-1}Y_{2n} - Y_{2n}X_{2n+1})$

- Acting D on Q^{W}

$$\mathsf{D}Q^{\mathrm{W}} = \frac{1}{2}Q^{\mathrm{M}}\mathsf{D}$$

$\implies Q^{\mathrm{W}}$ is a lattice winding charge*

* *Known conserved charge of the XX model [Vernier, O'Brien, Fendley '18]*

Lattice winding symmetry

What about the winding symmetry?

Symmetry transformation

$$e^{i\theta Q^{W}} X_j e^{-i\theta Q^{W}} = \begin{cases} X_j & j \text{ odd}\ , \\ X_j e^{-\frac{i\theta}{2} Y_j (X_{j-1} - X_{j+1})} & j \text{ even}\ , \end{cases}$$

$$e^{i\theta Q^{W}} Y_j e^{-i\theta Q^{W}} = \begin{cases} Y_j e^{\frac{i\theta}{2} X_j (Y_{j-1} - Y_{j+1})} & j \text{ odd} \\ Y_j & j \text{ even} \end{cases}$$

Unitarily equivalent to $U_{\mathrm{T}}^{-1} Q^{\mathrm{W}} U_{\mathrm{T}} = \frac{1}{4} \sum_{j=1}^{L} (-1)^j Z_j Z_{j+1}$

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Lattice winding symmetry

What about the winding symmetry?

The XX model has a lattice T-duality

- Isomorphism between H_{XX} and $H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}}$
- Conserved lattice Q^{M} and Q^{W} charges
- Non-invertible symmetry exchanging Q^{M} and Q^{W}

$$\mathsf{D}Q^{\mathrm{M}} = 2Q^{\mathrm{W}}\mathsf{D} \qquad \mathsf{D}Q^{\mathrm{W}} = \frac{1}{2}Q^{\mathrm{M}}\mathsf{D}$$

- Acting D on Q

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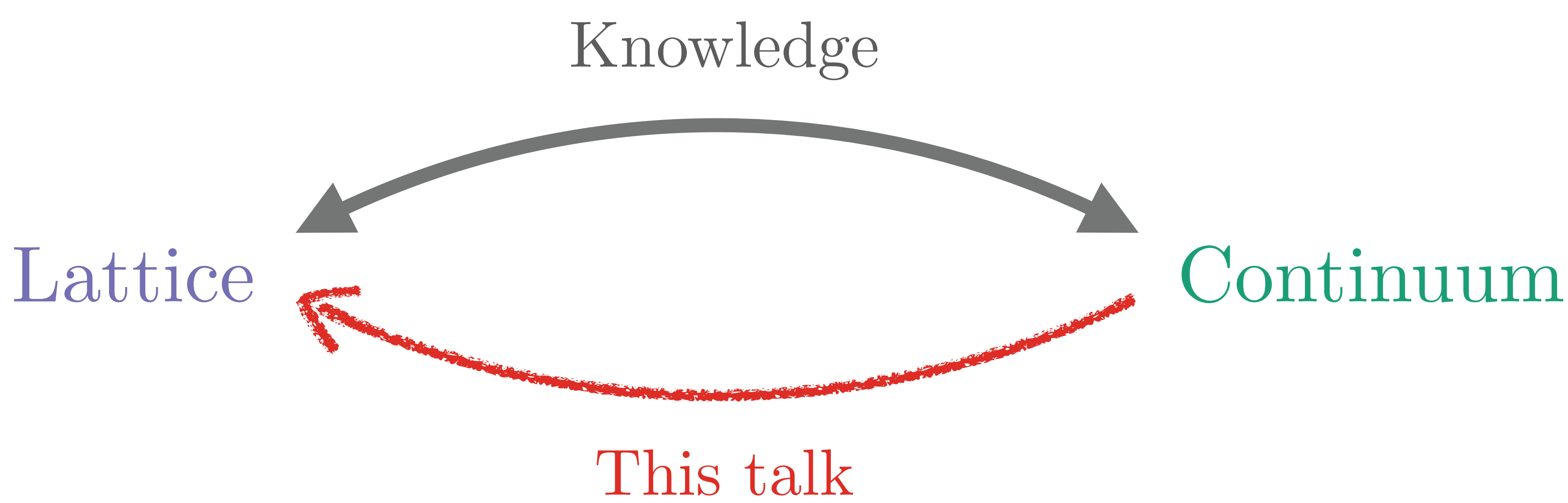
Knowledge stream

Recall the mutualistic relationship between lattice and continuum models



Motivated by the continuum, we found new symmetries and lattice T-duality in the XX model

- What else can we learn about the XX model using our continuum knowledge?

Onsager algebra

The charges Q^{M} and Q^{W} do not commute on the lattice

➤ Generate the Onsager algebra ($Q_0 = Q^{\mathrm{M}}$ and $Q_1 = 2Q^{\mathrm{W}}$)

[Onsager '44; Vernier, O'Brien, Fendley '18]

$$[Q_n, Q_m] = \mathrm{i} G_{m-n} \qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}\,(Q_{n-m} - Q_{n+m})$$

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$$[Q_n, G_m] = 2\mathrm{i}\,(Q_{n-m} - Q_{n+m})$$

- In the continuum

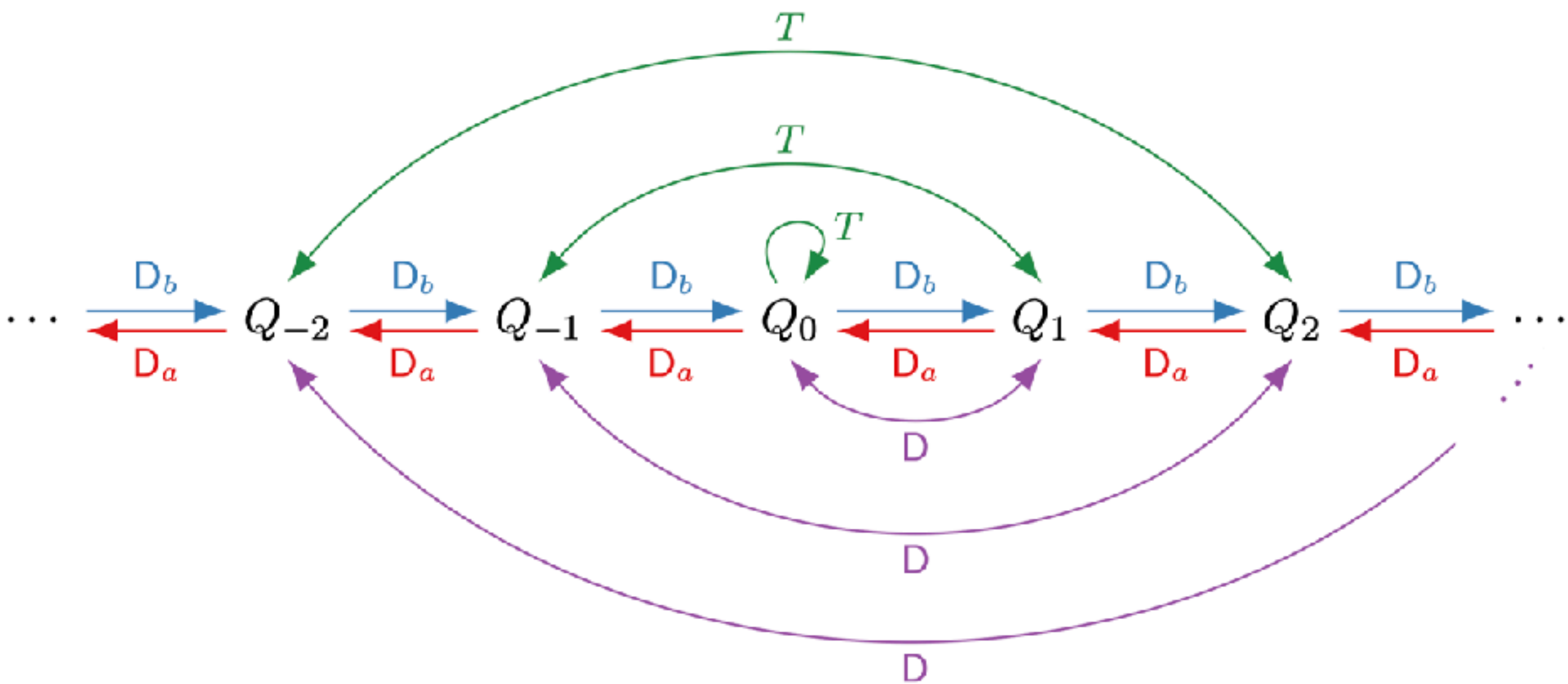
$$[Q^{\mathrm{M}}, Q^{\mathrm{W}}] \neq 0 \xrightarrow{\text{IR limit}} [\mathcal{Q}^{\mathcal{M}}, \mathcal{Q}^{\mathcal{W}}] = 0$$

$$Q_n \xrightarrow{\text{IR limit}} \begin{cases} 2\mathcal{Q}^{\mathcal{W}} & n \text{ odd} \\ \mathcal{Q}^{\mathcal{M}} & n \text{ even} \end{cases} \qquad G_n \xrightarrow{\text{IR limit}} 0$$

An ever richer algebraic structure

The Onsager charges have a rich interplay with other conserved operators of the XX model

- Let $\mathsf{D}_a = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}\mathsf{D}$, $\mathsf{D}_b = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}\mathsf{D}$, and T lattice translations



Anomalies

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Consider the symmetry operators*

$$\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \prod_{j=1}^{L} (-1)^j Z_j \qquad \mathrm{e}^{\mathrm{i}\theta Q^{\mathrm{W}}} \qquad C = \prod_{j=1}^{L} X_j$$

- Described by the group $\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}} \rtimes \mathbb{Z}_2^{\mathrm{C}}$
- These symmetries exist in the UV and IR, so their anomalies in the CFT must be realized on the lattice

* *Can show these unitaries are non-onsiteable*

Anomalies

How do the symmetries we found in the XX model realize the 't Hooft anomalies in the IR?

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➤ D

➤ T

a

$\mathbb{Z}_2^{\mathrm{C}} \times \mathbb{Z}_2^{\mathrm{M}} \times \mathbb{Z}_2^{\mathrm{W}}$

type III anomaly

$$\left\{ \begin{array}{l} \mathbb{Z}_2^{\mathrm{C}} \\ \mathbb{Z}_2^{\mathrm{M}} \\ \mathrm{U}(1)^{\mathrm{W}} \end{array} \right.$$

$$\left. \begin{array}{l} \mathbb{Z}_2^{\mathrm{M}} \\ \mathrm{U}(1)^{\mathrm{W}} \end{array} \right\}$$

$\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$

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type III anomaly

Projective algebras

$$\left\{ \begin{array}{l} \mathbb{Z}_2^{\mathrm{C}} \\ \\ \mathbb{Z}_2^{\mathrm{M}} \\ \\ \mathrm{U}(1)^{\mathrm{W}} \end{array} \right.$$

$\mathbb{Z}_2^{\mathrm{M}}$ and $\mathrm{U}(1)^{\mathrm{W}}$ $\Big\}$ $\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$
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$\mathbb{Z}_2^{\mathrm{C}} \times \mathbb{Z}_2^{\mathrm{M}} \times \mathbb{Z}_2^{\mathrm{W}}$
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Projective algebras

$\mathbb{Z}_2^{\mathrm{C}}$

$\mathbb{Z}_2^{\mathrm{M}}$

$\mathrm{U}(1)^{\mathrm{W}}$

Spectral flow

$\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$
mixed anomaly

* *Can show these unitaries are non-onsiteable*

Spectral flow in the XX model

Spectra flow from anomalous $\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$: inserting a $\theta = 2\pi$ $\mathrm{U}(1)^{\mathrm{W}}$ symmetry defect flips each $\mathbb{Z}_2^{\mathrm{M}}$ symmetry charge

➤ Corresponds to $\mathbb{Z}_2^{\mathrm{M}}$ charge pumping in $2+1$D SPT

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Inserting a $\theta \in \mathrm{U}(1)^{\mathrm{W}}$ symmetry defect at $\langle I-1, I\rangle$ (I odd)

$$H_{\mathrm{XX}} \longrightarrow H_{\mathrm{XX}}(\theta) \equiv H_{\mathrm{XX}} + (Y_{I-1}Y_I + Y_I Y_{I+1})\,(\mathrm{e}^{-\mathrm{i}\theta X_I \frac{1+Y_{I+1}}{2}} - 1)$$

$$\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} \longrightarrow \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}(\theta)} \equiv \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2} X_I}$$

$$\vdots$$

- Spectral flow: $\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}(\theta+2\pi)} = -\,\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}(\theta)}$

Perturbative anomalies in the IR

The mixed anomaly of $\mathrm{U}(1)^{\mathcal{M}} \times \mathrm{U}(1)^{\mathcal{W}}$ in the compact boson CFT is a "perturbative anomaly"

- Cannot be matched by gapped phases $\Longrightarrow$ enforces gaplessness [⋯ ; Córdova, Freed, Teleman '24]

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Do the lattice momentum and winding symmetries enforce gaplessness?

- In other words, does the Onsager algebra match the perturbative anomaly?

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Do the lattice momentum and winding symmetries enforce gaplessness?

- In other words, does the Onsager algebra match the perturbative anomaly?

Answer: Yes! Can show by fermionizing the XX model

Fermionizing the XX model

Writing the complex fermions c_j in terms of real fermions $c_j = (a_j + \mathrm{i}b_j)/2$, we use the fermionization map

$$Z_j \to \mathrm{i}a_j b_j \qquad\qquad X_j X_{j+1} \to \begin{cases} -\mathrm{i}a_j a_{j+1} & j \text{ odd} \\ -\mathrm{i}b_j b_{j+1} & j \text{ even} \end{cases}$$

$$H_{\mathrm{XX}} \xrightarrow{\text{Fermionize}} -\mathrm{i}\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1})$$

$$Q^{\mathrm{M}} \xrightarrow{\text{Fermionize}} \frac{1}{2}\sum_{j=1}^{L}\mathrm{i}a_j b_j \equiv Q^{\mathrm{V}}$$

$$2Q^{\mathrm{W}} \xrightarrow{\text{Fermionize}} \frac{1}{2}\sum_{j=1}^{L}\mathrm{i}a_j b_{j+1} \equiv Q^{\mathrm{A}}$$

Enforced gaplessness

We prove that the most general local Hamiltonian that commutes with Q^{V} and Q^{A} is

$$H_f = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n (a_j a_{j+n} + b_j b_{j+n})$$

- Is always gapless with dispersion $\omega_k = 4 \sum_n g_n \sin(2\pi k n / L)$

Bosonization: one-to-one correspondence between H_f and qubit Hamiltonians commuting with Q^{M} and Q^{W}

- Every qubit model with Q^{M} and Q^{W} conserved is gapless
- Q^{M} and Q^{W} enforce gaplessness

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> The perturbative anomaly of the compact boson CFT is matched by the Onsager algebra
>
> ➤ Exciting connections between Onsager algebras and anomalies [Chatterjee, **SP**, Shao '24; Gioia, Thorngren '25; **SP**, Kim, Chatterjee, Shao '25]

➤ … $L)$

Bo… and

qubit Hamiltonians commuting with Q^{M} and Q^{W}

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Enforced gaplessness

We prove that the most general local Hamiltonian that commutes with Q^{V} and Q^{A} is

When $L = 0 \bmod 4$, there is a unitary frame in which

$$Q^{\mathrm{M}} = -\frac{1}{2}\sum_{j=1}^{L} Z_j \qquad\qquad Q^{\mathrm{W}} = -\frac{1}{4}\sum_{j=1}^{L} X_j X_{j+1}$$

- Any qubit chain commuting with $\sum_j Z_j$ and $\sum_j X_j X_{j+1}$ is gapless

- Every qubit model with Q and Q conserved is gapless
- Q^{M} and Q^{W} enforce gaplessness

Symmetric deformations

Can find $\mathrm{U}(1)^{\mathrm{M}}$ and $\mathrm{U}(1)^{\mathrm{W}}$ symmetric deformations of the XX model by bosonizing H_f

$$H_j^{(1)} \xrightarrow{\text{bosonize}} X_j X_{j+1} + Y_j Y_{j+1}$$

$$H_j^{(2)} \xrightarrow{\text{bosonize}} Y_j Z_{j+1} X_{j+2} - X_j Z_{j+1} Y_{j+2}$$

$$H_j^{(3)} \xrightarrow{\text{bosonize}} X_j Z_{j+1} Z_{j+2} X_{j+3} + Y_j Z_{j+1} Z_{j+2} Y_{j+3}$$

$$\vdots$$

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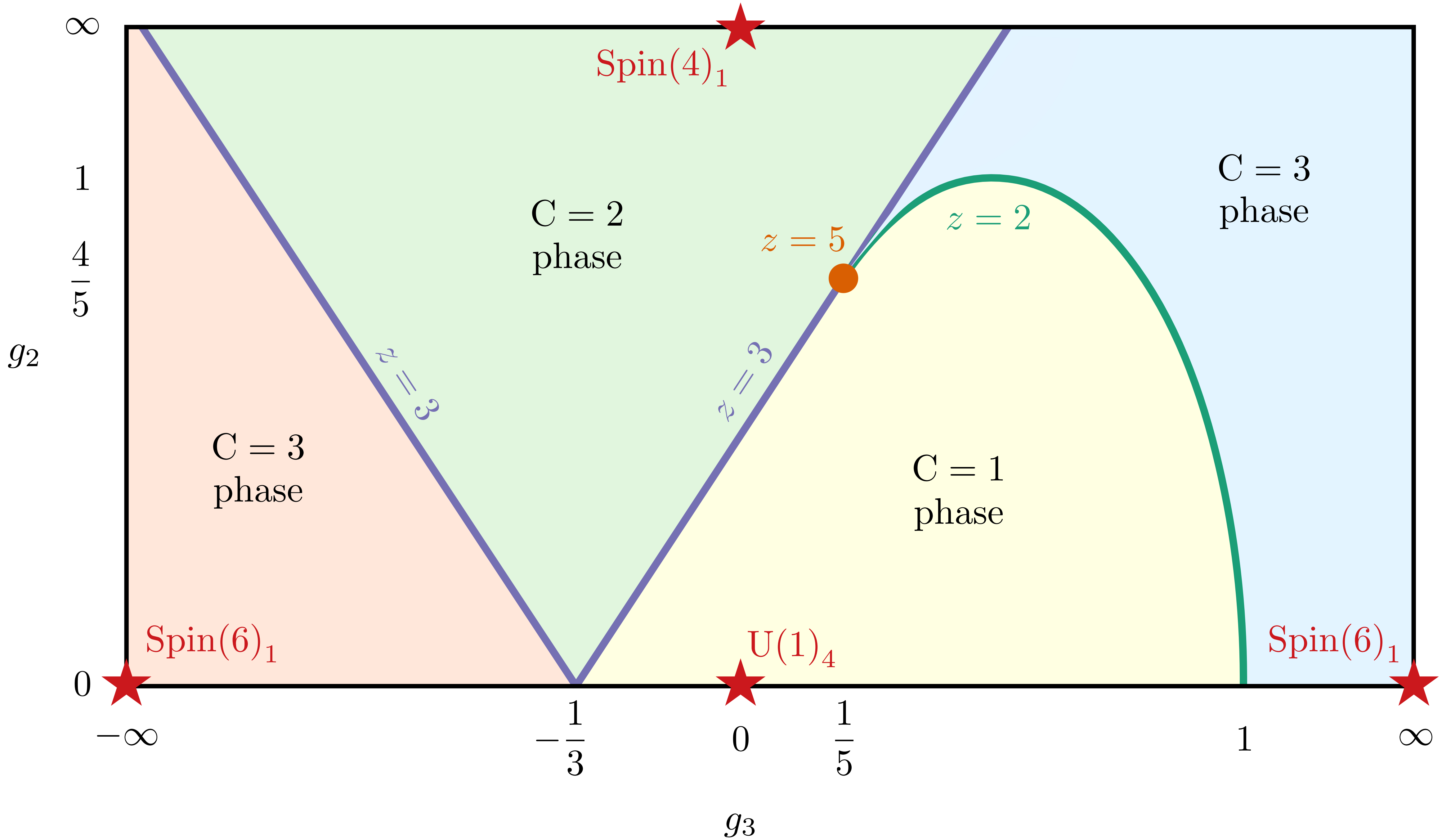
$$\vdots$$

Non-invertible symmetry D arises from $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}Q^{\mathrm{V}}} T_a$

- $\mathrm{U}(1)^{\mathrm{M}}$ and $\mathrm{U}(1)^{\mathrm{W}}$ guarantee the non-invertible symmetry and a lattice T-duality

Simplest 2-parameter phase diagram

$$H\left(g_2, g_3\right) = H_{\mathrm{XX}} + \sum_{j=1}^{L} \left(g_2 H_j^{(2)} + g_3 H_j^{(3)}\right)$$



Recap and outlook

Many aspects of the compact boson CFT surprisingly exist exactly in the XX model

1. Lattice T-duality and non-invertible symmetry
2. Lattice winding symmetry and 't Hooft anomalies
3. Symmetric deformations of the XX model

Recap and outlook

Many aspects of the compact boson CFT surprisingly exist exactly in the XX model

1. Lattice T-duality and non-invertible symmetry
2. Lattice winding symmetry and 't Hooft anomalies
3. Symmetric deformations of the XX model

Tip of an iceberg?

1. T-duality for other radii? S-duality in $3+1$D qubit models?
2. General relationship between perturbative anomalies and algebras? Between exact dualities of QFTs and unitary transformations in quantum lattice models?

Back-up slides

Lattice T-duality

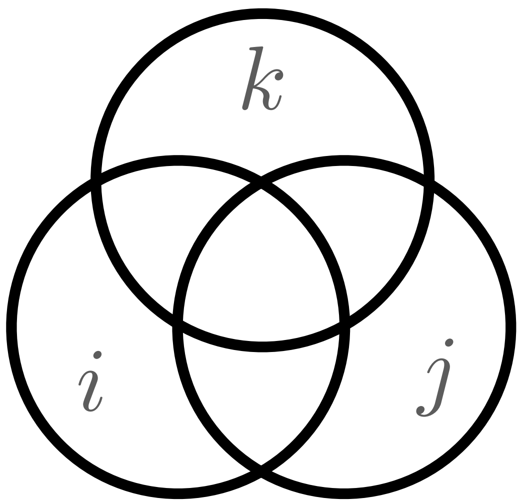
Can T-duality exist in lattice models that flow to the compact boson in the IR?

Yes: exists in the Modified Villain model

[Gross, Klebanov '90; Gorantla, Lam, Seiberg, Shao '21; Cheng, Seiberg '22; Fazza, Sulejmanpasic '22]

- Careful lattice regularization of the compact boson CFT

Patches $\{U_i\}$ $\quad \Phi_i : U_i \to \mathbb{R} \qquad n_{ij} : U_i \cap U_j \to \mathbb{Z}$



- $\Phi_i - \Phi_j = 2\pi n_{ij}$ on $U_i \cap U_j$
- Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

Lattice T-duality

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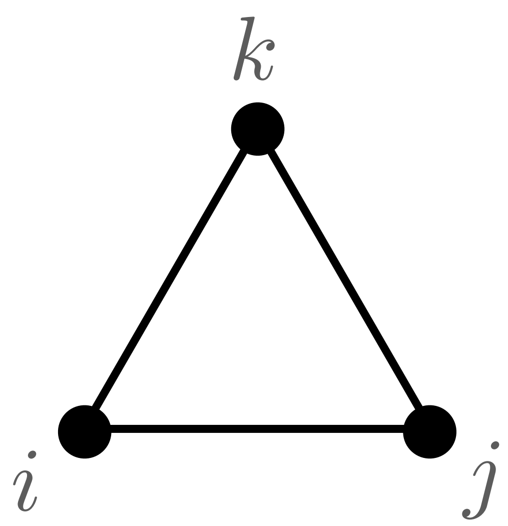
➤ Careful lattice regularization of the compact boson CFT

Spacetime lattice $\Phi_i \in \mathbb{R}$ $n_{ij} \in \mathbb{Z}$



➤ Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

➤ Infinite-dimensional local Hilbert space

Non-invertible symmetry algebra

The non-invertible symmetry operator D

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_{\mathrm{T}}} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

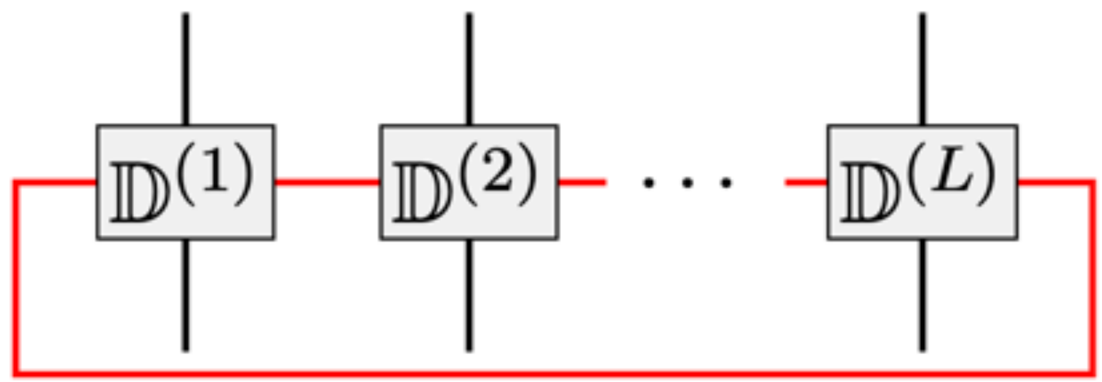
➤ Implies operator algebra

$$\mathsf{D}^2 = \left(1 + \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}}\right) T\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}, \qquad \mathsf{D}\,\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}}\mathsf{D} = \mathsf{D},$$

$$T\mathsf{D}T^{-1} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}\mathsf{D}, \qquad \mathsf{D}^\dagger = \mathsf{D}T^{-1}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}$$

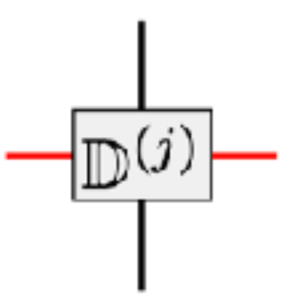
D as an matrix product operator

$$\mathsf{D} = \mathrm{Tr}\left(\prod_{j=1}^{L} \mathbb{D}^{(j)}\right) \equiv$$



where the MPO tensor

$$\mathbb{D}^{(j)} \equiv$$



$$= \begin{cases} \dfrac{1}{\sqrt{8}} \begin{pmatrix} \mathbf{1} - Z_j + X_j + \mathrm{i}Y_j & \mathbf{1} + Z_j + X_j - \mathrm{i}Y_j \\ -\mathbf{1} - Z_j + X_j - \mathrm{i}Y_j & \mathbf{1} - Z_j - X_j - \mathrm{i}Y_j \end{pmatrix} & j \text{ odd}, \\ \dfrac{\mathrm{i}}{\sqrt{8}} \begin{pmatrix} \mathbf{1} + Z_j - \mathrm{i}X_j - Y_j & -\mathbf{1} + Z_j - \mathrm{i}X_j + Y_j \\ \mathbf{1} - Z_j - \mathrm{i}X_j + Y_j & \mathbf{1} + Z_j + \mathrm{i}X_j + Y_j \end{pmatrix} & j \text{ even}. \end{cases}$$

Emergence of $\mathrm{TY}(\mathbb{Z}_2,+)$

The XX model has a continuous family of non-invertible symmetries

$$\mathsf{D}_{\phi,\theta} = \mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}\mathrm{e}^{\mathrm{i}\theta Q^{\mathrm{W}}}\mathsf{D}$$

- $(\mathsf{D}_{\phi,\theta})^2 = (1+\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}})\mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}\mathrm{e}^{\mathrm{i}(2\phi+\theta)Q^{\mathrm{W}}}\mathrm{e}^{\frac{\mathrm{i}}{2}(\theta-\pi)Q^{\mathrm{M}}}T$

The $R=\sqrt{2}$ compact boson CFT has an S^1-family of $\mathrm{TY}(\mathbb{Z}_2,+)$ symmetry operators $\mathcal{D}_\varphi$ [Thorngren, Wang '21]

In the IR, $T \xrightarrow{\text{IR limit}} \mathrm{e}^{\mathrm{i}\pi(\mathcal{Q}^{\mathcal{M}}+\mathcal{Q}^{\mathcal{W}})}$ [Metlitski, Thorngren '17; Cheng, Seiberg '22]

$$\mathsf{D}_{\phi,\pi-2\phi} \xrightarrow{\text{IR limit}} \mathcal{D}_\phi$$

Expressions of the Onsager charges 1

- The Onsager algebra. Formed by conserved charges $\{Q_n, G_n\}$

$$[Q_n, Q_m] = \mathrm{i}G_{m-n} \qquad\qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}\left(Q_{n-m} - Q_{n+m}\right)$$

The Onsager charges Q_n in terms of Q^{M} and Q^{W} are

$$Q_n = \begin{cases} 2S_n Q^{\mathrm{W}} S_n^{-1} & n \text{ odd} \\ S_n Q^{\mathrm{M}} S_n^{-1} & n \text{ even} \end{cases}$$

- Where $S_0 = S_1 = 1, \quad S_2 = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}, \quad S_3 = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}, \quad \cdots$
- S are the pivots of Onsager algebra [Jones, Prakash, Fendley '24]

Expressions of the Onsager charges 2

$$Q_n = \begin{cases} \frac{1}{2}\sum_{j=1}^{L} Z_j & n=0, \\ \frac{(-1)^{\frac{n+2}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j-1}\ \prod_{k=2j}^{2j+n-2} Z_k\ X_{2j+n-1} + Y_{2j}\ \prod_{k=2j+1}^{2j+n-1} Z_k\ Y_{2j+n}\right) & n>0 \text{ even}, \\ \frac{(-1)^{\frac{n-1}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j-1}\ \prod_{k=2j}^{2j+n-2} Z_k\ Y_{2j+n-1} - Y_{2j}\ \prod_{k=2j+1}^{2j+n-1} Z_k\ X_{2j+n}\right) & n>0 \text{ odd}, \\ \frac{(-1)^{\frac{n-2}{2}}}{2}\sum_{j=1}^{L/2}\left(Y_{2j+n-1}\ \prod_{k=2j+n}^{2j-2} Z_k\ Y_{2j-1} + X_{2j+n}\ \prod_{k=2j+n+1}^{2j-1} Z_k\ X_{2j}\right) & n<0 \text{ even}, \\ \frac{(-1)^{\frac{n+1}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j+n-1}\ \prod_{k=2j+n}^{2j-2} Z_k\ Y_{2j-1} - Y_{2j+n}\ \prod_{k=2j+n+1}^{2j-1} Z_k\ X_{2j}\right) & n<0 \text{ odd}, \end{cases}$$

$$G_n = \begin{cases} \mathrm{sign}(n)\frac{(-1)^{\frac{n}{2}}}{2}\sum_{j=1}^{L/2}(-1)^j\left(X_jY_{j+n} + Y_jX_{j+n}\right)\prod_{k=j+1}^{j+n-1} Z_k & n \text{ even}, \\ \mathrm{sign}(n)\frac{(-1)^{\frac{n-1}{2}}}{2}\sum_{j=1}^{L/2}(-1)^j\left(X_jX_{j+n} - Y_jY_{j+n}\right)\prod_{k=j+1}^{j+n-1} Z_k & n \text{ odd}. \end{cases}$$

Fermionizing by gauging

Gauss law

$$G_j = (-1)^j Z_j \,\mathrm{i} a_{j,j+1} b_{j,j+1}$$

Unitary transformation

$$Z_j \to Z_j \,\mathrm{i} a_{j,j+1} b_{j,j+1} \,, \qquad X_j \to \begin{cases} X_j & j \text{ odd}\,, \\ X_j \,\mathrm{i} a_{j,j+1} b_{j,j+1} & j \text{ even}\,. \end{cases}$$

$$a_{j,j+1} \to \begin{cases} X_j \, a_{j,j+1} & j \text{ odd}\,, \\ Y_j \, a_{j,j+1} & j \text{ even}\,, \end{cases} \qquad b_{j,j+1} \to \begin{cases} -X_j \, b_{j,j+1} & j \text{ odd}\,, \\ Y_j \, b_{j,j+1} & j \text{ even}\,. \end{cases}$$

Qubits now polarized $Z_j = 1$

Bosonizing by gauging

Gauss law

$$G_j = \begin{cases} X_{j-1,j}\,(\mathrm{i}a_j b_{j+1})\,Y_{j,j+1} & j \text{ odd}, \\ -Y_{j-1,j}\,(\mathrm{i}a_j b_{j+1})\,X_{j,j+1} & j \text{ even}. \end{cases}$$

Unitary transformation

$$a_j \to \begin{cases} -X_{j-1,j}\,a_j & j \text{ odd}, \\ Y_{j-1,j}\,a_j & j \text{ even}, \end{cases} \qquad b_j \to \begin{cases} -X_{j-1,j}\,b_j & j \text{ odd}, \\ -Y_{j-1,j}\,,b_j & j \text{ even}, \end{cases}$$

$$X_{j-1,j} \to \begin{cases} X_{j-1,j} & j \text{ odd}, \\ X_{j-1,j}\,(\mathrm{i}a_j\,b_j) & j \text{ even}, \end{cases} \qquad Z_{j-1,j} \to (-1)^{j-1}\,Z_{j-1,j}\,(\mathrm{i}a_j\,b_j).$$

Fermions now polarized $\mathrm{i}a_j b_j = 1$