NON-INVERTIBLE REFLECTION SYMMETRIES IN SPIN CHAINS

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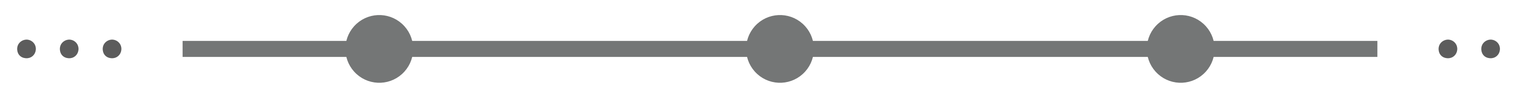
$\mathbb{Z}_N$ SPIN CHAINS

We consider 1+1D Hamiltonian lattice models of $\mathbb{Z}_N$ qudits

➤ Degrees of freedom labeled by $g \in \mathbb{Z}_N$ ($N = 2$ is qubit)

$$\mathscr{H} = \bigotimes_{j=1}^{L} \mathbb{C}^N$$

$|g_{j-1}\rangle \in \mathbb{C}^N$ $\quad |g_j\rangle \in \mathbb{C}^N$ $\quad |g_{j+1}\rangle \in \mathbb{C}^N$



➤ Acted on by generalized Pauli operators

$$X|g\rangle = |g+1\rangle \qquad Z|g\rangle = (\mathrm{e}^{2\pi\mathrm{i}/N})^g\,|g\rangle$$

$$ZX = \mathrm{e}^{2\pi\mathrm{i}/N}\,XZ \qquad X^N = Z^N = 1$$

Enforce periodic boundary conditions $j \sim j + L$

$$X_j = X_{j+L} \qquad Z_j = Z_{j+L}$$

$\mathbb{Z}_N$ SPIN CHAINS

We consider 1+1D Hamiltonian lattice models of $\mathbb{Z}_N$ qudits

Example: $\mathbb{Z}_N$ clock model

$$H_\lambda = -\sum_{j=1}^{L} Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1} + \lambda\,(X_j + X_j^\dagger)$$

Commutes with $U = \prod_j X_j$

➤ A $\mathbb{Z}_N$ symmetry since $U^N = 1$

Non-invertible symmetry at $\lambda = 1$

[*Seiberg, Shao (2024)*]

➤ Degrees ... qubit)

$\mathscr{H} = \bigotimes_{j=1}^{L}$ ... $_{j+1}\rangle \in \mathbb{C}^N$

➤ Acted o...

$X\,|$ ... $g\rangle$

$Z\,X$ ... $= 1$

Enforce periodic boundary conditions $j \sim j + L$

$$X_j = X_{j+L} \qquad Z_j = Z_{j+L}$$

A CURIOUS EXAMPLE

Consider a model of $\mathbb{Z}_5$ qudits and Hamiltonian

$$H_\lambda = -\sum_{j=1}^{L} Z_j Z_{j+1}^{\dagger\,2} + Z_j^\dagger Z_{j+1}^2 + \lambda\,(X_j + X_j^\dagger)$$

- $\mathbb{Z}_L$ translation symmetry: $T\,X_j\,T^{-1} = X_{j-1}$
- $\mathbb{Z}_5$ symmetry generated by*

$$U = \prod_{j=1}^{L} (X_j)^{3^j} = (X_1)^3\,(X_2)^4\,(X_3)^2\,X_4\,(X_5)^3\,(X_6)^4\,(X_7)^2\,X_8\,\cdots$$

- Does not have reflection symmetry $M\,X_j\,M^{-1} = X_{-j}$

Action of $\mathbb{Z}_L$ on $\mathbb{Z}_5$: $T\,U\,T^{-1} = U^3 \implies G_{\mathrm{sym}} = \mathbb{Z}_5 \rtimes \mathbb{Z}_L$

* *we assume* $L \in 4\mathbb{Z}_{>0}$

GAUGING THE SYMMETRY

What is the Kramers-Wannier transformation?

We gauge the modulated $\mathbb{Z}_5$ symmetry by

1. Introduce $\mathbb{Z}_5$ qudits on links acted on by $\widetilde{X}_{j,j+1}, \widetilde{Z}_{j,j+1}$.
2. Enforce Gauss's law $G_j = X_j\, \widetilde{Z}^{\dagger\,2}_{j-1,j}\, \widetilde{Z}_{j,j+1} = 1$

$$U = \prod_{j=1}^{L} (X_j)^{3^j} = \prod_{j=1}^{L} (G_j)^{3^j} = 1$$

Minimally couple $\widetilde{X}_{j,j+1}$:

$$H^{\vee}_{\lambda} = -\sum_{j=1}^{L} \widetilde{X}_{j,j+1}\, Z_j\, Z^{\dagger\,2}_{j+1} + \lambda\, X_j + \mathrm{hc}\,.$$

KRAMERS-WANNIER

Gauge fix using the unitary transformation

$$\widetilde{X}_{j,j+1} \to \widetilde{X}_{j,j+1}\, Z_j^\dagger\, Z_{j+1}^2 \qquad\qquad X_j \to X_j\, \widetilde{Z}_{j-1,j}^2\, \widetilde{Z}_{j,j+1}^\dagger$$

and using Gauss's law $X_j = 1$ causes

$$H_\lambda^\vee = -\sum_{j=1}^{L} \widetilde{X}_{j,j+1} + \lambda\, \widetilde{Z}_{j-1,j}^2\, \widetilde{Z}_{j,j+1}^\dagger + \text{hc}\,.$$

KRAMERS-WANNIER

Gauge fix using

$$\boxed{H_\lambda = -\sum_{j=1}^{L} Z_j\, Z_{j+1}^{\dagger\,2} + \lambda\, X_j + \mathrm{hc}\,.}$$

and using Gauss's law $X_j = 1$ causes

$$H_\lambda^\vee = -\sum_{j=1}^{L} \widetilde{X}_{j,j+1} + \lambda\, \widetilde{Z}^2_{j-1,j}\, \widetilde{Z}^\dagger_{j,j+1} + \mathrm{hc}\,.$$

- Kramers-Wannier (KW) transformation for H_λ

$$\mathsf{D}_{\mathrm{KW}}\, Z_j\, Z_{j+1}^{\dagger\,2} = X_j\, \mathsf{D}_{\mathrm{KW}} \qquad \mathsf{D}_{\mathrm{KW}}\, X_j = Z^2_{j-1}\, Z^\dagger_j\, \mathsf{D}_{\mathrm{KW}}$$

- D_{KW} is non-invertible because $\mathsf{D}_{\mathrm{KW}}\, U = \mathsf{D}_{\mathrm{KW}}$

NON-INVERTIBLE REFLECTION

$$\mathsf{D}_{\mathrm{KW}}: Z_j Z_{j+1}^{\dagger\,2} + \lambda X_j \to \lambda Z_{j-1}^2 Z_j^\dagger + X_j$$

Therefore, $\mathsf{D}_{\mathrm{KW}} H_{\lambda=1} = M H_{\lambda=1} M \mathsf{D}_{\mathrm{KW}}$, with $M: j \to -j$

- D_{KW} does *not* commute with $H_{\lambda=1}$ because $H_\lambda \neq M H_\lambda M$

$\mathsf{D}_{\mathrm{M}} := M \mathsf{D}_{\mathrm{KW}}$ does commute with $H_{\lambda=1}$

$$\mathsf{D}_{\mathrm{M}} Z_j Z_{j+1}^{\dagger\,2} = X_{-j} \mathsf{D}_{\mathrm{M}} \qquad \mathsf{D}_{\mathrm{M}} X_j = (Z_{-j} Z_{-j+1}^{\dagger\,2})^\dagger \mathsf{D}_{\mathrm{M}}$$

- Non-invertible reflection operator
- Similar non-invertible reflections exist in gauge theories

[*Choi, Lam, Shao (2023)*]

NON-INVERTIBLE REFLECTION

$$\mathsf{D}_{\mathrm{KW}}: Z_j Z_{j+1}^{\dagger\,2} + \lambda X_j \to \lambda Z_{j-1}^2 Z_j^\dagger + X_j$$

Therefore, $\mathsf{D}_{\mathrm{KW}} H_{\lambda,1} = M H_{\lambda,1} M \mathsf{D}_{\mathrm{KW}}$, with $M: j \to -j$

D_{M} operator algebra (where $\mathsf{C}: X_j, Z_j \to X_j^\dagger, Z_j^\dagger$)

$$\mathsf{D}_{\mathrm{M}} \mathsf{D}_{\mathrm{M}} = \left(1 + U + U^2 + U^3 + U^4\right) \mathsf{C}, \qquad \mathsf{D}_{\mathrm{M}}^\dagger = \mathsf{D}_{\mathrm{M}} \mathsf{C}$$

$$\mathsf{D}_{\mathrm{M}} U = U \mathsf{D}_{\mathrm{M}} = \mathsf{D}_{\mathrm{M}} \qquad \mathsf{D}_{\mathrm{M}} T = T^{-1} \mathsf{D}_{\mathrm{M}}$$

- Non-invertible reflection operator
- Similar non-invertible reflections exist in gauge theories

[Choi, Lam, Shao (2023)]

NON-INVERTIBLE REFLECTION

$$\mathsf{D}_{\mathrm{M}} = \sqrt{5}\, P_{U=1}\, M\, W\, \mathfrak{H}_L\, \mathrm{CZ}_L\, \mathfrak{H}_{L-1}\, \mathrm{CZ}_{L-1} \cdots \mathfrak{H}_3\, \mathrm{CZ}_3\, \mathfrak{H}_2\, \mathrm{CZ}_2$$

$$\mathrm{CZ}_j = \frac{1}{5} \sum_{\alpha,\beta=0}^{4} (\mathrm{e}^{-2\pi \mathrm{i}/5})^{\alpha\beta}\, Z_j^{\alpha}\, Z_{j-1}^{2\beta}$$

$$\mathfrak{H}_j = \frac{1}{5^{3/2}} \sum_{\alpha,\beta,\gamma=0}^{4} (\mathrm{e}^{-2\pi \mathrm{i}/5})^{\beta(\alpha+\gamma)}\, X_j^{\alpha-\beta}\, Z_j^{\gamma}$$

$$W = \frac{1}{5} \sum_{\alpha,\beta=0}^{4} (\mathrm{e}^{-2\pi \mathrm{i}/5})^{\alpha\beta}\, Z_1^{\alpha-\beta}\, Z_L^{2\beta}$$

$$M = \prod_{j=1}^{L-1} \mathsf{S}_{j,L-j} \quad \text{where} \quad \mathsf{S}_{i,j} = \frac{1}{5} \sum_{\alpha,\beta=0}^{4} (\mathrm{e}^{2\pi \mathrm{i}/5})^{\alpha\beta}\, X_i^{\alpha}\, Z_i^{\beta}\, X_j^{-\alpha}\, Z_j^{-\beta},$$

MODULATED SYMMETRIES

$\mathcal{S}$ is a (spatially) modulated symmetry if its symmetry transformation is position-dependent

$$\mathcal{S}_{\text{total}} = \text{“}\mathcal{S} \rtimes_{\varphi} \mathcal{S}_{\text{space}}\text{”} \qquad \varphi : \mathcal{S}_{\text{space}} \to \mathrm{Aut}(\mathcal{S})$$

➤ $\mathcal{S}$ can be a generalized symmetry [*Oh, **SP**, Han, You, Lee (2023)*]

➤ Standard example is dipole symmetry:

$$[Q, H] = [\vec{P}, H] = 0 \qquad T e^{\mathrm{i}\,\vec{P}}\, T^{-1} = e^{\mathrm{i}\,Q} e^{\mathrm{i}\,\vec{P}}$$

1) Lifshitz-type theories

[*Griffin et.al. (2013), Seiberg (2020), Gorantla et. al. (2022), ⋯*]

2) Generalized Bose Hubbard models

[*Lake et.al (2022), Zechmann et. al. (2023), Sala et. al. (2024)*]

WHAT WE DO IN 2406.12962?

Gauge finite abelian modulated symmetries in $\mathbb{Z}_N$ spin chains

$$G_{\text{total}} = A \rtimes_{\varphi} G_{\text{space}} \xrightarrow{\quad Gauge\ A \quad} G^{\vee}_{\text{total}} = A \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

- Dual symmetry can have different modulation (i.e., $\varphi^{\vee} \neq \varphi$)

We establish sufficient conditions for the existence of an isomorphism between G_{total} and $G^{\vee}_{\text{total}}$, naturally implemented by lattice reflections.

- Gives rise to non-invertible reflection symmetries, and for ordinary-reflection symmetric models, KW symmetries
- The isomorphism always exists for translation-invariant $\mathbb{Z}_p$ spin chains (p prime), but does *not* exist for all G_{total}

A GENERALIZED ISING MODEL

Consider a model of $\mathbb{Z}_N$ qudits on sites $j \sim j + L$ with

$$H_\lambda = -\sum_j \prod_k Z_k^{\Delta_{j,k}} + \lambda X_j + \mathrm{hc}$$

Assumptions on $\Delta_{j,k} \in \mathbb{Z}_N$:

1. Translation invariance: $\Delta_{j,k} = \Delta_{0,k-j}$

2. Locality: finite range r

$$\Delta_{0,k-j} = 0 \quad \text{if } k - j < 0 \quad \text{or} \quad k - j > r - 1$$

$$\Delta_{0,0},\ \Delta_{0,r-1} \neq 0 \mod N$$

3. Coprime condition: $\gcd(\Delta_{0,0}, N) = \gcd(\Delta_{0,r-1}, N) = 1$

A GENERALIZED ISING MODEL

Consider a model of $\mathbb{Z}_N$ qudits on sites $j \sim j+L$ with

$$H_\lambda = -\sum_j \prod_k Z_k^{\Delta_{j,k}} + \lambda\, X_j + \mathrm{hc}$$

Assumptions on $\Delta_{j,k} \in \mathbb{Z}_N$:

1. T

2. L

Examples

$\mathbb{Z}_N$ clock model: $\Delta_{j,k} = \delta_{k-j,\,0} - \delta_{k-j,\,1}$

Our curious example: $\Delta_{j,k} = \delta_{k-j,\,0} - 2\,\delta_{k-j,\,1}$

Dipole symmetry: $\Delta_{j,k} = \delta_{k-j,\,0} - 2\,\delta_{k-j,\,1} + \delta_{k-j,\,2}$

3. Coprime condition: $\gcd(\Delta_{0,0}, N) = \gcd(\Delta_{0,r-1}, N) = 1$

GENERALIZED ISING MODEL

Under these assumptions, we prove that:

- H has a modulated $\mathbb{Z}_N^{\times r-1}$ symmetry

$$U_f = \prod_j (X_j)^{f_j}$$

where $f_j \in \mathbb{Z}_N$ and $\sum_k \Delta_{j,k} f_k = 0 \mod N.$

- The KW transformation related to gauging $\mathbb{Z}_N^{\times r-1}$ is

$$\mathsf{D}_{\mathrm{KW}} \prod_k Z_k^{\Delta_{j,k}} = X_j \mathsf{D}_{\mathrm{KW}} \qquad\qquad \mathsf{D}_{\mathrm{KW}} X_j = \prod_k Z_k^{\dagger\, \Delta_{-j,-k}} \mathsf{D}_{\mathrm{KW}}$$

NON-INVERTIBLE REFLECTIONS

$$\mathsf{D}_{\mathrm{KW}} : H_\lambda \to \lambda\, M\, H_{\lambda^{-1}}\, M$$

- $H_{\lambda=1}$ generally not invariant under the KW transformation
- $H_{\lambda=1}$ has a non-invertible reflection symmetry $\mathsf{D}_{\mathrm{M}} := M\,\mathsf{D}_{\mathrm{KW}}$

$$\mathsf{D}_{\mathrm{M}} \prod_k Z_k^{\Delta_{j,k}} = X_{-j}\,\mathsf{D}_{\mathrm{M}} \qquad \mathsf{D}_{\mathrm{M}}\, X_j = \prod_k Z_k^{\dagger\,\Delta_{-j,k}}\,\mathsf{D}_{\mathrm{M}}$$

D_{M} satisfies the operator algebra

$$\mathsf{D}_{\mathrm{M}}\,\mathsf{D}_{\mathrm{M}} = \mathsf{C} \sum_f U_f, \qquad \mathsf{D}_{\mathrm{M}}^\dagger = \mathsf{D}_{\mathrm{M}}\,\mathsf{C}$$

$$\mathsf{D}_{\mathrm{M}}\, U_f = U_f\,\mathsf{D}_{\mathrm{M}} = \mathsf{D}_{\mathrm{M}} \qquad \mathsf{D}_{\mathrm{M}}\, T = T^{-1}\,\mathsf{D}_{\mathrm{M}}$$

KRAMERS-WANNIER SYMMETRY

H_λ is reflection-symmetric if

$$\Delta_{j,-k} = \sigma\, \Delta_{-(j+r-1),k} \qquad (\sigma = \pm 1)$$

If satisfied, $H_{\lambda=1}$ is invariant under the KW transformation.

- Canonically defined D_{KW} satisfies operator algebra*

$$\mathsf{D}_{\mathrm{KW}}\,\mathsf{D}_{\mathrm{KW}} = \mathfrak{H}^{1+\sigma}\, T^{(r-1 \mod 2)} \sum_f U_f, \qquad \mathsf{D}^\dagger_{\mathrm{KW}} = \mathsf{D}_{\mathrm{KW}}\, \mathfrak{H}^{1+\sigma}\, T^{\dagger\,(r-1 \mod 2)}$$

$$\mathsf{D}_{\mathrm{KW}}\, U_f = U_f\, \mathsf{D}_{\mathrm{KW}} = \mathsf{D}_{\mathrm{KW}} \qquad \mathsf{D}_{\mathrm{KW}}\, T = T\, \mathsf{D}_{\mathrm{KW}}$$

*Recovers known dipole D_{KW}

Yan, Li 2403.16017

Cao, Li, Yamazaki 2406.05454

Seifnashri, Shao 2404.01369

Gorantla, Shao, Tantivasadakarn 2404.01369

THANKS FOR LISTENING :-)

Gauging modulated symmetries: Kramers-Wannier dualities and non-invertible reflections

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