TOPOLOGICAL HOLOGRAPHY AND SPACETIME SYMMETRY

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SIMONS FOUNDATION

A SYMMETRY RENAISSANCE

Our understanding of global symmetry in QFT and quantum lattice models is currently undergoing a dramatic revolution

Higher symmetries

Subsystem symmetry

Non-invertible symmetry

⋮

- Applications in classifying phases of matter, providing selection rules, new 't Hooft anomalies, naturalness, … [*Sungwoo's talk*]
- Topological holography has emerged as a powerful framework for studying/applying (generalized) symmetries [*Emily's talk*]

THIS TALK

1) Review topological holography

- Emphasis on simple examples from both field theory and quantum code points of view

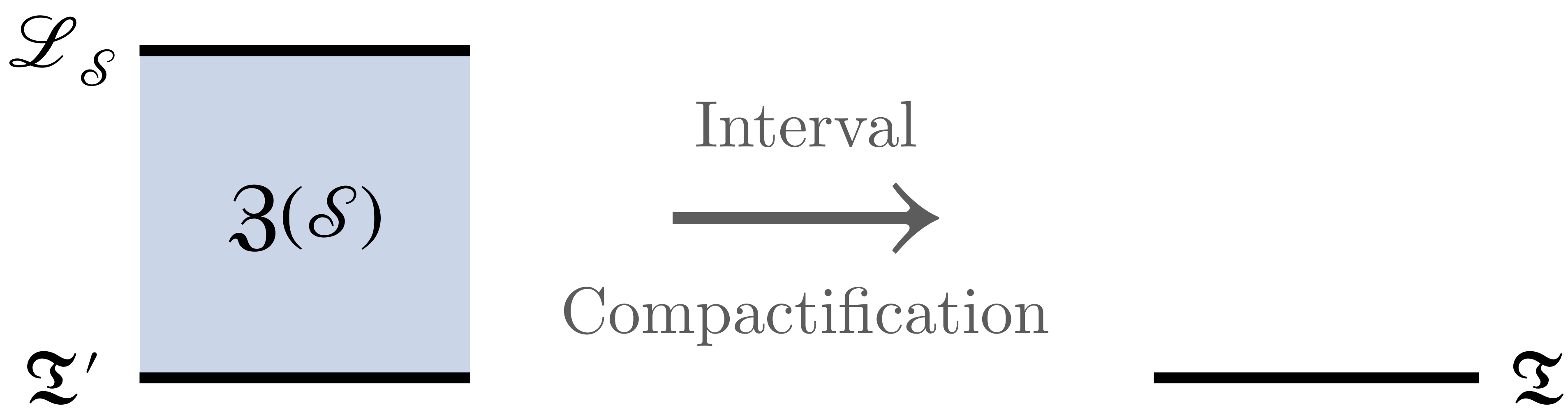
2) Discuss our proposal for including spacetime symmetries

3) Show applications to discrete spatial translation symmetry

- LSM anomalies and dipole symmetries in topological holography

TOPOLOGICAL HOLOGRAPHY

Given a $d+1$D theory $\mathfrak{T}$ with symmetry $\mathcal{S}$, topological holography is a framework for separating the kinematics $\mathcal{S}$ from the dynamics of $\mathfrak{T}$ using a $d+2$D topological theory $\mathfrak{Z}(\mathcal{S})$



- $\mathscr{L}_{\mathcal{S}}$ is a boundary whose defects are described by $\mathcal{S}$
- $\mathfrak{Z}(\mathcal{S})$ is a topological field theory called SymTFT
- $\mathfrak{T}'$ is a boundary realizing the dynamics of $\mathfrak{T}$

EXAMPLE (FIELD THEORY)

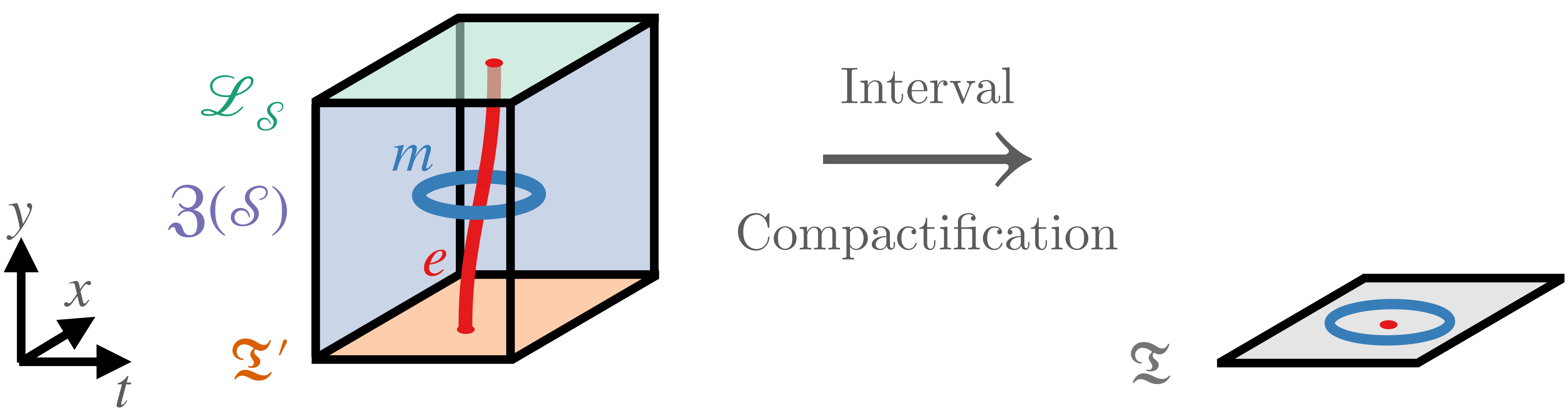
Example: $\mathcal{S} = \mathbb{Z}_2$ in $d + 1 = 1 + 1\mathrm{D}$

The SymTFT $\mathfrak{Z}(\mathcal{S})$ is the $2 + 1\mathrm{D}$ BF theory $L = \dfrac{2\mathrm{i}}{2\pi}\epsilon^{\mu\nu\rho}\, b_\mu \partial_\nu a_\rho$

- Topological defect lines $e(\gamma) = \mathrm{e}^{\mathrm{i}\int_\gamma a_\mu \mathrm{d}x^\mu}$ $\quad m(\gamma) = \mathrm{e}^{\mathrm{i}\int_\gamma b_\mu \mathrm{d}x^\mu}$

$$\langle e(\gamma_1)\, m(\gamma_2)\rangle = (-1)^{\mathrm{Link}(\gamma_1,\gamma_2)}$$

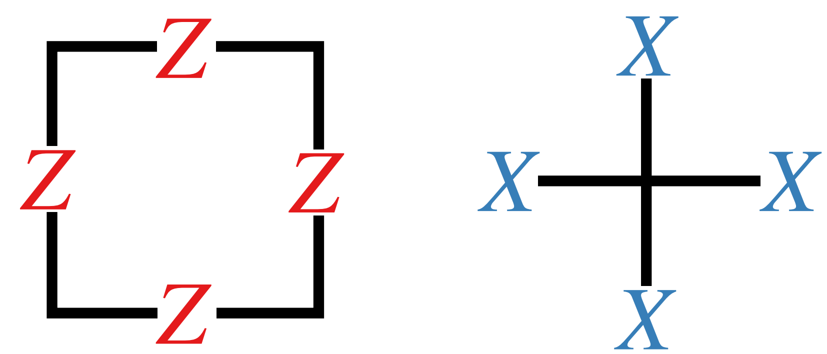
- $\mathscr{L}_\mathcal{S}$ boundary condition $a_0 = 0,\ a_x = 0$ (Dirichlet)



EXAMPLE (QUANTUM CODE)

Example: $\mathcal{S} = \mathbb{Z}_2$ in $d + 1 = 1 + 1$D

$\mathfrak{Z}(\mathcal{S})$ is the ground state space of $2 + 1$D toric code



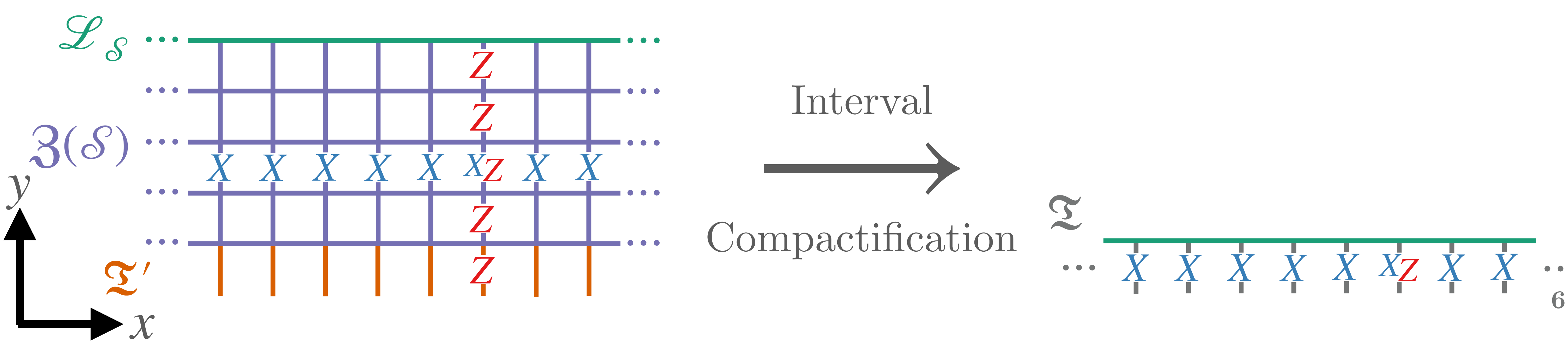
$X^2 = Z^2 = 1$

$XZ = -ZX$

- 1-form symmetry operators

$$e(\gamma) = \prod_{l \in \gamma} Z_l \qquad m(\gamma^\vee) = \prod_{l \perp \gamma^\vee} X_l$$

- $\mathcal{L}_{\mathcal{S}}$ is $Z = 1$ "rough" boundary



TOPOLOGICAL HOLOGRAPHY

Topological defects of SymTFT $\mathfrak{Z}(\mathcal{S})$ label $\mathcal{S}$-symmetry sectors $\mathscr{H}_q^{(s)}$ of s-twisted Hilbert space of $\mathfrak{T}$:

- $\mathcal{S} = \mathbb{Z}_2$ example: $1 = \mathscr{H}_{\text{id}}^{(0)} \qquad e = \mathscr{H}_{\text{sign}}^{(0)} \qquad m = \mathscr{H}_{\text{id}}^{(1)} \qquad f = \mathscr{H}_{\text{sign}}^{(1)}$
- Discrete gauging implemented by changing $\mathscr{L}_{\mathcal{S}}$

  ($\mathcal{S}_1$ and $\mathcal{S}_2$ related by discrete gauging have the same SymTFT)

TOPOLOGICAL HOLOGRAPHY

⋮

1 *Kong, Wen, Zheng (2014);*

1 *Freed, Teleman (2018);*

1 *Ji, Wen (2019);*

3 *Lichtman, Thorngren, Lindner, Stern, Berg (2020); Kong, Lan, Wen, Zhang, Zheng (2020); Gaiotto, Kulp (2020);*

1 *Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki (2021);*

7 *Chatterjee, Wen (2022); Apruzzi (2022); Chatterjee, Wen (2022); Moradi, Moosavian, Tiwari (2022); Freed, Moore, Teleman (2022); Kaidi, Ohmori, Zheng (2022); Chatterjee, Ji, Wen (2022);*

18 *Kaidi, Nardoni, Zafrir, Zheng (2023); Zhang, Córdova (2023); Lan, Zhou (2023); Bhardwaj, Schafer-Nameki (2023); Chen, Cui, Haghighat, Wang (2023); Apruzzi, Bonetti, Gould, Schafer-Nameki (2023); Bah, Leung, Waddleton (2023); Córdova, Hsin, Zhang (2023); Cao, Jia (2023);* ***SP*** *(2023); Baume, Heckman, Hübner, Torres, Turner, Yu (2023); Huang, Cheng (2023); Wen, Potter (2023); Inamura, Wen (2023); Schuster, Tantivasadakarn, Vishwanath, Yao (2023); Bhardwaj, Bottini, Pajer, Schafer-Nameki (2023);* ***SP****, Zhu, Beaudry, Wen (2023); Motamarri, McLauchlan, Béri (2023);*

19 *Brennan, Sun (2024); Antinucci, Benini (2024); Bonetti, Del Zotto, Minasian (2024); Apruzzi, Bedogna, Dondi (2024); Del Zotto, Nadir Meynet, Moscrop (2024); Bhardaj, Pajer, Schafer-Nameki, Warman (2024); Argurio, Benini, Bertolini, Galati, Niro (2024); Wen, Ye, Potter (2024); Franco, Yu (2024); Putrov, Radhakrishnan (2024); Chatterjee, Aksoy, Wen (2024); Bhardwaj, Bottini, Schafer-Nameki, Tiwari (2024); Arbalestrier, Arguio, Tizzano (2024); Huang (2024); Bhardwaj, Inamura, Tiwari (2024); Hasan, Meynet, Migliorati (2024); Nardoni, Sacchi, Sela, Zafrir, Zheng (2024); Heckman, Hübner (2024); Ji, Chen (2024)*

… and probably even more … 😅

INCLUDING SPACETIME SYMMETRIES

Proposal: incorporate effects from spacetime symmetries on $\mathcal{S}$ by enriching $\mathfrak{Z}(\mathcal{S})$ with spacetime symmetries

- Topological defects of $\mathfrak{Z}(\mathcal{S})$ can have nontrivial interplay with spacetime symmetry
- i.e., a symmetry enriched topological order (SET)

Why? Suppose $1 \to \mathcal{S} \to \mathcal{S}_{\text{total}} \to \mathcal{S}_{\text{spacetime}} \to 1$ in $d+1$D

- Gauging $\mathcal{S}$ of $d+2$D $\mathcal{S}$-SPT gives $\mathfrak{Z}(\mathcal{S})$
- Gauging $\mathcal{S}$ of $d+2$D $\mathcal{S}_{\text{total}}$-SPT gives $\mathfrak{Z}(\mathcal{S})$ enriched by $\mathcal{S}_{\text{spacetime}}$

DISCRETE TRANSLATIONS

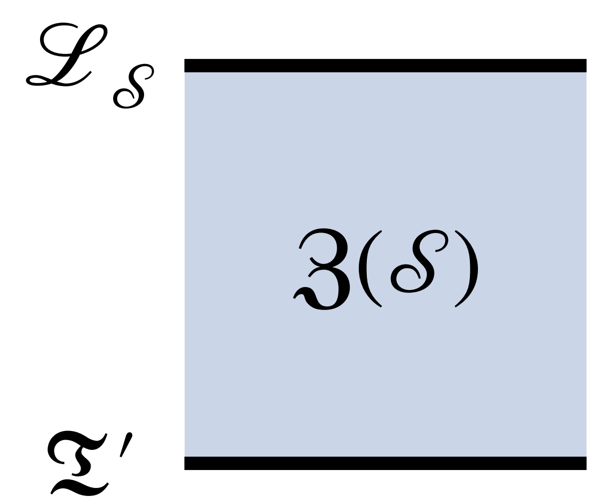
Here we explore including 1D discrete spatial translations

- SymTFT is a 2+1D translation SET

  *Wen (2003); Kitaev (2006); Essin, Hermele (2012); Oh, Kim, Moon, Hoon Han (2021); Rao, Sodemann (2021); **SP**, Wen (2022); Song, Senthil (2022); Delfino, Fontana, Gomes, Chamon (2022); Tam, Venderbos, Kane (2022); Oh, **SP**, Han, You, Lee (2023); Delfino, Chamon, You (2023)*

- SymTFT is generally only a foliated TFT.

  Slagle, Aasen, Williamson (2018); Gorantla, Lam, Seiberg, Shao (2020); Ohmori, Shimamura (2022); Spieler (2023); Cao, Jia (2023); Ebisu, Honda, Nakanishi (2023); Ebisu, Honda, Nakanishi (2024)



Interplay of discrete translations and $\mathscr{S}$:

- Modulated symmetry
- Lieb-Schultz-Mattis (LSM) anomalies

$\vdots$

MODULATED SYMMETRY

$\mathcal{S}$ is a (spatially) modulated symmetry if its symmetry transformation is spatially position-dependent

$$\mathcal{S}_{\text{total}} = \text{``}\mathcal{S} \rtimes_{\varphi} \mathcal{S}_{\text{space}}\text{''} \qquad \varphi : \mathcal{S}_{\text{space}} \to \mathrm{Aut}(\mathcal{S})$$

➤ $\mathcal{S}$ can be a generalized symmetry *Oh, **SP**, Han, You, Lee (2023)*

➤ Standard example is $1+1$D dipole symmetry:

$$[\mathrm{e}^{\mathrm{i}a\,Q}, H] = [\mathrm{e}^{\mathrm{i}b\,P}, H] = 0 \qquad T\mathrm{e}^{\mathrm{i}\,P}T^{-1} = \mathrm{e}^{\mathrm{i}\,Q}\mathrm{e}^{\mathrm{i}\,P}$$

E.g., Lifshitz-type theories

Griffin, Grosvenor, Horava, Yam (2013); Seiberg (2020); Gorantla, Lam, Seiberg, Shao (2022), ⋯

$$L = (\partial_x^2\varphi)^2 - (\partial_t\varphi)^2 \ \text{ invariant under } \varphi \to \varphi + a + b\,x$$

LSM ANOMALIES

Lieb-Schultz-Mattis (LSM) anomaly is an obstruction to a gapped SPT phase due to an interplay of internal and lattice symmetries *Lieb, Schultz, Mattis (1961); Oshikawa (1999); Hastings (2003); Hastings (2004); ... Yao, Oshikawa (2019)*

- Match 't Hooft anomalies in the continuum

  Cho, Hsieh, Ryu (2017); Cheng, Seiberg (2022); Aksoy, Mudry, Furusaki, Tiwari (2023); Seifnashri (2023)

Example: LSM anomaly in the $1+1$D XY model ($j \sim j + L$)

$$H = -\sum_j X_j X_{j+1} + Y_j Y_{j+1}$$

- Symmetry operators $U_x = \prod_j X_j$ and $U_y = \prod_j Y_j$ obey

$$U_x U_y = (-1)^L U_y U_x$$

SYMTFT FOR DIPOLE SYMMETRY

What is the SymTFT for a $1+1$D $\mathbb{Z}_2$ dipole symmetry?

1. Start with trivial $\mathcal{S}_{\text{total}} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_{x-\text{dir}}$ SPT in $2+1$D
2. Gauge the $\mathbb{Z}_2 \times \mathbb{Z}_2$ sub-symmetry of $\mathcal{S}_{\text{total}}$ to get $\mathfrak{Z}(\mathcal{S})$
3. SymTFT is $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory nontrivially enriched by $\mathbb{Z}_{x-\text{dir}}$ translations (symmetry operators form a 2-group)

SYMTFT (FIELD THEORY)

$\mathfrak{Z}(\mathscr{S})$ is level 2 "x-dipole" BF theory

$L = \frac{}{2} \ldots a_{xx})\Big)$

m_2 and e_1 are modulated 1-form symmetries

- Under discrete translation $x \to x + \Lambda^{-1}$

$$m_2 \to m_1\, m_2$$
$$e_1 \to e_2\, e_1$$

- Gaug…

$a_0 \ldots f^{(a)}$

$b_0 \sim b_0 + \partial_t f^{(b)}$ $\quad$ $b_{xx} \sim b_{xx} + \partial_x^2 f^{(b)}$ $\quad$ $b_y \sim b_y + \partial_y f^{(b)}$

- Topological defect lines

$$m_1(C) = \mathrm{e}^{\mathrm{i}\Lambda^{-1}\int_C [b_{xx}]\mathrm{d}x + [\partial_x b_y]\mathrm{d}y + [\partial_x b_0]\mathrm{d}t} \qquad m_2(C) = \mathrm{e}^{\mathrm{i}\int_C [x b_{xx}]\mathrm{d}x + [x\partial_x b_y - b_y]\mathrm{d}y + [x\partial_x b_0 - b_0]\mathrm{d}t}$$

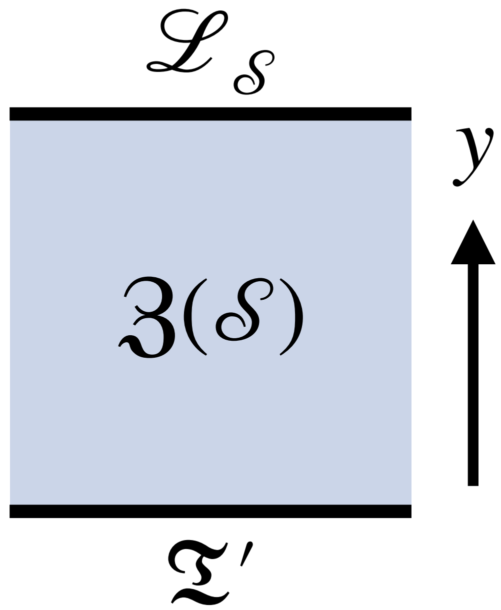
$$e_2(C) = \mathrm{e}^{\mathrm{i}\Lambda^{-1}\int_C [a_{xx}]\mathrm{d}x + [\partial_x a_y]\mathrm{d}y + [\partial_x a_0]\mathrm{d}t} \qquad e_1(C) = \mathrm{e}^{\mathrm{i}\int_C [x a_{xx}]\mathrm{d}x + [x\partial_x a_y - a_y]\mathrm{d}y + [x\partial_x a_0 - a_0]\mathrm{d}t}$$

SYMTFT AS A FOLIATED TFT

$\mathfrak{Z}(\mathscr{S})$ is level 2 "x-dipole" BF theory

$$L = \frac{2\mathrm{i}}{2\pi\Lambda}\Big(b_y\partial_t a_{xx} + a_y\partial_t b_{xx} + a_0(\partial_x^2 b_y - \partial_y b_{xx}) - b_0(\partial_x^2 a_y - \partial_y a_{xx})\Big)$$

This is a foliated TQFT



- No local dof, only defect lines
- All defects are topological in the y,t directions
- Not all defects are topological in x-direction (i.e., $\mathrm{e}^{\mathrm{i}\oint a_y\,\mathrm{d}y + a_0\,\mathrm{d}t}$ and $\mathrm{e}^{\mathrm{i}\oint b_y\,\mathrm{d}y + b_0\,\mathrm{d}t}$)

Can be constructed by gauging the foliated $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT

$$L = \frac{2\mathrm{i}}{2\pi} A \wedge B \wedge \mathrm{d}x$$

Ebisu, Honda, Nakanishi (2024)

DIPOLE BOUNDARY (FIELD THEORY)

Dipole symmetry boundary $\mathscr{L}_{\mathcal{S}}$:

- Gauge fields satisfy Dirichlet boundary condition

$$a_0 = 0 \qquad\qquad a_{xx} = 0$$

Topological defects e_i and m_i on this boundary become

$$m_1(C) = \mathrm{e}^{\mathrm{i}\Lambda^{-1}\int_C [b_{xx}]\mathrm{d}x + [\partial_x b_0]\mathrm{d}t} \qquad\qquad m_2(C) = \mathrm{e}^{\mathrm{i}\int_C [x b_{xx}]\mathrm{d}x + [x\partial_x b_0 - b_0]\mathrm{d}t}$$

$$e_2(C) = 1 \qquad\qquad e_1(C) = 1$$

- Discrete translations T: $x \to x + \Lambda^{-1}$ causes

$$(m_1, m_2) \to (m_1, m_1 m_2)$$

LSM BOUNDARY (FIELD THEORY)

LSM symmetry boundary $\mathscr{L}_{\mathcal{S}}$:

➤ Introduce boundary fields $\tilde{a}_x \sim \tilde{a}_x + \partial_x f^{(a)}, \;\; \tilde{b}_x \sim \tilde{b}_x + \partial_x f^{(b)}$

➤ Boundary conditions:

$$\partial_x a_0 = \partial_t \tilde{a}_x \qquad\qquad \partial_x b_0 = \partial_t \tilde{b}_x$$

$$a_{xx} = \partial_x \tilde{a}_x \qquad\qquad b_{xx} = \partial_x \tilde{b}_x$$

Topological defects e_i and m_i on this boundary become

$$m_1(C) = 1 \qquad\qquad m_2(C) = \mathrm{e}^{-\mathrm{i}\int_C \tilde{b}_x\,\mathrm{d}x + b_0\,\mathrm{d}t}$$

$$e_2(C) = 1 \qquad\qquad e_1(C) = \mathrm{e}^{-\mathrm{i}\int_C \tilde{a}_x\mathrm{d}x + a_0\mathrm{d}t}$$

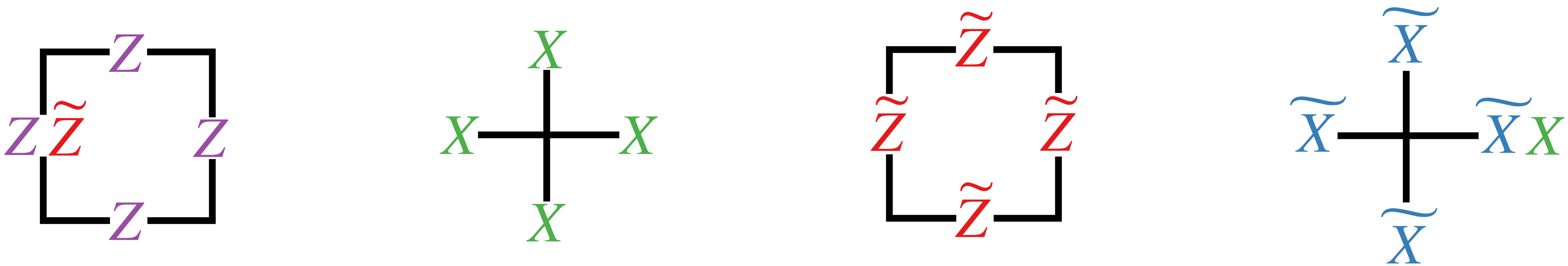
➤ Using this $\mathscr{L}_{\mathcal{S}}$, there are no translation-preserving gapped SPT phases (follows from e_1 and m_2 being modulated)

SYMTFT (QUANTUM CODE)

$\mathfrak{Z}(\mathcal{S})$ is stabilizer code with two qubits on each link of the square lattice

$$X^2 = Z^2 = \widetilde{X}^2 = \widetilde{Z}^2 = 1$$
$$XZ = -ZX$$
$$\widetilde{X}\widetilde{Z} = -\widetilde{Z}\widetilde{X}$$

- Stabilizers:



1-form symmetry operators

$$m_1(\gamma^\vee) = \prod_{(\boldsymbol{r},i)\perp\gamma^\vee} X_{\boldsymbol{r},i} \qquad m_2\left(\gamma^\vee\right) = \prod_{(\boldsymbol{r},i)\subset\gamma^\vee} \left[\widetilde{X}_{\boldsymbol{r},y} X^{x-1}_{\boldsymbol{r},y}\right]\delta_{i,y} + \left[\widetilde{X}_{\boldsymbol{r},x} X^{x}_{\boldsymbol{r},x}\right]\delta_{i,x}$$

$$e_2(\gamma) = \prod_{(\boldsymbol{r},i)\subset\gamma} \widetilde{Z}_{\boldsymbol{r},i} \qquad e_1(\gamma) = \prod_{(\boldsymbol{r},i)\in\gamma} \left[Z_{\boldsymbol{r},x} \widetilde{Z}^{x}_{\boldsymbol{r},x}\right]\delta_{i,x} + \left[Z_{\boldsymbol{r},y} \widetilde{Z}^{x-1}_{\boldsymbol{r},y}\right]\delta_{i,y}$$

DIPOLE BOUNDARY (QUANTUM CODE)

Dipole symmetry boundary stabilizers (rough boundary)

On this symmetry boundary, the 1-form symmetry operators become

$$e_1 = e_2 = 1 \qquad m_1 = \prod_{j=1}^{L_x} X_{(j,L_y-1),y} \qquad m_2 = \prod_{j=1}^{L_x} \widetilde{X}_{(j,L_y-1),y} X^{j-1}_{(j,L_y-1),y}$$

➤ Satisfies $T m_2 T^{-1} = m_1 m_2 \implies$ dipole symmetry

LSM BOUNDARY (QUANTUM CODE)

LSM symmetry boundary stabilizers (boundary dof: qubits)

On this symmetry boundary, the 1-form symmetry operators become

$$m_1 = e_2 = 1 \qquad e_1 = \prod_{j=1}^{L_x} Z_{(j,L_y),x} \qquad m_2 = \prod_{j=1}^{L_x} \widetilde{X}_{(j,L_y-1),y} X_{(j,L_y),x}$$

➤ Satisfies $e_1 m_2 = (-1)^{L_x} m_2 e_1 \implies$ LSM anomaly

GENERAL FINITE GROUP G

More generally, we construct the SymTFT for a modulated $G \times Z(G)$ symmetry

<u>Gauging web</u>

$$
\begin{tikzcd}
(G \times Z(G)) \rtimes \mathbb{Z} \arrow[r, shift left, "Z(G)"] \arrow[d, shift left, "G"] & G \times Z(G) \times \textit{Non-invertible translations} \arrow[l, shift left, "Z(G)"] \arrow[d, shift left, "G"] \\
\textit{LSM:}\ \mathsf{Rep}\,(G) \times Z(G) \times \mathbb{Z} \arrow[r, shift left, "Z(G)"] \arrow[u, shift left, "\mathsf{Rep}\,(G)"] & (\mathsf{Rep}\,(G) \times Z(G)) \rtimes \mathbb{Z} \arrow[l, shift left, "Z(G)"] \arrow[u, shift left, "\mathsf{Rep}\,(G)"]
\end{tikzcd}
$$

- Non-invertible symmetry with LSM anomaly
- Non-invertible modulated symmetry

THANKS FOR LISTENING :-)

Two open questions:

- What about other spacetime symmetries and different types of symmetry enrichment for the SymTFT
- What is the general relationship between foliated SPTs (a.k.a. weak SPTs) and foliated SymTFTs for modulated symmetries?