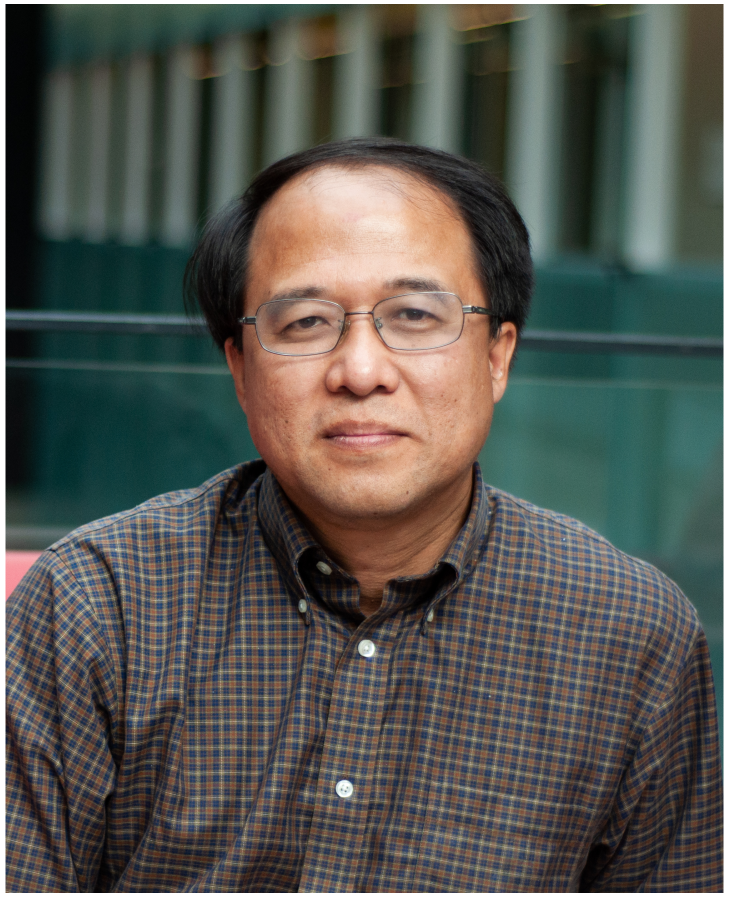
UV/IR MIXING IN THE $\mathbb{Z}_N$ RANK-2 TORIC CODE

Sal Pace (MIT)

arXiv:2204.07111

WORK IN COLLABORATION WITH

Xiao-Gang Wen (MIT)



Position-Dependent Excitations and UV/IR Mixing in the $\mathbb{Z}_N$ Rank-2 Toric Code and its Low-Energy Effective Field Theory

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(Dated: April 15, 2022)

arXiv:2204.07111

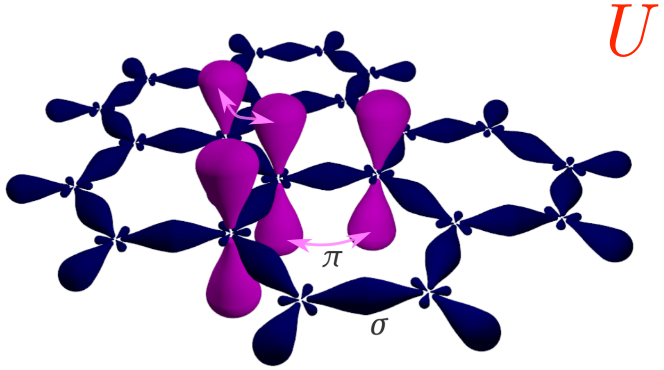
WILSON'S PERSPECTIVE

Short Distances (UV)

Long Distances (IR)

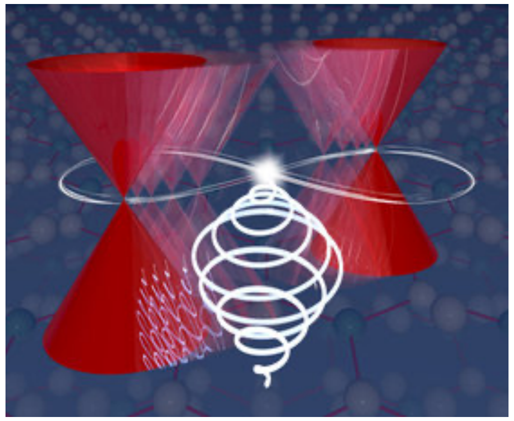
e.g., graphene

UV



$$H = \sum_{\langle i,j \rangle} t_{ij}\, c_i^{\dagger} c_j$$

IR



$$\mathscr{L} = \bar{\psi}\ \mathrm{i}\partial_\mu \gamma^\mu\ \psi$$

The IR *decouples* from the UV

WILSON'S *PROPAGANDA*?

Our Hero: Wilson

The Villain: UV/IR Mixing

UV/IR Mixing: The IR *depends* on the UV

- Long-distance sensitivity to short-distance details
- Mixing between small momenta and low energy with large momenta
- Cannot even define the IR theory without referring to the UV theory

WILSON'S *PROPAGANDA*?

Our Hero: Wilson

The Villain: UV/IR Mixing

UV/IR Mixing: The IR *depends* on the UV

- Long-distance sensitivity to short-distance details

In high-energy:

- Noncommutative field theory
- Quantum Gravity

Minwalla, Van Raamsdonk, & Seiberg, JHEP 2000.02 (2000): 020
Douglas & Nekrasov, Rev. Mod. Phys. 73.4 (2001): 977.
Grosse, Steinacker, & Wohlgenannt. JHEP 2008.04 (2008): 023
Berglund et al. arXiv:2202.06890 (2022).

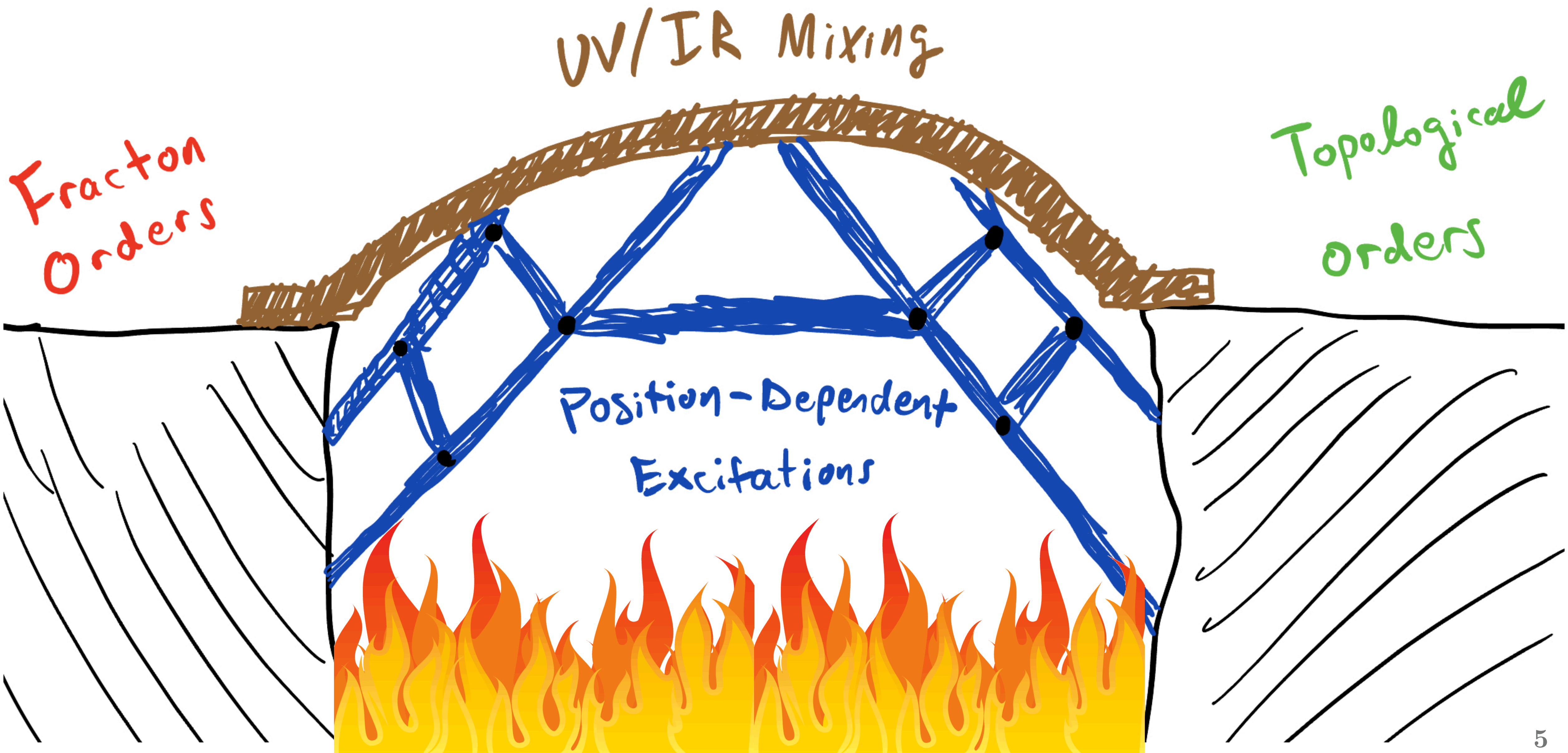
In quantum matter:

- Fracton topological order
- Exciton Bose liquid
- Marginal Fermi liquids

Gorantla et al. PRB 104.23 (2021): 235116.
You & Moessner. arXiv:2106.07664 (2021).
Lake, PRB 105.7 (2022): 075115.
Ye, Lee, & Zou, PRL 128.10 (2022): 106402.

DESTINATION FOR THIS TALK

Befriend UV/IR mixing and conceptually bridge some fracton orders and some topological orders



THE AGENDA

1. Review Topological and Fracton Order
2. Position-Dependent Excitations
3. $\mathbb{Z}_N$ Rank-2 Toric Code (R2TC)
4. UV/IR mixing in R2TC from the UV
5. UV/IR mixing in R2TC from the IR

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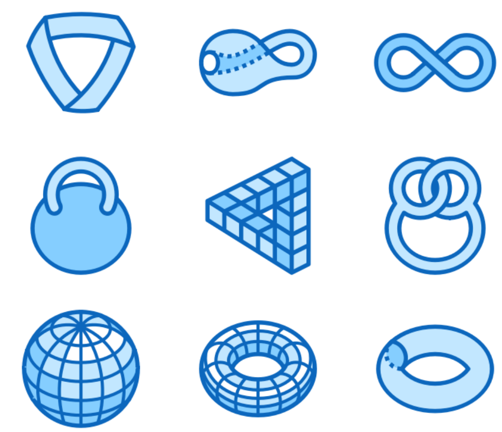
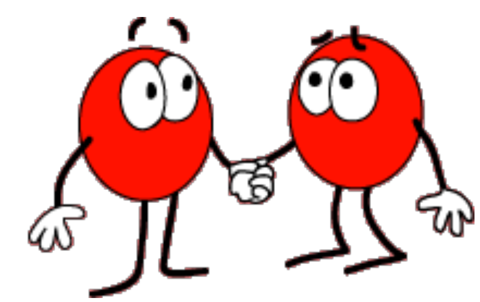
TOPOLOGICAL ORDER

Topological orders occur in gapped, long-range entangled quantum phases. Two key characteristics in our story:

Topological Degeneracy: number of ground states depends on the topology of spacetime



Topological Excitations: particles that (1) cannot be created alone and (2) can be anyons



The non-local order of a discrete higher-symmetry SSB phase?

Wen, Int. J. Mod. Phys. B 4.02 (1990)
Kitaev & Preskill, PRL 96.11 (2006): 110404.
Levin & Wen, PRL 96.11 (2006): 110405.
Chen ,Gu, & Wen, PRB 82.15 (2010): 155138.
Wen, Science 363.6429 (2019)
McGreevy, arXiv:2204.03045 (2022).

TOPOLOGICAL ORDER

Our Hero: Wilson

UV: Lattice Models

IR: Topological Quantum Field Theories

e.g., 2+1D $\mathbb{Z}_2$ topological order

UV: toric code

$$H = -\sum_{s} \prod_{e \in \delta s} \sigma_e^z - \sum_{p} \prod_{e \in \partial p} \sigma_e^x$$

IR: mutual Chern-Simons theory

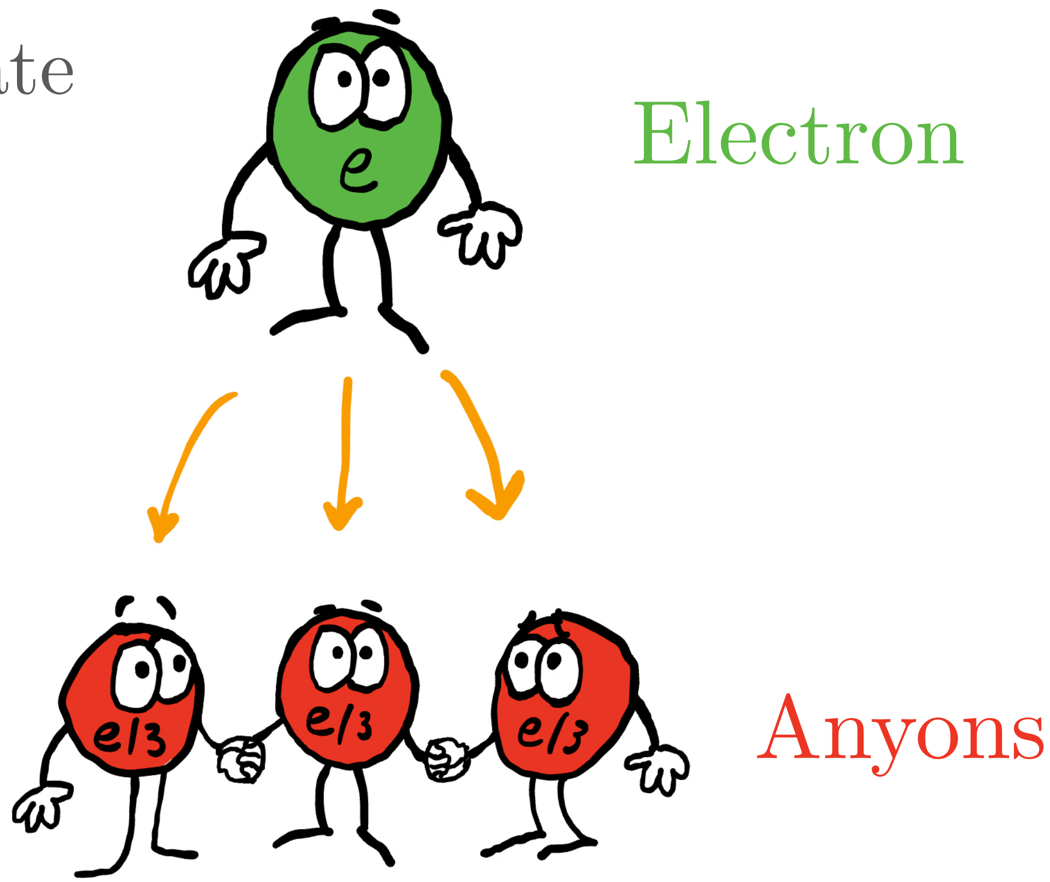
$$S[a, b] = \frac{1}{\pi} \int_{X^3} a \wedge \mathrm{d}b$$

TOPOLOGICAL ORDER + SYMMETRY

The interplay between topological order and symmetry leads to symmetry-enriched topological (SET) order

Conventional SET: Excitations carry fractional quantum numbers of the symmetry (symmetry fractionalization)

- e.g., $\nu = \frac{1}{3}$ FQH state



Wen, PRB 65.16 (2002): 165113.
Kou & Wen, PRB 80.22 (2009): 224406.
Essin & Hermele, PRB 87.10 (2013): 104406.
Mesaros & Ran, PRB 87.15 (2013): 155115.

TOPOLOGICAL ORDER + SYMMETRY

The interplay between topological order and symmetry leads to symmetry-enriched topological (SET) order

Unconventional SET: Excitations also <u>change type</u> under symmetry transformation

- Two excitations are of the same type if they can be transformed into each other using local operators
- 

may involve UV/IR mixing

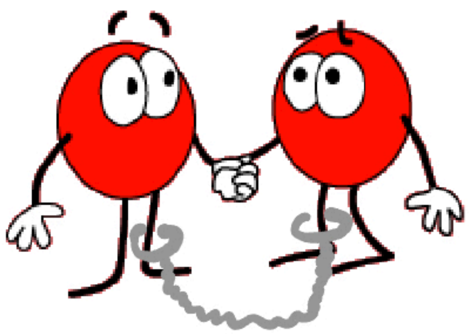
Kou, Levin, & Wen, PRB 78.15 (2008): 155134.
Lu & Vishwanath, PRB 93.15 (2016): 155121.
Tarantino, Lindner, & Fidkowski, New J. Phys. 18.3 (2016): 035006.
Barkeshli et al., PRB 100.11 (2019): 115147.

FRACTON ORDER

Fracton topological orders occur in gapped, long-range entangled 3+1D quantum phases. Two key characteristics in our story:

Subextensive Topological Degeneracy: number of grounds states depends on the topology and geometry of lattice

Subdimensional Topological Excitations: particles (1) cannot be created alone and (2) cannot move alone

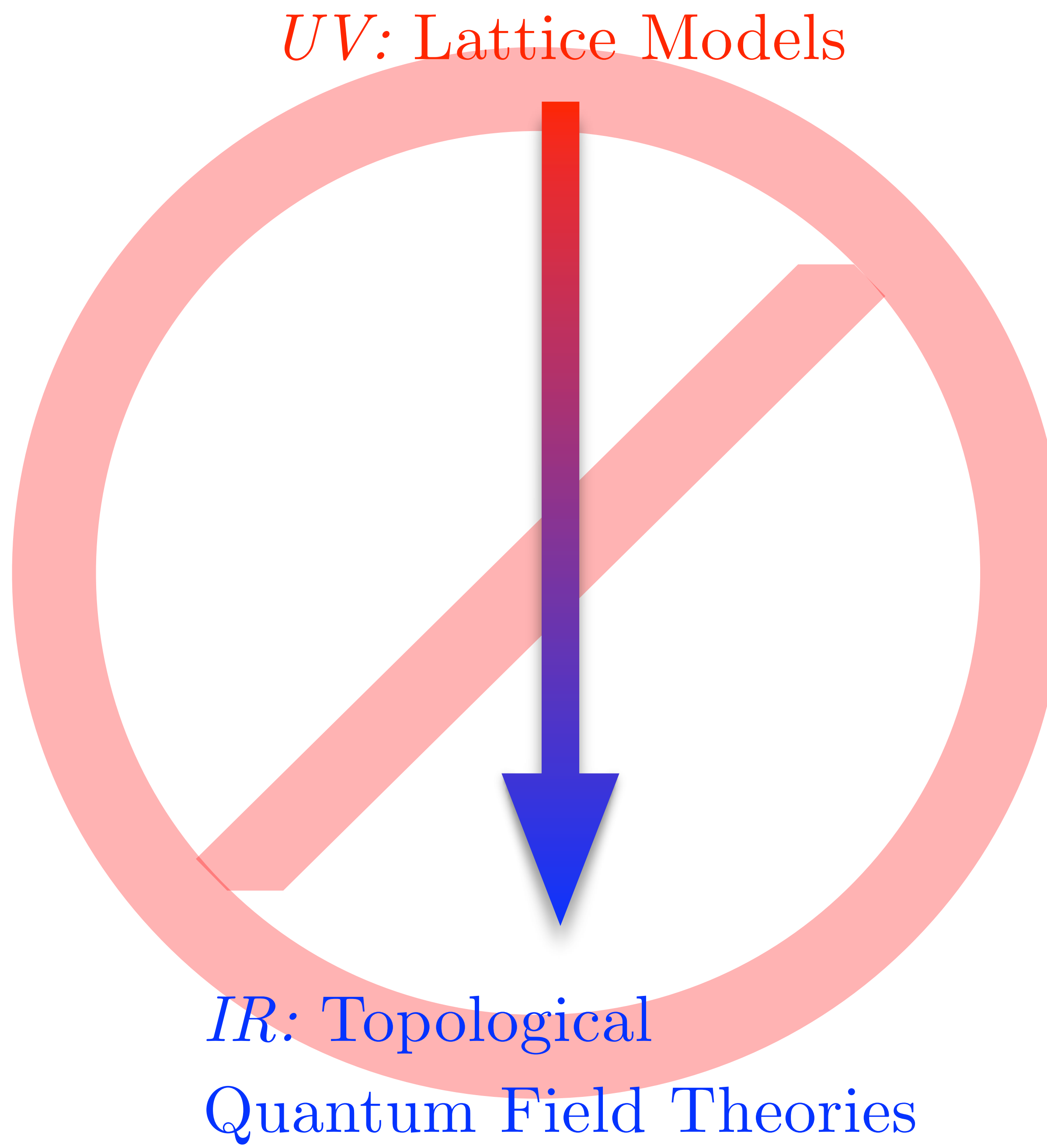


Fractons: completely immobile
Lineons: can move along lines
Planons: can move within planes

Chamon, PRL 94.4 (2005): 040402.
Haah, PRA 83.4 (2011): 042330.
Yoshida, PRB 88.12 (2013): 125122.
Vijay, Haah, & Fu. PRB 92.23 (2015): 235136.
Nandkishore & Hermele, Annu. Rev. Condens. Matter Phys. 10 (2019): 295-313.
Pretko, Chen, & You, Int. J. Mod. Phys. A 35.06 (2020): 2030003.

FRACTON ORDER

The Villain:
UV/IR Mixing



Slagle & Kim, PRB 96.19 (2017): 195139.
Radicevic, arXiv:1910.06336 (2019).
Slagle, Aasen, & Williamson, SciPost Physics 6.4 (2019): 043.
You et al., Phys. Rev. Res. 2.2 (2020): 023249.
Rudelius, Seiberg, & Shao. PRB 103.19 (2021): 195113.
Fontana, Gomes, & Chamon, SciPost Physics Core 4.2 (2021): 012.
Gorantla et al., arXiv:2201.10589 (2022).

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POSITION DEPENDENT EXCITATIONS

Position dependent excitations: excitations at different lattice sites are different types.

- Excitations that change type under lattice translations

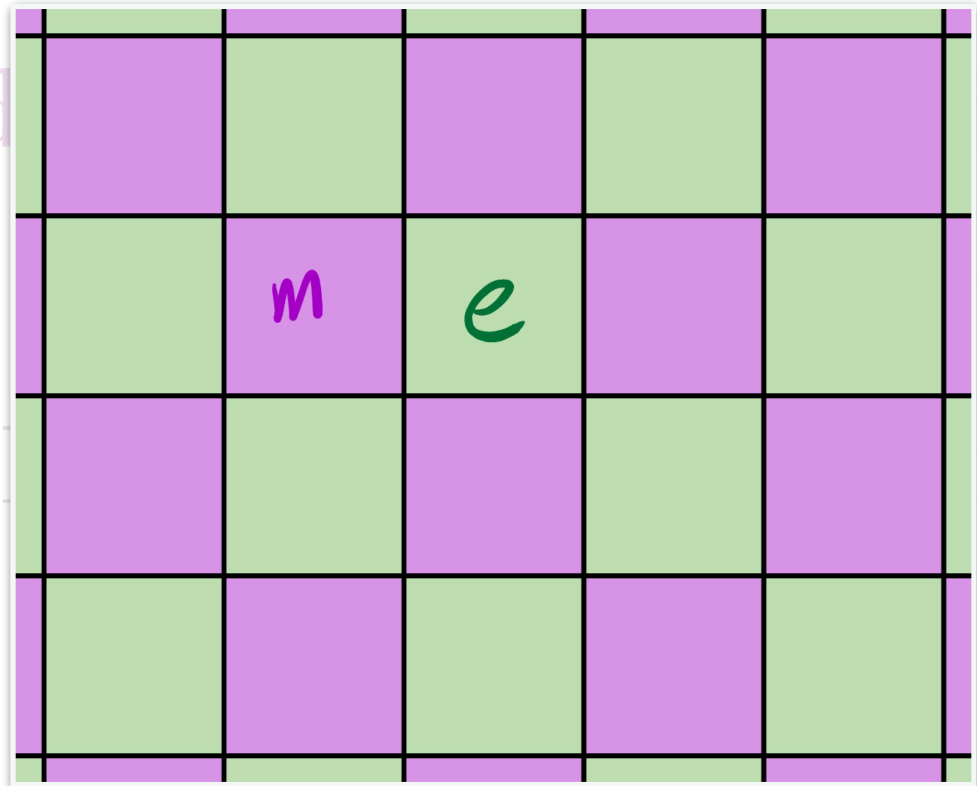
Unconventional SET orders: $(T_i)^n : a \mapsto a_n \neq a$ for $n < N$ where $N < L_i$ s.t. $(T_i)^N : a \mapsto a$ $(i.e., a_N = a)$

e.g., Wen's Plaquette model: $T_{x,y} : e \mapsto m$ and $T_{x,y} : m \mapsto e$ but $T^2_{x,y} : e \mapsto e$ and $T^2_{x,y} : m \mapsto m$. (Also $T_xT_y : e \mapsto e$, $T_xT_y : m \mapsto m$)

POSITION DEPENDENT EXCITATIONS

Position depend itations at different lattice pes.

- Excitations t der lattice translations



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POSITION DEPENDENT EXCITATIONS

Position dependent excitations: excitations at different lattice sites are different types.

- Excitations that change type under lattice translations

Fracton orders: $(T_i)^n : a \mapsto a_n \neq a$ for all $n < L_i$

Fractons: completely immobile so $i = x, y, z$

Lineons: e.g., can move in z direction: $i = x, y$ but $T_z : \ell \mapsto \ell$

Planons: e.g., can move in xy plane: $i = z$ but $T_{x,y} : p \mapsto p$

POSITION DEPENDENT EXCITATIONS

All subdimensional particles are position dependent excitations, but not all position dependent excitations are subdimensional particles

- Key point: all position dependent excitations have restricted mobility

Claim: position dependent excitations causes UV/IR mixing

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HIGGS AND TOPOLOGICAL ORDER

The Higgs phases of $U(1)$ gauge theory can have topological order

- Higgs phase of vector $U(1)$ gauge theory where $U(1) \to \mathbb{Z}_N$: exactly solvable point corresponds to the $\mathbb{Z}_N$ toric code

What about Higgs phase of symmetric rank-2 tensor $U(1)$ gauge theory?

Fradkin & Shenker, PRD 19.12 (1979): 3682.

SYMMETRIC TENSOR GAUGE THEORY

Deconfined phase of symmetric tensor $U(1)$ gauge theories are gapless fracton phases.

- Gapped subdimensional particles with gapless "photon" mode

Example: tensor gauge field A^{ij} and electric field E^{ij} in 2+1D with Gauss law $\rho = \partial_i \partial_j E^{ij}$

Gauge charges are fractons

"Flux loops" are lineons

Pretko, PRB 95.11 (2017): 115139.

2+1D RANK-2 TORIC CODE

Square lattice with two $\mathbb{Z}_N$ quantum rotors on each site and one $\mathbb{Z}_N$ quantum rotors on each plaquette

- Rank-2 Toric Code (R2TC): Exactly solvable point in the Higgs phase of symmetric rank-2 tensor $U(1)$ gauge theory

Ma, Hermele, & Chen, PRB 98.3 (2018): 035111.
Bulmash & Barkeshli, PRB 97.23 (2018): 235112. *Oh et al., PRB 105.4 (2022): 045128.*

2+1D RANK-2 TORIC CODE

Square lattice with two $\mathbb{Z}_N$ quantum rotors on each site and one $\mathbb{Z}_N$ quantum rotors on each plaquette

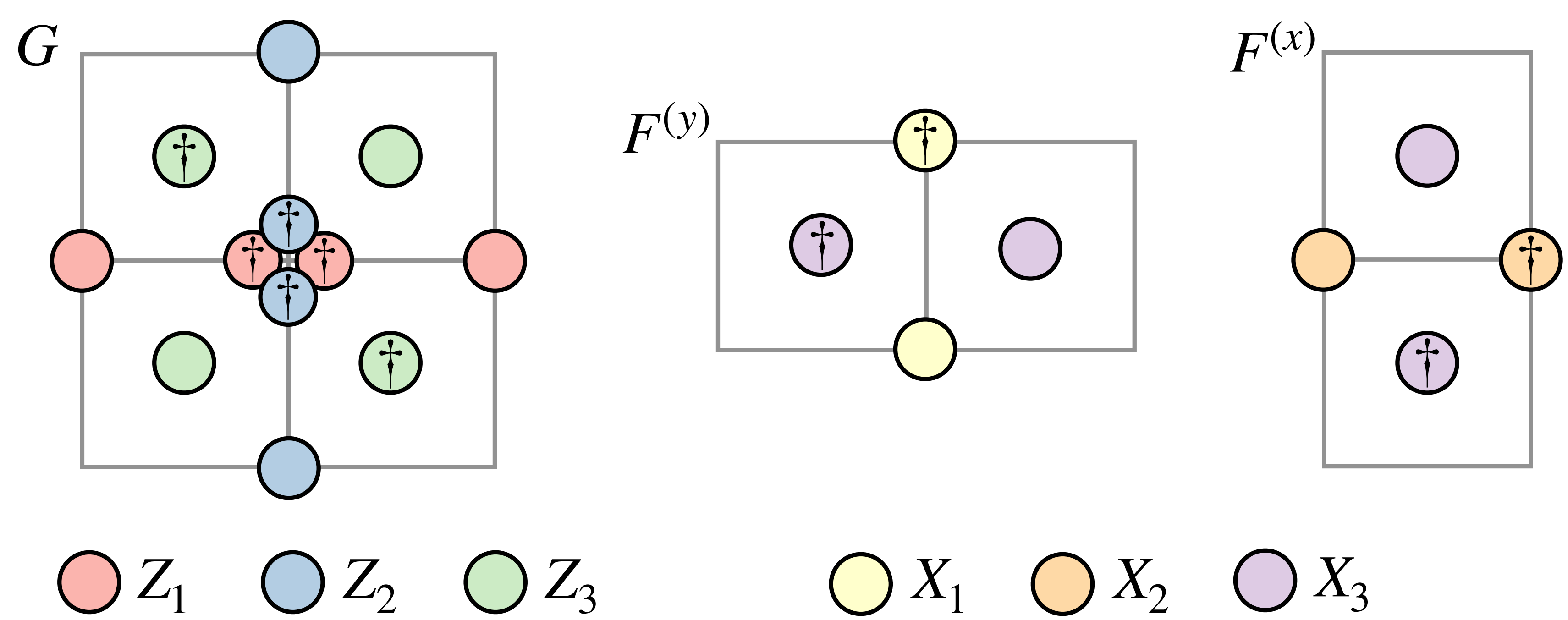
$$H = -\sum_{x,y}\left(G_{x,y} + F^{(x)}_{x,y} + F^{(y)}_{x,y} + \mathrm{h.c.}\right)$$

$\mathbb{Z}_N$ clock operators:

$$Z_j X_i = \omega^{\delta_{i,j}} X_i Z_j$$

$$\omega = \exp[2\pi \mathrm{i}/N]$$

$$X_i^N = Z_i^N = 1$$



Ma, Hermele, & Chen, PRB 98.3 (2018): 035111.
Bulmash & Barkeshli, PRB 97.23 (2018): 235112.
Oh et al., PRB 105.4 (2022): 045128.

R2TC IS EXACTLY SOLVABLE

$$H = -\sum_{x,y}\left(G_{x,y} + F^{(x)}_{x,y} + F^{(y)}_{x,y} + \text{h.c.}\right)$$

Exactly solvable because: $[G_{x,y}, F^{(x)}_{x',y'}] = [G_{x,y}, F^{(y)}_{x',y'}] = 0$

- The ground state $|\text{vac}\rangle$ satisfies:

$$G_{x,y}|\text{vac}\rangle = |\text{vac}\rangle \qquad F^{(x)}_{x,y}|\text{vac}\rangle = |\text{vac}\rangle \qquad F^{(y)}_{x,y}|\text{vac}\rangle = |\text{vac}\rangle$$

- A gapped excited states $|\psi\rangle$ satisfies:

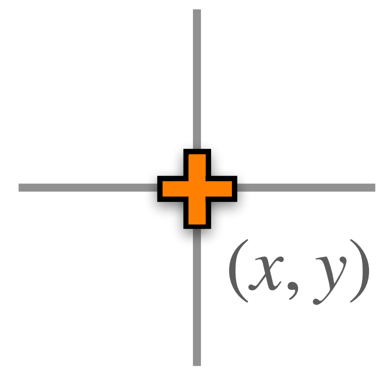
$$G_{x,y}|\psi\rangle = \omega^{\mathfrak{e}_{x,y}}|\psi\rangle \qquad F^{(x)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(x)}_{x,y}}|\psi\rangle \qquad F^{(y)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(y)}_{x,y}}|\psi\rangle$$

EXCITATIONS IN THE R2TC

Three species of excitations:

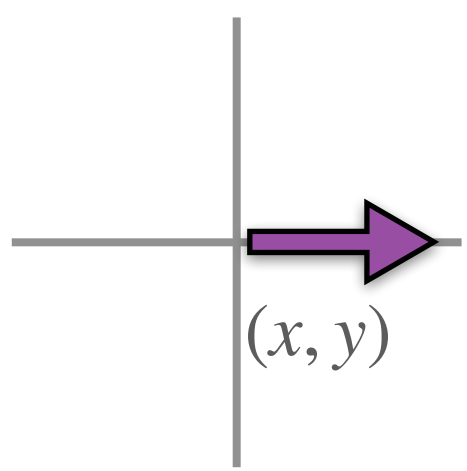
$\mathbb{Z}_N$ charge: $G_{x,y}|\psi\rangle = \omega^{\mathfrak{e}_{x,y}}|\psi\rangle$

$e_{x,y}$

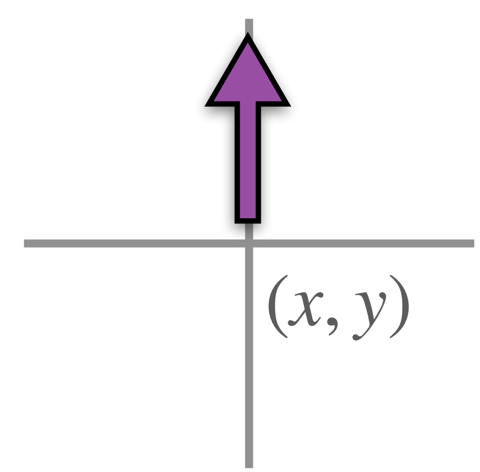


$\mathbb{Z}_N$ fluxes: $F^{(x)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(x)}_{x,y}}|\psi\rangle$ and $F^{(y)}_{x,y}|\psi\rangle = \omega^{\mathfrak{m}^{(y)}_{x,y}}|\psi\rangle$

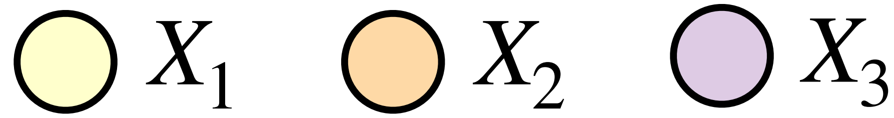
$m^{(x)}_{x,y}$

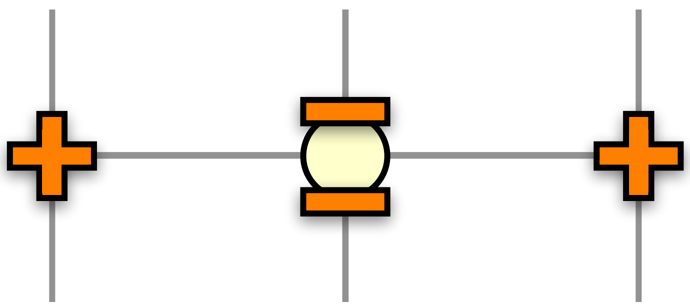


$m^{(y)}_{x,y}$

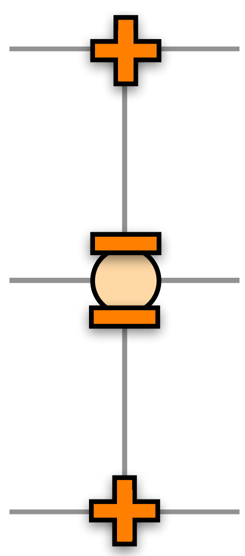


GENERATING FUSION RULES

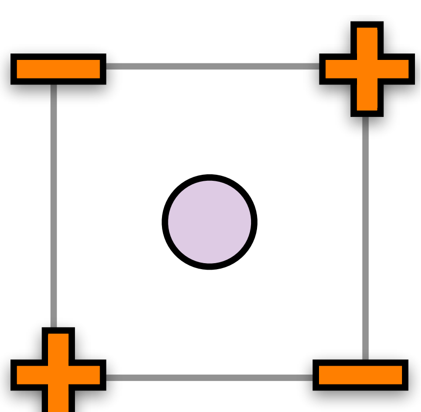




$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y}$$



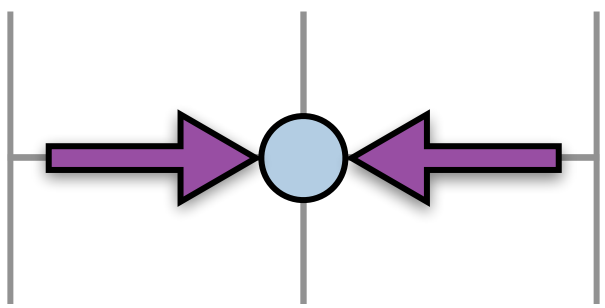
$$\mathbf{1} = e_{x,y-1} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x,y+1}$$



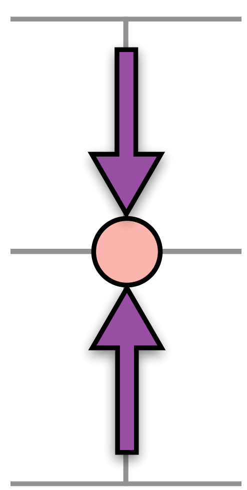
$$\mathbf{1} = e_{x,y} \otimes \bar{e}_{x+1,y} \otimes e_{x+1,y+1} \otimes \bar{e}_{x,y+1}$$

GENERATING FUSION RULES

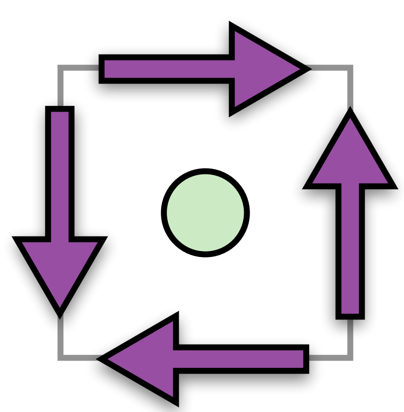
Z_1 Z_2 Z_3



$$\mathbf{1} = m^{(x)}_{x-1,y} \otimes \bar{m}^{(x)}_{x,y}$$



$$\mathbf{1} = m^{(y)}_{x,y-1} \otimes \bar{m}^{(y)}_{x,y}$$



$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y}$$

EMERGENT CONSERVATION LAWS

<u>Fusion rules</u> $\Longrightarrow$ <u>conservation laws</u>

$$\mathbf{1} = m^{(x)}_{x-1,y} \otimes \bar{m}^{(x)}_{x,y} \qquad 0 = \mathfrak{m}^{(x)}_{x-1,y} - \mathfrak{m}^{(x)}_{x,y},$$

$$\mathbf{1} = m^{(y)}_{x,y-1} \otimes \bar{m}^{(y)}_{x,y} \qquad 0 = \mathfrak{m}^{(y)}_{x,y-1} - \mathfrak{m}^{(y)}_{x,y},$$

$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y} \qquad 0 = -\,\mathfrak{m}^{(x)}_{x,y} - \mathfrak{m}^{(y)}_{x,y} + \mathfrak{m}^{(x)}_{x,y+1} + \mathfrak{m}^{(y)}_{x+1,y},$$

$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y} \qquad 0 = \mathfrak{e}_{x-1,y} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x+1,y}$$

$$\mathbf{1} = e_{x,y-1} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y-1} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = e_{x,y} \otimes \bar{e}_{x+1,y} \otimes e_{x+1,y+1} \otimes \bar{e}_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y} - \mathfrak{e}_{x+1,y} + \mathfrak{e}_{x+1,y+1} - \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = (m^{(x)}_{x,y})^{\otimes N} \qquad 0 = N\,\mathfrak{m}^{(x)}_{x,y}$$

$$\mathbf{1} = (m^{(y)}_{x,y})^{\otimes N} \qquad 0 = N\,\mathfrak{m}^{(y)}_{x,y}$$

$$\mathbf{1} = (e_{x,y})^{\otimes N} \qquad 0 = N\,\mathfrak{e}_{x,y}$$

EMERGENT CONSERVATION LAWS

Fusion rules $\Longrightarrow$ conservation laws

$$\mathbf{1} = m^{(x)}_{x-1,y} \otimes \bar{m}^{(x)}_{x,y}$$

$$\mathbf{1} = m^{(y)}_{x,y-1} \otimes \bar{m}^{(y)}_{x,y}$$

$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y}$$

$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y}$$

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$$\mathbf{1} = e_{x,y} \otimes \bar{e}_{x+1,y} \otimes e_{x+1,y+1} \otimes \bar{e}_{x,y+1}$$

$$\mathbf{1} = (m^{(x)}_{x,y})^{\otimes N}$$

$$\mathbf{1} = (m^{(y)}_{x,y})^{\otimes N}$$

$$\mathbf{1} = (e_{x,y})^{\otimes N}$$

$$0 = \mathfrak{m}^{(x)}_{x-1,y} - \mathfrak{m}^{(x)}_{x,y},$$

$$0 = \mathfrak{m}^{(y)}_{x,y-1} - \mathfrak{m}^{(y)}_{x,y},$$

$$0 = -\,\mathfrak{m}^{(x)}_{x,y} - \mathfrak{m}^{(y)}_{x,y} + \mathfrak{m}^{(x)}_{x,y+1} + \mathfrak{m}^{(y)}_{x+1,y},$$

$$0 = \mathfrak{e}_{x-1,y} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x+1,y}$$

$$0 = \mathfrak{e}_{x,y-1} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x,y+1}$$

$$0 = \mathfrak{e}_{x,y} - \mathfrak{e}_{x+1,y} + \mathfrak{e}_{x+1,y+1} - \mathfrak{e}_{x,y+1}$$

$$0 = N\ \mathfrak{m}^{(x)}_{x,y}$$

$$0 = N\ \mathfrak{m}^{(y)}_{x,y}$$

$$0 = N\ \mathfrak{e}_{x,y}$$

EMERGENT CONSERVATION LAWS

<u>Fusion rules</u> $\Longrightarrow$ <u>conservation laws</u>

$$\mathbf{1} = m^{(x)}_{x-1,y} \otimes \bar{m}^{(x)}_{x,y} \qquad 0 = \mathfrak{m}^{(x)}_{x-1,y} - \mathfrak{m}^{(x)}_{x,y},$$

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$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y} \qquad 0 = -\,\mathfrak{m}^{(x)}_{x,y} - \mathfrak{m}^{(y)}_{x,y} + \mathfrak{m}^{(x)}_{x,y+1} + \mathfrak{m}^{(y)}_{x+1,y},$$

$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y} \qquad 0 = \mathfrak{e}_{x-1,y} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x+1,y}$$

$$\mathbf{1} = e_{x,y-1} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y-1} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = e_{x,y} \otimes \bar{e}_{x+1,y} \otimes e_{x+1,y+1} \otimes \bar{e}_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y} - \mathfrak{e}_{x+1,y} + \mathfrak{e}_{x+1,y+1} - \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = (m^{(x)}_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{m}^{(x)}_{x,y}$$

$$\mathbf{1} = (m^{(y)}_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{m}^{(y)}_{x,y}$$

$$\mathbf{1} = (e_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{e}_{x,y}$$

EMERGENT CONSERVATION LAWS

Fusion rules $\Longrightarrow$ conservation laws

$$\mathbf{1} = m^{(x)}_{x-1,y} \otimes \bar{m}^{(x)}_{x,y} \qquad 0 = \mathfrak{m}^{(x)}_{x-1,y} - \mathfrak{m}^{(x)}_{x,y},$$

$$\mathbf{1} = m^{(y)}_{x,y-1} \otimes \bar{m}^{(y)}_{x,y} \qquad 0 = \mathfrak{m}^{(y)}_{x,y-1} - \mathfrak{m}^{(y)}_{x,y},$$

$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y} \qquad 0 = -\,\mathfrak{m}^{(x)}_{x,y} - \mathfrak{m}^{(y)}_{x,y} + \mathfrak{m}^{(x)}_{x,y+1} + \mathfrak{m}^{(y)}_{x+1,y},$$

$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y} \qquad 0 = \mathfrak{e}_{x-1,y} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x+1,y}$$

$$\mathbf{1} = e_{x,y-1} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y-1} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = e_{x,y} \otimes \bar{e}_{x+1,y} \otimes e_{x+1,y+1} \otimes \bar{e}_{x,y+1} \qquad 0 = \mathfrak{e}_{x,y} - \mathfrak{e}_{x+1,y} + \mathfrak{e}_{x+1,y+1} - \mathfrak{e}_{x,y+1}$$

$$\mathbf{1} = (m^{(x)}_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{m}^{(x)}_{x,y}$$

$$\mathbf{1} = (m^{(y)}_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{m}^{(y)}_{x,y}$$

$$\mathbf{1} = (e_{x,y})^{\otimes N} \qquad 0 = N\ \mathfrak{e}_{x,y}$$

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$$\mathbf{1} = \bar{m}^{(x)}_{x,y} \otimes \bar{m}^{(y)}_{x,y} \otimes m^{(x)}_{x,y+1} \otimes m^{(y)}_{x+1,y} \qquad 0 = -\,\mathfrak{m}^{(x)}_{x,y} - \mathfrak{m}^{(y)}_{x,y} + \mathfrak{m}^{(x)}_{x,y+1} + \mathfrak{m}^{(y)}_{x+1,y},$$

$$\mathbf{1} = e_{x-1,y} \otimes \bar{e}_{x,y} \otimes \bar{e}_{x,y} \otimes e_{x+1,y} \qquad 0 = \mathfrak{e}_{x-1,y} - 2\mathfrak{e}_{x,y} + \mathfrak{e}_{x+1,y}$$

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$$\mathbf{1} = (m^{(x)}_{x,y})^{\otimes N} \qquad 0 = N\, \mathfrak{m}^{(x)}_{x,y}$$

$$\mathbf{1} = (m^{(y)}_{x,y})^{\otimes N} \qquad 0 = N\, \mathfrak{m}^{(y)}_{x,y}$$

$$\mathbf{1} = (e_{x,y})^{\otimes N} \qquad 0 = N\, \mathfrak{e}_{x,y}$$

The conservation laws make life simpler!

POSITION DEPENDENT EXCITATIONS

Basis fluxes: $\mathfrak{m}^x = \mathfrak{m}^{(x)}_{0,0}$, $\quad \mathfrak{m}^y = \mathfrak{m}^{(y)}_{0,0}$, $\quad \mathfrak{g} = \mathfrak{m}^{(x)}_{0,1} - \mathfrak{m}^{(x)}_{0,0}$

Basis charges: $\mathfrak{e} = \mathfrak{e}_{0,0}$, $\quad \mathfrak{p}^x = \mathfrak{e}_{1,0} - \mathfrak{e}_{0,0}$, $\quad \mathfrak{p}^y = \mathfrak{e}_{0,1} - \mathfrak{e}_{0,0}$

$$\boxed{\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^x + y\ \mathfrak{g} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^y - x\ \mathfrak{g} \qquad \mathfrak{e}_{x,y} = \mathfrak{e} + x\ \mathfrak{p}^x + y\ \mathfrak{p}^y}$$

Carry position dependent charge/flux $\Longrightarrow$ are position dependent excitations

- If i.e., $\mathfrak{e}_{x,y} = \mathfrak{e}_{x',y'}$: e can move from (x, y) to (x', y')

  <u>Conservation laws view</u>: because charge is conserved

  <u>Fusion rules view</u>: because $\exists$ a string operator

POSITION DEPENDENT EXCITATIONS

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^x + y\ \mathfrak{g} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^y - x\ \mathfrak{g} \qquad \mathfrak{e}_{x,y} = \mathfrak{e} + x\ \mathfrak{p}^x + y\ \mathfrak{p}^y$$

$$N\ \mathfrak{m}^x = N\ \mathfrak{m}^y = N\ \mathfrak{g} = N\ \mathfrak{e} = N\ \mathfrak{p}^x = N\ \mathfrak{p}^y = 0$$

Restricted Mobility

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^{(x)}_{x',y} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^{(y)}_{x+N,y} \qquad \mathfrak{e}_{x,y} = \mathfrak{e}_{x+N,y}$$

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^{(x)}_{x,y+N} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^{(y)}_{x,y'} \qquad \mathfrak{e}_{x,y} = \mathfrak{e}_{x,y+N}$$

- $m^{(i)}$ hops by 1 in longitudinal direction, by N in transverse direction → pseudo-lineon
- e hops by N in all directions → pseudo-fracton

THE AGENDA

1. Review Topological and Fracton Order
2. Position-Dependent Excitations
3. $\mathbb{Z}_N$ Rank-2 Toric Code (R2TC)
4. UV/IR mixing in R2TC from the UV
5. UV/IR mixing in R2TC from the IR

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UV/IR MIXING

To see UV/IR mixing, need to consider an observable

- Topological degeneracy

GSD on torus = number of anyon types

General excitation carries:

$$\ell = \ell_1 \mathfrak{m}^x + \ell_2 \mathfrak{m}^y + \ell_3 \mathfrak{g} + \ell_4 \mathfrak{e} + \ell_5 \mathfrak{p}^x + \ell_6 \mathfrak{p}^y$$

So GSD $= N^6$? Where's UV/IR mixing?

Barkeshli et al, PRB 100.11 (2019): 115147.
Ji & Wen. Phys. Rev. Res. 1.3 (2019): 033054.

PERIODIC BOUNDARY CONDITIONS

Periodic boundary conditions give rise to global equivalence relations:

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^{(x)}_{x+L_x,y} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^{(y)}_{x+L_x,y} \qquad \mathfrak{e}_{x,y} = \mathfrak{e}_{x+L_x,y}$$

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^{(x)}_{x,y+L_y} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^{(y)}_{x,y+L_y} \qquad \mathfrak{e}_{x,y} = \mathfrak{e}_{x,y+L_y}$$

This combined with the fact that these are $\mathbb{Z}_N$ charges/fluxes:

$$\gcd(L_x, N)\ \mathfrak{p}^x = 0 \qquad \gcd(L_y, N)\ \mathfrak{p}^y = 0$$

$$\gcd(L_x, L_y, N)\ \mathfrak{g} = 0$$

PERIODIC BOUNDARY CONDITIONS

Without PBC: $\mathfrak{m}^x$, $\mathfrak{m}^y$, $\mathfrak{g}$, $\mathfrak{e}$, $\mathfrak{p}^x$, $\mathfrak{p}^y$ are $\mathbb{Z}_N$ fluxes/charges

With PBC: $\mathfrak{m}^x$, $\mathfrak{m}^y$, and $\mathfrak{e}$ still $\mathbb{Z}_N$ fluxes/charges, but

$\mathfrak{p}^x$ is a $\mathbb{Z}_{\gcd(L_x,N)}$ charge

$\mathfrak{p}^y$ is a $\mathbb{Z}_{\gcd(L_y,N)}$ charge

$\mathfrak{g}$ is a $\mathbb{Z}_{\gcd(L_x,L_y,N)}$ flux

➤ First consequence: restricted non-local mobility

$$\mathfrak{m}^{(x)}_{x,y} = \mathfrak{m}^{(x)}_{x,y+\gcd(L_x,L_y,N)} \qquad \mathfrak{m}^{(y)}_{x,y} = \mathfrak{m}^{(y)}_{x+\gcd(L_x,L_y,N),y}$$

$$\mathfrak{e}_{x,y} = \mathfrak{e}_{x+\gcd(L_x,N),y} \qquad \mathfrak{e}_{x,y} = \mathfrak{e}_{x,y+\gcd(L_y,N)}$$

NUMBER OF EXCITATION TYPES

Second Consequence: for a general excitation

$$\ell = \ell_1 \mathfrak{m}^x + \ell_2 \mathfrak{m}^y + \ell_3 \mathfrak{g} + \ell_4 \mathfrak{e} + \ell_5 \mathfrak{p}^x + \ell_6 \mathfrak{p}^y$$

Without PBC: $\ell \in \mathbb{Z}_N^6 \implies N^6$ excitation types

With PBC: $\ell \in \mathbb{Z}_N^3 \otimes \mathbb{Z}_{\gcd(L_x,N)} \otimes \mathbb{Z}_{\gcd(L_y,N)} \otimes \mathbb{Z}_{\gcd(L_x,L_y,N)}$

and so $N^3 \gcd(L_x, N) \gcd(L_y, N) \gcd(L_x, L_y, N)$ excitation types

TOPOLOGICAL DEGENERACY

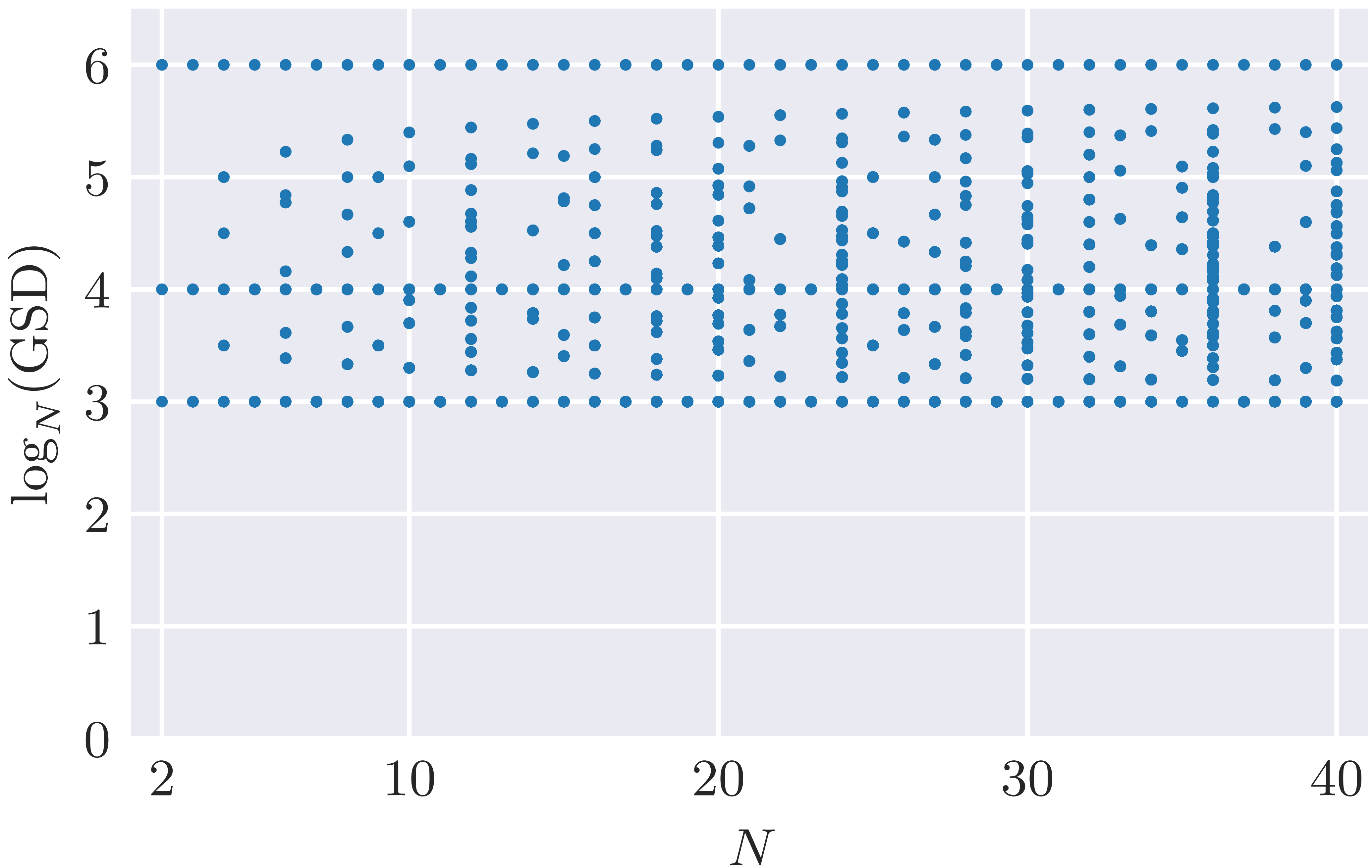
Conjecture: on a torus, GSD equals number of topological excitations that are globally distinguishable

$$\boxed{\mathrm{GSD} = N^3 \gcd(L_x, N) \gcd(L_y, N) \gcd(L_x, L_y, N)}$$

- We'll independently check this correct in a bit

TOPOLOGICAL DEGENERACY

$$\mathrm{GSD} = N^3 \gcd(L_x, N)\gcd(L_y, N)\gcd(L_x, L_y, N)$$



UV/IR MIXING

Topological degeneracy (IR) depends on lattice (UV) details

Manifestation of UV/IR mixing

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Interplay between topological order and lattice symmetry

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Consequence of position dependent excitations

Interplay between topological order and lattice symmetry

Claim: position dependent excitations cause UV/IR mixing

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MUTUAL CHERN-SIMONS THEORY

Powerful theoretical tool to describe/characterize 2+1D abelian topological orders

Mutual Chern-Simons (MCS) theory in 2+1D flat spacetime

$$S_{MCS} = \frac{K_{ij}}{4\pi} \int_{X^3} a^{(i)} \wedge \mathrm{d}a^{(j)} + \ldots$$

- $a^{(i)}$ are dynamical 1-form fields with $U(1)$ gauge redundancy $\Omega : a^{(i)} \mapsto a^{(i)} + \mathrm{d}\omega^{(i)}$.
- K is a symmetric integer matrix

Wen & Zee, PRB 46.4 (1992): 2290.

FROM R2TC TO MCS

Each field $a^{(i)}$ corresponds to a basis anyon of the abelian topological order

- R2TC:

basis anyons	MCS fields
$\mathfrak{m}^x$	$a^{(1)}$
$\mathfrak{m}^y$	$a^{(2)}$
$\mathfrak{g}$	$a^{(3)}$
$\mathfrak{e}$	$a^{(4)}$
$\mathfrak{p}^x$	$a^{(5)}$
$\mathfrak{p}^y$	$a^{(6)}$

FROM R2TC TO MCS

How to find K matrix?

- Braiding statistics between an anyon carrying $a^{(i)}$ unit-charge and an anyon carrying $a^{(j)}$ unit-charge:

$$\theta_{ij} = 2\pi \left(K^{-1}\right)_{ij}$$

From the UV lattice model:

Braiding phase θ	$\mathfrak{m}^x$	$\mathfrak{m}^y$	$\mathfrak{g}$
$\mathfrak{e}$	0	0	$2\pi/N$
$\mathfrak{p}^x$	0	$2\pi/N$	0
$\mathfrak{p}^y$	$-2\pi/N$	0	0

MCS THEORY AND R2TC

Using θ_{ij} can find K^{-1} and thus K:

$$K = \begin{pmatrix} 0_3 & C \\ C^\top & 0_3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & -N \\ 0 & N & 0 \\ N & 0 & 0 \end{pmatrix}$$

$$S = \frac{N}{2\pi} \int_{X^3} \left(a^{(3)} \wedge \mathrm{d}a^{(4)} + a^{(2)} \wedge \mathrm{d}a^{(5)} - a^{(1)} \wedge \mathrm{d}a^{(6)} \right) + \ldots$$

- looks like three $\mathbb{Z}_N$ toric codes

Is this the IR theory? NO! Need zero modes of $a^{(i)}$

OUR HERO WILSON'S LOOPS

IR observables are Wilson loops $W_i = \exp[\mathrm{i}\ \Gamma_i]$ where Γ_i are a set of basis holonomies

Construction of $\{\Gamma_i\}$ are complicated. See arXiv:2204.07111

Why?

- Position dependent excitations sometimes only hop by N (restricted mobility!)
- Gauge fields $a^{(i)}$ and gauge parameters $\omega^{(i)}$ satisfy twisted periodic boundary conditions
- W_i are physical, so Γ_i must be gauge invariant

OUR HERO WILSON'S LOOPS

IR observables are Wilson loops $W_i = \exp[\mathrm{i}\ \Gamma_i]$ where Γ_i are a set of basis holonomies

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nice holonomies 😃

$$\Gamma_1(y,t) = \oint_0^{L_x} \mathrm{d}x\ a_x^{(3)} \qquad \Gamma_7(y,t) = \oint_0^{L_x} \mathrm{d}x\ a_x^{(5)}$$

$$\Gamma_2(x,t) = \oint_0^{L_y} \mathrm{d}y\ a_y^{(3)} \qquad \Gamma_8(x,t) = \oint_0^{L_y} \mathrm{d}y\ a_y^{(5)}$$

$$\Gamma_3(x,t) = \oint_0^{L_y} \mathrm{d}y\ a_y^{(2)} \qquad \Gamma_9(y,t) = \oint_0^{L_x} \mathrm{d}x\ a_x^{(6)}$$

$$\Gamma_4(y,t) = \oint_0^{L_x} \mathrm{d}x\ a_x^{(1)} \qquad \Gamma_{10}(x,t) = \oint_0^{L_y} \mathrm{d}y\ a_y^{(6)}$$

fancy holonomies 😎

$$\Gamma_5(x,t) = \oint_0^{\mathrm{lcm}(L_y,N)} \mathrm{d}y\ a_y^{(1)}$$

$$\Gamma_{11}(y,t) = \oint_0^{\mathrm{lcm}\left(L_x,N\right)} \mathrm{d}x\ a_x^{(4)} \qquad \Gamma_{12}(x,t) = \oint_0^{\mathrm{lcm}\left(L_y,N\right)} \mathrm{d}y\ a_y^{(4)}$$

the evil holonomy 😈

$$\Gamma_6(x,y,t) = \oint_0^{\mathrm{lcm}(L_x,\mathrm{gcd}(L_y,N))} \mathrm{d}x\ a_x^{(2)} + \oint_0^{nL_y} \mathrm{d}y\ a_y^{(1)}$$

ZERO MODES

Gauss-law constraint $\mathrm{d}a^{(i)} = 0$ enforces that in the ground states Γ_i are spatially independent

$$\Gamma_i(r,t) = \varphi_i(t) \in \mathbb{R}/2\pi\mathbb{Z}$$

nice zero modes 😃

$$a_x^{(3)}(x,y,t) = \frac{\varphi_1(t)}{L_x} \qquad a_x^{(5)}(x,y,t) = \frac{\varphi_7(t)}{L_x}$$

$$a_y^{(3)}(x,y,t) = \frac{\varphi_2(t)}{L_y} \qquad a_y^{(5)}(x,y,t) = \frac{\varphi_8(t)}{L_y}$$

$$a_y^{(2)}(x,y,t) = \frac{\varphi_3(t)}{L_x} \qquad a_x^{(6)}(x,y,t) = \frac{\varphi_9(t)}{L_x}$$

$$a_x^{(1)}(x,y,t) = \frac{\varphi_4(t)}{L_y} \qquad a_y^{(6)}(x,y,t) = \frac{\varphi_{10}(t)}{L_y}$$

fancy zero modes 😎

$$a_y^{(1)}(x,y,t) = \frac{\varphi_5(t)}{\mathrm{lcm}(L_y, N)}$$

$$a_x^{(4)}(x,y,t) = \frac{\varphi_{11}(t)}{\mathrm{lcm}(L_x, N)} \qquad a_y^{(4)}(x,y,t) = \frac{\varphi_{12}(t)}{\mathrm{lcm}(L_y, N)}$$

the evil zero mode 😈

$$a_x^{(2)}(x,y,t) = \frac{N\,\varphi_6(t) - n\,\mathrm{gcd}(L_y,N)\;\varphi_5(t)}{N\,\mathrm{lcm}(L_x, \mathrm{gcd}(L_y,N))}$$

IR EFFECTIVE THEORY

Plugging zero modes into MCS action:

$$S_{\mathrm{IR}} = \frac{b^{ij}}{2\pi}\int \mathrm{d}t\ \varphi_i \frac{\mathrm{d}\varphi_j}{\mathrm{d}t} \qquad b = \begin{pmatrix} 0_6 & B \\ -B^\top & 0_6 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & N_y \\ 0 & 0 & 0 & 0 & -N_x & 0 \\ -N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -N & 0 & 0 \\ 0 & -n\ N_{xy} & N_y & 0 & 0 & 0 \\ 0 & N\, N_{xy}/N_y & 0 & 0 & 0 & 0 \end{pmatrix}$$

Key point: The IR action S_{IR} depends on the UV lattice

- ➤ direct manifestation of UV/IR mixing

$$N_x \equiv \gcd(L_x, N),$$
$$N_y \equiv \gcd(L_y, N),$$
$$N_{xy} \equiv \gcd(L_x, L_y, N)\,.$$

GROUND STATE DEGENERACY

Ground states form representation of the algebra satisfied by the Wilson loops:

$$W_i W_j = \exp\left[-2\pi \mathrm{i} \left(b^{-1}\right)_{ij}\right] W_j W_i$$

Ground state degeneracy given by size of smallest faithful representation:

$$\mathrm{GSD} = |\,\mathrm{pf}(b)\,| = N^3 \gcd(L_x, N) \gcd(L_y, N) \gcd(L_x, L_y, N)$$

➤ Matches ground state degeneracy found before!

Wesolowski, Hosotani, & Ho, Int. J. Mod. Phys. A 9.06 (1994): 969-989.

In the R2TC, position dependent excitations induced restricted mobility and UV/IR mixing in the form of lattice dependent topological degeneracy