Recurrences in the β-FPUT Chain

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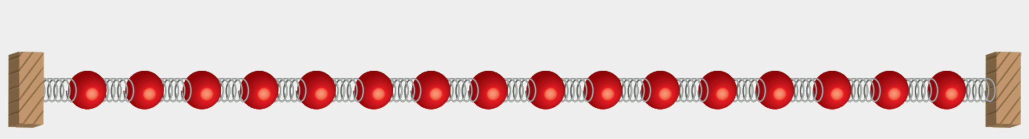
Abstract

One of the most remarkable phenomena in nonlinear dynamics is the Fermi-Pasta-Ulam-Tsginou (FPUT) problem [1]. I have investigated the β-FPUT chain and studied the scaling of the recurrence time with the number of masses, N, the system energy, E, and the nonlinear parameter β. Using computer simulations, a rescaled recurrence time, $T=t(N+1)^{-3}$, was found to depend, for large N, only on the parameter S=Eβ(N+1). For large |S|, T was proportional to $|S|^{-1/2}$. Then, using analytical methods, I have extended previous results and found a closed form for the recurrence time for small |S|. Furthermore, I showed that the recurrences to the initial state for large |S| can be predicted by solitary waves, called solitons, in the continuum limit. These results suggest opportunities for further detailed studies of other important, open questions in the β-FPUT chain and have potential applications to experiments on ultra-cold atoms.

Model

- A simple model for a solid is an array of masses connected by springs.

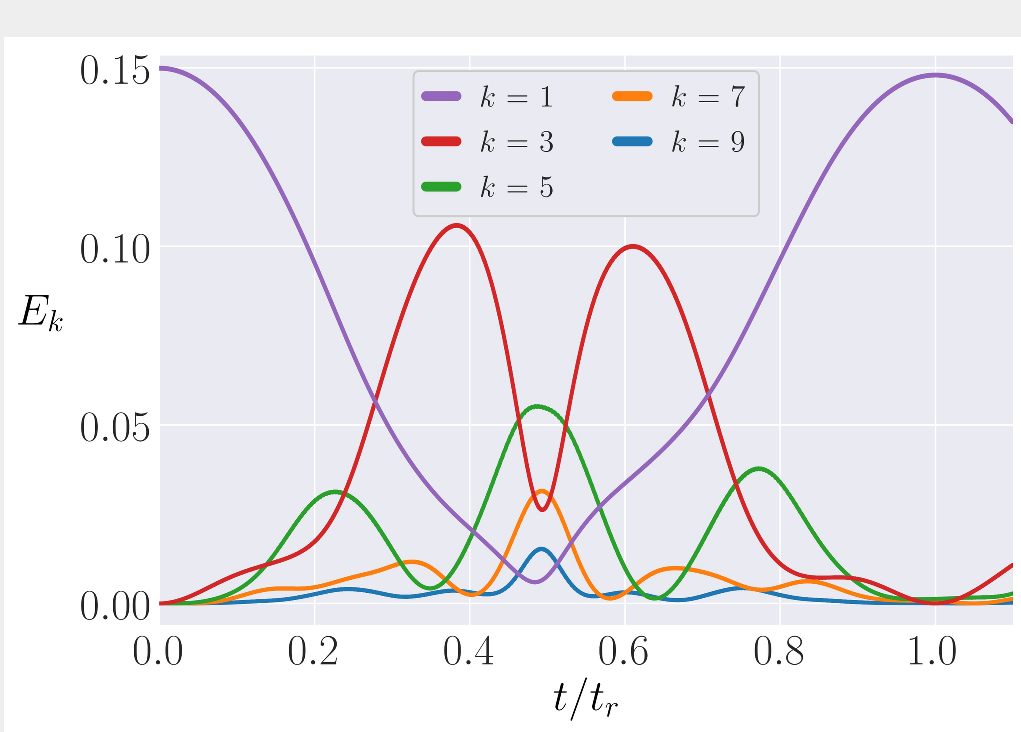


Source: Porter, Mason A., et al. "Fermi, Pasta, Ulam and the Birth of Experimental Mathematics: A numerical experiment that Enrico Fermi, John Pasta, and Stanislaw Ulam reported 54 years ago continues to inspire discovery." American Scientist 97.3 (2009): 214-221, figure 2

- β-FPUT chain is such a model:

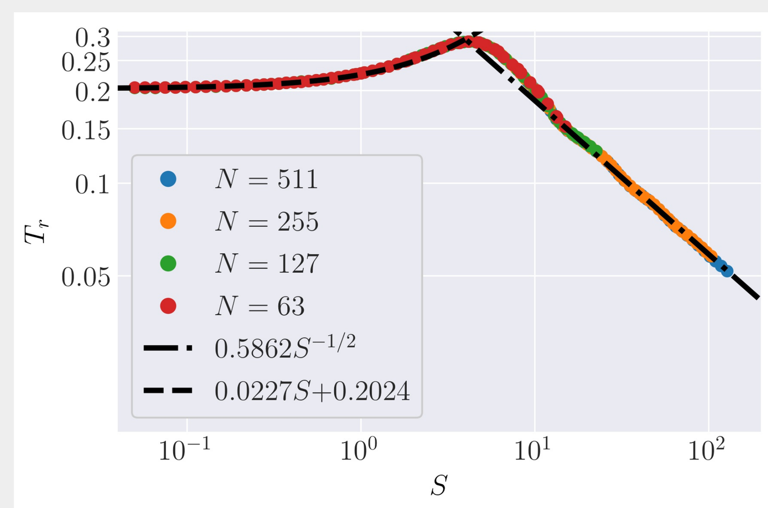
$$H=\sum_{n=1}^{N}\frac{p_n^2+(q_{n+1}-q_n)^2}{2}+\beta\frac{(q_{n+1}-q_n)^4}{4}$$

FPUT Recurrences:

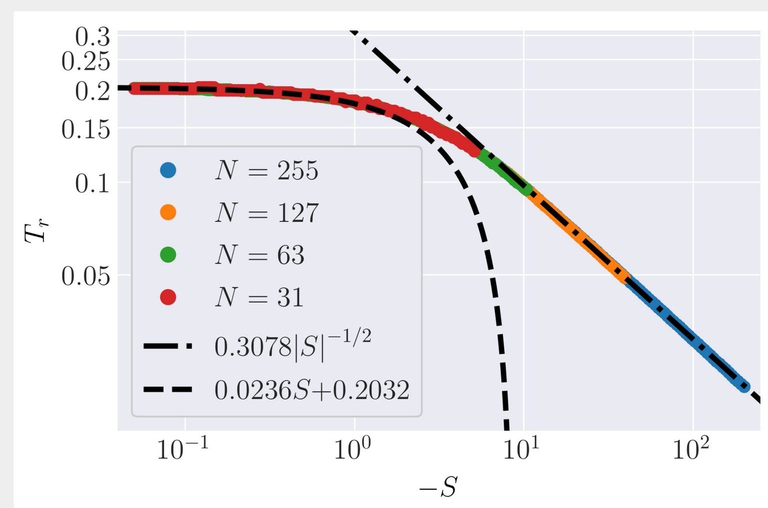


Numerical Results

β > 0:



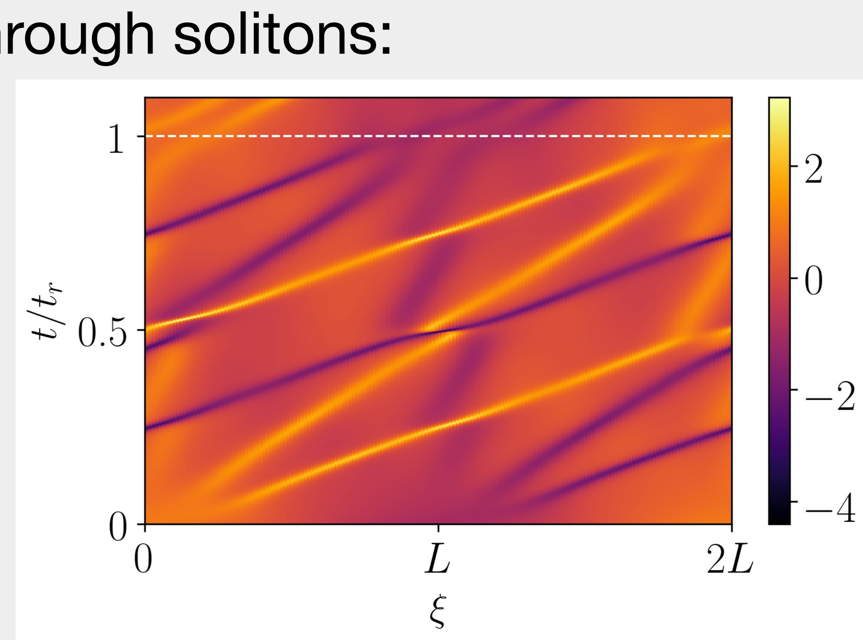
β < 0:



In the continuum limit, dynamics can be approximated by the modified Korteweg-de Vries equation:

$$2\phi_t+\beta\phi^2\phi_x+\frac{1}{12}\phi_{xxx}=0.$$

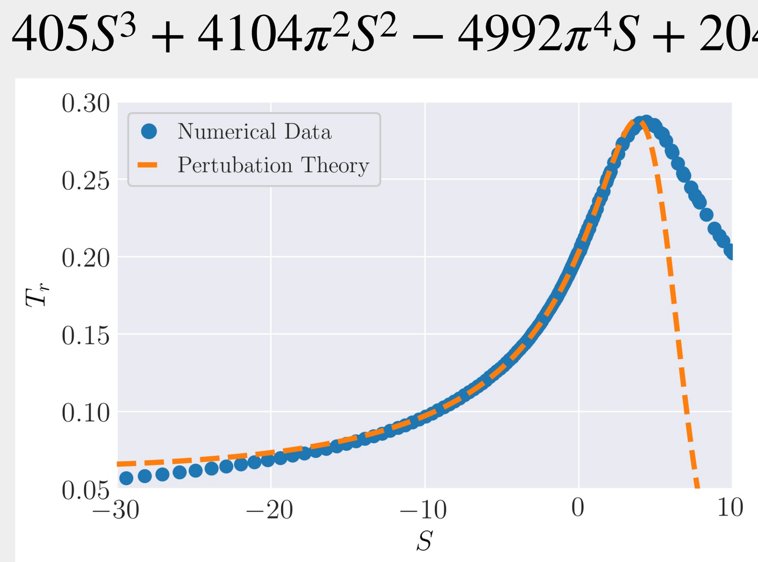
Recurrences can be understood through solitons:



Analytical Results

Small |S|: Solve lattice dynamics perturbatively:

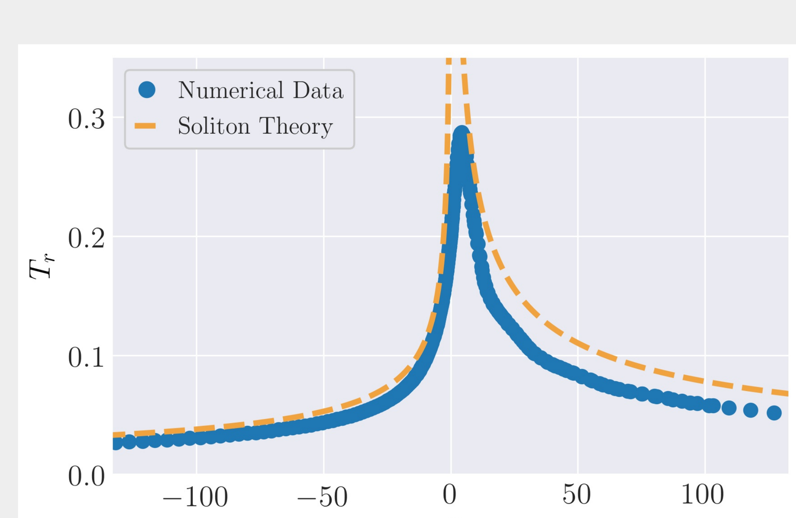
$$T_r=\frac{864S^2-5376\pi^2S+4096\pi^4}{405S^3+4104\pi^2S^2-4992\pi^4S+2048\pi^6}.$$



Large |S|: Approximate Recurrence time in continuum using solitons velocities:

$$\beta>0: T_r=\frac{2\sqrt{6}}{\pi^2\sqrt{|\beta|}A}.$$

$$\beta<0: T_r=\frac{3\sqrt{2}}{\pi\sqrt{12|S|+\pi\sqrt{6|S|}}}.$$



Numerical Method

I solved Hamilton's equations using the symplectic SABA2C integrator [2]. We computed them on BU's Shared Computer Cluster (SCC), and created the graphics using Matplotlib in Python. This integration scheme introduced a relative error in the energy of about 10^{-9}, ensuring numerical energy conservation. Furthermore, to test the time-reversal symmetry of classical mechanics, we ran the codes backwards in time to ensure that energy was properly returned to the initial state.

Future Directions

The results strongly suggest that the β-FPUT chain needs to be further investigated for β<0, where few previous studies have occurred. Just as the FPUT recurrences behaved and scaled differently based on the sign of β, I expect other features of the dynamics to change with the sign of β. This includes the timescale to energy equipartition and also the intermittent dynamics at equilibrium. Furthermore, while there have been numerous studies on the stability and localizing properties of q-breathers, there has yet to be a quantitative comparison of their periods to the FPUT recurrence time. The results presented here for the β-FPUT chain provide an opportunity for a detailed study of this important open question. As a final remark, it has recently been shown that the dynamics of a one-dimensional Bose gas in the quantum rotor regime maps onto the β-FPUT chain with β<0. This suggests that our results may be of interest in studies of ultra-cold bosons confined in optical lattices.

References

[1] Fermi, E., Pasta, J. and Ulam, S "Studies of the nonlinear problems," No. LA-1940. Los Alamos Scientific Lab., N. Mex., 1955

[2] Laskar, Jacques, and Philippe Robutel. "High order symplectic integrators for perturbed Hamiltonian systems." Celestial Mechanics and Dynamical Astronomy 80.1 (2001):