Tutorial 1

→ Plan : 1) Higher-form sym in quantum lattice models (QLMs)
2) QLM examples of higher-form SPTs

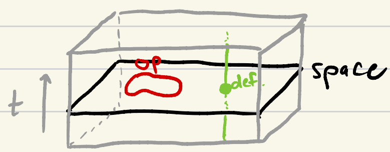
Higher-form Sym in QLM

→ An n-form sym. in a $(d+1)$D QLM $(n = 0, 1, \cdots, d)$

1) Sym. operators act on $(d-n)$-cycles of space

2) Sym defects supported on $(d-n-1)$ cycles of space

⇒ Ordinary sym. have $n=0$, higher-form sym have $n>0$.
⇒ e.g. 1-form sym in 2+1D.



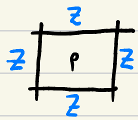
→ Two types of higher-form sym in QLMs.

	Non-topological	topological
Sym operators topological ?	No	Yes
Sym defects topological ?	Yes	Yes
Unitary, Abelian G n-form sym op (assuming $\partial\Lambda = \emptyset$)	$Z_{d-n}(\Lambda, G) \to \mathcal{U}(\mathcal{H})$	$H_{d-n}(\Lambda, G) \to \mathcal{U}(\mathcal{H})$
In relativistic QFT	Impossible	Possible
In QLM with $\mathcal{H} = \bigotimes_a \mathcal{H}_a$	Possible	Impossible

Examples

→ Let Λ be the square lattice with a qubit on each edge

1) $\mathbb{Z}_2$ gauge theory

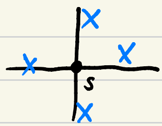
→ Denote by $B_p =$ 

the "plaquette operator"

Flatness condition.

→ $\mathcal{H} = \bigotimes_{e \in \Lambda_1} \mathbb{C}^2 / \sim$ where $\sim$ is $B_p |\psi\rangle = |\psi\rangle$.

→ Hamiltonian $H = - \sum_{s \in \Lambda_0} A_s$

with $A_s =$ 

the "Star operator"

↱ notation : 1-form Sym.

→ Has the topological $\mathbb{Z}_2^{(1)}$ sym $W(\gamma) = \prod_{e \subset \gamma} Z_e$

- Symmetry: $[H, W(\gamma)] = 0 \quad \forall \quad \gamma \in Z_1(\Lambda, \mathbb{Z}_2)$

- $\mathbb{Z}_2$ Sym: $W(\gamma) \times W(\gamma) = \mathbb{1}$

- top. Sym op:

$$W(\gamma + \partial\Sigma)|\psi\rangle = W(\gamma)\left(\prod_{p \subset \Sigma} B_p\right)|\psi\rangle = W(\gamma)\,|\psi\rangle$$

- top. Sym defect:

$$H \xmapsto[\text{@ site } r]{W \text{ defect}} H_r = H + 2A_r .$$

⇒ topological b/c $\quad H_{r_1} = U_{12} H_{r_2} U_{12}$ with Unitary
$U_{12} = \prod_{1 \to 2} Z.$

2) Toric Code Model

$\rightarrow \mathcal{H} = \bigotimes_{e\in\Lambda_1} \mathbb{C}^2$ and $H = -\sum_{s\in\Lambda_0} A_s - \sum_{p\in\Lambda_2} B_p$

$\rightarrow W(\gamma)$ now generates a non-top $\mathbb{Z}_2^{(1)}$ Sym.

- Now non-topological b/c there are $|\psi\rangle \in \mathcal{H}$ with $B_p|\psi\rangle = -|\psi\rangle$.

- Becomes a topological 1-form sym @ low-energy

Higher-form SPTs

Higher-form sym can characterize SPTs.

Our Case Study: Bosonic $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ SPTs in 2+1D

$\rightarrow$ Classified by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{$paramagnet, LG, 2d-cluster, LG $\boxtimes$ 2d-cluster$\}$

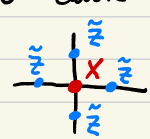
QLMs for these SPTs/(LG SPT)

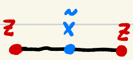
$\rightarrow$ Let Λ be square lattice with a qubit on each site and edge

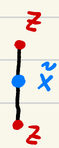
$$\mathcal{H} = \left(\bigotimes_{s\in\Lambda_0} \mathbb{C}^2\right) \otimes \left(\bigotimes_{e\in\Lambda_1} \mathbb{C}^2\right)$$

$\rightarrow$ SPT 1: 2+1D paramagnet $H_p = -\sum_s X_s - \sum_e \tilde{X}_e$

$\rightarrow$ SPT 2: 2+1D cluster state model

$H_c = -\sum_s$ 

$-\sum_{e_x}$ 

$-\sum_{e_y}$ 

$\equiv A_s$ $\equiv B_e$

Both QLMs

1) have $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ symmetries generated by

$$U = \prod_{s \in \Lambda_0} X_s \qquad V(\gamma) = \prod_{e \subset \gamma} \tilde{X}_e$$

2) are exactly solvable

	Unique gapped ground state
H_p	$\lvert P\rangle = X_s \lvert P\rangle = \tilde{X}_e \lvert P\rangle$
H_c	$\lvert C\rangle = A_s \lvert C\rangle = B_e \lvert C\rangle$

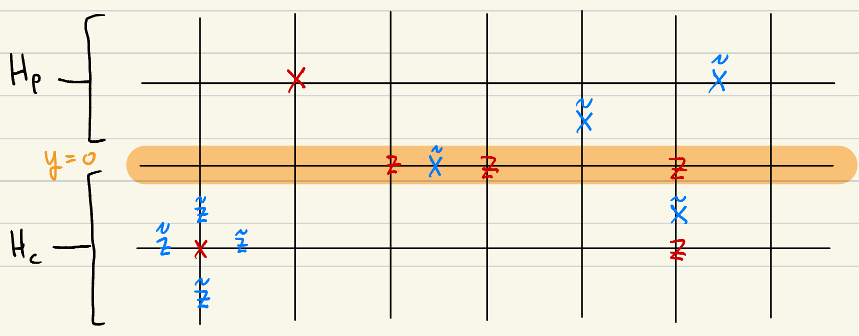
3) In $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ SPT phases

$$U \lvert P/C\rangle = V(\gamma) \lvert P/C\rangle = \lvert P/C\rangle$$

H_p and H_c are in different SPTs

1) Anomaly inflow / Interface perspective
(different SPTs b/c nontrivial degeneracies @ interfaces)

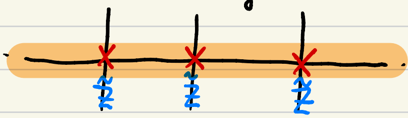
Consider Hamiltonian H_{int} that is H_c for $y<0$ and H_p for $y>0$
(Can be viewed as boundary of cluster state model)

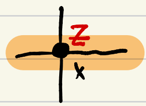


→ U and $V(\gamma)$ still commute with H_{int}

→ H_{int} is gapped with two ground states $|gs_{1,2}\rangle$.

$\Rightarrow$ U and $V(\gamma)$ act nontrivially on ground states:

Let: $U_{int} = \cdots$  $\cdots$

$V_{int}^{(x)} =$ 

Then $U|gs_{1,2}\rangle = U_{int}|gs_{1,2}\rangle$

$$V(\gamma)|gs_{1,2}\rangle = \prod_{x:\gamma\cap int.} V_{int}^{(x)} |gs_{1,2}\rangle$$

$\Rightarrow$ $\mathbb{Z}_2^{(0)}$ is SSB'd @ Interface

$\Rightarrow$ In ground state subspace, $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ act as an anomalous sym on interface bc

$$\left. U_{int} V_{int}^{(x)} = - V_{int}^{(x)} U_{int} \right\} \boxed{\text{anomaly inflow}}$$

2) Sym defect perspective (different SPTs b/c different defect responses)

→ Inserting a $\mathbb{Z}_2^{(1)}$ sym defect @ site r

$$H_p \longmapsto H_p$$
$$H_c \longmapsto H_c + 2A_r$$

$|P\rangle$ unchanged, but $|C\rangle$ is modified to $|C_r\rangle$:

$$A_s|C_r\rangle = \begin{cases} |C_r\rangle & s \neq r \\ -|C_r\rangle & s = r \end{cases}$$

$$\Rightarrow U|C_r\rangle = \prod_{s\in\Lambda_0} A_s|C_r\rangle = -|C_r\rangle$$

$$\boxed{\mathbb{Z}_2^{(1)} \text{ defect carries } \left\{\begin{matrix}\text{trivial} \\ \text{non-trivial}\end{matrix}\right\} \mathbb{Z}_2^{(0)} \text{ charge in } \left\{\begin{matrix}\text{SPT 1} \\ \text{SPT 2}\end{matrix}\right\}}$$

→ Inserting a $\mathbb{Z}_2^{(0)}$ Sym defect @ dual 1-cycle $\gamma^\vee$

$$H_P \longmapsto H_P$$
$$H_C \longmapsto H_C + 2\sum_{e\in\gamma^\perp} B_e$$

$|P\rangle$ unchanged but $|C\rangle$ changed to $|C_{\gamma^\vee}\rangle$ satisfying

$$\Rightarrow V(\gamma)|C_{\gamma^\vee}\rangle = (-1)^{\#(\gamma,\gamma^\vee)}|C_{\gamma^\vee}\rangle$$

$$\boxed{\mathbb{Z}_2^{(0)} \text{ defect carries } \left\{\begin{matrix}\text{trivial} \\ \text{non-trivial}\end{matrix}\right\} \mathbb{Z}_2^{(1)} \text{ charge in } \left\{\begin{matrix}\text{SPT 1} \\ \text{SPT 2}\end{matrix}\right\}}$$

SPT 1 and 2 are different SPTs since sym defects have different sym charge decorations

Tutorial 2

Plan: 1) Condensation defects in QLMs
2) Higher-gauging in QLMs

In lecture 2: 2+1D 0-form SPT $\xrightarrow{\text{gauge}}$ 2+1D TO

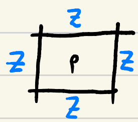
→ Our case study: 2+1 D $\mathbb{Z}_2$ gauge theory on square lattice Λ

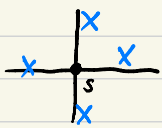
Recall:

$$\mathcal{H} = \bigotimes_{s \in \Lambda_0} \mathbb{C}^2 \qquad\qquad H_p = -\sum_{s \in \Lambda_0} X_s$$

gauge $\mathbb{Z}_2$ sym $U = \prod_s X_s$

$$\mathcal{H}^\vee = \bigotimes_{e \in \Lambda_1} \mathbb{C}^2 / \underbrace{\sim B_p = 1}_{\text{Was energetically enforced in lecture}} \qquad\qquad H_p^\vee = -\sum_{s \in \Lambda_0} A_s$$

where $B_p =$ 

and $A_s =$ 

→ What about an SPT with a defect?
⇒ get a TO with a defect.

Insert a $\mathbb{Z}_2$ SSB defect in H_p along $x=0$ line:

$$H_{def} = H_p + \sum_y \left(X_{(0,y)} - Z_{(0,y)} Z_{(0,y+1)} \right)$$

→ Still commutes with $U = \prod_s X_s$

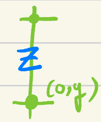
→ Has 2 ground states $|gs_{1,2}\rangle$ and

$$U|gs_{1,2}\rangle = \prod_y X_{(0,y)} |gs_{1,2}\rangle \neq |gs_{1,2}\rangle$$

⇒ $\mathbb{Z}_2$ SSB'd along $x=0$ line.

Gauge $\mathbb{Z}_2$ sym in H_{def} :

$$\mathcal{H}^\vee = \bigotimes_e \mathbb{C}^2 / \{B_p = 1\}$$



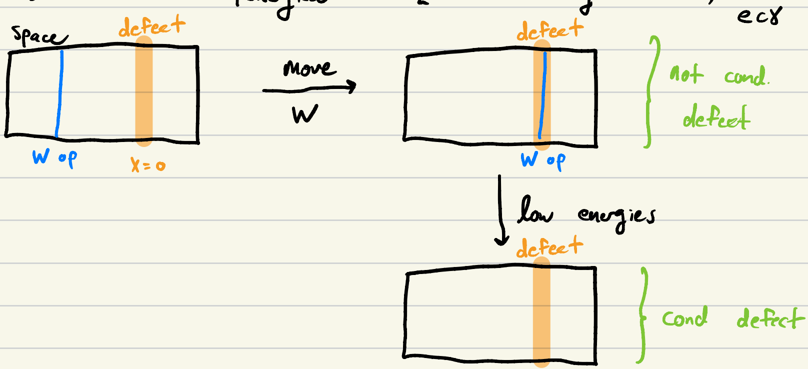
$$H^\vee_{def} = -\sum_s A_s + \sum_y \left(A_{(0,y)} - Z_{(0,y),y}\right)$$

→ $\mathbb{Z}_2$ gauge theory with a defect

→ Becomes a "condensation defect" in the low-energy Hilbert space where $Z_{(0,y),y}|\psi\rangle = |\psi\rangle$.

⇒ condenses the e anyons along $x=0$ line

⇒ "Eats" the topological $\mathbb{Z}_2$ 1-form sym $W(\gamma) = \prod_{e\in\gamma} Z_e$



⇒ This condensation defect is non-invertible.

⇒ Is a topological defect

(See Tantivasadakarn, Chen '23 for unitary op.)

Condensation defects can be inserted by higher gauging

→ gauging higher-form sym within a codim. >0 spatial subspace

→ q-gauging = gauge on codim-q subspace.

→ Condensation defects are top. defects that can end (Plausible definition)

The $W(\gamma)$ cond. defect can be inserted by higher gauging $W(\gamma)$ along $x=0$ line γ_y in $\mathbb{Z}_2$ gauge theory

Recall:

$$\mathcal{H} = \bigotimes_e \mathbb{C}^2 /\sim_{B_p=1} \qquad\qquad H = -\sum_s A_s$$

let's first gauge $\mathbb{Z}_2^{(1)}$ normally (just to see how that works)

1) Introduce a qubit on each lattice site s, acted on by $(\tilde{X}_s, \tilde{Z}_s)$ (like $\mathbb{Z}_2$ 2-form gauge field)

$$\mathcal{H} \longmapsto \mathcal{H} \otimes \bigotimes_s \mathbb{C}^2 = \mathcal{H}_{big}$$

2) Enfore gauss law $G_e = \tilde{Z}\, Z\, \tilde{Z} = 1$

arbitrary edge

$$\mathcal{H}_{big} \longmapsto \mathcal{H}_{big} / \{G_e = 1\} \equiv \mathcal{H}^\vee$$

$$\Rightarrow W(\gamma) = \prod_{e\in\gamma} G_e = 1 \quad \forall\ \gamma \text{ on } \mathcal{H}^\vee.$$

3) Minimal couple: $A_s \longmapsto \tilde{X}_s A_s$

$$H_{MC} = -\sum_s \tilde{X}_s A_s$$

4) Unitary transformation

$$\tilde{X}_s \longrightarrow \tilde{X}_s A_s \qquad \tilde{Z}_s \rightarrow \tilde{Z}_s$$

$$X_e \longrightarrow X_e \qquad Z_e \rightarrow G_e$$

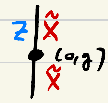
now gauss law $G_e = 1 \rightarrow Z_e = 1$

$$\Rightarrow \mathcal{H}^\vee = \bigotimes_s \mathbb{C}^2 \quad \text{and} \quad H^\vee = -\sum_s \tilde{X}_s$$

Higher gauging procedure: (1-gauge $W(\gamma)$ on γ_y)

1) Introduce new qubits on vertical edges $\langle (0,y),(0,y+1)\rangle$
(act as the $\mathbb{Z}_2$ gauge fields)

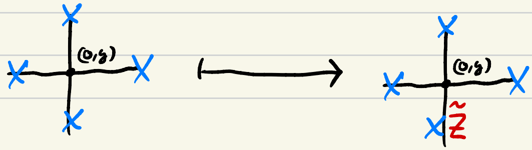
$$\mathcal{H} \longmapsto \mathcal{H} \otimes \bigotimes_y \mathbb{C}^2 \equiv \mathcal{H}_{big}$$

2) Enforce gauss law $G_y =$  $= 1$

$$\mathcal{H}_{big} \longmapsto \mathcal{H}_{big} / \{G_y = 1\} \equiv \mathcal{H}^\vee$$

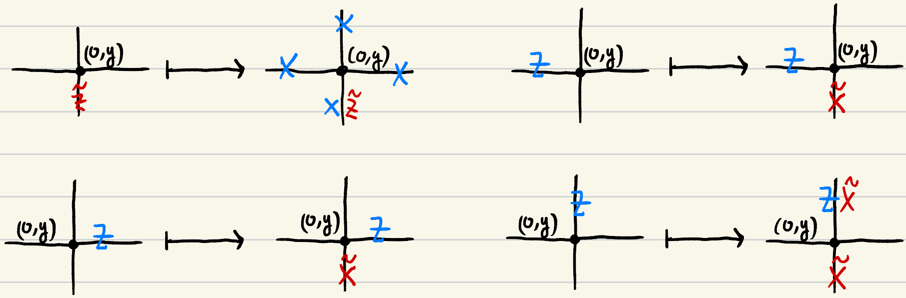
$$\Rightarrow W(\gamma_y) = \prod_{y \subset \gamma_y} G_y = 1 \quad \text{on} \quad \mathcal{H}^\vee$$

3) Minamal coupling to make all B_p and A_s
gauge invariant (commute with G_y).

$A_{(0,y)} =$ 

$$\Rightarrow H_{MC} = -\sum_{s \neq (0,y)} A_s - \sum_{s=(0,y)} A_s \tilde{Z}_{(0,y-1),y}$$

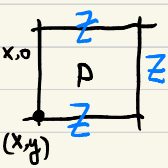
4) Unitary change of basis:



→ all other ops unchanged.

→ B_p unchanged but now $G_y = Z_{(0,y),y} = 1$.

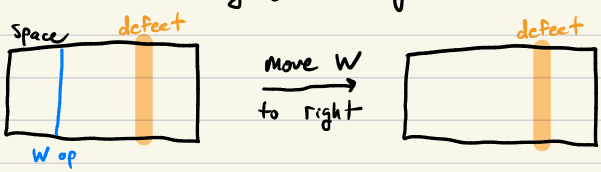
$$\Rightarrow \mathcal{H}^{\vee} = \bigotimes_e \mathbb{C}^2 \Big/ \{\tilde{B}_p = 1\} \quad \text{with}$$

$$\tilde{B}_p = Z^{1-\delta_{x,0}}$$  $$Z^{1-\delta_{x,-1}}$$

→ Hamiltonian (droping tildes)

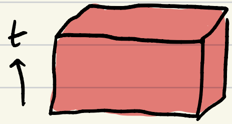
$$H^{\vee} = -\sum_{s \neq (0,y)} A_s - \sum_{s=(0,y)} Z_{(0,y),y}$$

⇒ this is $\mathbb{Z}_2$ gauge theory w/ the cond. defect

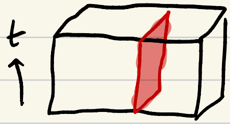


Higher gauging on a line is like gauging on a surface in space time

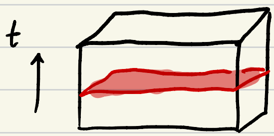
→ Gauging $\mathbb{Z}_2^{(1)}$: all of 2+1 D spacetime



→ $\mathbb{Z}_2^{(1)}$ cond defect : Surface extended in time



→ $\mathbb{Z}_2^{(1)}$ cond. op : Surface within time slice



The condensation op for $W(\gamma)$ constructed by summing over all W.

$$C = \frac{1}{\sqrt{|H_1(\Lambda, \mathbb{Z}_2)|}} \sum_{[\gamma] \in H_1(\Lambda, \mathbb{Z}_2)} W(\gamma)$$

This is what cond def look like in qft.

For Λ on a torus in $\mathbb{Z}_2$ gauge theory without defects:

$$C = \frac{1}{2}\left(1 + W(\gamma_x) + W(\gamma_y) + W(\gamma_x) W(\gamma_y)\right)$$

→ $[H, C] = 0$ bc $[H, W(\gamma)] = 0$

→ $C^2 = 2C$ ⟹ C is a non-invertible op.

$\rightarrow \quad C\,W(\gamma) = W(\gamma)\,C = C \quad (C \text{ eats } W(\gamma))$

$\mathbb{Z}_2$ gauge theory in $A_s = 1$ subspace has five non-trivial Condensation defects / operators

	higher gauge	Invertible ?	Sym transformation
e cond defect	$\mathbb{Z}_2^{(1),e}$ (ΠZ)	No	$e \mapsto 1 \oplus e$ $m \mapsto 0$
m cond def.	$\mathbb{Z}_2^{(1),m}$ (ΠX)	No	$e \mapsto 0$ $m \mapsto 1 \oplus m$
$e \otimes m$ cond defect	$\mathbb{Z}_2^{(1),\mathrm{diag}}$	Yes	$e \mapsto m$ $m \mapsto e$
"em" cond defect	$\mathbb{Z}_2^{(1)e} \times \mathbb{Z}_2^{(1)m}$	No	$e \mapsto 0$ $m \mapsto 1+e$
"me" cond defect	twisted gauge $\mathbb{Z}_2^{(1),e} \times \mathbb{Z}_2^{(1),m}$	No	$e \mapsto 1+m$ $m \mapsto 0$