Intro to SPTs and anomalies

Outline: 1) Basics of SPTs and Symmetry defects.
2) Symmetry anomalies
3) Symmetry-enforced entanglement (SPT-LSM theorems)

Cultural Background

Theme of todays talks: what can quantum matter do?

→ Applied questions, eg) carbon forming graphite or diamonds

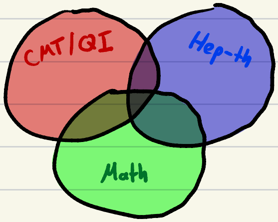
→ Theoretical questions arising from abstractification (this talk's style)

<u>Assumptions:</u> Unitarity and locality

<u>Input data:</u> Microscopic dof, dimension of spacetime, Symmetries

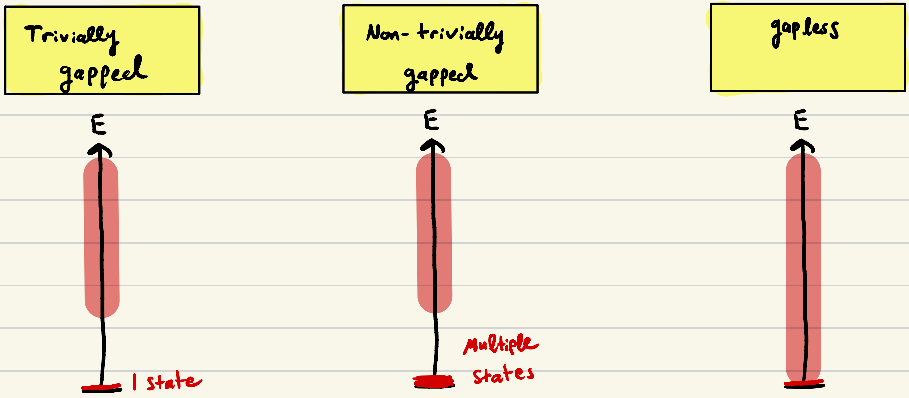
<u>Question:</u> Possible quantum phases?

Topic with incredible interdisciplinary synergy



To answer, need to characterize macroscopic behaviors and define an equivalence relation Very Hard!

→ A 0^{th} order question: what is the low-energy spectrum like?



→ Can then further refine after answering this question.

Symmetry Protected Topological Phases (SPTs)

An SPT is a trivially gapped phase (has one ground state on all closed spaces) protected by a symmetry.

↝ the simplest, most boring quantum phase, but there are many ways to be boring

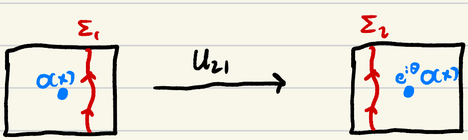
↝ Interesting physics can arise @ boundaries / Interfaces

SPTs are characterized by response to static probes
e.g.) Symmetry defects, Background gauge fields, magnetic monopoles, ...

<u>Aside</u>: Symmetry defects are localized modifications to the theory

$$H \xrightarrow[\text{defect}]{\text{insert}} H(\Sigma) = H + \delta H(\Sigma), \qquad Q \rightarrow Q(\Sigma) = Q + \delta Q(\Sigma), \quad \ldots$$

- Implement sym across space



- Moved via unitaries (are topological defects)

$$H(\Sigma_2) = U_{21}\, H(\Sigma_1)\, U_{21}^\dagger$$

- Twisted boundary conditions

$$(T_\perp)^L = \text{Sym op.}$$

<u>Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs</u>

- 1d closed chain w/ 2 qubits per site

$$H_p = -\sum_{j=1}^{L} (X_j + \tilde{X}_j)$$

$$|gs\rangle = |++\cdots+\rangle$$

Commute.

$$H_c = -\sum_{j=1}^{L} (\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1})$$

$$|gs\rangle = (\quad)|gs\rangle = (\quad)|gs\rangle$$

- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ sym

$$U = \prod_{j=1}^{L} X_j \qquad \tilde{U} = \prod_{j=1}^{L} \tilde{X}_j$$

- Both models have unique sym. gapped gs

H_p and H_c are in $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

- Are H_p and H_c different SPTs?
  $\rightsquigarrow$ check by inserting U sym defect: gives rise to twisted boundary conditions

$$Z_{j+L} = -Z_j$$

1) H_p unaffected:

$$\mathcal{U}|gs\rangle = |gs\rangle \qquad \tilde{\mathcal{U}}|gs\rangle = |gs\rangle$$

2) H_c becomes $H_c + 2 Z_L \tilde{X}_L Z_1$

$$\mathcal{U}|gs\rangle = |gs\rangle \qquad \tilde{\mathcal{U}}|gs\rangle = -|gs\rangle$$

$\rightsquigarrow$ Different responses $\Rightarrow$ different SPTs

$$Z_p[A,\tilde{A}] = 1 \qquad Z_c[A,\tilde{A}] = (-1)^{\int A \cup \tilde{A}}$$

Remarks on SPTs

- SPTs realized experimentally: ordinary vs topological insulators
- Another popular perspective: FDLUs $\rightsquigarrow$ has short comings
- Various proposed classifications for invertible symmetries

Symmetry anomalies

A symmetry is anomalous if it is incompatible with a trivial gapped phase (ie, it has no SPTs) $\Rightarrow$ all phases must be nontrivially gapped or gapless.

- Related to obstruction to gauging
- called an 't Hooft anomaly for internal symmetries
- called a Lieb-Schultz-Mattis (LSM) anomaly for internal + spatial syms.

A trivial example with a single qubit

→ Hilbert space $\mathcal{H} = \mathbb{C}^2 \simeq \text{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$

→ Most general Hamiltonian $H = \alpha X + \beta Y + \gamma Z$ $(\alpha, \beta, \gamma \in \mathbb{R})$

↳ $Y = iXZ$

→ Suppose H has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by X and Z:

$$[H, X] = [H, Z] = 0$$

- Forces $\alpha = \beta = \gamma = 0 \Rightarrow H = 0$, which has a 2-fold degeneracy.
  $\Rightarrow$ Forbids unique ground state
- More abstractly, Unique ground state forbiden by projective algebra $XZ = -ZX$.

$$\boxed{\therefore \text{ this } \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ symmetry is anomalous.}}$$

→ Remark: All symmetry anomalies in QM manifest through projective representations. More subtle in $d+1 > 0+1$.

Example 2

1d closed chain w/ 1 qubit per site and symmetries:

→ lattice translations $T: O_j \to O_{j+1}$

→ $\mathbb{Z}_2^x \times \mathbb{Z}_2^y$ symmetry generated by $U_x = \prod_{j=1}^{L} X_j$ and $U_y = \prod_{j=1}^{L} Y_j$

Infinitely-many Hamiltonians with this symmetry

eg) XY model: $H = \sum_{j=1}^{L} \left(J_x X_j X_{j+1} + J_y Y_j Y_{j+1} \right)$

U_x and U_y furnish a local projective rep

$$U_x U_y = (-1)^L U_y U_x .$$

→ Manifestation of an LSM anomaly between translations and $\mathbb{Z}_2^x \times \mathbb{Z}_2^y$

"Physics Proof" of LSM anomaly (See Ogata & Tasaki '19 for rigorous proof)

1) Insert U_x Sym. defect

$$H \rightarrow H_{defect} = H + \delta H_{\langle L,1 \rangle}$$

$$T \rightarrow T_{defect} = X_1 T$$

→ Now have projective algebra $T_{defect} U_y = -U_y T_{defect}$

(Translations $\times \mathbb{Z}_2^y$ projectively represented in U_x defect $\mathcal{H}$.
Like a type III anomaly.)

2) all states are doubly-degenerate in presence of a U_x symmetry defect

3) Can now argue this implies an obstruction to an SPT in absence of sym. defect

→ Proof by contradiction.

- Suppose H has unique gapped g.s. $|gs\rangle$

$$U_x |gs\rangle = |gs\rangle$$

- In $L \rightarrow \infty$, $U_x(a) = \prod_{j=a}^{\infty} X_j$ acts on $|gs\rangle$ by

$$U_x(a)|gs\rangle = U_a|gs\rangle \quad \Big\}\ \text{b/c } |gs\rangle \text{ is symmetric}$$

→ U_a is a unitary localized around $j=a$.

- U_a creates a U_x symmetry defect, so

$$H_{\text{defect}} = U_a H U_a^\dagger$$

→ Contradiction b/c H has $\text{gsd}=1$ by assumption while H_{defect} has $\text{gsd} \geq 2$.

∴ U_x, U_y, and translations have an LSM anomaly

Remarks on anomalies

- Constraints from anomalies are non-perturbative
- Symmetry anomalies do not signal inconsistent theory or require a bulk theory in one-higher dimension.
- Other types of similar anomalies : gravitational anomalies, Family anomalies.

Symmetry enforced entanglement

Bosonic SPTs in $(d+1)$D protected by a G Symmetry are classified by

$$H^{d+1}(BG, U(1)) - \text{torsor}$$

→ There is no mathematically canonical trivial SPT

→ Often times, $\mathcal{H} = \otimes_j \mathcal{H}_j$ and $\exists$ a Product state SPT $\otimes_j |\psi_j\rangle$ chosen as a non-canonical trivial SPT.

There can be obstructions to product state SPTs while preserving onsite structure of symmetry

→ a type of SPT-LSM constraint

→ e.g., SPTs of symmetries w/ a local projective representation.

<u>Example</u> [Jiang, Cheng, Qi, Lu '21]

1+1D system of $\mathbb{Z}_4 \times \tilde{\mathbb{Z}}_4$ qudits on $L \in 2\mathbb{Z}$ sites j.

→ $\mathbb{Z}_4$ qudit is 4-level system w/ clock and shift ops.

$$Z^4 = X^4 = 1 \qquad\qquad ZX = i\, XZ$$

→ Consider $\mathbb{Z}_4 \times \mathbb{Z}_4$ Symmetry operators

$$U = \prod_j X_j \tilde{X}_j \qquad\qquad V = \prod_j \left(Z_j \tilde{Z}_j \right)^{2j+1}$$

→ Local projective algebra

$$U_j V_j = - V_j U_j \qquad\qquad (UV = VU)$$

LSM anomaly?

→ Insert a U symmetry defect

$$T \longrightarrow \tilde{T} = U_1 T \qquad\qquad V \longrightarrow V$$

→ Now

$$\tilde{T}\, V = W\, V\, \tilde{T} \qquad \text{w/} \qquad W = -\prod_j Z_j^2 \tilde{Z}_j^2$$

No LSM anomaly!

→ Not a projective algebra, instead a non-abelian group

→ Compatible w/ unique gapped gs if $W|gs\rangle = |gs\rangle$.

Projectivity still has consequenses ⟹ SPT - LSM theorem

→ Assume SPT ground state $|\psi\rangle = \bigotimes_j |\psi_j\rangle$

$$U|\psi\rangle = |\psi\rangle \qquad V|\psi\rangle = |\psi\rangle$$

→ bc U and V are onsite:

$$U_j|\psi\rangle = |\psi\rangle \qquad V_j|\psi\rangle = |\psi\rangle$$

→ But U_j and V_j anticommute ⟹ contradiction to SPT state req.

$\therefore \bigotimes_j |\psi_j\rangle$ cannot be an SPT state.