

Lattice T-duality from non-invertible symmetries in quantum spin chains

Sal Pace

MIT

Caltech CMT Seminar





Arkya
Chatterjee

MIT → YITP



Shu-Heng
Shao

MIT

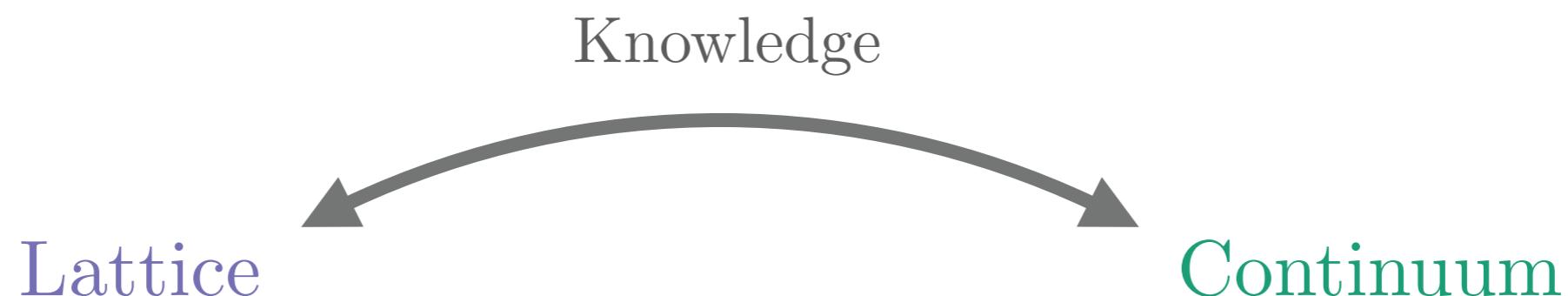
arXiv:2409.12220 [PRL 134, 021601 (2025)]
arXiv:2412.18606 [SciPost Phys. 18, 121 (2025)]

The lattice and the continuum

The interplay between **lattice** and **continuum** theories has a long and fruitful history in physics

- Lattice → **continuum**: extract universal properties
- **Continuum** → lattice: a laboratory for QFTs to do numerics and have an intrinsic UV cutoff

The two continually push each other forward.



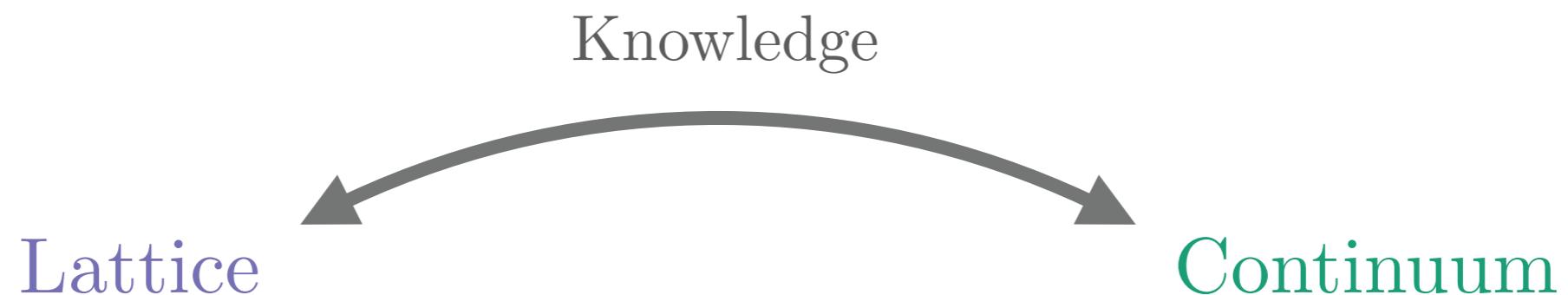
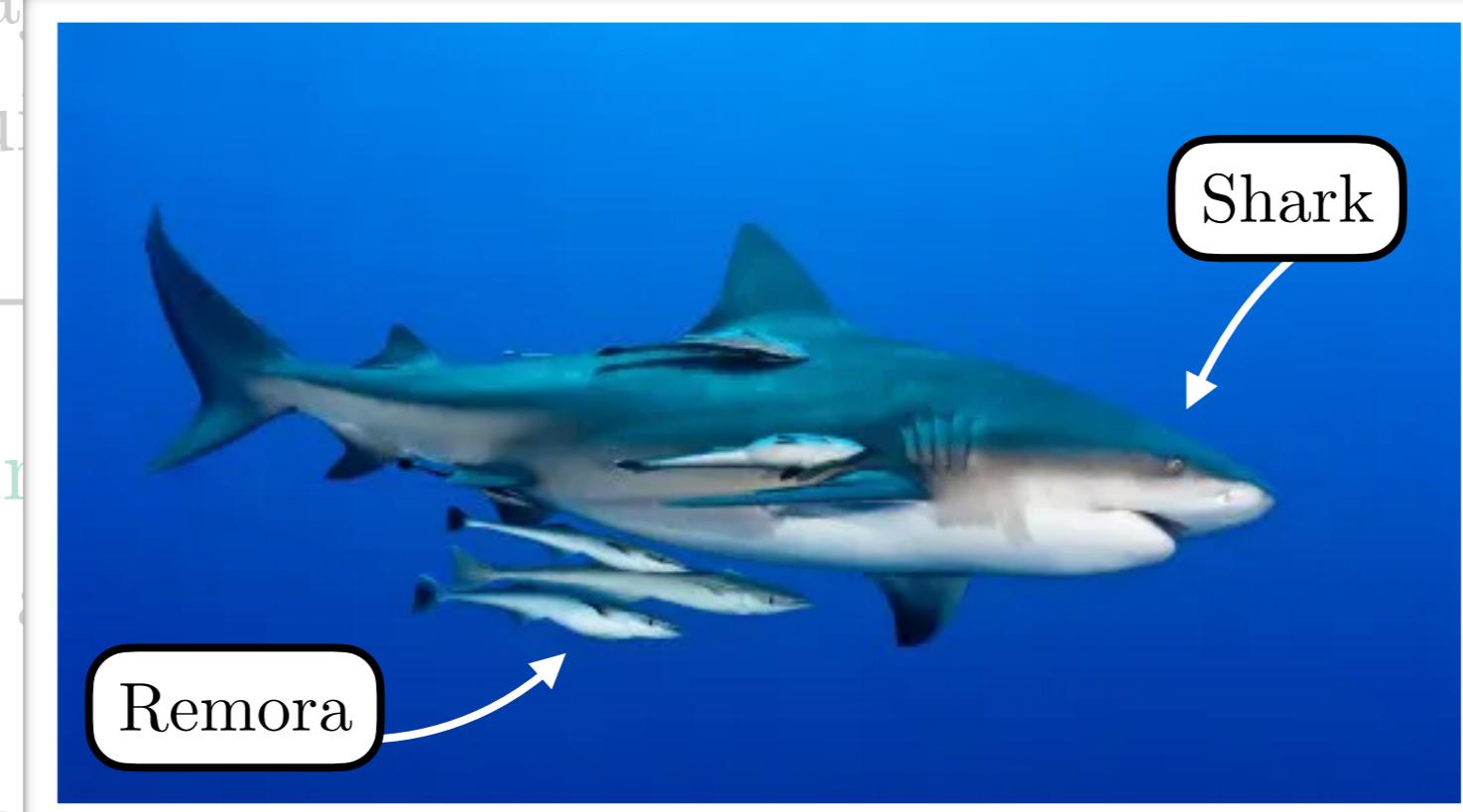
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- Lattice –
- Continuum – numerics

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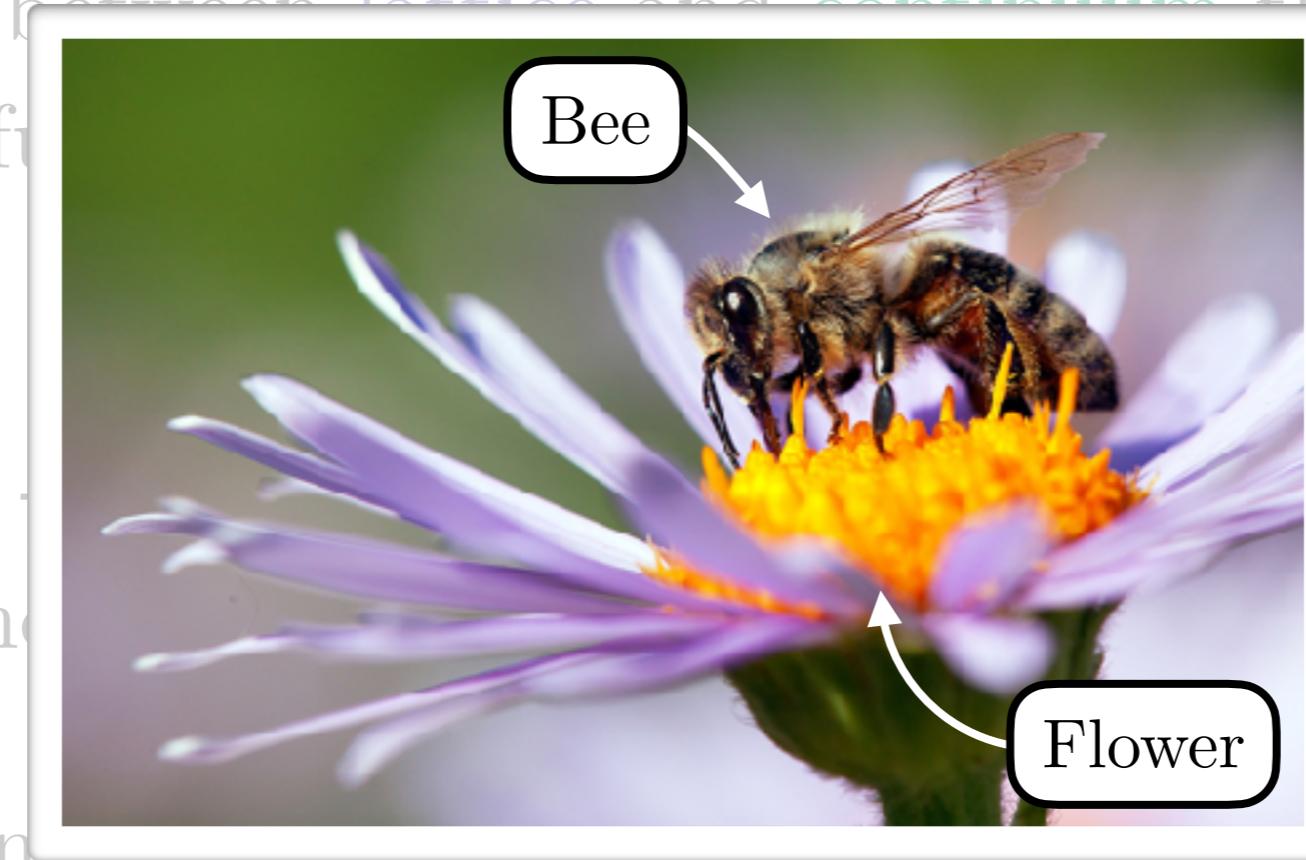
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► Lattice →

► Continuum →

numerics and



The two continua push each other forward.

Knowledge

Lattice

Continuum



► This is a **mutualistic relationship!**

Dualities in quantum systems

A theme in both lattice and continuum models is **duality**

- Dualities are maps between two seemingly distinct theories that are “secretly the same.”
- Both conceptually and practically useful

Dualities in quantum systems

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Ask people on the street their favorite **duality** and hear:

T-duality, Particle-Vortex duality,

Kramers-Wannier transformation

- These are not all the same notion of **duality!**
- Need to be more precise with “secretly the same.”

Three* types of dualities

1. **Exact duality**: relates two different presentations of the same quantum system (is an isomorphism).
- Electromagnetic duality, T-duality, Level-rank duality

* there are certainly more than just three

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 - Particle-vortex duality, Seiberg duality
3. **Discrete gauging**: relates two distinct quantum systems by gauging a discrete symmetry.
 - Kennedy-Tasaki (gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$), Kramers-Wannier (gauging \mathbb{Z}_2), bosonization (gauging $(-1)^F$)

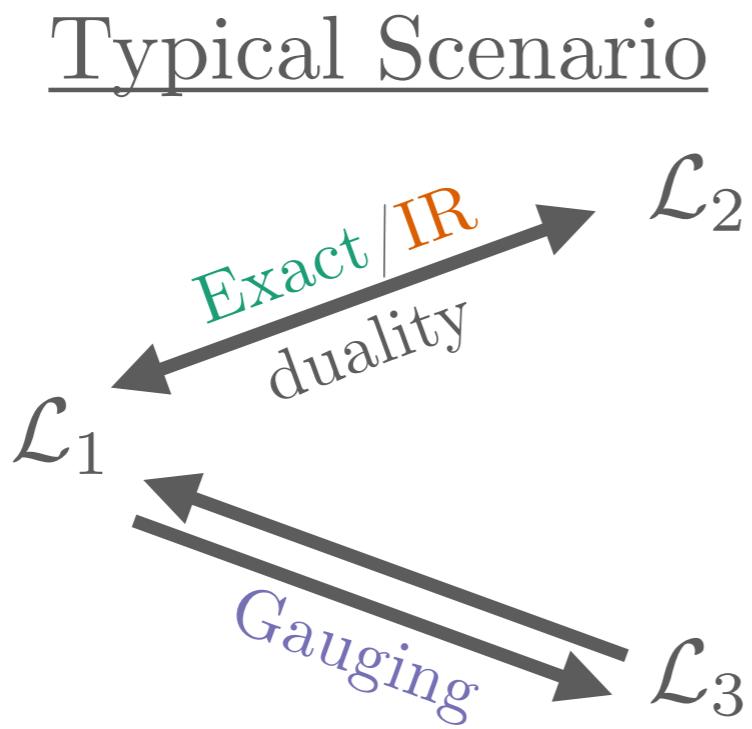
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Three* types of dualities

1. Exact/IR duality and discrete gauging are generally unrelated to each other

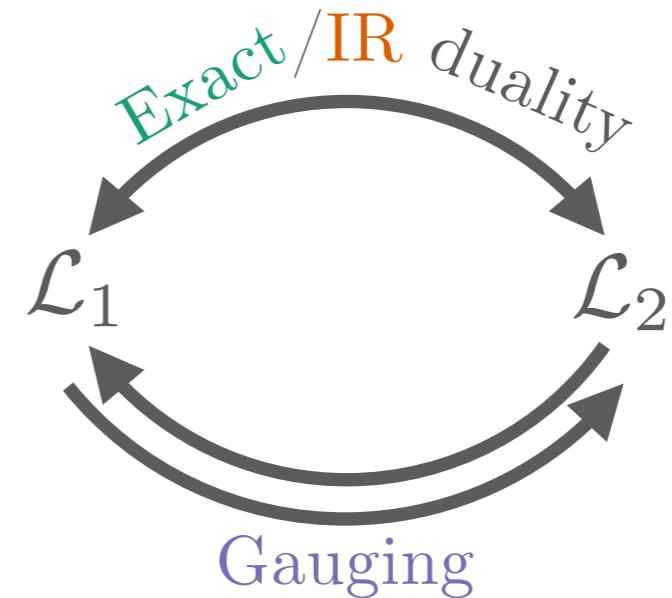
Discrete gauging is generally unrelated to exact duality and IR duality!

2. IR duality in



3. Discrete gauging

Special Scenario



- Exact/IR dualities and discrete gauging always implement different maps

* there are certainly more than just three

T-duality in the compact boson CFT

The **compact boson CFT** at radius R is a 1 + 1D CFT with

$$\mathcal{L}_R = \frac{R^2}{4\pi} \partial_\mu \Phi \partial^\mu \Phi, \quad \Phi \sim \Phi + 2\pi$$

- **T-duality:** \mathcal{L}_R and $\mathcal{L}_{1/R}$ are the same CFT:

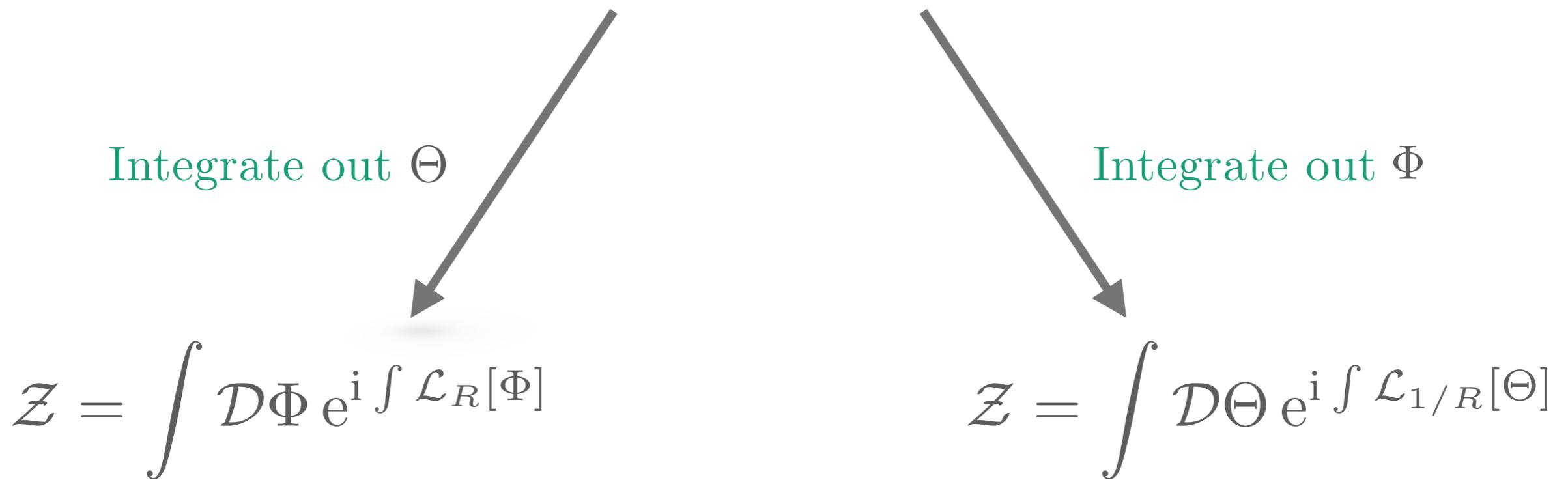
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$$\mathcal{Z} = \int \mathcal{D}\Phi \mathcal{D}\Theta e^{i \int \frac{1}{2\pi} \partial_t \Phi \partial_x \Theta - \frac{1}{4\pi} [R^{-2}(\partial_x \Theta)^2 + R^2(\partial_x \Phi)^2]}$$



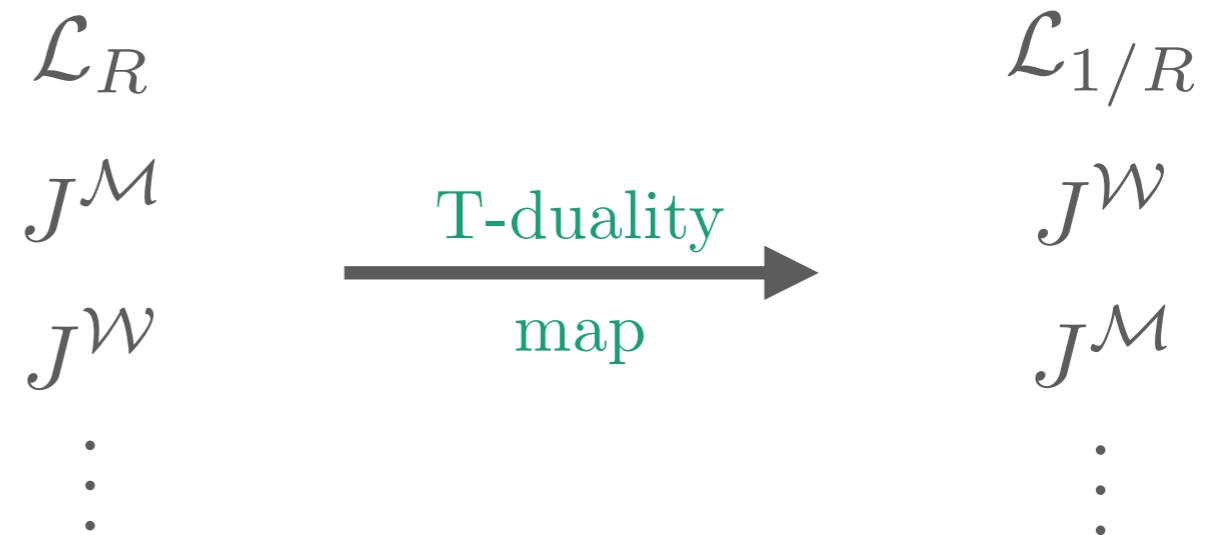
T-duality in the compact boson CFT

T-duality is an isomorphism of *all* operators & *all* states of \mathcal{L}_R and $\mathcal{L}_{1/R}$ (it is an exact duality). Preserves symmetries!

- Example: The CFT has $U(1)^{\mathcal{M}}$ momentum and $U(1)^{\mathcal{W}}$ winding symmetries

$$J_{\mu}^{\mathcal{M}} = \frac{R^2}{2\pi} \partial_{\mu} \Phi$$

$$J_{\mu}^{\mathcal{W}} = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial^{\nu} \Phi$$

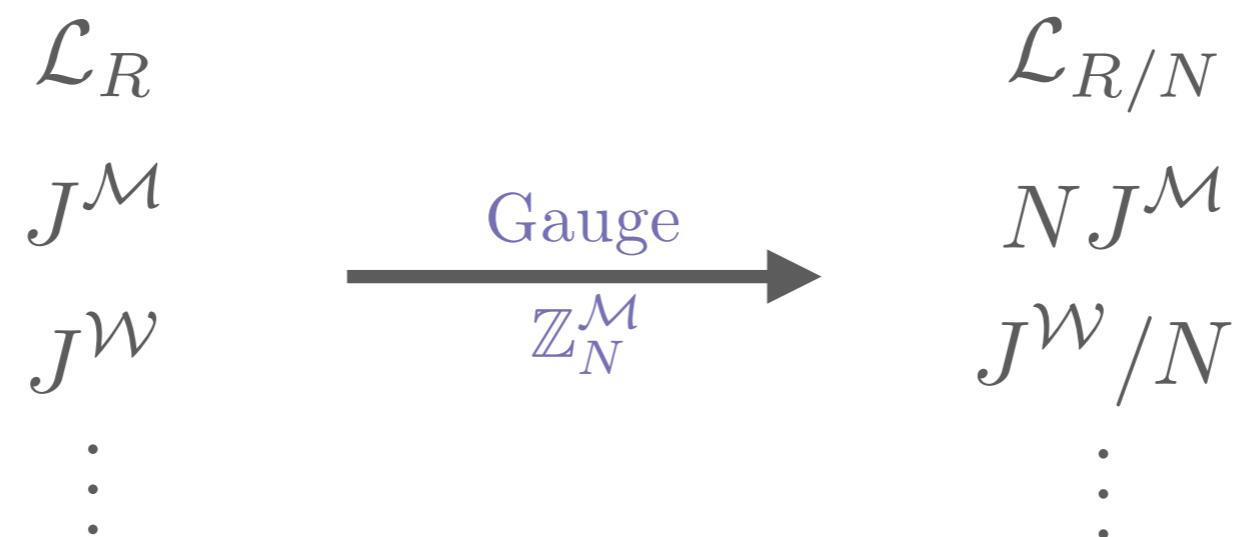


- Exchange of momentum and winding symmetries is a hallmark of T-duality!

Gauging in the compact boson CFT

The $\text{U}(1)^{\mathcal{M}}$ symmetry transformation $\Phi \rightarrow \Phi + C$

- Gauging $\mathbb{Z}_N^{\mathcal{M}} \subset \text{U}(1)^{\mathcal{M}}$ causes $\Phi \sim \Phi + 2\pi/N$, which is equivalent to preserving $\Phi \sim \Phi + 2\pi$ while $R \rightarrow R/N$

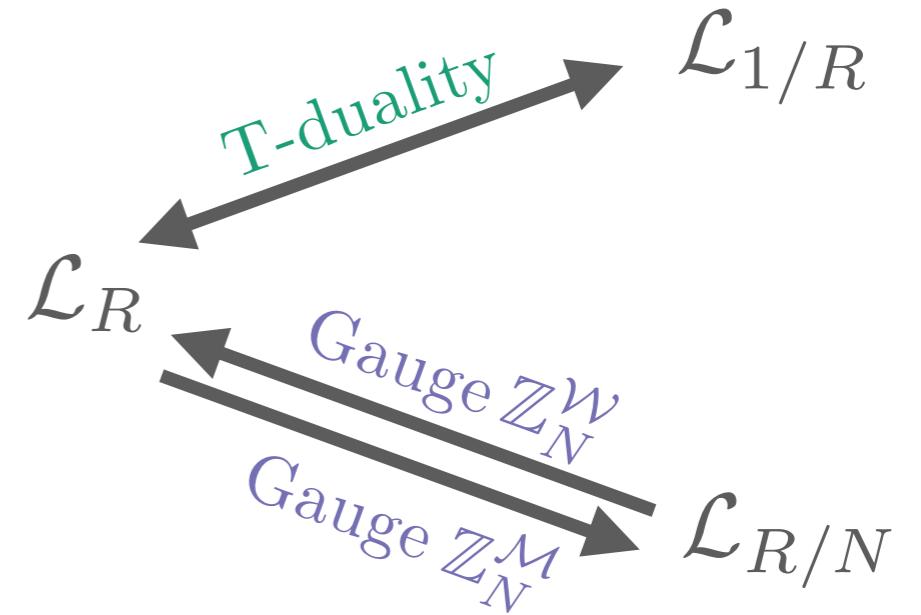


- Has a nontrivial kernel spanned by $Q^{\mathcal{M}} \notin N\mathbb{Z}$ states

Gauging maps are non-invertible (not bijective)!

Non-invertible symmetry at $R = \sqrt{N}$

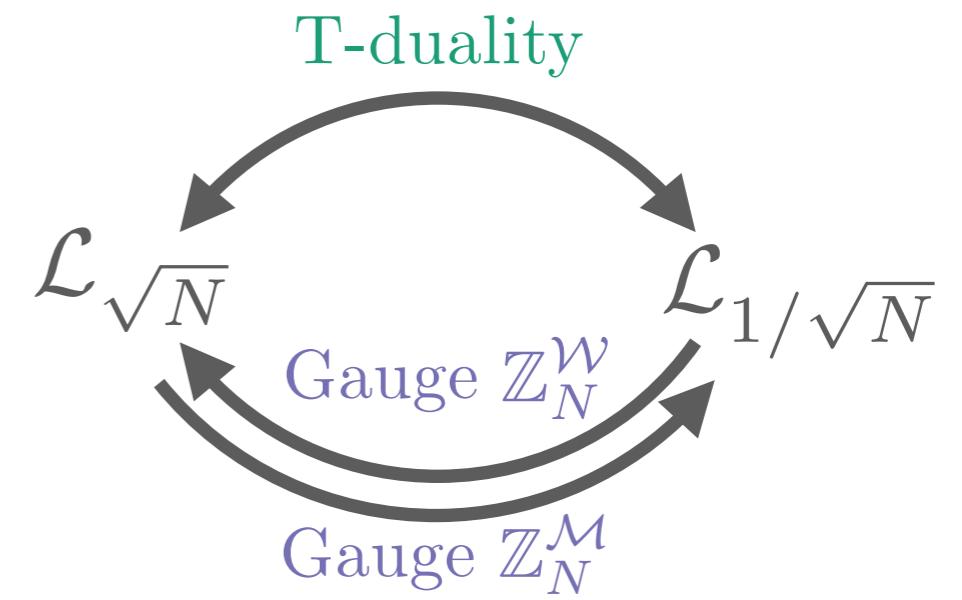
When $R \neq \sqrt{N}$, the image of \mathcal{L}_R under **T-duality** and **Gauging $\mathbb{Z}_N^{\mathcal{M}}$** is different.



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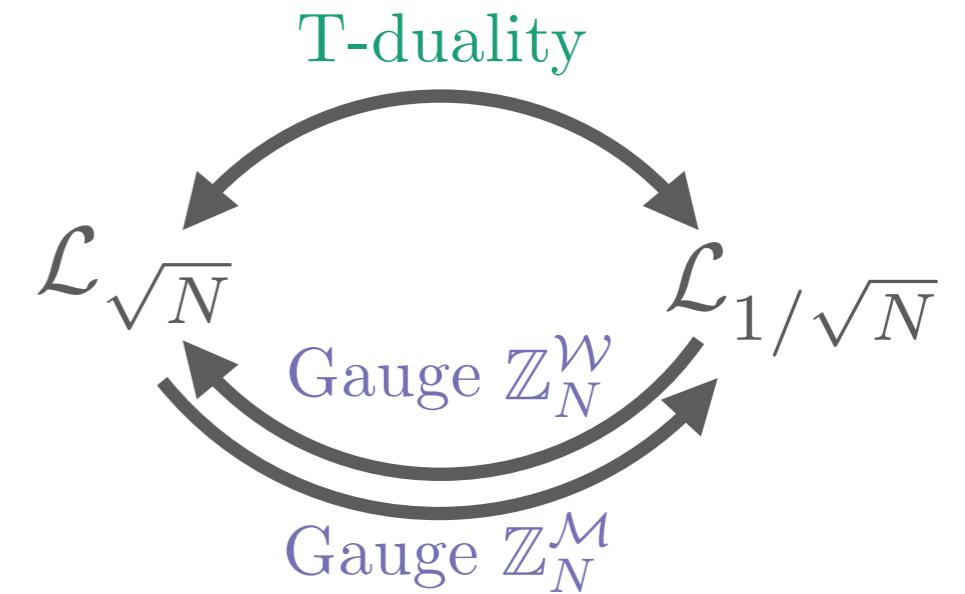
- **T-duality:** Isomorphism of $\mathcal{L}_{\sqrt{N}}$ and its $\mathbb{Z}_N^{\mathcal{M}}$ -gauged theory



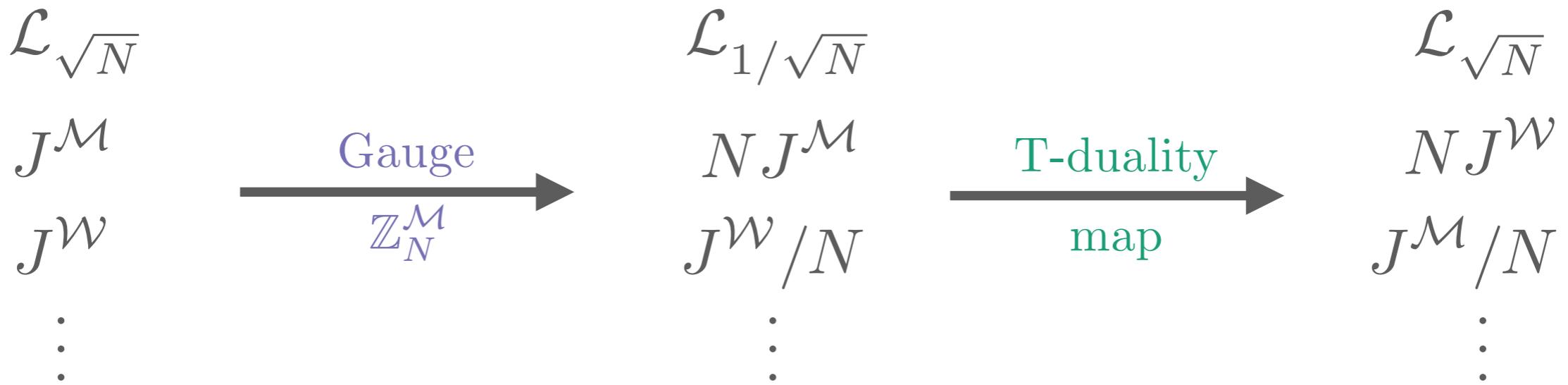
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- **Non-invertible symmetry** [Thorngren, Wang '21; Choi, Córdova, Hsin, Lam, Shao '21]



Non-invertible symmetry at $R = \sqrt{N}$

When $R = \sqrt{N}$, the image of \mathcal{L}_R

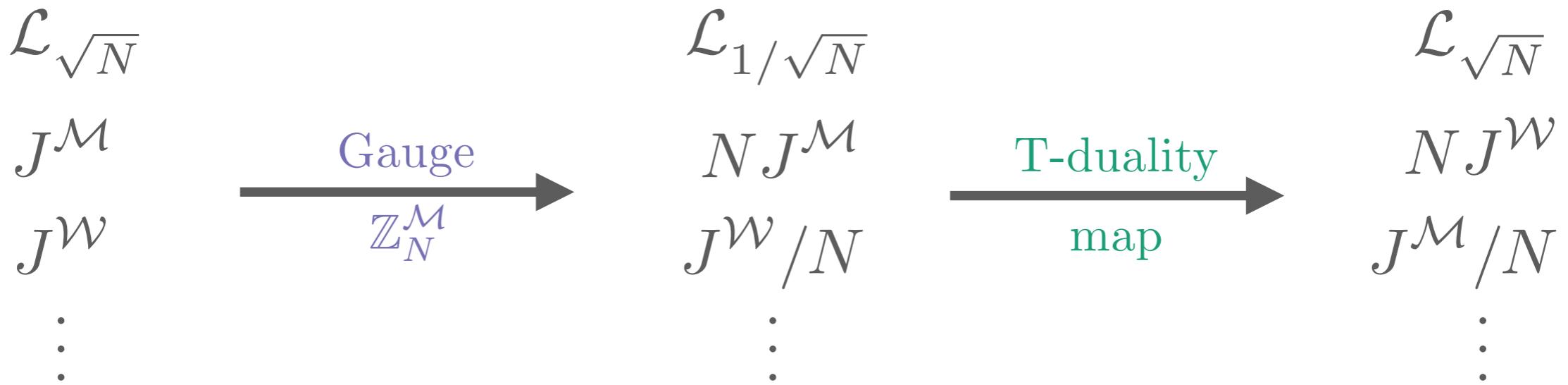
T-duality

- The existence of U(1) **momentum**, U(1) **winding**, & is t
this **non-invertible** symmetry provides an invariant
definition of T-duality.

and its $\mathbb{Z}_N^{\mathcal{M}}$ -gauged theory

Gauge $\mathbb{Z}_N^{\mathcal{M}}$

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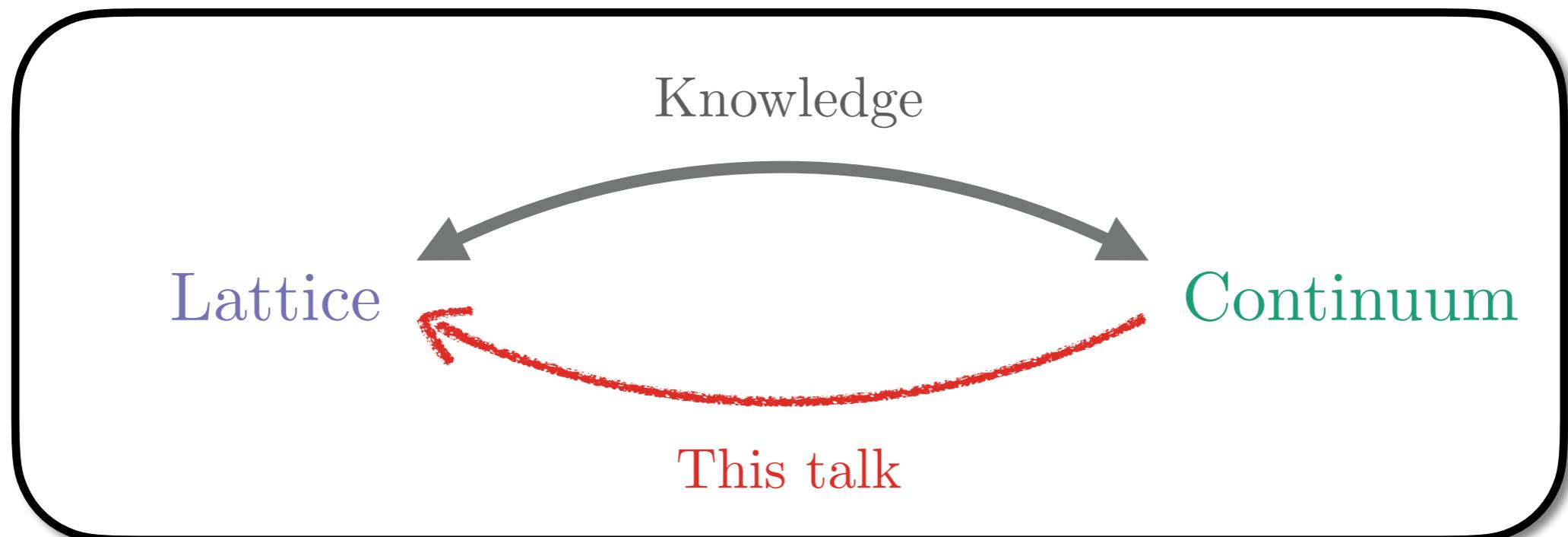


T-duality and qubits?

Can **T-duality** exist in lattice models that flow to the
compact boson in the IR? How about in **qubit** models?

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T-duality and qubits?

Can T-duality exist in lattice models that flow to the compact boson in the IR? How about in qubit models?

tl;dr

1. In the XX spin chain, there is a non-invertible symmetry and corresponding lattice T-duality
2. Encounter conserved charges forming the Onsager algebra. We'll discuss 't Hooft anomalies and prove a gaplessness constraint
3. Explore phase diagram from symmetric deformations of the XX model

The XX model

Consider 1 + 1D quantum lattice model on a finite ring with a **qubit** residing on each site j

- The number of sites L is even
- Pauli operators satisfy $X_{j+L} = X_j$ and $Z_{j+L} = Z_j$

XX model Hamiltonian [Lieb, Schultz, Mattis '61; Baxter '71; ...]

$$H_{\text{XX}} = \sum_{j=1}^L (X_j X_{j+1} + Y_j Y_{j+1})$$

- Spin rotation $\text{U}(1)^M$ symmetry

$$Q^M = \frac{1}{2} \sum_{j=1}^L Z_j$$

The XX model

$$H_{\text{XX}} = 2 \sum_{j=1}^L (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) \quad \sigma_j^\pm = \frac{1}{2} (X_j \pm iY_j)$$

► $e^{i\phi Q^M}$ transforms $\sigma_j^\pm \rightarrow e^{\pm i\phi} \sigma_j^\pm$

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► Spin rotation $U(1)^M$ symmetry

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IR limit of the XX model

The **IR** of the **XX** model is described by the **compact boson CFT** at $R = \sqrt{2}$ *

[Alcaraz, Barber, Batchelor '87; Baake, Christe, Rittenberg '88]

- The **IR limit**: focus on low-energy states within an $\mathcal{O}(L^0)$ energy window above the ground state and take $L \rightarrow \infty$

$$\begin{array}{ccc} \sigma_j^+ & & \exp[i\Phi] \\ Q^M & \xrightarrow{\text{IR limit}} & \mathcal{Q}^M = \int J_0^M \\ \vdots & & \vdots \end{array}$$

- Q^M generates a $U(1)$ **momentum symmetry** on the lattice

* *Reminder: in our convention, $R = 1$ is the self-dual radius*

Gauging \mathbb{Z}_2^M in the XX model

Does the XX model have a lattice T-duality?

- In the IR: implements an isomorphism between the $R = \sqrt{2}$ compact boson CFT and its \mathbb{Z}_2^M gauged theory

Let's gauge the \mathbb{Z}_2^M symmetry $e^{i\pi Q^M} = \prod_{j=1}^L (-1)^j Z_j$ in the XX model

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$$\begin{pmatrix} (-1)^j Z_j \\ X_j X_{j+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^M} \begin{pmatrix} Z_j Z_{j+1} \\ X_{j+1} \end{pmatrix}$$

- \mathbb{Z}_2^M gauged Hamiltonian

$$H_{\text{XX}/\mathbb{Z}_2^M} = \sum_{j=1}^L (X_j + Z_{j-1} X_j Z_{j+1})$$

Lattice T-duality

Hamiltonians are **unitarily equivalent**: $H_{\text{XX}} = U_{\text{T}} H_{\text{XX}/\mathbb{Z}_2^M} U_{\text{T}}^{-1}$

$$U_{\text{T}} = \prod_{n=1}^{L/2} \left(e^{i\frac{\pi}{4} Z_{2n+1}} e^{i\frac{\pi}{4} X_{2n+1}} e^{-i\frac{\pi}{4} X_{2n}} \text{CZ}_{2n,2n+1} \right)$$

- U_{T} implements an **isomorphism** between the XX model and its \mathbb{Z}_2^M gauged theory

T-duality from the **isomorphism** pov!

What about the **symmetry** pov?

Non-invertible symmetry of the XX model

Unitary equivalence $H_{\text{XX}} = U_T H_{\text{XX}/\mathbb{Z}_2^M} U_T^{-1}$ implies there is a non-invertible symmetry D

$$D H_{\text{XX}} = (U_T H_{\text{XX}/\mathbb{Z}_2^M} U_T^{-1}) D = H_{\text{XX}} D$$

► Non-invertible because $D e^{i\pi Q^M} = D$

$$e^{i\pi Q^M} |\psi\rangle = -|\psi\rangle \implies D|\psi\rangle = 0 \implies D^{-1} \text{ does not exist}$$

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► Symmetry transformation:*

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^M} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_T} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

* this is a non-invertible modulated symmetry

Lattice winding symmetry

What about the winding symmetry?

Lattice winding symmetry

What about the **winding symmetry**?

- Acting D on Q^M

$$DQ^M = 2Q^W D$$

where $Q^W = \frac{1}{4} \sum_{n=1}^{L/2} (X_{2n-1}Y_{2n} - Y_{2n}X_{2n+1})$

⇒ There is a lattice **winding charge***

- Acting D on Q^W

$$DQ^W = \frac{1}{2} Q^M D$$

* Known conserved charge of the XX model [Vernier, O'Brien, Fendley '18; Miao '21; Popkov, Zhang, Göhmann, Klümper '23]

Lattice winding symmetry

What about the winding symmetry?

The XX model has a lattice T-duality

- Isomorphism between H_{XX} and H_{XX}/\mathbb{Z}_2^M
- Conserved lattice Q^M and Q^W charges
- Non-invertible symmetry exchanging Q^M and Q^W

$$DQ^M = 2Q^W D \quad DQ^W = \frac{1}{2}Q^M D$$

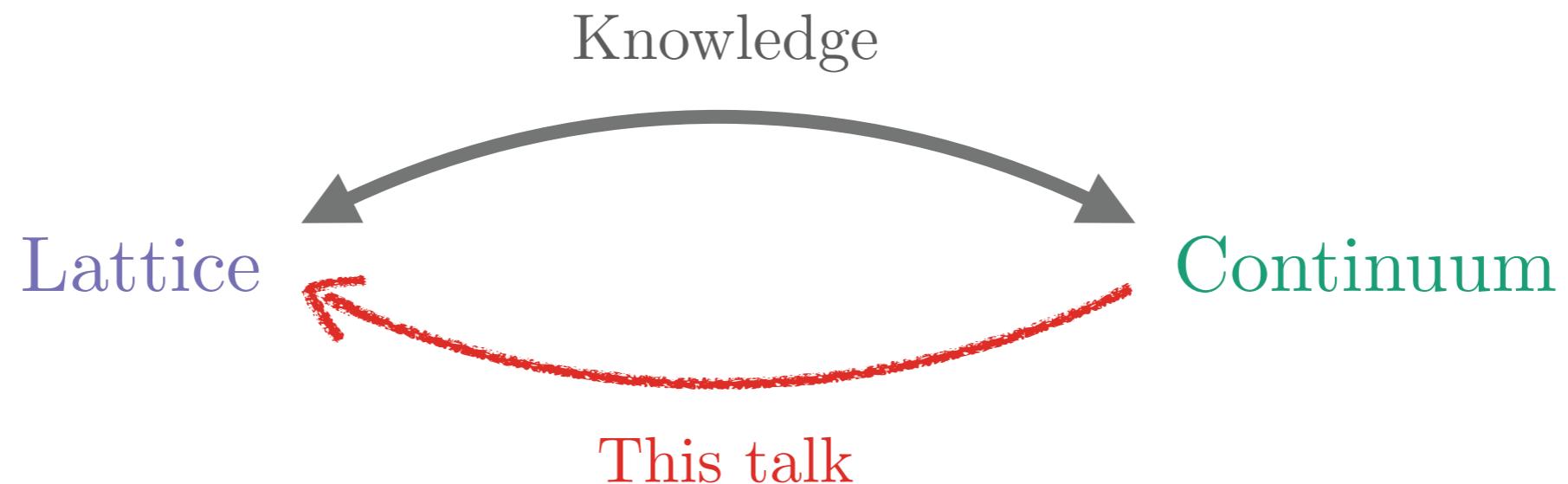
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Knowledge stream

Recall the **mutualistic relationship** between **lattice** and **continuum** models



Motivated by the **continuum**, we found new symmetries and lattice T-duality in the **XX** model

- What else can we learn about the **XX** model using our **continuum** knowledge?

Onsager algebra

The **charges** Q^M and Q^W do not commute on the lattice

- Generate the Onsager algebra ($Q_0 = Q^M$ and $Q_1 = 2Q^W$)

[Onsager '44; Vernier, O'Brien, Fendley '18; Miao '21]

$$[Q_n, Q_m] = iG_{m-n} \quad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

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$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

- In the continuum

$$[Q^M, Q^W] \neq 0 \xrightarrow{\text{IR limit}} [Q^M, Q^W] = 0$$

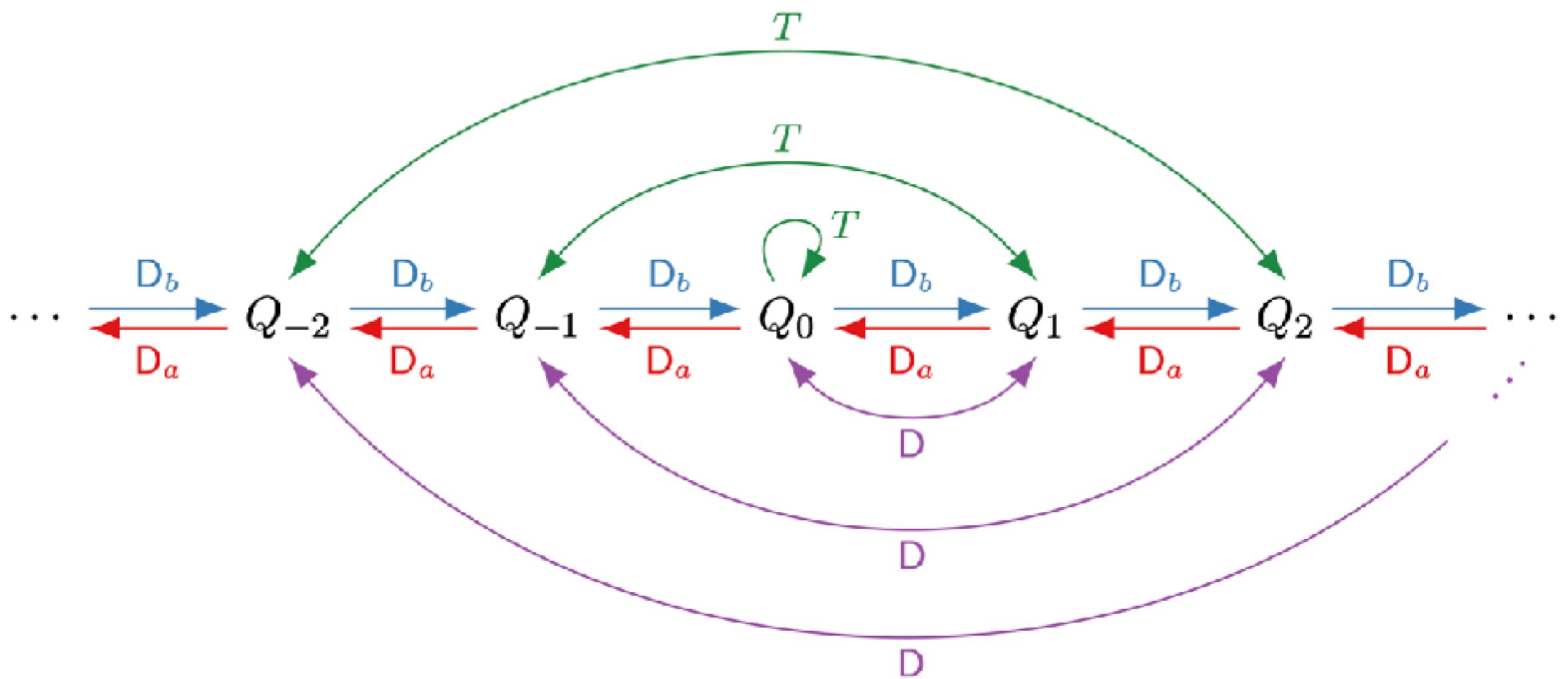
$$Q_n \xrightarrow{\text{IR limit}} \begin{cases} 2Q^W & n \text{ odd} \\ Q^M & n \text{ even} \end{cases}$$

$$G_n \xrightarrow{\text{IR limit}} 0$$

An ever richer algebraic structure

The Onsager charges have a rich interplay with other conserved operators of the XX model

- Let $D_a = e^{i\frac{\pi}{2}Q^M} D$ and $D_b = e^{i\pi Q^W} D$



Anomalies

How do the symmetries we found in the XX model realize the
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How do the symmetries we found in the **XX** model realize the '**t Hooft anomalies** in the **IR**?

Consider the symmetry operators*

$$e^{i\pi Q^M} = \prod_{j=1}^L (-1)^j Z_j \quad e^{i\theta Q^W} \quad C = \prod_{j=1}^L X_j$$

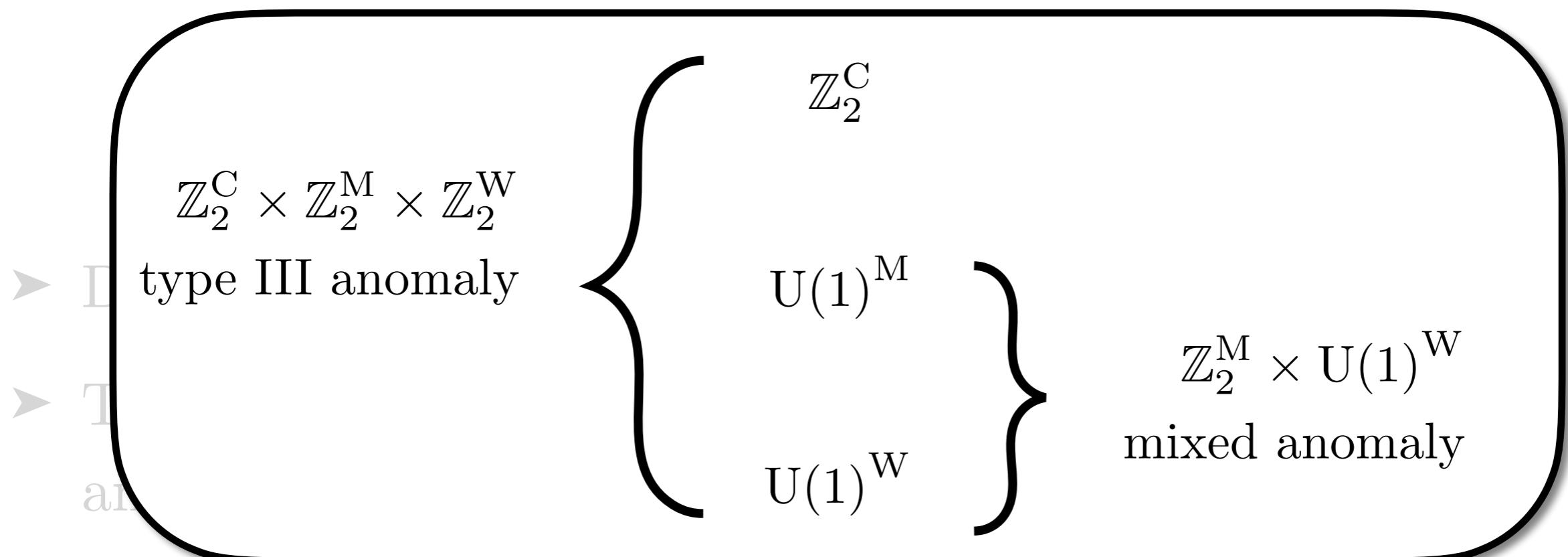
- Described by the group $\mathbb{Z}_2^M \times U(1)^W \rtimes \mathbb{Z}_2^C$
- These symmetries exist in the **UV** and **IR**, so their anomalies in the **CFT** must be realized on the **lattice**

* Can show these unitaries are non-onsiteable

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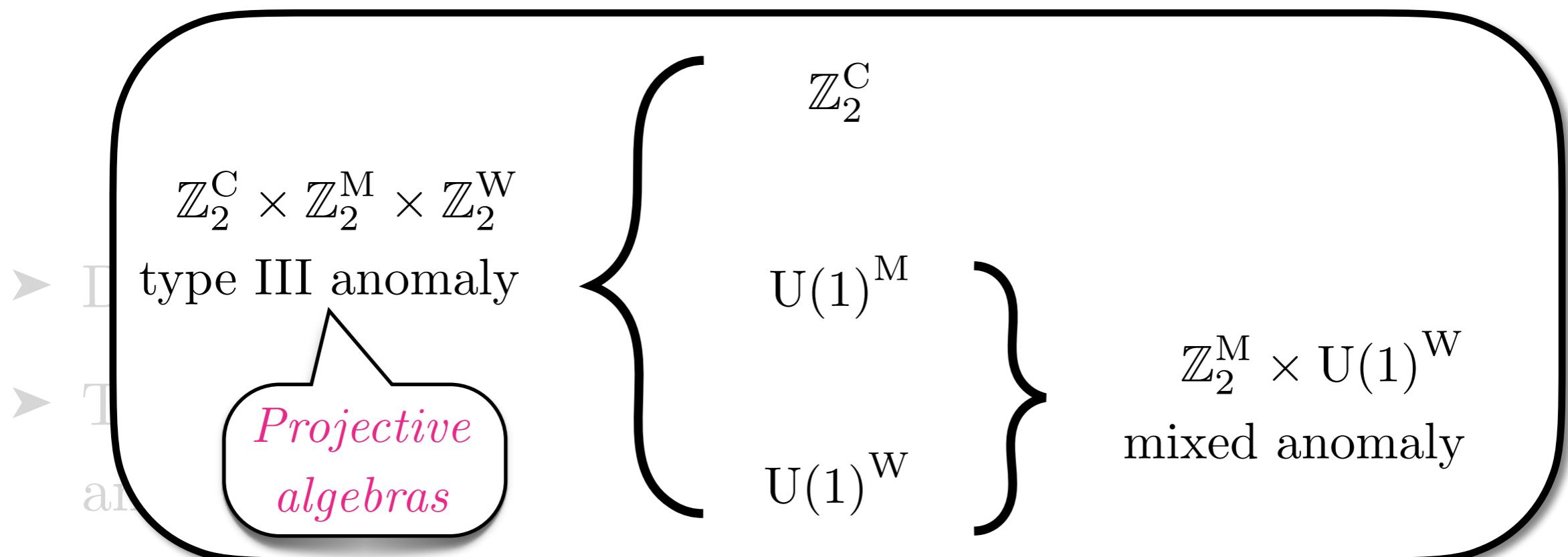


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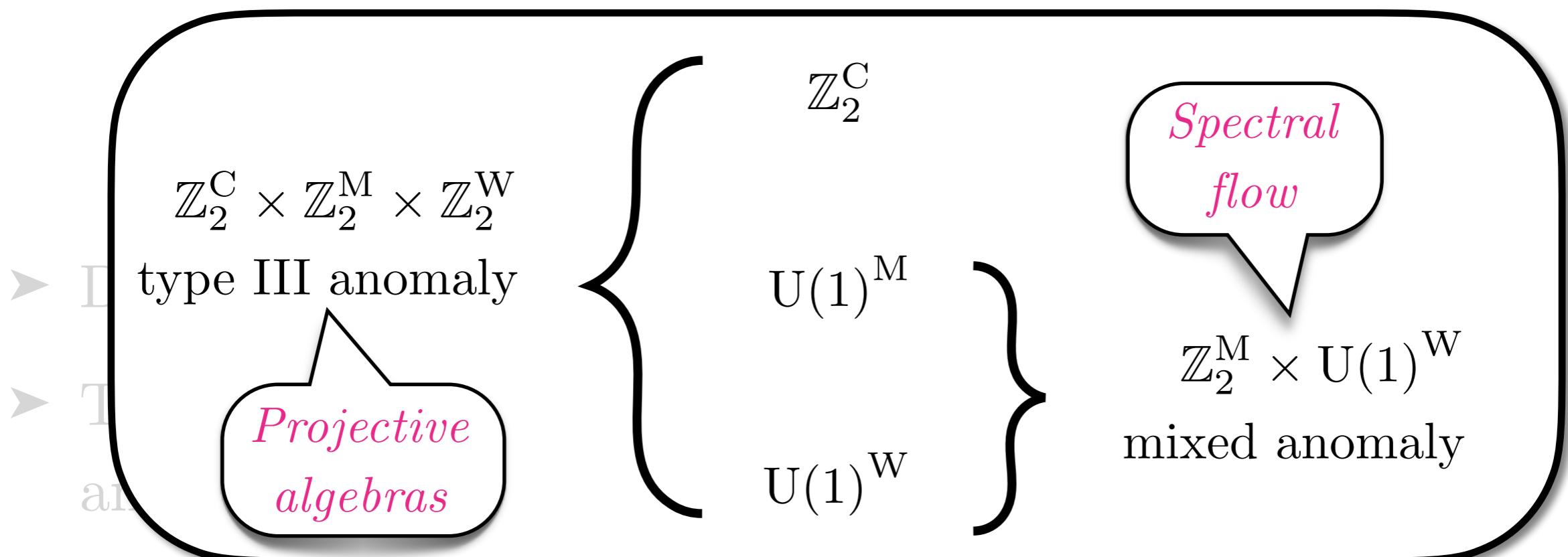


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Perturbative anomalies in the IR

The **mixed anomaly** of $U(1)^{\mathcal{M}} \times U(1)^{\mathcal{W}}$ in the **compact boson CFT** is a “perturbative anomaly”

- Cannot be matched by gapped phases \implies enforces **gaplessness** [… ; Córdova, Freed, Teleman '24]

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Answer: Yes! Can show by fermionizing the **XX model**

Fermionizing the XX model

We **fermionize** the XX model by gauging the \mathbb{Z}_2^M symmetry using complex fermion operators c_j and c_j^\dagger

[...; Radičević '18; Borla, Verresen, Shah, Moroz '20; Seiberg, Shao '23; Aksoy, Mudry, Furusaki, Tiwari '23]

- In terms of **real fermions** $c_j = (a_j + i b_j)/2$

$$\{a_j, b_{j'}\} = 0 \quad \{a_j, a_{j'}\} = 2\delta_{j,j'} \quad \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

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Gauging implemented using the **Gauss law**

$$G_j = (-1)^j Z_j i a_j b_j = 1$$

- Map to gauged theory summarized by

$$Z_j \rightarrow i a_j b_j$$

$$X_j X_{j+1} \rightarrow \begin{cases} -i a_j a_{j+1} & j \text{ odd} \\ -i b_j b_{j+1} & j \text{ even} \end{cases}$$

Fermionizing the XX model

We fermionize the XX model using complex fermions

[...; Radičević '18] by using complex fermions

$$H_{\text{XX}} \xrightarrow{\text{Fermionize}} -i \sum_{j=1}^L (a_j a_{j+1} + b_j b_{j+1})$$

► In terms of fermions $\{a_j, b_j\}$

$$Q^M \xrightarrow{\text{Fermionize}} \frac{1}{2} \sum_{j=1}^L i a_j b_j \equiv Q^V$$

Gauging the fermion charge $2\delta_{j,j'}$

$$2Q^W \xrightarrow{\text{Fermionize}} \frac{1}{2} \sum_{j=1}^L i a_j b_{j+1} \equiv Q^A$$

► Map the XX model to a free fermion theory

To show gaplessness, let's consider the most general symmetric Hamiltonian

$Z_j = (-1)^{a_j a_{j+1}}$ if j even

Symmetric Q^V and Q^A Hamiltonians

We assume the Hamiltonian is local:

$$H_f = \sum_n \sum_{j=1}^L g_{j,n} H_j^{(n)}$$

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We assume the **Hamiltonian** is local:

$$H_f = \sum_n \sum_{j=1}^L g_{j,n} H_j^{(n)}$$

1. $e^{-i\frac{\pi}{2}Q^A} e^{i\frac{\pi}{2}Q^V} : (a_j, b_j) \rightarrow (a_{j-1}, b_{j+1})$ invariance requires $H_j^{(n)}$ to not have terms **mixing** a_j and b_j and $g_{j,n} = g_n$

Symmetric Q^V and Q^A Hamiltonians

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2. Under the $e^{i\phi Q^V}$ transformation

$$a_j \rightarrow \cos(\phi) a_j + \sin(\phi) b_j \quad b_j \rightarrow \cos(\phi) b_j - \sin(\phi) a_j$$

\implies Symmetric $H_j^{(n)}$ are quadratic

$$H_j^{(n)} = i a_j a_{j+n} + i b_j b_{j+n}$$

Enforced gaplessness

$$H_f = i \sum_n \sum_{j=1}^L g_n (a_j a_{j+n} + b_j b_{j+n})$$

The Q^V and Q^A symmetric Hamiltonians are always **gapless**

- In momentum space:

$$H_f = \sum_{k \in \text{BZ}} \omega_k c_k^\dagger c_k, \quad \omega_k = 4 \sum_n g_n \sin(2\pi k n / L)$$

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Bosonization: one-to-one correspondence between H_f and qubit Hamiltonians commuting with Q^M and Q^W

- Because H_f is gapless, Q^M and Q^W enforce **gaplessness**

Enforced gaplessness

$$H_f = i \sum_n \sum_{j=1}^L g_n (a_j a_{j+n} + b_j b_{j+n})$$

- The perturbative anomaly of the compact boson CFT
is matched by the Onsager algebra
- Exciting connections between Onsager algebras and anomalies [Chatterjee, SP, Shao '24; Gioia, Thorngren '25; SP, Kim, Chatterjee, Shao '25]

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Enforced gaplessness

$$H_f = i \sum_n \sum_{j=1}^L g_n (a_j a_{j+n} + b_j b_{j+n})$$

- The Hamiltonian H_f is gapless.
- When $L = 0 \bmod 4$, there is a **unitary frame** in which
- $$Q^M = -\frac{1}{2} \sum_{j=1}^L Z_j \quad Q^W = -\frac{1}{4} \sum_{j=1}^L X_j X_{j+1}$$
- Any **qubit chain** commuting with $\sum_j Z_j$ and $\sum_j X_j X_{j+1}$ is **gapless**
 - Because H_f is gapless, Q^M and Q^W enforce **gaplessness**

Symmetric deformations

Can find $U(1)^M$ and $U(1)^W$ **symmetric deformations** of the XX model by bosonizing H_f

$$H_j^{(1)} \xrightarrow{\text{bosonize}} X_j X_{j+1} + Y_j Y_{j+1}$$

$$H_j^{(2)} \xrightarrow{\text{bosonize}} Y_j Z_{j+1} X_{j+2} - X_j Z_{j+1} Y_{j+2}$$

$$H_j^{(3)} \xrightarrow{\text{bosonize}} X_j Z_{j+1} Z_{j+2} X_{j+3} + Y_j Z_{j+1} Z_{j+2} Y_{j+3}$$

⋮

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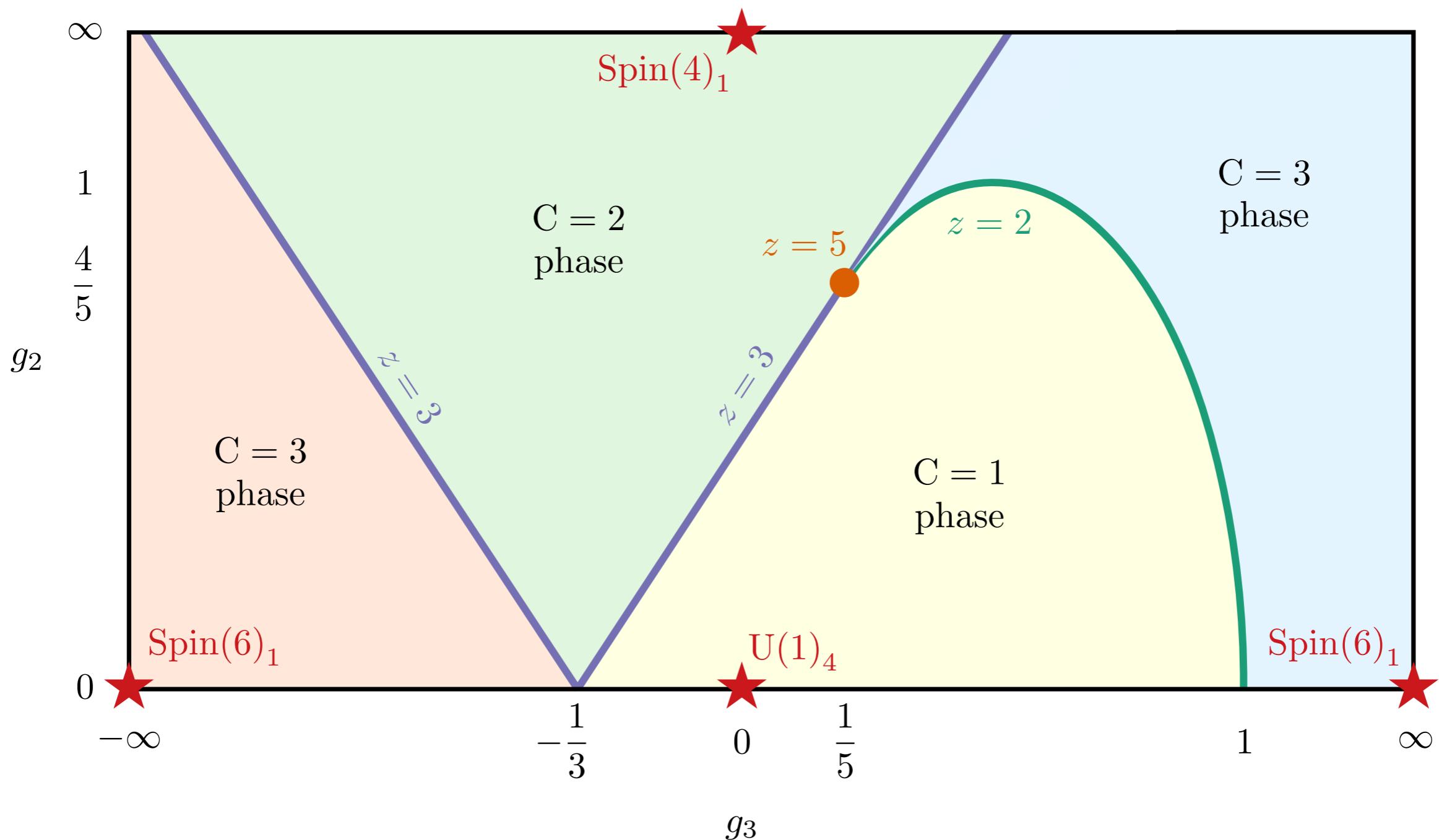
⋮

Non-invertible symmetry D arises from $e^{-i\frac{\pi}{2}Q^V} T_a$

- $U(1)^M$ and $U(1)^W$ guarantee the **non-invertible symmetry** and a lattice **T-duality**

Simplest 2-parameter phase diagram

$$H(g_2, g_3) = H_{\text{XX}} + \sum_{j=1}^L \left(g_2 H_j^{(2)} + g_3 H_j^{(3)} \right)$$



Recap and outlook

Many aspects of the **compact boson CFT** surprisingly exist exactly in the **XX** model

1. Lattice **T-duality** and non-invertible symmetry
2. Lattice **winding symmetry** and 't Hooft anomalies
3. Symmetric deformations of the **XX** model

Recap and outlook

Many aspects of the **compact boson CFT** surprisingly exist exactly in the **XX** model

1. Lattice **T-duality** and non-invertible symmetry
2. Lattice **winding symmetry** and 't Hooft anomalies
3. Symmetric deformations of the **XX** model

Tip of an iceberg?

1. T-duality for other radii? S-duality in 3 + 1D qubit models?
2. General relationship between perturbative **anomalies** and algebras? Between **exact dualities** of QFTs and unitary transformations in quantum lattice models?

Back-up slides

Lattice T-duality

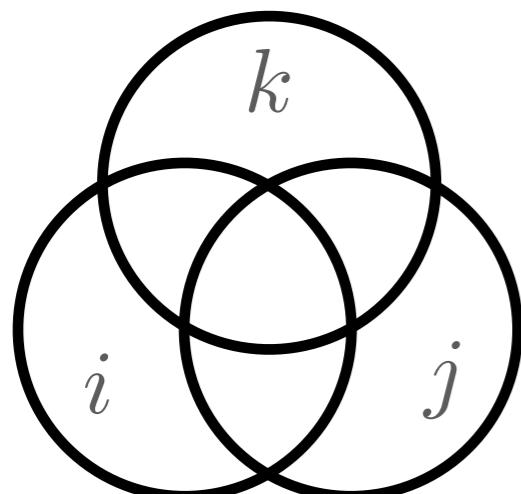
Can T-duality exist in lattice models that flow to the compact boson in the IR?

Yes: exists in the Modified Villain model

[Gross, Klebanov '90; Gorantla, Lam, Seiberg, Shao '21; Cheng, Seiberg '22; Fazza, Sulejmanpasic '22]

- Careful lattice regularization of the compact boson CFT

$$\text{Patches } \{U_i\} \quad \Phi_i: U_i \rightarrow \mathbb{R} \quad n_{ij}: U_i \cap U_j \rightarrow \mathbb{Z}$$



- $\Phi_i - \Phi_j = 2\pi n_{ij}$ on $U_i \cap U_j$
- Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

Lattice T-duality

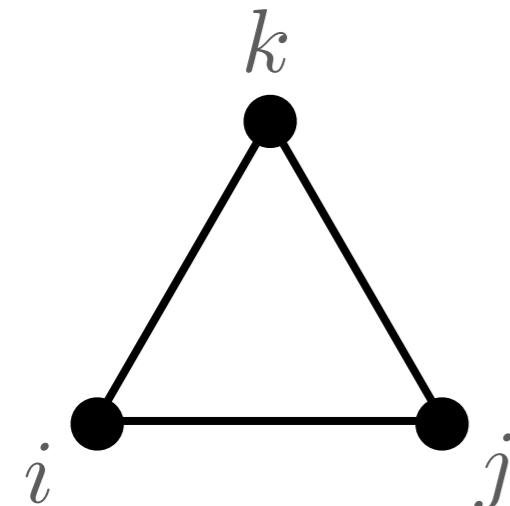
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- Careful lattice regularization of the compact boson CFT

Spacetime lattice $\Phi_i \in \mathbb{R}$ $n_{ij} \in \mathbb{Z}$



- Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

- Infinite-dimensional local Hilbert space

Non-invertible symmetry algebra

The **non-invertible symmetry** operator D

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^M} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_T} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

- Implies operator algebra

$$D^2 = (1 + e^{i\pi Q^M}) T e^{-i\frac{\pi}{2}Q^M}, \quad D e^{i\pi Q^M} = e^{i\pi Q^M} D = D,$$

$$TDT^{-1} = e^{i\frac{\pi}{2}Q^M} e^{i\pi Q^W} D, \quad D^\dagger = DT^{-1} e^{i\frac{\pi}{2}Q^M}$$

D as an matrix product operator

$$D = \text{Tr} \left(\prod_{j=1}^L D^{(j)} \right) \equiv \boxed{\begin{array}{c} | \\ \boxed{D^{(1)}} \\ | \\ \hline | \\ \boxed{D^{(2)}} \\ | \\ \hline | \\ \dots \\ | \\ \boxed{D^{(L)}} \\ | \\ \hline \end{array}}$$

where the MPO tensor

$$D^{(j)} \equiv \boxed{D^{(j)}} = \begin{cases} \frac{1}{\sqrt{8}} \begin{pmatrix} 1 - Z_j + X_j + iY_j & 1 + Z_j + X_j - iY_j \\ -1 - Z_j + X_j - iY_j & 1 - Z_j - X_j - iY_j \end{pmatrix} & j \text{ odd,} \\ \frac{i}{\sqrt{8}} \begin{pmatrix} 1 + Z_j - iX_j - Y_j & -1 + Z_j - iX_j + Y_j \\ 1 - Z_j - iX_j + Y_j & 1 + Z_j + iX_j + Y_j \end{pmatrix} & j \text{ even.} \end{cases}$$

Emergence of $\text{TY}(\mathbb{Z}_2, +)$

The XX model has a continuous family of **non-invertible symmetries**

$$D_{\phi,\theta} = e^{i\phi Q^M} e^{i\theta Q^W} D$$

► $(D_{\phi,\theta})^2 = (1 + e^{i\pi Q^M}) e^{i\phi Q^M} e^{i(2\phi+\theta)Q^W} e^{\frac{i}{2}(\theta-\pi)Q^M} T$

The $R = \sqrt{2}$ compact boson CFT has an S^1 -family of $\text{TY}(\mathbb{Z}_2, +)$ **symmetry** operators \mathcal{D}_φ [Thorngren, Wang '21]

In the IR, $T \xrightarrow{\text{IR limit}} e^{i\pi(Q^M + Q^W)}$ [Metlitski, Thorngren '17; Cheng, Seiberg '22]

$$D_{\phi,\pi-2\phi} \xrightarrow{\text{IR limit}} \mathcal{D}_\phi$$

Details on lattice winding symmetry

Lattice winding charge

$$Q^W = \frac{1}{4} \sum_{n=1}^{L/2} (X_{2n-1}Y_{2n} - Y_{2n}X_{2n+1})$$

- Unitarily equivalent to $U_T^{-1} Q^W U_T = \frac{1}{4} \sum_{j=1}^L (-1)^j Z_j Z_{j+1}$
- Symmetry transformation

$$e^{i\theta Q^W} X_j e^{-i\theta Q^W} = \begin{cases} X_j & j \text{ odd} , \\ X_j e^{-\frac{i\theta}{2} Y_j (X_{j-1} - X_{j+1})} & j \text{ even} , \end{cases}$$

$$e^{i\theta Q^W} Y_j e^{-i\theta Q^W} = \begin{cases} Y_j e^{\frac{i\theta}{2} X_j (Y_{j-1} - Y_{j+1})} & j \text{ odd} \\ Y_j & j \text{ even} \end{cases}$$

Expressions of the Onsager charges 1

- The Onsager algebra. Formed by **conserved charges** $\{Q_n, G_n\}$

$$[Q_n, Q_m] = iG_{m-n} \quad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

The Onsager charges Q_n in terms of Q^M and Q^W are

$$Q_n = \begin{cases} 2S_n Q^W S_n^{-1} & n \text{ odd} \\ S_n Q^M S_n^{-1} & n \text{ even} \end{cases}$$

- Where $S_0 = S_1 = 1$, $S_2 = e^{i\pi Q^W}$, $S_3 = e^{i\pi Q^W} e^{i\frac{\pi}{2} Q^M}$, ...
- S are the pivots of Onsager algebra [Jones, Prakash, Fendley '24]

Expressions of the Onsager charges 2

$$Q_n = \begin{cases} \frac{1}{2} \sum_{j=1}^L Z_j & n = 0, \\ \frac{(-1)^{\frac{n+2}{2}}}{2} \sum_{j=1}^{L/2} \left(X_{2j-1} \prod_{k=2j}^{2j+n-2} Z_k X_{2j+n-1} + Y_{2j} \prod_{k=2j+1}^{2j+n-1} Z_k Y_{2j+n} \right) & n > 0 \text{ even}, \\ \frac{(-1)^{\frac{n-1}{2}}}{2} \sum_{j=1}^{L/2} \left(X_{2j-1} \prod_{k=2j}^{2j+n-2} Z_k Y_{2j+n-1} - Y_{2j} \prod_{k=2j+1}^{2j+n-1} Z_k X_{2j+n} \right) & n > 0 \text{ odd}, \\ \frac{(-1)^{\frac{n-2}{2}}}{2} \sum_{j=1}^{L/2} \left(Y_{2j+n-1} \prod_{k=2j+n}^{2j-2} Z_k Y_{2j-1} + X_{2j+n} \prod_{k=2j+n+1}^{2j-1} Z_k X_{2j} \right) & n < 0 \text{ even}, \\ \frac{(-1)^{\frac{n+1}{2}}}{2} \sum_{j=1}^{L/2} \left(X_{2j+n-1} \prod_{k=2j+n}^{2j-2} Z_k Y_{2j-1} - Y_{2j+n} \prod_{k=2j+n+1}^{2j-1} Z_k X_{2j} \right) & n < 0 \text{ odd}, \end{cases}$$

$$G_n = \begin{cases} \text{sign}(n) \frac{(-1)^{\frac{n}{2}}}{2} \sum_{j=1}^{L/2} (-1)^j (X_j Y_{j+n} + Y_j X_{j+n}) \prod_{k=j+1}^{j+n-1} Z_k & n \text{ even}, \\ \text{sign}(n) \frac{(-1)^{\frac{n-1}{2}}}{2} \sum_{j=1}^{L/2} (-1)^j (X_j X_{j+n} - Y_j Y_{j+n}) \prod_{k=j+1}^{j+n-1} Z_k & n \text{ odd}. \end{cases}$$

Fermionizing by gauging

Gauss law

$$G_j = (-1)^j Z_j \ i a_{j,j+1} b_{j,j+1}$$

Unitary transformation

$$Z_j \rightarrow Z_j \ i a_{j,j+1} b_{j,j+1},$$

$$X_j \rightarrow \begin{cases} X_j & j \text{ odd}, \\ X_j \ i a_{j,j+1} b_{j,j+1} & j \text{ even}. \end{cases}$$

$$a_{j,j+1} \rightarrow \begin{cases} X_j \ a_{j,j+1} & j \text{ odd}, \\ Y_j \ a_{j,j+1} & j \text{ even}, \end{cases}$$

$$b_{j,j+1} \rightarrow \begin{cases} -X_j \ b_{j,j+1} & j \text{ odd}, \\ Y_j \ b_{j,j+1} & j \text{ even}. \end{cases}$$

Qubits now polarized $Z_j = 1$

Bosonizing by gauging

Gauss law

$$G_j = \begin{cases} X_{j-1,j} (\mathrm{i} a_j b_{j+1}) Y_{j,j+1} & j \text{ odd}, \\ -Y_{j-1,j} (\mathrm{i} a_j b_{j+1}) X_{j,j+1} & j \text{ even}. \end{cases}$$

Unitary transformation

$$a_j \rightarrow \begin{cases} -X_{j-1,j} a_j & j \text{ odd}, \\ Y_{j-1,j} a_j & j \text{ even}, \end{cases}$$

$$b_j \rightarrow \begin{cases} -X_{j-1,j} b_j & j \text{ odd}, \\ -Y_{j-1,j} b_j & j \text{ even}, \end{cases}$$

$$X_{j-1,j} \rightarrow \begin{cases} X_{j-1,j} & j \text{ odd}, \\ X_{j-1,j} (\mathrm{i} a_j b_j) & j \text{ even}, \end{cases}$$

$$Z_{j-1,j} \rightarrow (-1)^{j-1} Z_{j-1,j} (\mathrm{i} a_j b_j).$$

Fermions now polarized $\mathrm{i} a_j b_j = 1$