

An SPT-LSM theorem for weak SPTs with non-invertible symmetry

Salvatore Pace

MIT

UCSD Seminar



SIMONS
FOUNDATION





Ho Tat Lam



Ömer Aksoy

arXiv:2409.18113 [SciPost]

Quantum phases and symmetry.....

A fundamental problem in QFT/CMT/HEP is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

Quantum phases and symmetry.....

A fundamental problem in QFT/CMT/HEP is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

Sometimes, phases are characterized by a symmetry

- Superfluids by $U(1)$ boson number conservation
- Topological insulators by $U(1)_f$ and \mathbb{Z}_2^T symmetries

For such phases, symmetries provide answers to questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in generalizing the notion of symmetries

- Ordinary symmetries transform local operators in an invertible manner (e.g., $c_r^\dagger \rightarrow e^{i\theta} c_r^\dagger$)
- So-called generalized symmetries modify this definition

Generalized symmetries

There has been a recent flurry of interest in **generalizing** the notion of **symmetries**

- **Ordinary** symmetries transform **local** operators in an **invertible** manner (e.g., $c_r^\dagger \rightarrow e^{i\theta} c_r^\dagger$)
- So-called **generalized symmetries** modify this definition

Non-invertible symmetries have non-invertible transformations

[Bhardwaj, Tachikawa '17; Chang, Lin, Shao, Wang, Yin '18; ...]

- Can arise at **critical points** from Kramers-Wannier dualities

[Thorngren, Yang '21 ; Choi, Córdova, Hsin, Lam, Shao '21; ...]

- Can emerge in **ordered phases** (are symmetries of nonlinear sigma models) [Hsin '22; **SP** '23; **SP**, Zhu, Beaudry, X-G Wen '23]

Generalized symmetries

Th
no
Q: Why should we consider these as symmetries?



No



e

ns

Generalized symmetries

Q: Why should we consider these as **symmetries**?

A: They pass the **duck test**!



If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- Have conservation laws
- Can constrain phase diagrams (be anomalous)
- Can characterize **SSB** and **SPT** phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by
generalized symmetries?

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$a_1 - a_2 - a_3 - \dots$ Symmetry

(2) Specify SSB and SPT pattern

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$a_1 - a_2 - a_3 - \dots$ Symmetry

(2) Specify SSB and SPT pattern

Ordered phases

Topological insulators

Topological order

Maxwell phases

Higgs phases

Fracton phases

Phases we have yet to name!

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Why care?

1. Provides a novel and unifying perspective of quantum phases
2. Guides us towards new quantum phases and models
3. Further develops a classification of quantum phases based on symmetries (a “generalized Landau paradigm”) [McGreevy '22]

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Why care?

1. Provides a novel and unifying perspective of quantum phases
2. Guides us towards new quantum phases and models
3. Further develops a classification of quantum phases based on symmetries (a “generalized Landau paradigm”) [McGreevy '22]

TL;DR for this talk

This talk: 1 + 1D SPT phases characterized by translation and non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries on the lattice

TL;DR for this talk

This talk: 1 + 1D SPT phases characterized by translation and non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries on the lattice

Outline

1. Review SPTs from a symmetry defect perspective
2. Simple example of an entangled weak SPT characterized by a projective non-invertible symmetry
3. General discussion on projective $Z(G) \times \text{Rep}(G)$ symmetry and SPT-LSM theorems

What are SPTs

An **SPT phase** is a gapped quantum phase protected by a **symmetry** with a **unique ground state** on all closed spatial manifolds [Chen, Gu, Liu, Wen '11; ...]

- Interesting physics can arise on **boundaries** and **interfaces** between **SPTs** (e.g., topological order, gapless excitations)

What are SPTs

An **SPT phase** is a gapped quantum phase protected by a **symmetry** with a **unique ground state** on all closed spatial manifolds [Chen, Gu, Liu, Wen '11; ...]

- Interesting physics can arise on **boundaries** and **interfaces** between **SPTs** (e.g., topological order, gapless excitations)

SPTs are characterized by their bulk **response to static probes**

- Background gauge fields and **symmetry defects**

What are SPTs

An **SPT phase** is a gapped quantum phase protected by a **symmetry** with a **unique ground state** on all closed spatial manifolds [Chen, Gu, Liu, Wen '11; ...]

- Interesting physics can arise on **boundaries** and **interfaces** between **SPTs** (e.g., topological order, gapless excitations)

SPTs are characterized by their bulk **response to static probes**

- Background gauge fields and **symmetry defects**

Ordinary insulator

$$S_0[A_\mu] = \frac{1}{2} \int (F_{\mu\nu})^2 d^3x dt$$

Topological insulator

[Qi, Hughes, Zhang '08; ...]

$$S_\pi[A_\mu] = S_0[A_\mu] + \frac{\pi}{4\pi} \int F \wedge F$$

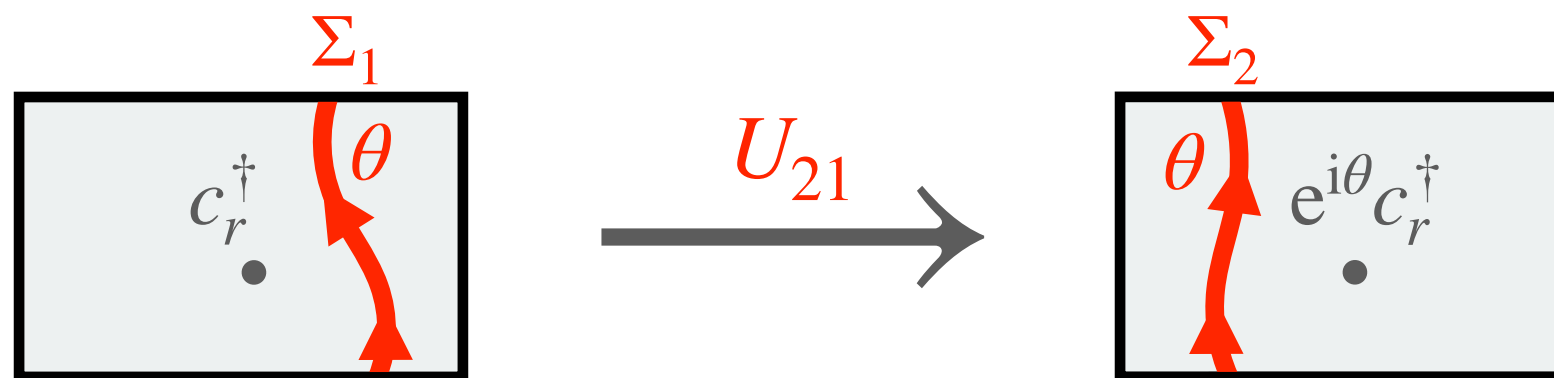
Symmetry defects

Symmetry defects are localized modifications to the Hamiltonian $H_{\text{defect}}^{(\Sigma)} = H + \delta H(\Sigma)$ and other operators

- Moved using **unitary operators** (are **topological** defects)

$$H_{\text{defect}}^{(\Sigma_2)} = U_{21} H_{\text{defect}}^{(\Sigma_1)} U_{21}^\dagger$$

- Implement the **symmetry** transformation across space



- **Twisted** boundary conditions $(T_\perp)^L = \text{Symmetry operator}$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d closed chain in space with **two qubits** on each site $j \sim j + L$
acted on by **Pauli operators** X_j, Z_j and \tilde{X}_j, \tilde{Z}_j .

$$\begin{array}{l|l} H_p = - \sum_{j=1}^L (X_j + \tilde{X}_j) & H_c = - \sum_{j=1}^L (\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1}) \\ \hline |\text{GS}_p\rangle = |++ \cdots +\rangle & |\text{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\text{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\text{GS}_c\rangle \end{array}$$

- Both models have a **unique symmetric gapped ground state**
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ **symmetry** $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$
with $U|\text{GS}_\bullet\rangle = \tilde{U}|\text{GS}_\bullet\rangle = |\text{GS}_\bullet\rangle$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d closed chain in space with **two qubits** on each site $j \sim j + L$
acted on by **Pauli operators** X_j, Z_j and \tilde{X}_j, \tilde{Z}_j .

$$H_p = - \sum_{j=1}^L (X_j + \tilde{X}_j) \quad \Bigg| \quad H_c = - \sum_{j=1}^L (\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1})$$

$$|\text{GS}_p\rangle = |++\cdots+\rangle \quad \Bigg| \quad |\text{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\text{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\text{GS}_c\rangle$$

➤ Both models have a **unique symmetric gapped ground state**

H_p and H_c are both in a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phase

➤ There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ **symmetry** $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$
with $U|\text{GS}_\bullet\rangle = \tilde{U}|\text{GS}_\bullet\rangle = |\text{GS}_\bullet\rangle$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a U symmetry defect at $\langle L, 1 \rangle$

► Gives rise to U -twisted boundary conditions: $Z_{j+L} = -Z_j$

1. H_p is unaffected, so its ground state still satisfies

$$U |\text{GS}_{p;U}\rangle = + |\text{GS}_{p;U}\rangle \qquad \tilde{U} |\text{GS}_{p;U}\rangle = + |\text{GS}_{p;U}\rangle$$

2. H_c becomes $H_c + 2Z_L \tilde{X}_L Z_1$, and its ground state satisfies

$$U |\text{GS}_{c;U}\rangle = + |\text{GS}_{c;U}\rangle \qquad \tilde{U} |\text{GS}_{c;U}\rangle = - |\text{GS}_{c;U}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Different **responses** imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

[Chen, Lu, Vishwanath '13; Gaiotto, Johnson-Freyd '17; Wang, Ning, Cheng '21]

Low-energy EFTs of H_p and H_c

$$Z_p[A, \tilde{A}] = 1 \qquad Z_c[A, \tilde{A}] = (-1)^{\int A \cup \tilde{A}}$$

1. H_p is unaffected, so its ground state still satisfies

$$U |\text{GS}_{p;U}\rangle = + |\text{GS}_{p;U}\rangle \qquad \tilde{U} |\text{GS}_{p;U}\rangle = + |\text{GS}_{p;U}\rangle$$

2. H_c becomes $H_c + 2Z_L \tilde{X}_L Z_1$, and its ground state satisfies

$$U |\text{GS}_{c;U}\rangle = + |\text{GS}_{c;U}\rangle \qquad \tilde{U} |\text{GS}_{c;U}\rangle = - |\text{GS}_{c;U}\rangle$$

Example: \mathbb{Z}_2 weak SPTs

1d periodic lattice with a **qubit** on each site $j \sim j + L$

$$H_+ = - \sum_j X_j \quad \text{vs.} \quad H_- = + \sum_j X_j$$

- Both have a unique gapped ground state $|\text{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- **Symmetries:** $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \rightarrow j + 1$

H_+ and H_- are both in $\mathbb{Z}_2 \times \mathbb{Z}_L$ SPT phases

Example: \mathbb{Z}_2 weak SPTs

1d periodic lattice with a **qubit** on each site $j \sim j + L$

$$H_+ = - \sum_j X_j \quad \text{vs.} \quad H_- = + \sum_j X_j$$

- Both have a unique gapped ground state $|\text{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- **Symmetries:** $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \rightarrow j + 1$

H_+ and H_- are both in $\mathbb{Z}_2 \times \mathbb{Z}_L$ SPT phases

SPTs characterized $G \times$ translations are called weak G SPTs

H_+ and H_- are both in \mathbb{Z}_2 weak SPT phases

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

	Even L	Even L , \mathbb{Z}_2 symmetry defect
$U \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$+ \text{GS}_{\pm} \rangle$
$T \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$\pm \text{GS}_{\pm} \rangle$

*Different \mathbb{Z}_2
weak SPTs*

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

	Even L	Even L , \mathbb{Z}_2 symmetry defect	Odd L
$U \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$+ \text{GS}_{\pm} \rangle$	$\pm \text{GS}_{\pm} \rangle$
$T \text{GS}_{\pm} \rangle =$	$+ \text{GS}_{\pm} \rangle$	$\pm \text{GS}_{\pm} \rangle$	$+ \text{GS}_{\pm} \rangle$

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Translation defect carries \mathbb{Z}_2 symmetry charge in $|\text{GS}_-\rangle$

➤ Inserting a translation defect is done by

$$T^L = 1 \rightarrow T^L = T \implies L \rightarrow L - 1$$

➤ Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

	Even L	Even L , \mathbb{Z}_2 symmetry defect	Odd L
$U \text{GS}_\pm\rangle =$	$+ \text{GS}_\pm\rangle$	$+ \text{GS}_\pm\rangle$	$\pm \text{GS}_\pm\rangle$
$T \text{GS}_\pm\rangle =$	$+ \text{GS}_\pm\rangle$	$\pm \text{GS}_\pm\rangle$	$+ \text{GS}_\pm\rangle$

A curious Hamiltonian

1d periodic lattice with a single **qubit** and \mathbb{Z}_4 **qudit** on each site $j \sim j + L$ [SP, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on **qubits**: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on \mathbb{Z}_4 **qudits**: $X^4 = Z^4 = 1$ and $ZX = i XZ$

A curious Hamiltonian

1d periodic lattice with a single **qubit** and \mathbb{Z}_4 **qudit** on each site $j \sim j + L$ [SP, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on **qubits**: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on \mathbb{Z}_4 **qudits**: $X^4 = Z^4 = 1$ and $ZX = i XZ$

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

- C acts as $X \rightarrow X^\dagger$ and $Z \rightarrow Z^\dagger$
- Is a sum of commuting terms and has a **unique gapped ground state**

A curious Hamiltonian

1d periodic lattice with a single **qubit** and \mathbb{Z}_4 **qudit** on each site $j \sim j + L$ [SP, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on **qubits**: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on \mathbb{Z}_4 **qudits**: $X^4 = Z^4 = 1$ and $ZX = i XZ$

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

➤ $C_j = \frac{1}{4} (Z_j - Z_j^\dagger) (Z_{j+1} - Z_{j+1}^\dagger)$

➤ Is

g1

$$|\text{GS}\rangle = \sum_{\substack{\{\varphi_j = 0,1\} \\ \{\alpha_j = 0,2\}}} i^{\sum_j \alpha_j (\varphi_j - \varphi_{j-1})} \bigotimes_j | \sigma_j^x = (-1)^{\varphi_j}, Z_j = i^{\alpha_j+1} \rangle$$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the **symmetries** of H ?

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the **symmetries** of H ?

➤ \mathbb{Z}_L lattice **translations** $T: j \rightarrow j + 1$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the **symmetries** of H ?

- \mathbb{Z}_L lattice **translations** $T: j \rightarrow j + 1$
- Three **\mathbb{Z}_2** symmetry operators

$$U = \prod_j X_j^2, \quad R_1 = \prod_j \sigma_j^z, \quad R_2 = \prod_j Z_j^2$$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the **symmetries** of H ?

- \mathbb{Z}_L lattice **translations** $T: j \rightarrow j + 1$
- Three **\mathbb{Z}_2** symmetry operators

$$U = \prod_j X_j^2, \quad R_1 = \prod_j \sigma_j^z, \quad R_2 = \prod_j Z_j^2$$

- 🙋 symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

R_E can be written as a $\chi = 2$ matrix product operator

$$R_E = \text{Tr} \left(\prod_{j=1}^L M_j \right) \equiv \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \quad | \\ \boxed{M_1} \text{---} \boxed{M_2} \text{---} \dots \text{---} \boxed{M_L} \\ | \quad | \quad | \quad | \quad | \end{array}$$

➤ MPO tensor

$$M_j = \frac{1}{2} \begin{pmatrix} Z_j + Z_j^\dagger & i(Z_j - Z_j^\dagger) \sigma_j^z \\ -i(Z_j - Z_j^\dagger) & (Z_j + Z_j^\dagger) \sigma_j^z \end{pmatrix}$$

➤ 🙋 symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the symmetries of H ?

R_E is a **non-invertible symmetry** operator

- $R_1 |\psi\rangle = -|\psi\rangle$ or $R_2 |\psi\rangle = -|\psi\rangle \implies R_E |\psi\rangle = 0$
- R_E have zero-eigenvalues $\implies R_E$ is non-invertible

➤ 🙋 symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$U R_E = (-1)^L R_E U$$

► Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$U R_E = (-1)^L R_E U$$

► Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry

Ground state satisfies:

$$T|\text{GS}\rangle = +|\text{GS}\rangle \quad U|\text{GS}\rangle = +|\text{GS}\rangle \quad R_1|\text{GS}\rangle = +|\text{GS}\rangle$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$U R_E = (-1)^L R_E U$$

► Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry

Ground state satisfies:

$$T|\text{GS}\rangle = +|\text{GS}\rangle \quad U|\text{GS}\rangle = +|\text{GS}\rangle \quad R_1|\text{GS}\rangle = +|\text{GS}\rangle$$

$$R_2|\text{GS}\rangle = \begin{cases} +|\text{GS}\rangle, & L \text{ even} \\ -|\text{GS}\rangle, & L \text{ odd} \end{cases}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$U R_E = (-1)^L R_E U$$

► Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry

Ground state satisfies:

$$T|\text{GS}\rangle = +|\text{GS}\rangle \quad U|\text{GS}\rangle = +|\text{GS}\rangle \quad R_1|\text{GS}\rangle = +|\text{GS}\rangle$$

$$R_2|\text{GS}\rangle = \begin{cases} +|\text{GS}\rangle, & L \text{ even} \\ -|\text{GS}\rangle, & L \text{ odd} \end{cases} \quad R_E|\text{GS}\rangle = \begin{cases} +2|\text{GS}\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad H \text{ is in a } \mathbb{Z}_2 \times \text{Rep}(D_8) \text{ weak SPT phase} \quad R_i R_E = R_E$$

- Translation defects carry $\text{Rep}(D_8)$ symmetry charge in $|\text{GS}\rangle$

➤ Form a $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_8$ symmetry group

Ground state satisfies:

$$T|\text{GS}\rangle = +|\text{GS}\rangle \quad U|\text{GS}\rangle = +|\text{GS}\rangle \quad R_1|\text{GS}\rangle = +|\text{GS}\rangle$$

$$R_2|\text{GS}\rangle = \begin{cases} +|\text{GS}\rangle, & L \text{ even} \\ -|\text{GS}\rangle, & L \text{ odd} \end{cases} \quad R_E|\text{GS}\rangle = \begin{cases} +2|\text{GS}\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U R_E = -R_E U \quad (\text{odd } L)$$

Projective unitary symmetries $U_1 U_2 = e^{i\theta} U_2 U_1$ forbid SPTs

► Assume non-degenerate symmetric ground state:

$$\left. \begin{array}{l} 1. \quad U_1 U_2 |\psi\rangle = |\psi\rangle \\ 2. \quad U_1 U_2 |\psi\rangle = e^{i\theta} U_2 U_1 |\psi\rangle = e^{i\theta} |\psi\rangle \end{array} \right\} \begin{array}{l} \text{Contradicts} \\ \text{assumption} \end{array}$$

A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U R_E = -R_E U \quad (\text{odd } L)$$

Projective unitary symmetries $U_1 U_2 = e^{i\theta} U_2 U_1$ forbid SPTs

➤ Assume non-degenerate symmetric ground state:

$$\left. \begin{array}{l} 1. \quad U_1 U_2 |\psi\rangle = |\psi\rangle \\ 2. \quad U_1 U_2 |\psi\rangle = e^{i\theta} U_2 U_1 |\psi\rangle = e^{i\theta} |\psi\rangle \end{array} \right\} \begin{array}{l} \text{Contradicts} \\ \text{assumption} \end{array}$$

Projective non-invertible symmetries are compatible with SPTs

➤ **Loophole**: symmetry operator has zero-eigenvalues

➤ $U R_E = (-1)^L R_E U \implies R_E |\text{GS}_{\text{SPT}}\rangle = 0$ when L is odd

Non-invertible symmetry and SPTs.....

SPTs protected by internal **invertible** versus **non-invertible** symmetry [Thorngren, Wang '19; Inamura '21; Bhardwaj, Bottini, Pajer, Schäfer-Nameki '23; Fechisin, Tantivasadakarn, Albert '23; Seifnashri, Shao '24; Inamura, Ohyama '24]

	Invertible	Non-invertible
Stacking	Yes	No
Entanglers	Yes	No
Classification	Cohomology	Fiber functors
Edge/interface modes	Yes	Yes
Defect characterization	Yes	Yes

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projective** $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry is a **special case** of a more general **projective** $Z(G) \times \text{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projective** $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry is a **special case** of a more general **projective** $Z(G) \times \text{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

$Z(G)$ **symmetry** operator U_z , with $z \in Z(G)$, satisfies

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\text{Rep}(G)$ **symmetry** operator R_Γ , with Γ an irrep of G , satisfies

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- **Non-invertible symmetry** when G is non-Abelian

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projectivity** arises through the relation

$$R_{\Gamma} U_z = (e^{i\phi_{\Gamma}(z)})^L U_z R_{\Gamma} \quad \text{with} \quad e^{i\phi_{\Gamma}(z)} = \text{Tr}[\Gamma(z)] / d_{\Gamma}$$

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projectivity** arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma \quad \text{with} \quad e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

e.g., $e^{i\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ ($Z(\mathbb{Z}_2) = \mathbb{Z}_2$)

$z \backslash \Gamma$	1	sign
+1	+1	+1
-1	+1	-1

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projectivity** arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma \quad \text{with} \quad e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

e.g., $e^{i\phi_\Gamma(z)}$ when $G = D_8$ ($Z(D_8) = \mathbb{Z}_2$)

$z \backslash \Gamma$	1	1₁	1₂	1₃	E
+1	+1	+1	+1	+1	+1
-1	+1	+1	+1	+1	-1

Projective $Z(G) \times \text{Rep}(G)$ symmetry.....

The **projectivity** arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma \quad \text{with} \quad e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

e.g., $e^{i\phi_\Gamma(z)}$ when $G = D_8$ ($Z(D_8) = \mathbb{Z}_2$)

$z \backslash \Gamma$	1	1₁	1₂	1₃	E
+1	+1	+1	+1	+1	+1
-1	+1	+1	+1	+1	-1

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \dots, zg_L\rangle \langle g_1, \dots, g_L| \quad R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \dots, g_L\rangle \langle g_1, \dots, g_L|$$

(SPT)-LSM theorems

$$R_{\Gamma} U_z = (e^{i\phi_{\Gamma}(z)})^L U_z R_{\Gamma}$$

There is an **Lieb-Schultz-Mattis (LSM) theorem** when $e^{i\phi_{\Gamma}(z)}$ is non-trivial for a unitary R_{Γ}

[...; Matsui '08; Chen, Gu, Wen '10; Yao, Oshikawa '20; Ogata, Tasaki '21; Seifnashri '23; Kapustin, Sopenko '24]

➤ The **LSM theorem** forbids **SPT phases**

(SPT)-LSM theorems

$$R_{\Gamma} U_z = (e^{i\phi_{\Gamma}(z)})^L U_z R_{\Gamma}$$

There is an **Lieb-Schultz-Mattis (LSM) theorem** when $e^{i\phi_{\Gamma}(z)}$ is non-trivial for a unitary R_{Γ}

[...; Matsui '08; Chen, Gu, Wen '10; Yao, Oshikawa '20; Ogata, Tasaki '21; Seifnashri '23; Kapustin, Sopenko '24]

➤ The **LSM theorem** forbids **SPT phases**

When there is no **LSM theorem**, the **projective algebra** gives rise to an **SPT-LSM theorem** [Lu '17; Yang, Jiang, Vishwanath, Ran '17; Lu, Ran, Oshikawa '17; Else, Thorngren '20]

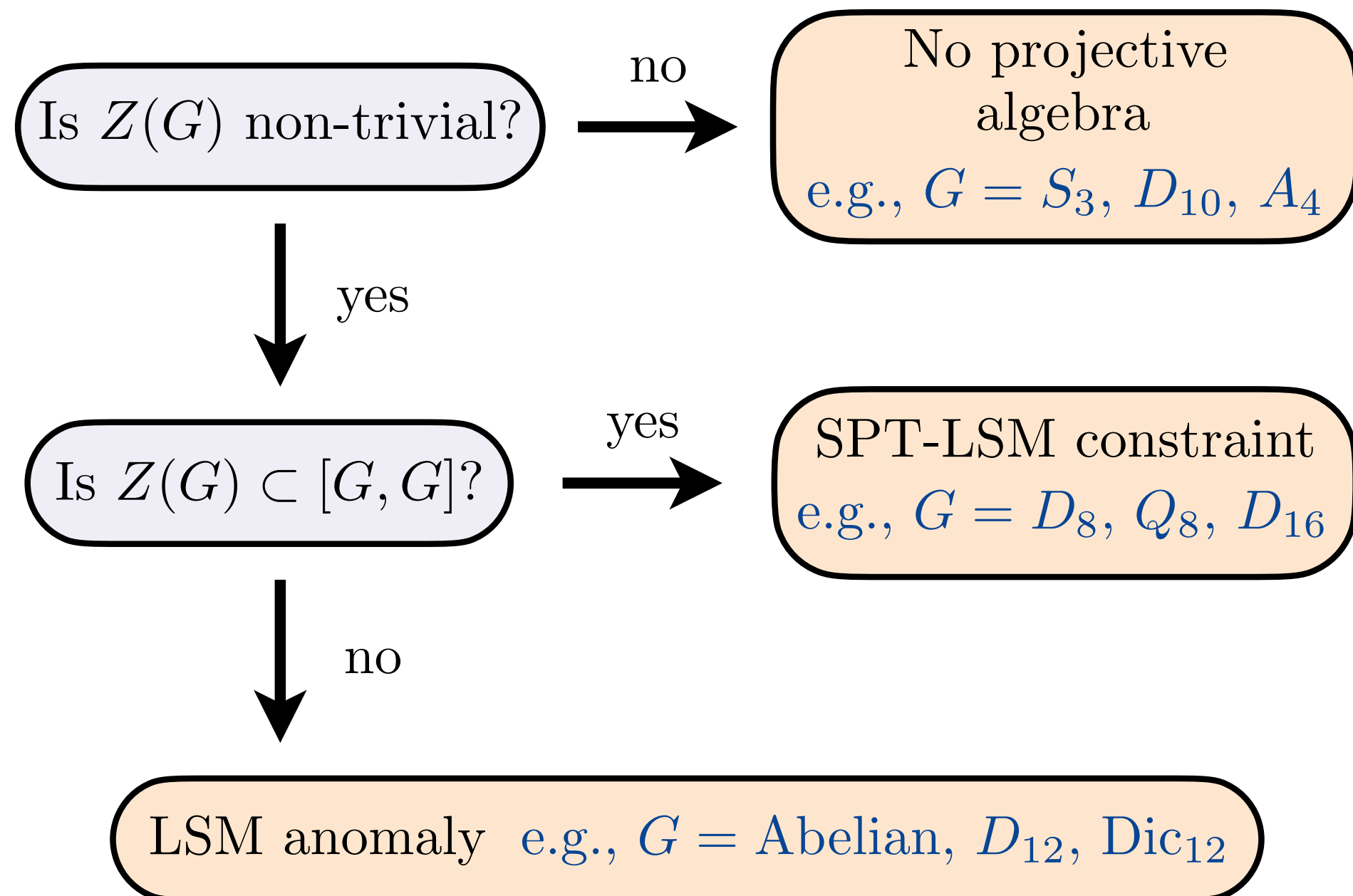
➤ $R_{\Gamma} U_z = (e^{i\phi_{\Gamma}(z)})^L U_z R_{\Gamma}$ forces any **SPT state** to satisfy

$$R_{\Gamma} |\text{GS}\rangle = 0 \text{ for nontrivial } (e^{i\phi_{\Gamma}(z)})^L$$

➤ Any **SPT state** must have non-zero entanglement

(SPT)-LSM theorems

Whether there is an (SPT)-LSM theorem depends on G :



SPT-LSM theorem

To prove this **SPT-LSM theorem**, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

2. Assume that any **unique gapped ground state** $|\text{GS}\rangle$
satisfies $R_\Gamma |\text{GS}\rangle \neq 0$ for some $L = L^*$ $\left(e^{i\phi_\Gamma(z)L^*} = 1 \right)$

SPT-LSM theorem

To prove this **SPT-LSM theorem**, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

2. Assume that any **unique gapped ground state** $|\text{GS}\rangle$
satisfies $R_\Gamma |\text{GS}\rangle \neq 0$ for some $L = L^*$ $\left(e^{i\phi_\Gamma(z)L^*} = 1 \right)$

Easy to prove assumption for product states in $\bigotimes_j \mathbb{C}^{|G|}$, where

$$R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \cdots, g_L\rangle \langle g_1, \cdots, g_L|$$

*but it is true as long as there is an IR **TQFT** description*

SPT-LSM theorem

If there is a **unique gapped** $|\text{GS}\rangle$ that is a **product state**:

$$\blacktriangleright U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$$

Using the assumption, $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$ at $L = L^*$:

$$\left. \begin{aligned} 1. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle &= R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \\ 2. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle &= e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma |\text{GS}\rangle = \lambda_\Gamma e^{i\phi_\Gamma(z)} |\text{GS}\rangle \end{aligned} \right\} \text{Contradiction}$$

SPT-LSM theorem

If there is a **unique gapped** $|\text{GS}\rangle$ that is a **product state**:

$$\blacktriangleright U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$$

Using the assumption, $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$ at $L = L^*$:

$$\left. \begin{aligned} 1. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle &= R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \\ 2. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle &= e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma |\text{GS}\rangle = \lambda_\Gamma e^{i\phi_\Gamma(z)} |\text{GS}\rangle \end{aligned} \right\} \text{Contradiction}$$

\implies Cannot be an **SPT state** that is a **product state** at $L = L^*$

SPT-LSM theorem

If there is a **unique gapped** $|\text{GS}\rangle$ that is a **product state**:

$$\blacktriangleright U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$$

Using the assumption, $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$ at $L = L^*$:

$$\left. \begin{array}{l} 1. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle = R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \\ 2. \quad R_\Gamma U_j^{(z)} |\text{GS}\rangle = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma |\text{GS}\rangle = \lambda_\Gamma e^{i\phi_\Gamma(z)} |\text{GS}\rangle \end{array} \right\} \text{Contradiction}$$

\implies Cannot be an **SPT state** that is a **product state** at $L = L^*$

\implies By locality, there cannot be an **SPT state** that is a **product state** for any L

SPT-LSM theorem

If there is a unique gapped $|\text{GS}\rangle$ that is a product state:

$$\triangleright U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$$

Using

Therefore, the projective non-invertible symmetry
1. R prevents a product state SPT

2. R \triangleright All SPTs must have non-zero entanglement

\implies Cannot be an SPT state that is a product state at $L = L^*$

\implies By locality, there cannot be an SPT state that is a product state for any L

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces **its**
ground state to satisfy $R_\Gamma |\text{GS}\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces **its ground state** to satisfy $R_\Gamma |\text{GS}\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

Two possibilities:

1. An **SPT state** satisfies $R_\Gamma |\text{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(e^{i\phi_\Gamma(z)})^{L^*} = 1$, an **SPT state** satisfies $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$, but $R_\Gamma |\text{GS}\rangle = 0$ for $L \neq L^*$

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces **its ground state** to satisfy $R_\Gamma |\text{GS}\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

Two possibilities:

1. An **SPT state** satisfies $R_\Gamma |\text{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(e^{i\phi_\Gamma(z)})^{L^*} = 1$, an **SPT state** satisfies $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$, but $R_\Gamma |\text{GS}\rangle = 0$ for $L \neq L^*$

The first is incompatible with an IR TQFT

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its ground state to satisfy $R_\Gamma |\text{GS}\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

At $L = L^*$, SPTs satisfy $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle$

At $L = L^* + 1$, SPTs satisfy $R_\Gamma |\text{GS}\rangle = 0$

➤ All SPT states have translation defects dressed by non-trivial $\text{Rep}(G)$ symmetry charge

➤ \nexists a trivial SPT \implies SPT-LSM theorem

The first is incompatible with an IR TQFT

Outlook

We found a new class of entangled weak SPTs characterized by a projective $Z(G) \times \text{Rep}(G)$ non-invertible symmetry

1. An exactly solvable model in a weak SPT phase characterized by a projective $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry
2. General discussion on projective $Z(G) \times \text{Rep}(G)$ weak SPTs \implies an SPT-LSM theorem

New quantum phases and models can be discovered using generalized symmetries as a guide!

SP, Lam, Aksoy arXiv:2409.18113 [SciPost]

Back-up slides

The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the projective algebras

U symmetry defect	R_E symmetry defect
$R_E T = - T R_E$	$T U = - U T$

For invertible symmetries, such projective algebras imply an 't Hooft anomaly (e.g., the type III anomaly $(-1)^{\int_{M_3} a \cup b \cup c}$)

[Matsui '08; Yao, Oshikawa '20; Seifnashri '23; Kapustin, Sopenko '24]

➤ This is not true for non-invertible symmetries!

The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the projective algebras

U symmetry defect	R_E symmetry defect
$R_E T = -T R_E$	$T U = -U T$

For invertible symmetries, such projective algebras imply an 't Hooft anomaly of the type III anomaly $(-1)^{\int_{M_3} a \cup b \cup c}$

Fails because of $R_E = 0$ loophole

[Kapustin, Sopenko '24]

THIS IS TRUE for non-invertible symmetries!

The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the projective algebras

U symmetry defect	R_E symmetry defect
$R_E T = -T R_E$	$T U = -U T$

Fails because of
 $R_E = 0$ loophole

Fails because the degeneracy
is encoded in the defect's
quantum dimension

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^L J \sigma_j^x \sigma_{j+1}^x + K \sigma_j^y \sigma_{j+1}^y$$

- There is an **LSM anomaly** involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the **projective algebras** [Cheng, Seiberg 2023]

<i>Translation defects</i>	\mathbb{Z}_2^x defect	\mathbb{Z}_2^y defect
$U_x U_y = (-1)^L U_y U_x$	$U_y T = -T U_y$	$T U_x = -U_x T$

GROUP BASED QUDITS

A **G -qudit** is a $|G|$ -level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

➤ G is a **finite group**, e.g. \mathbb{Z}_2 , S_3 , D_8 , SmallGroup(32,49)

Group based **Pauli operators** [Brell 2014]

➤ X operators labeled by **group elements**

$$\vec{X}^{(g)} = \sum_h |gh\rangle\langle h|$$

$$\overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

$$\bar{g} \equiv g^{-1}$$

➤ Z operators are MPOs labeled by **irreps** $\Gamma: G \rightarrow \text{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv \alpha \text{---} \boxed{Z^{(\Gamma)}} \text{---} \beta \quad (\alpha, \beta = 1, 2, \dots, d_\Gamma)$$

GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{1, 1'\}$

$$\vec{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(1)}]_{11} = 1$$

$$\vec{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \qquad [Z^{(1')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

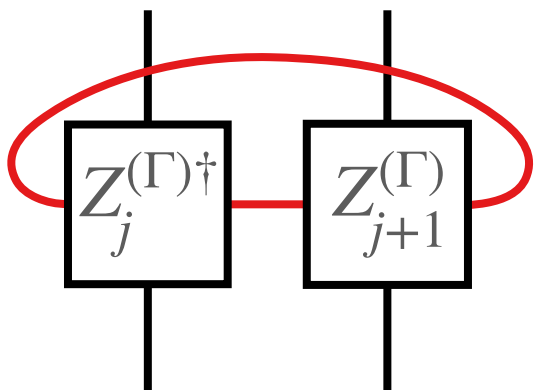
1. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(gh)}$, $\overleftarrow{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\vec{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)} \vec{X}^{(g)}$
2. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(h)} \vec{X}^{(g)}$ iff g and h commute
3. $\vec{X}^{(g)} [Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma} [Z^{(\Gamma)}]_{\gamma\beta} \vec{X}^{(g)}$
4. **Unitarity:** $\vec{X}^{(g)\dagger} = \vec{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger} Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

GROUP BASED XY MODEL

Group based **Pauli operators** are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based *XY* model: Consider a **periodic 1d lattice** of L sites. On each site j resides a **G -qudit** and its Hamiltonian

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$$\text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_{\Gamma}(\bar{g}_j g_{j+1}) |\{g\}\rangle \langle \{g\}| \equiv$$


► For $G = \mathbb{Z}_2$, this is the ordinary **quantum *XY* model**

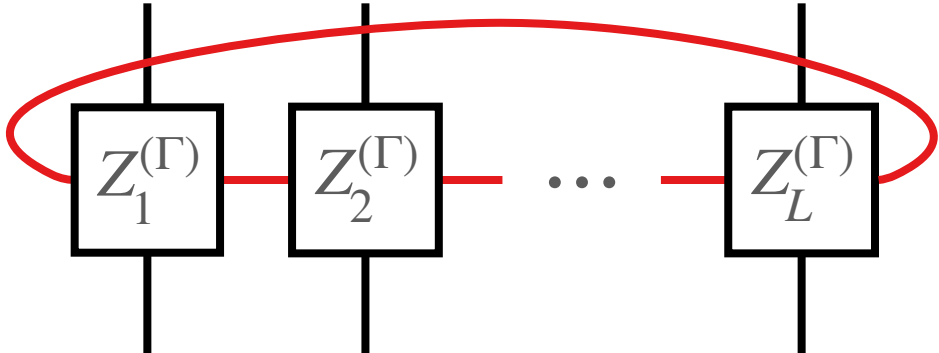
SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

\mathbb{Z}_L lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

► $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$

► $\text{Rep}(G)$ symmetry $R_{\Gamma} = \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$\begin{aligned}
 U_z &= \prod_j \vec{X}_j^{(z)} & R_\Gamma &= \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right) \\
 T_{\text{tw}}^{(z)} &= \vec{X}_I^{(z)} T & T_{\text{tw}}^{(\Gamma)} &= \hat{Z}_I^{(\Gamma)} (T \otimes \mathbf{1})
 \end{aligned}$$

Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

<i>Translation defects</i>	$z \in Z(G)$ <i>defect</i>	$\Gamma \in \text{Rep}(G)$ <i>defect</i>
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$	$R_\Gamma T_{\text{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\text{tw}}^{(z)} R_\Gamma$	$T_{\text{tw}}^{(\Gamma)} U_z = e^{i\phi_\Gamma(z)} U_z T_{\text{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ **projective algebra** of the ordinary quantum XY model

GAUGING WEB

