Behavior and Breakdown of Higher-Order FPUT Recurrences

Salvatore D. Pace and David K. Campbell

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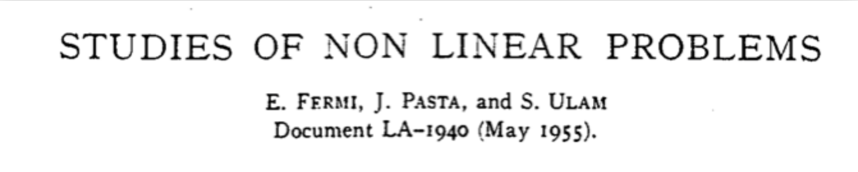
Fermi-Pasta-Ulam-Tsingou (FPUT) Problem


STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM
Document LA-1940 (May 1955).

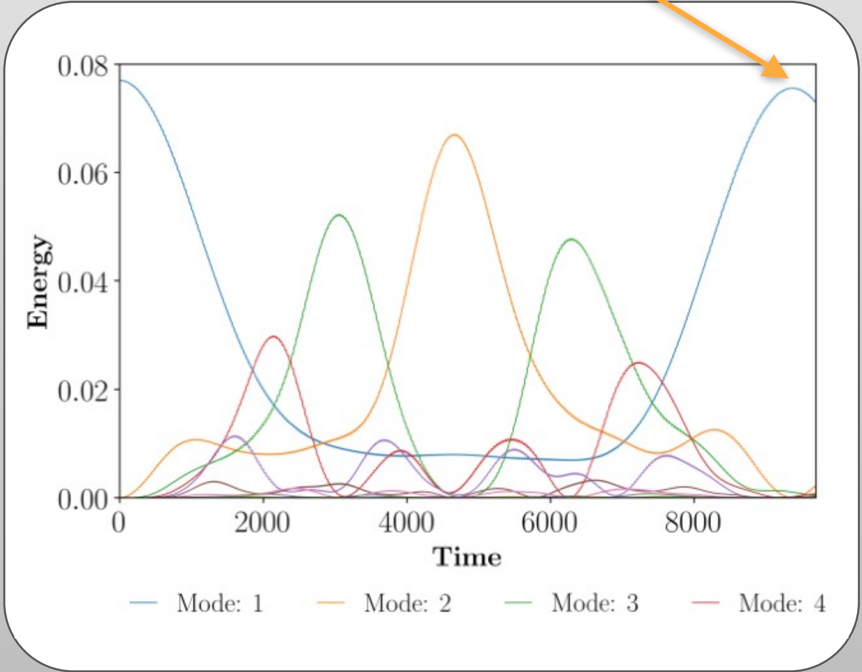
α-FPUT: $H_\alpha = \sum_{n=1}^{N} \frac{p_n^2}{2} + \sum_{n=0}^{N} \frac{1}{2}\left(q_{n+1} - q_n\right)^2 + \frac{\alpha}{3}\left(q_{n+1} - q_n\right)^3$

β-FPUT: $H_\beta = \sum_{n=1}^{N} \frac{p_n^2}{2} + \sum_{n=0}^{N} \frac{1}{2}\left(q_{n+1} - q_n\right)^2 + \frac{\beta}{4}\left(q_{n+1} - q_n\right)^4$

Expectation: System would thermalize and achieve energy equipartition among normal modes.

Observation: For long-wavelength, low-energy initial conditions, energy shared among only lowest normal modes and remarkable *near-recurrences* to the initial state

FPUT Recurrence at t~9400



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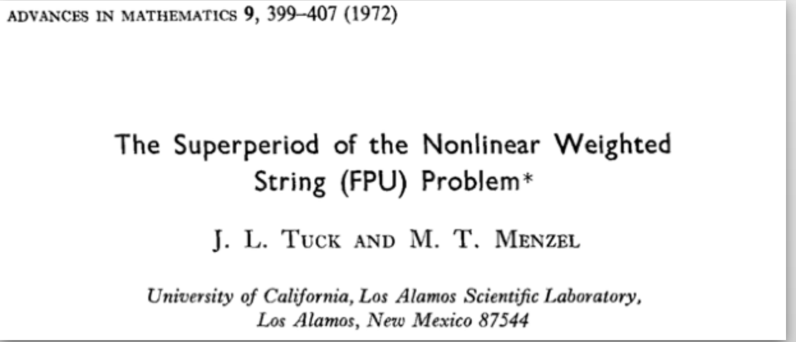
Tuck and Menzel's Super-Recurrences



ADVANCES IN MATHEMATICS 9, 399–407 (1972)

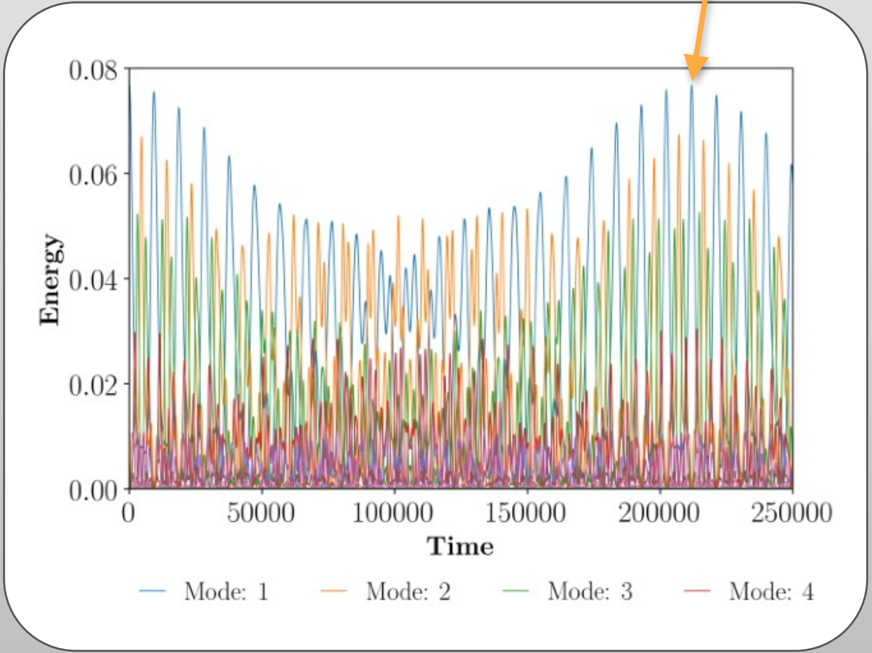
The Superperiod of the Nonlinear Weighted String (FPU) Problem*

J. L. Tuck and M. T. Menzel

University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544

Longer numerical integration times showed the existence of *super-recurrences*, which amount to nearly periodic modulations of the FPUT recurrences.

Super-recurrence at t~220,000



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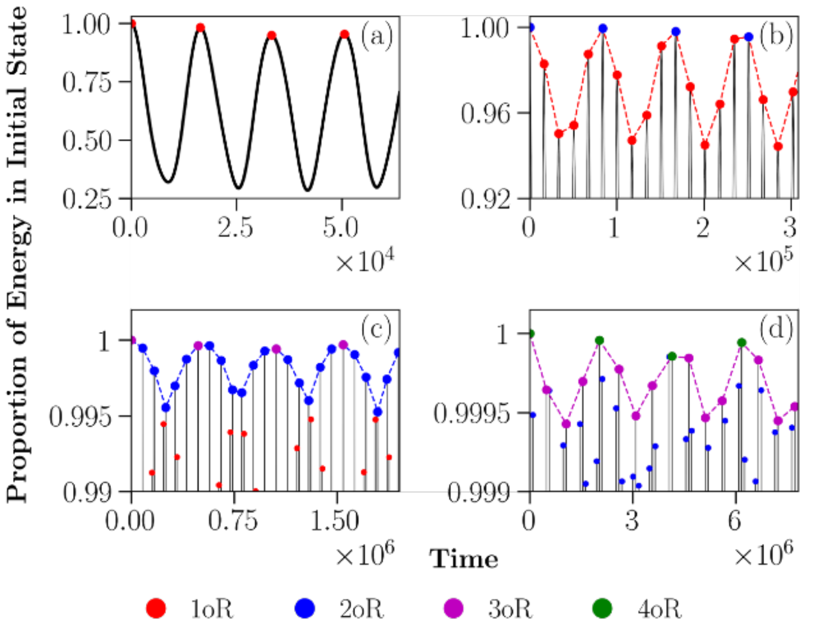
Higher-Order Recurrences (HoRs)

- Terminology:
  - 1st order recurrence (1oR) - Original FPUT recurrence
  - 2nd order recurrence (2oR) - Tuck and Menzel's super-recurrence
  - 3rd order recurrence (3oR) - "super-super-recurrence"
- An n^{th} order recurrence amounts to nearly periodic modulations of the $(n-1)^{th}$ order recurrences.
- HoRs are seen in both the α and β-FPUT systems



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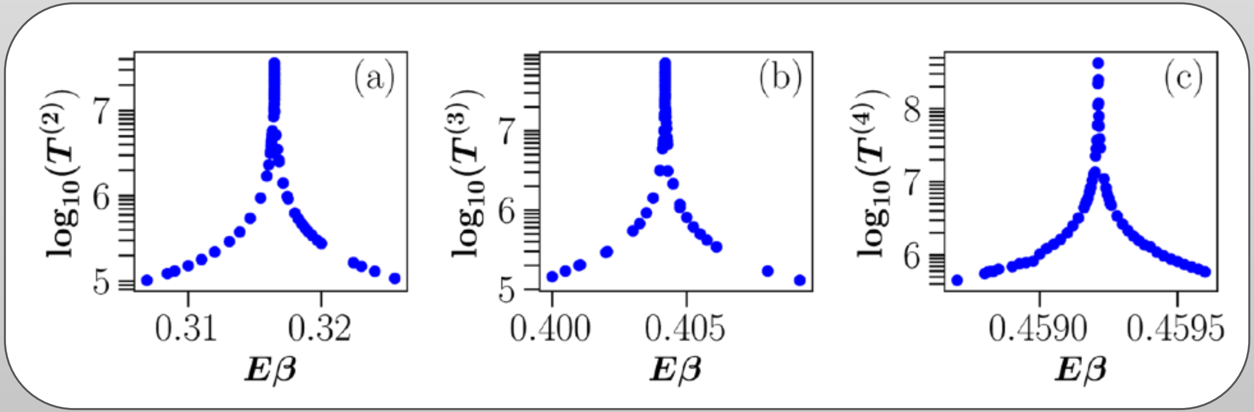
Nontrivial Scaling of HoR Times

Unlike FPUT recurrences at low energies, HoR times do not have a straightforward scaling. There exists energies at which HoRs simply do not exist, and other energies at which the HoR time blows up

- The higher the order of the recurrence, the region of energy where the apparent singularity exists becomes increasingly narrower.
- Interestingly, the α-FPUT system's 2oRs do not appear to exhibit singularities, while β-FPUT system does.



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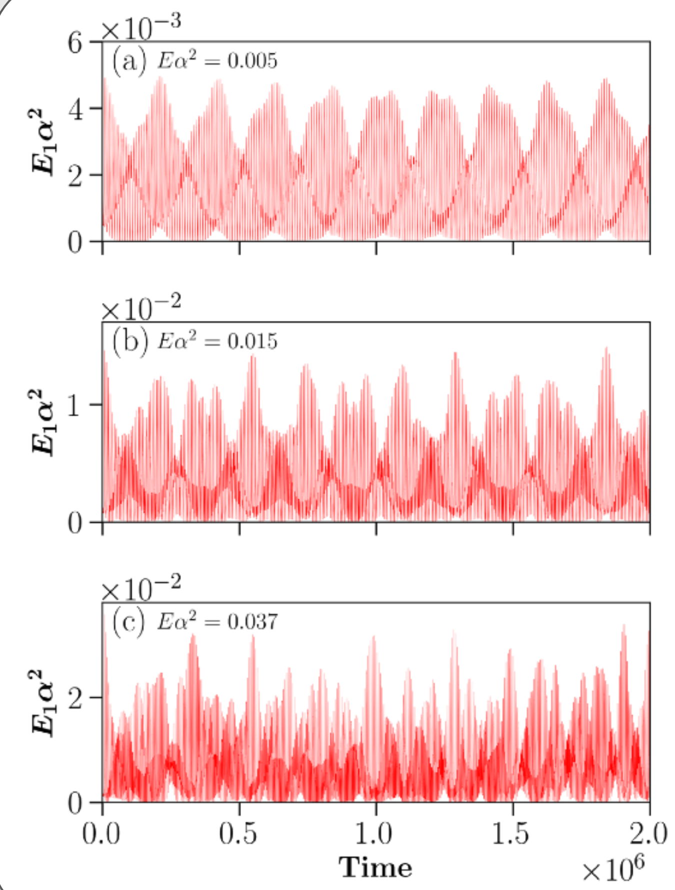
Breakdown of 2oRs in α-FPUT System

At larger enough energies, both FPUT systems quickly thermalize. Thus what happens to the HoRs as energy is increased to this regime?

- Past a critical energy, increasing energy causes 2oRs structure to degrade after a very short timescale.
- This deformation is due to the FPUT recurrences themselves to become poor.



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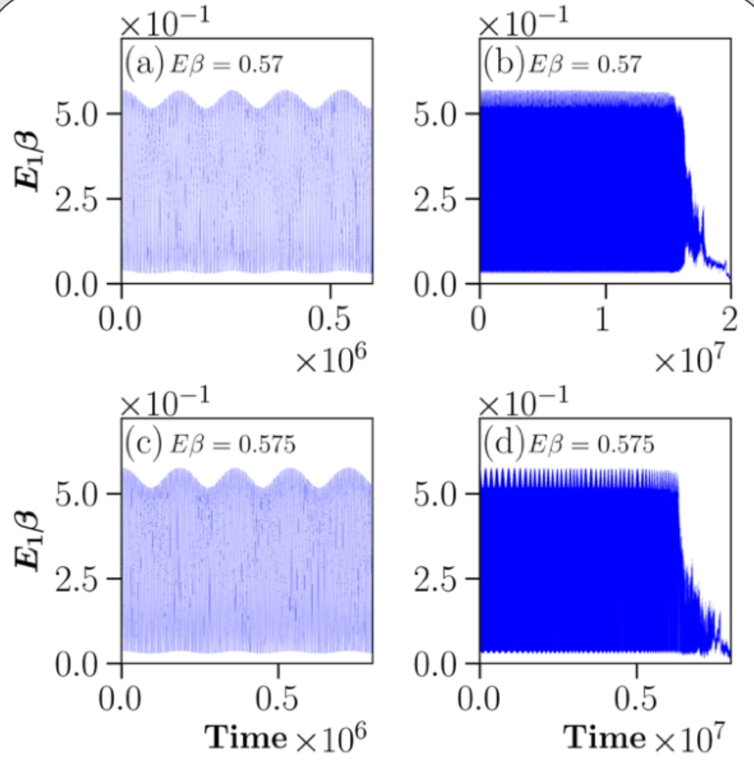
Breakdown of 2oRs in β-FPUT system

The 2oR breakdown mechanism is found to be very different from that in the α-FPUT system

- 2oRs break down abruptly such that they completely retain their form before their breakdown
- Increasing energy, even slightly, causes the 2oR breakdown to happen much sooner in time



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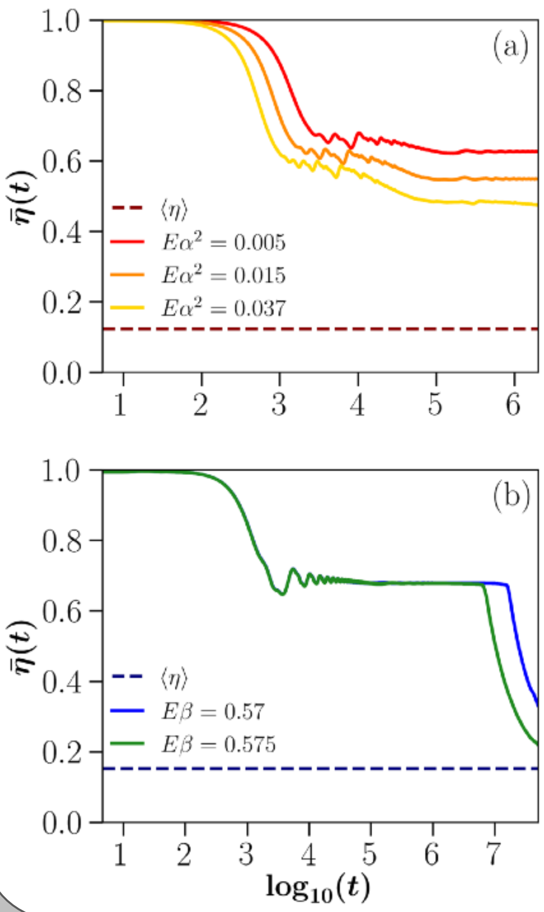
2oRs Breakdown and Thermalization

- Breakdown of 2oRs in the α-FPUT system occurs while the lattice is still in its quasi-stationary, *metastable*, state.

- Breakdown of 2oRs in the β-FPUT system is associated with the destruction of this metastable state and hence is associated with relaxation towards equilibrium.



Same values of $E\alpha^2$ and $E\beta$ as shown in previous two slides which demonstrated the 2oR breakdown mechanics.

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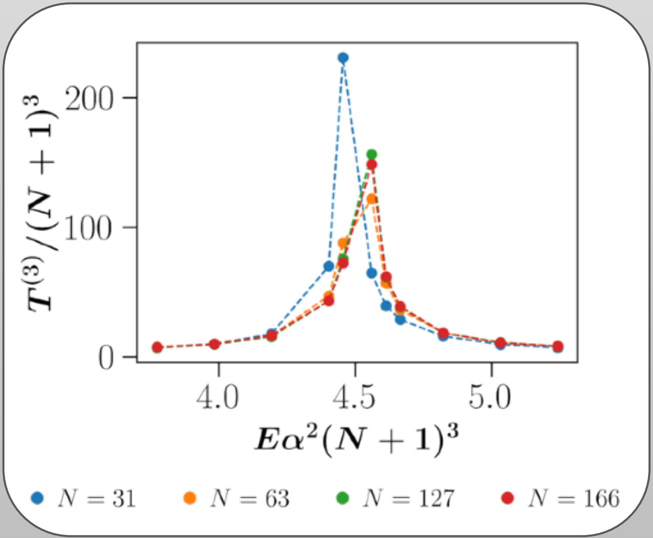
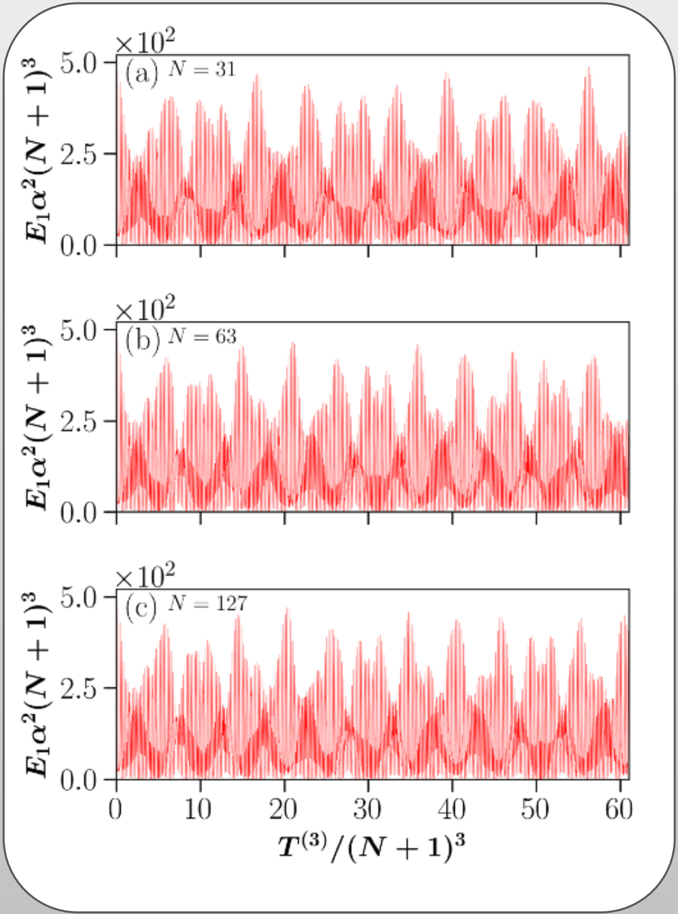
Remarks on System Size

- In the β-FPUT system, results have been reproduced for various system sizes, N.
- In the α-FPUT system, using a rescaling of time and energy, results seem to be general for all large system sizes.





N. J. Zabusky, Phys. Soc. Jpn. J. Suppl. 26, 196 (1969).

Recap

- HoRs Exist in both the α and β-FPUT system.
- HoR times scale non-trivially with energy because of apparent singularities:
  - The β-FPUT system has singularities for 2oRs and greater.
  - The α-FPUT system has singularities for 3oRs and greater.
- HoRs breakdown mechanisms and their correspondence to thermalization are different between α and β-FPUT system:
  - β-model 2oRs breakdown abruptly alongside breakdown of metastable state.
  - α-model 2oRs breakdown on small timescale while lattice is still metastable.
- Results presented appear to be general for different system sizes, N.

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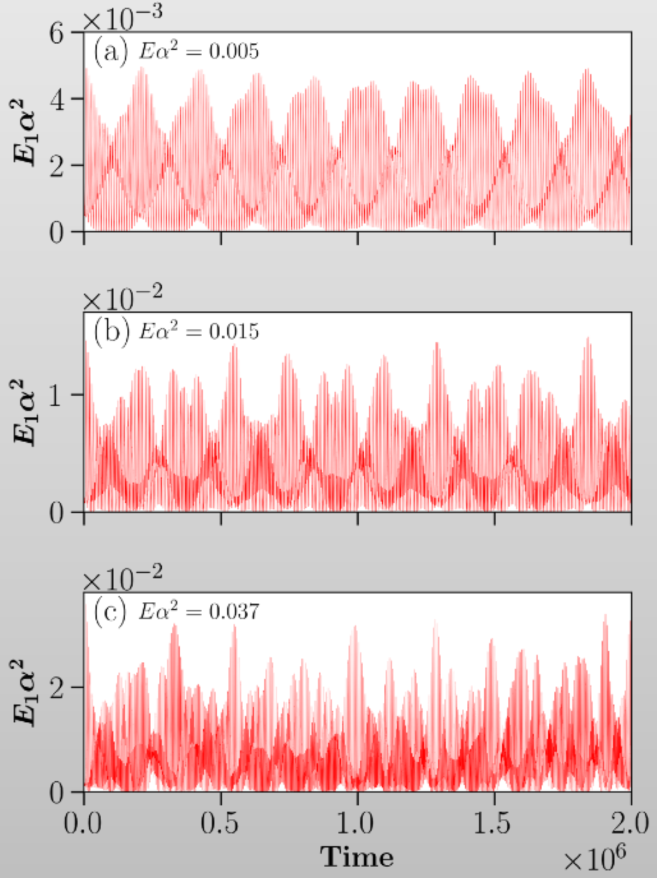
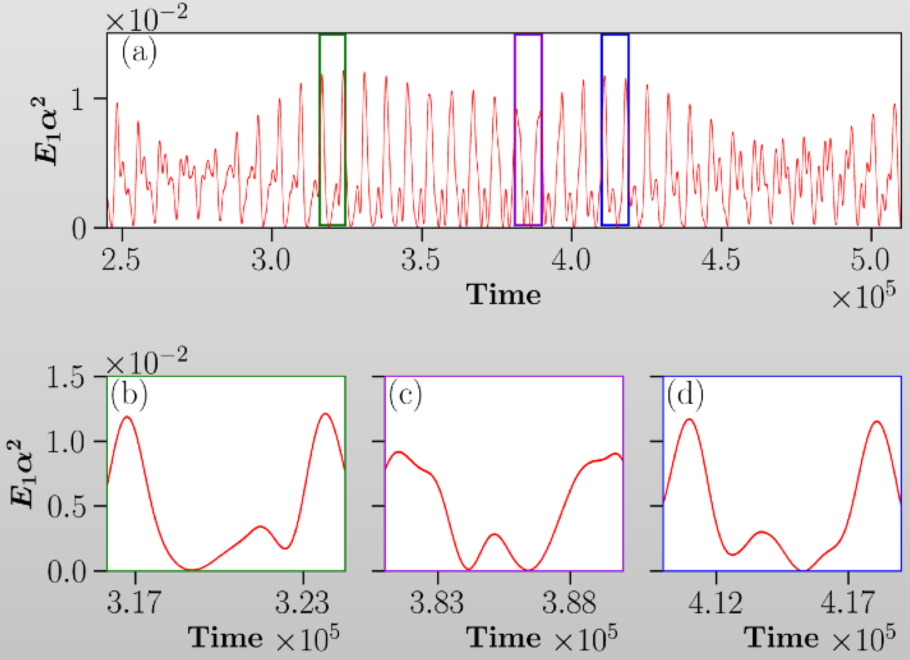
Mini-Recurrences





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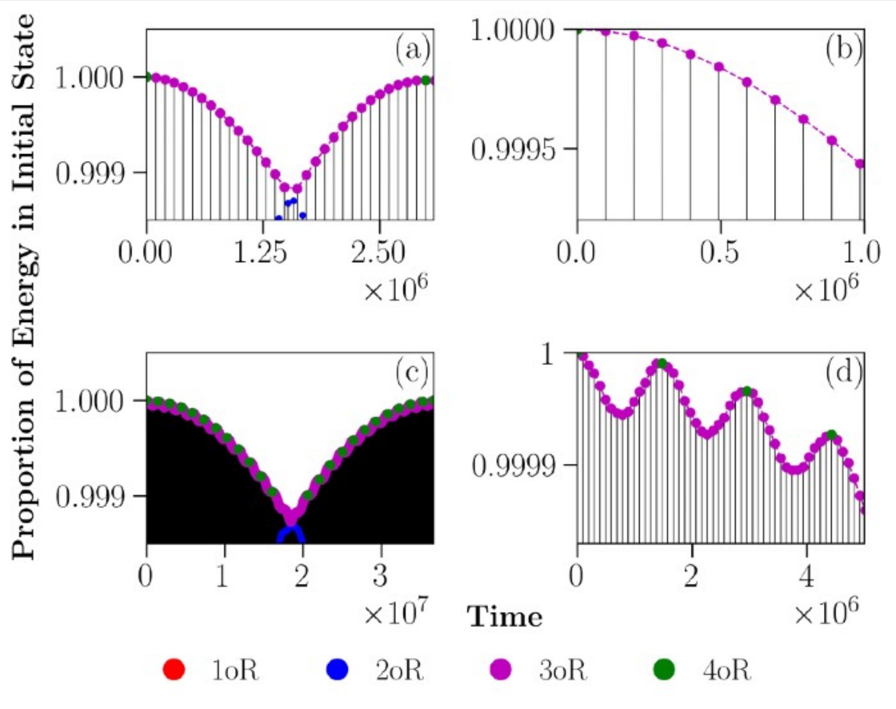
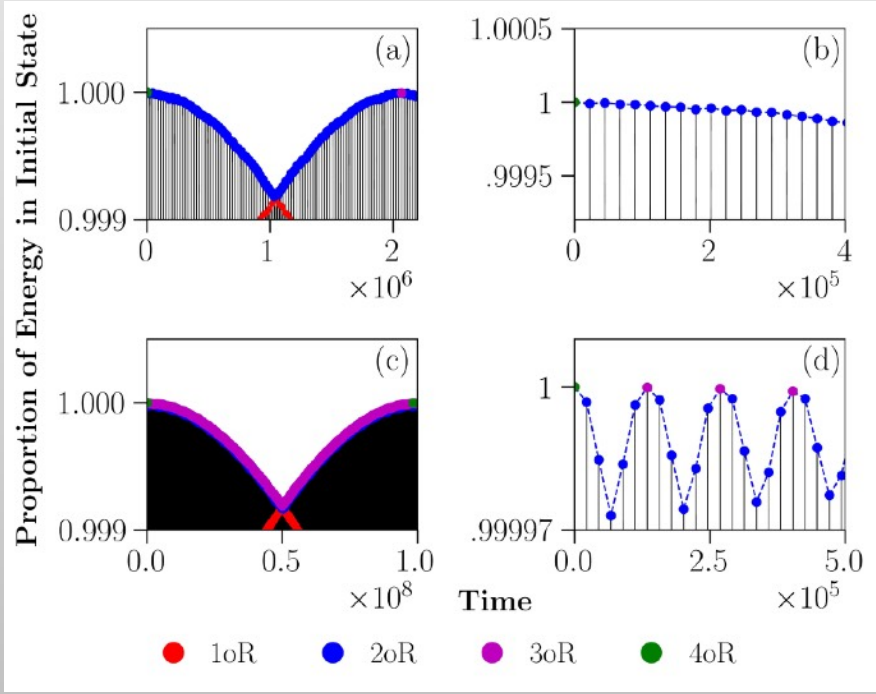
Nested Recurrences at HoR blow-up times





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Spectral Entropy as an Equipartition Indicator

- Spectral Entropy definition: $S(t) = -\sum_{k=1}^{N} e_k \ln\left(e_k\right)$, where $e_k(t) = E_k(t) / \sum_k E_k(0)$

- Rescale Spectral Entropy: $\eta(t) = \dfrac{S(t) - S_{\max}}{S(0) - S_{\max}}$

Euler-Mascheroni constant.

- Compare thermal average, $\langle \eta \rangle = \dfrac{1}{Z} \int_{\mathbb{R}} \prod_{k=1}^{N} \left(dQ_k dP_{,k}\right) \eta(\boldsymbol{Q}, \boldsymbol{P}) e^{-\beta H(\boldsymbol{Q}, \boldsymbol{P})} \sim \dfrac{1-\gamma}{S_{\max} - S(0)}$

  to time average, $\overline{\eta}(t) = \dfrac{1}{t} \int_0^t ds \eta(s)$, to probe for ergodicity.

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Scaling of FPUT Recurrences

Alpha-FPUT system:

$$\frac{T_r}{(N+1)^3} = \frac{1.026}{(E\alpha^2(N+1)^3)^{1/4}}$$

Beta-FPUT system:

$$\frac{T_r}{(N+1)^3} = \frac{0.595}{(E\beta(N+1))^{1/2}}$$

C. Y. Lin, C. G. Goedde, and S. Lichter, Phys. Lett. A 229, 367 (1997).

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Singularities

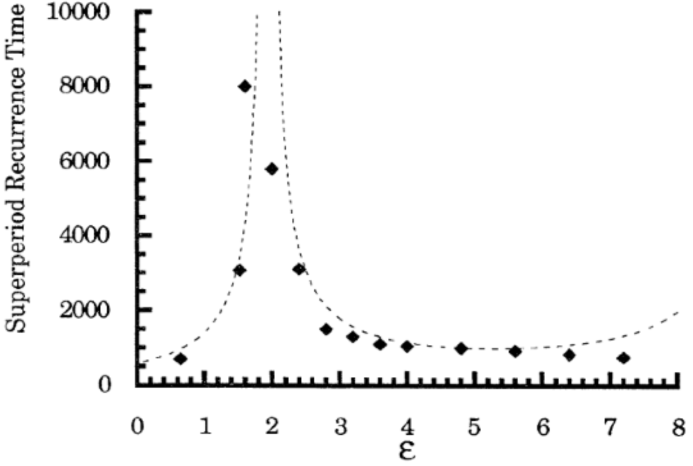
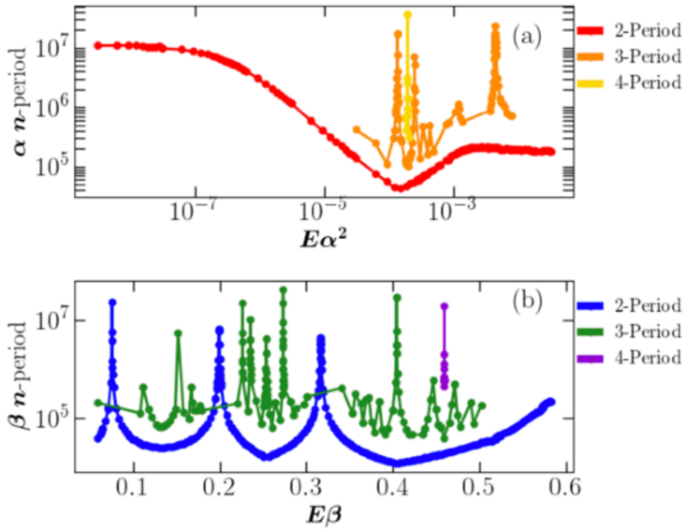


FIG. 7. The superperiod recurrence time in the quartic chain with $N=7$ as a function of ϵ. The dashed line shows theoretical results from Eq. (42); measurements from numerical simulations are represented by ♦.

$$T^{(2)} = \frac{2\pi}{5\Omega_1 - \Omega_7} = \frac{2\pi}{5\sqrt{\omega_1^2 + \mu_{1,1}\beta + \mu_{1,2}\beta^2 + \mu_{1,3}\beta^3} - \omega_7}$$



Sholl, D. S., & Henry, B. I. (1991). Physical Review A, 44(10), 6364–6374.