CS2040S Data Structures and Algorithms
Lecture Note #7

Map ADT – HashTable

For efficient look-up in a table

Objectives

1

- To know Map ADT and one efficient implementation – hashing/hashtable

2

- To understand how **hashing** is used to accelerate table lookup

3

- To study the issue of **collision** and techniques to resolve it

Outline

1. Map ADT and Hashing
2. Direct Addressing Table
3. Hash Table
4. Hash Functions
    - Good/bad/perfect/uniform hash function
5. Collision Resolution
    - Separate Chaining
    - Linear Probing
    - Quadratic Probing
    - Double Hashing
    - Performance of hash table operations
6. Set ADT
7. Summary
8. Java HashMap Class

1 Map ADT and Hashing

An ADT to map values to keys

1. Map ADT and Hashing

- Map ADT
  - An abstract data type that contains a collection of <key,value> pairs/mappings
  - It associates a key to a value in a one-to-one or many-to-one relation
  - There cannot be duplicate keys in the map
  - There are 3 basic operations
    - Retrieval – retrieve the value using the given key
    - Insertion – insert/replace a value using the given key
    - Deletion – delete the <key,value> pair using the given key

1. Map ADT and Hashing

- Hashing which is performed via a hash function is one concrete way for Map ADT to map a key to its value.
- A hash table (or hash map) is a data structure that uses a hash function to efficiently map keys to values, for efficient search and retrieval.
- Widely used in many kinds of computer software, particularly for associative arrays, database indexing and caches.

1. Map ADT Operations

	Sorted List (Array impl. By sorting key)	**Balanced BST**	**HashTable**
Insertion	O(*n*)	O(log *n*)	O(1) avg
Deletion	O(*n*)	O(log *n*)	O(1) avg
Retrieval	O(log *n*)	O(log *n*)	O(1) avg

Note: Balanced Binary Search Tree (bBST) will be covered in later lectures.

- Hence, hash table supports the Map ADT in constant time on average for the above operations. It has many applications.

2 Direct Addressing Table

A simplified version of hash table

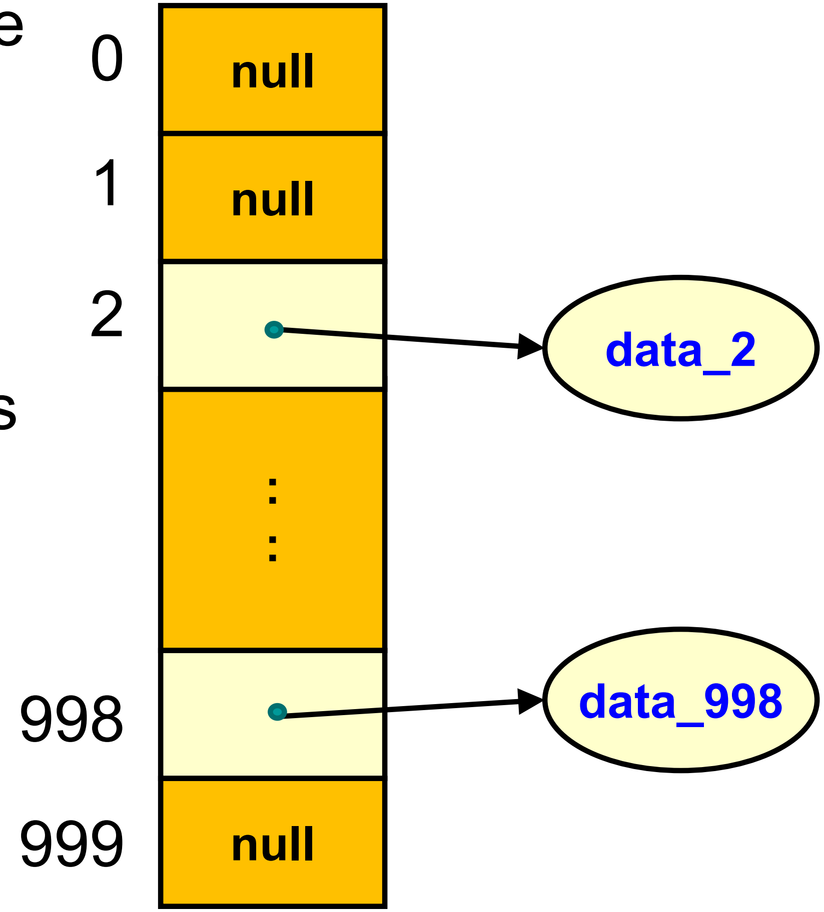
2 SBS Transit Problem

- Retrieval: find(*num*)
  - Find the bus route of bus service number *num*
- Insertion: insert(*num*)
  - Introduce a new bus service number *num*
- Deletion: delete(*num*)
  - Remove bus service number *num*

2 Direct Addressing Table

Assume that bus numbers are integers between 0 and 999, we can create an array of 1000 slots, each is a **reference** to an object which contains the details of the bus route (one-to-one mapping).

Note: You will want to store the key values, i.e. bus numbers, also.



2 Direct Addressing Table: Operations

insert (key, data)

a[key] = data // where a[] is an array – the table

delete (key)

a[key] = null

find (key)

return a[key]

2 Direct Addressing Table: Restrictions

- Keys must be non-negative integer values
  - What happens for key values 151A and NR10?
- Range of keys must be small
- Keys must be dense, i.e. not many gaps in the key values.
- How to overcome these restrictions?

3 Hash Table

Hash Table is a generalization of direct addressing table, to remove the latter's restrictions.

3 Origins of the term Hash

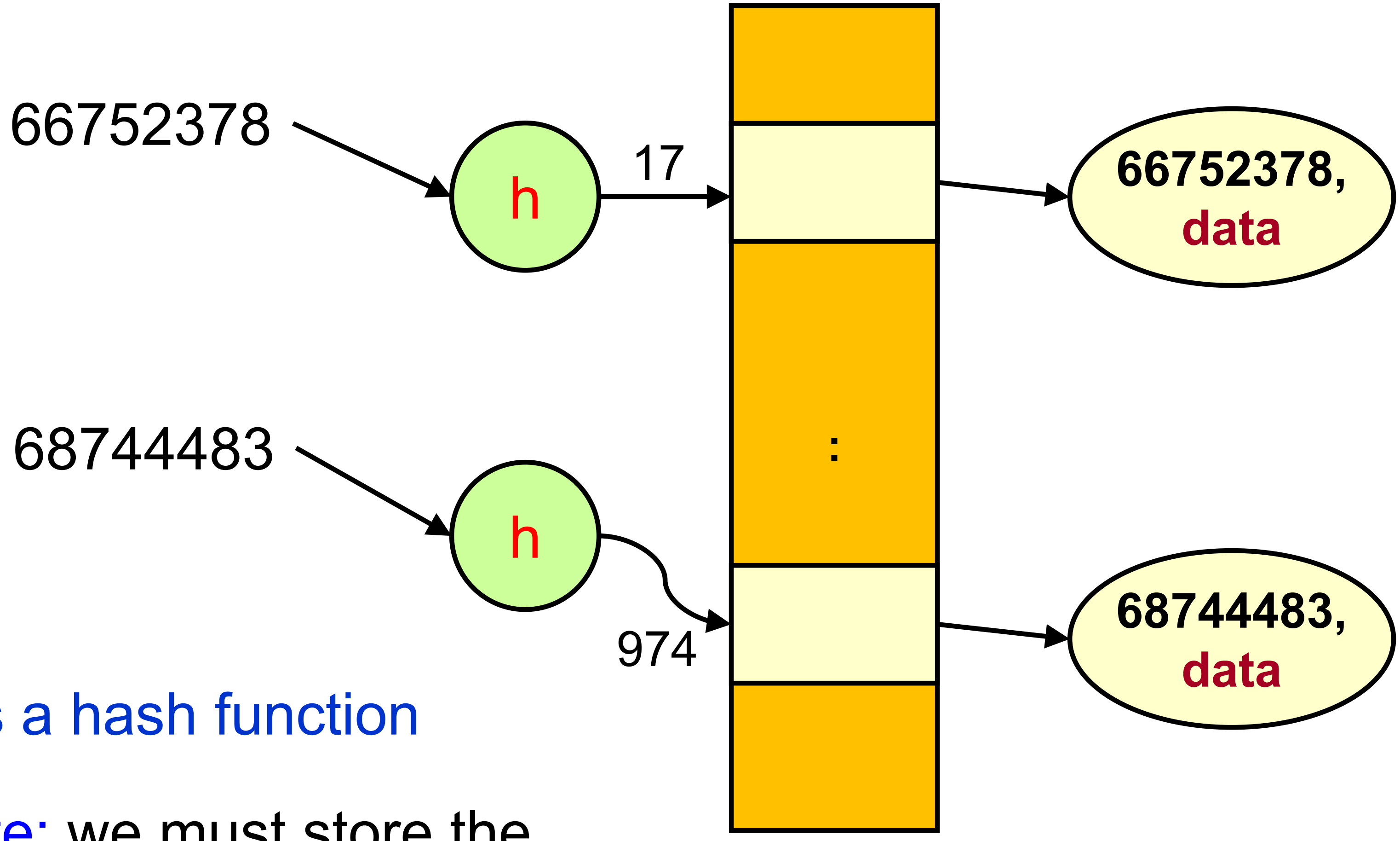
- The term "hash" comes by way of analogy with its standard meaning in the physical world, to "chop and mix".
- Indeed, typical hash functions, like the **mod** operation, “chop” the input domain into many sub-domains that get “mixed” into the output range.
- Donald Knuth notes that Hans Peter Luhn of IBM appears to have been the first to use the concept, in a memo dated January 1953, and that Robert Morris used the term in a survey paper in CACM which elevated the term from technical jargon to formal terminology.

3 Ideas

- Map large integers to smaller integers
- Map non-integer keys to integers

HASHING

3 Hash Table



h is a hash function

Note: we must store the key values. Why?

3 Hash Table: Operations

insert (key, data)

a[h(key)] = data // h is a hash function and a[] is an array

delete (key)

a[h(key)] = null

find (key)

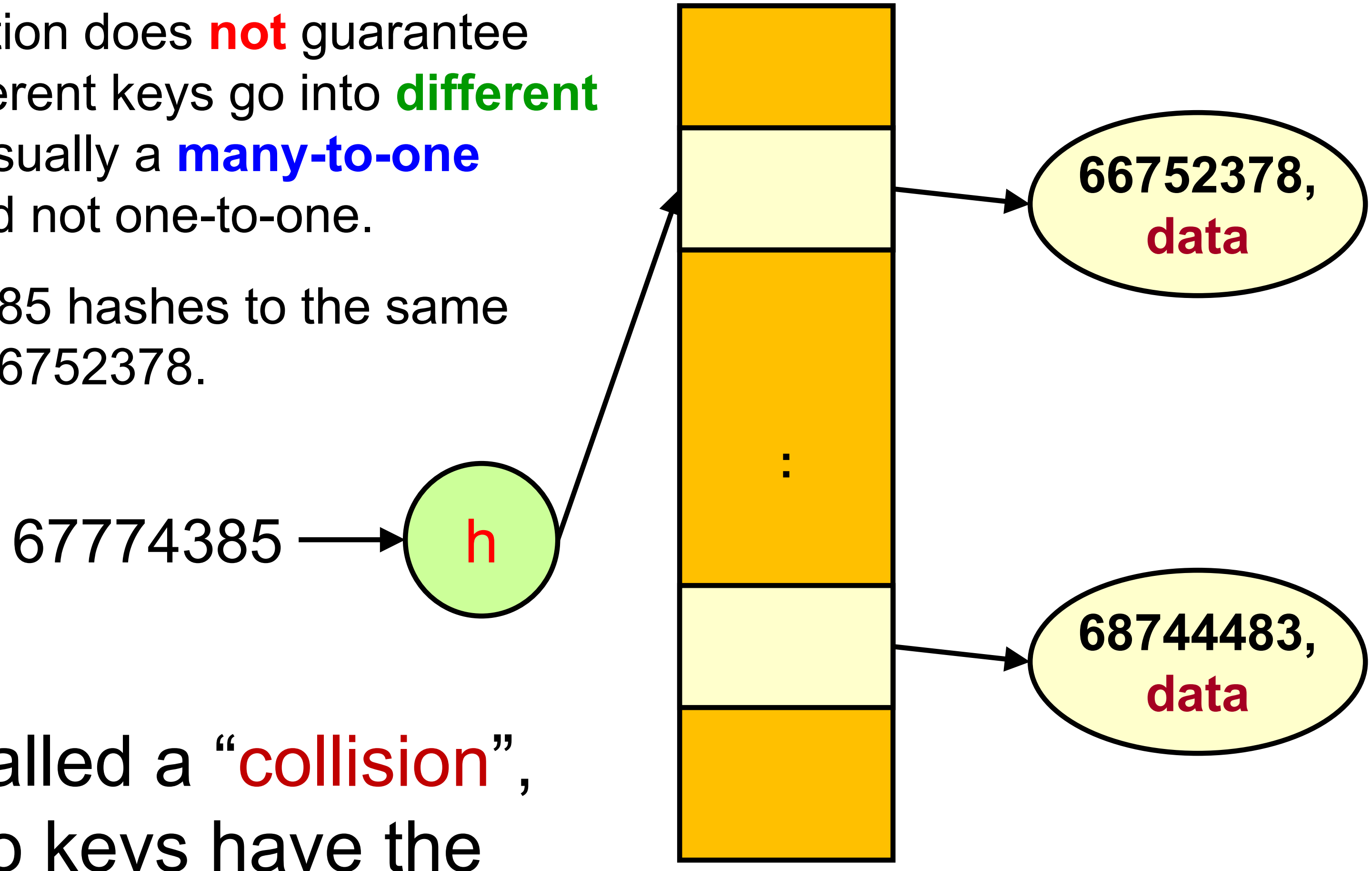
return a[h(key)]

However, this does **not** work for **all** cases! (Why?)

3 Hash Table: Collision

A hash function does **not** guarantee that two different keys go into **different** slots! It is usually a **many-to-one** mapping and not one-to-one.

E.g. 67774385 hashes to the same location of 66752378.



This is called a "collision", when two keys have the same hash value.

3 Two Important Issues

- How to hash?
- How to resolve collisions?
- These are important issues that can affect the efficiency of hashing

4 Hash Functions

4 Criteria of Good Hash Functions

- Fast to compute
- Scatter keys evenly throughout the hash table
- Less collisions
- Need less slots (space)

4 Example of Bad Hash Function

- Select Digits – e.g. choose the 4th and 8th digits of a phone number
  - hash(67754378) = 58
  - hash(63497820) = 90
- What happen when you hash Singapore's house phone numbers by selecting the first three digits?

4 Perfect Hash Functions

- Perfect hash function is a **one-to-one** mapping between keys and hash values. So no collision occurs.
- Possible if all keys are known beforehand.
- Applications: compiler and interpreter search for reserved words; shell interpreter searches for built-in commands.
- GNU gperf is a freely available perfect hash function generator written in C++ that automatically constructs perfect functions (a C++ program) from a user supplied list of keywords.
- Minimal perfect hash function: The table size is the same as the number of keywords supplied.

4 Uniform Hash Functions

- Distributes keys evenly in the hash table
- Example
  - If k integers are uniformly distributed among 0 and X-1, we can map the values to a hash table of size ***m*** (m < X) using the hash function below

$$k \in [0, X)$$

$$hash(k) = \left\lfloor \frac{km}{X} \right\rfloor$$

k is the key value
[ ]: close interval
( ): open interval
Hence, $0 \le k < X$
$\lfloor \ \rfloor$ is the *floor* function

4 Division method (mod operator)

- Map into a hash table of ***m*** slots.
- Use the modulo operator (**%** in Java) to map an integer to a value between 0 and m-1.
- n mod m = remainder of n divided by m, where n and m are positive integers.

$$hash(k) = k \% m$$

The most popular method.

4 Multiplication method

1. Multiply by a constant real number **A** between 0 and 1
2. Extract the fractional part
3. Multiply by *m*, the hash table size

$$hash(k) = \left\lfloor m\left(k\mathbf{A} - \lfloor k\mathbf{A} \rfloor\right)\right\rfloor$$

The reciprocal of the golden ratio
= (sqrt(5) - 1)/2 = 0.618033 seems to be a good choice for **A** (recommended by Knuth).

4 How to pick *m*?

- The choice of m (or hash table size) is important. If m is power of two, say 2^n, then key modulo of m is the same as extracting the last n bits of the key.
- If m is 10^n, then our hash values is the last n digit of keys.
- Both are no good.
- Rule of thumb:
  - Pick a prime number close to a power of two to be m.

4 Hashing of strings (1/4)

- An example hash function for strings:

```
hash(s) {    //  s is a string
   sum = 0
   for each character c in s {
      sum += c      //  sum up the ASCII values of all characters
   }
   return sum % m    //  m is the hash table size
}
```

4 Hashing of strings: Examples (2/4)

hash("Tan Ah Teck")

= ("T" + "a" + "n" + " " +
"A" + "h" + " " +
"T" + "e" + "c" + "k") % 11 // hash table size is 11

= (84 + 97 + 110 + 32 +
65 + 104 + 32 +
84 + 101 + 99 + 107) % 11

= 825 % 11

= 0

4 Hashing of strings: Examples (3/4)

- All 3 strings below have the same hash value! Why?
  - Lee Chin Tan
  - Chen Le Tian
  - Chan Tin Lee
- Problem: This hash function value does not depend on positions of characters! – Bad

4 Hashing of strings (4/4)

- A better hash function for strings is to “shift” the sum after each character, so that the positions of the characters affect the hash value.

```
hash(s)
    sum = 0
    for each character c in s {
        sum = sum*31 + c
    }
    return sum % m      // m is the hash table size
```

Java’s String.hashCode() uses *31 as well.

5 Collision Resolution

5 Probability of Collision (1/2)

- **von Mises Paradox (The Birthday Paradox)**:
  "How many people must be in a room before the probability that some share a birthday, ignoring the year and leap days, becomes at least 50 percent?"

Q(n) = Probability of unique birthday for n people

$$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \cdots \frac{365-n+1}{365}$$

P(n) = Probability of collisions (same birthday) for n people

= 1 – Q(n)

P(**23**) = 0.507

Hence, you need only 23 people in the room!

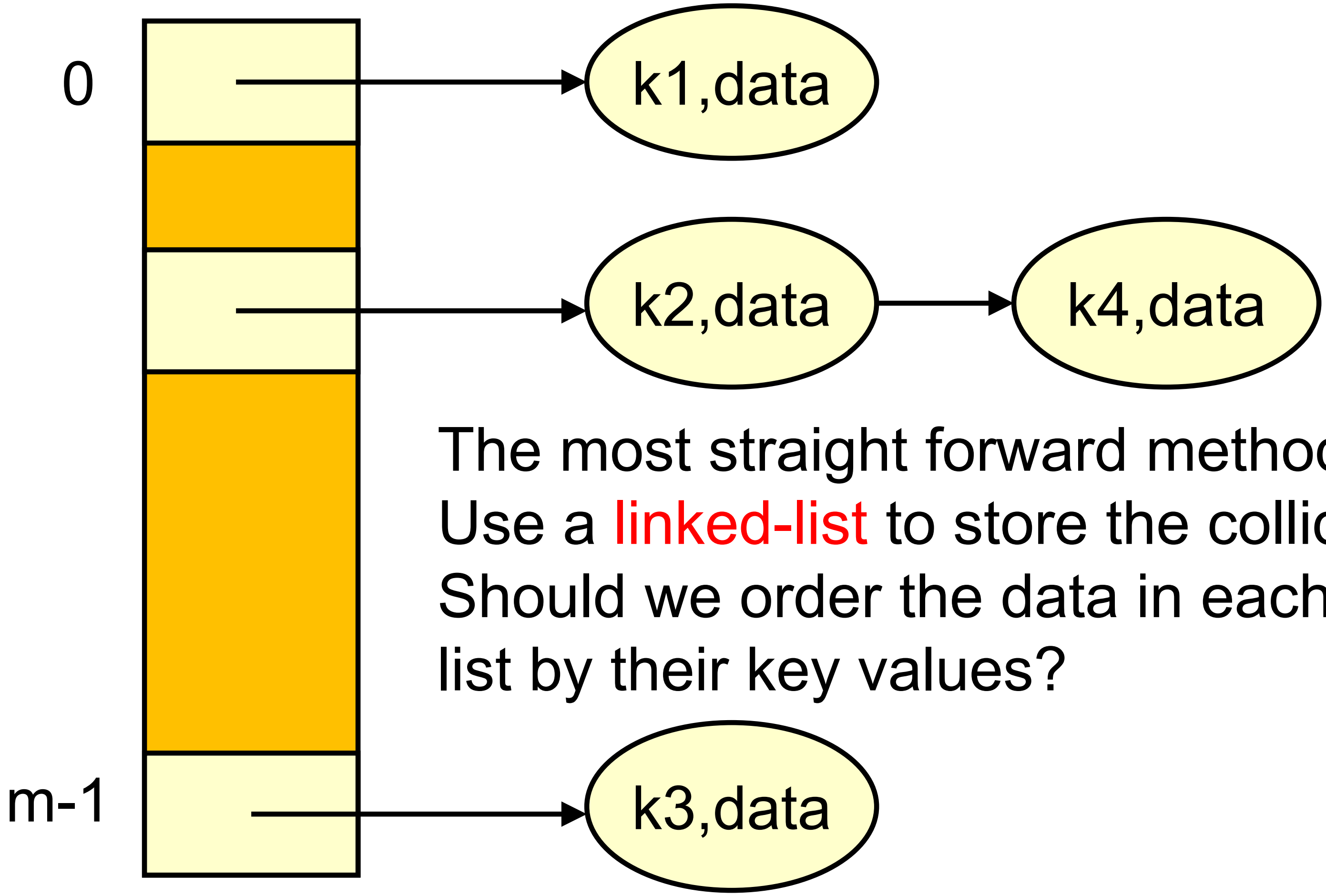
5 Probability of Collision (2/2)

- This means that if there are 23 people in a room, the probability that some people share a birthday is 50.7%!
- In the hashing context, if we insert 23 keys into a table with 365 slots, <u>more than half of the time</u> we will get collisions! Such a result is counter-intuitive to many.
- So, collision is very likely!

5 Collision Resolution Techniques

- Separate Chaining
- Open Addressing
  - Linear Probing + Modified Linear Probing
  - Quadratic Probing
  - Double Hashing

5.1 Separate Chaining



The most straight forward method.
Use a linked-list to store the collided keys.
Should we order the data in each linked list by their key values?

5.1 Hash operations

insert (key, data)

Insert data into the list a[h(key)]

Takes O(1) time

find (key)

Find key from the list a[h(key)]

Takes O(n) time, where n is length of the chain

delete (key)

Delete data from the list a[h(key)]

Takes O(n) time, where n is length of the chain

5.1 Analysis: Performance of Hash Table

- n: number of keys in the hash table
- m: size of the hash tables – number of slots
- α: load factor

  $$\alpha = n/m$$

  a measure of how full the hash table is. If table size is the number of linked lists, then α is the average length of the linked lists.
- Using a linked list for the chains also means separate chaining is not cache friendly

5.1 Reconstructing Hash Table

- To keep α bounded, we may need to reconstruct the whole table when the load factor exceeds the bound.
- Whenever the load factor exceeds the bound, we need to rehash all keys into a bigger table (increase m to reduce α), say a prime close to double the table size m.

5.2 Linear Probing

hash(*k*) = *k* mod 7

Here the table size m=7

Note: 7 is a prime number.

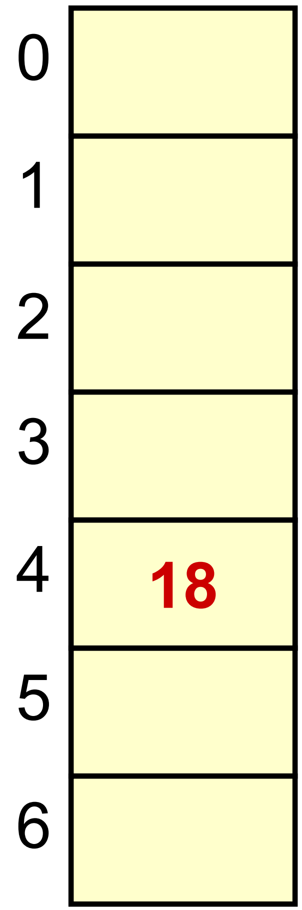
0	
1	
2	
3	
4	
5	
6	

In linear probing, when we get a collision, we scan through the table looking for the next empty slot (wrapping around when we reach the last slot).

5.2 Linear Probing: Insert 18

hash(*k*) = *k* mod 7

hash(18) = 18 mod 7 = 4



5.2 Linear Probing: Insert 14

hash(*k*) = *k* mod 7

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

0	14
1	
2	
3	
4	18
5	
6	

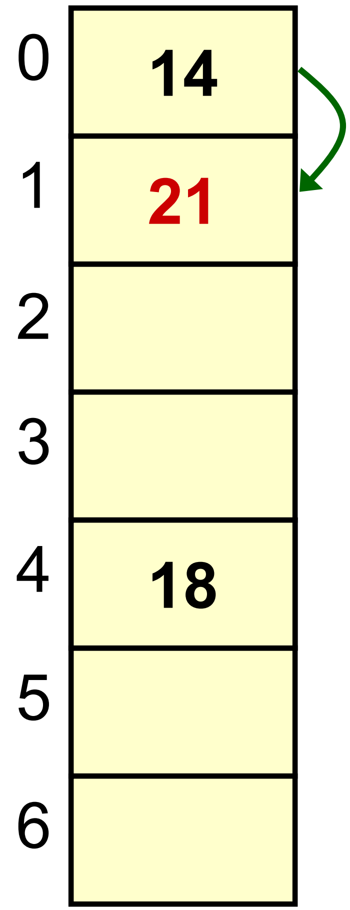
5.2 Linear Probing: Insert 21

hash(*k*) = *k* mod 7

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0



Collision occurs!

What should we do?

Look for next empty slot.

5.2 Linear Probing: Insert 1

hash(*k*) = *k* mod 7

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0

hash(1) = 1 mod 7 = 1

0	**14**
1	**21**
2	**1**
3	
4	**18**
5	
6	

Collides with 21 (hash value 0).

What should we do?

Look for next empty slot.

5.2 Linear Probing: Insert 35

hash(*k*) = *k* mod 7

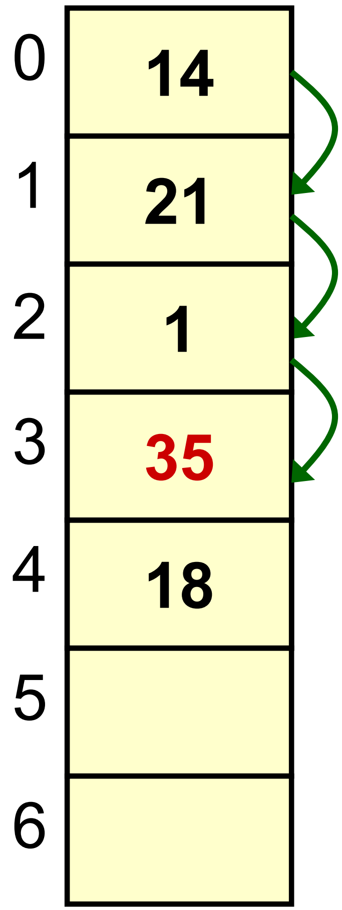
hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0

hash(1) = 1 mod 7 = 1

hash(35) = 35 mod 7 = 0

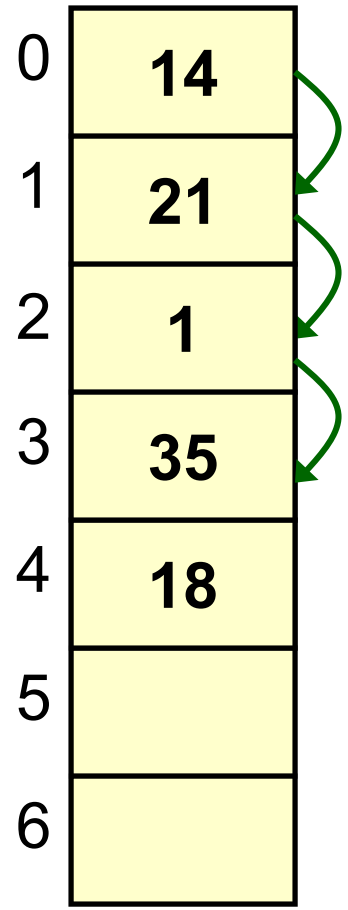


Collision, need to check next 3 slots.

5.2 Linear Probing: Find 35

hash(k) = k mod 7

hash(35) = 0

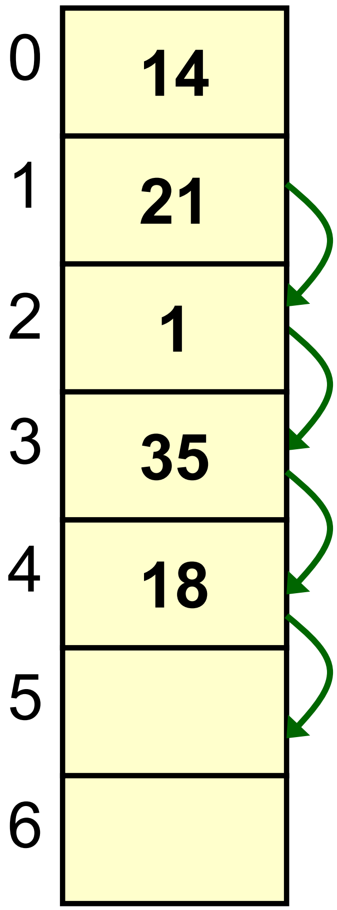


Found 35, after 4 probes.

5.2 Linear Probing: Find 8

hash(k) = k mod 7

hash(8) = 1



8 NOT found.
Need 5 probes!

5.2 Linear Probing: Delete 21

hash(*k*) = *k* mod 7

hash(21) = 0

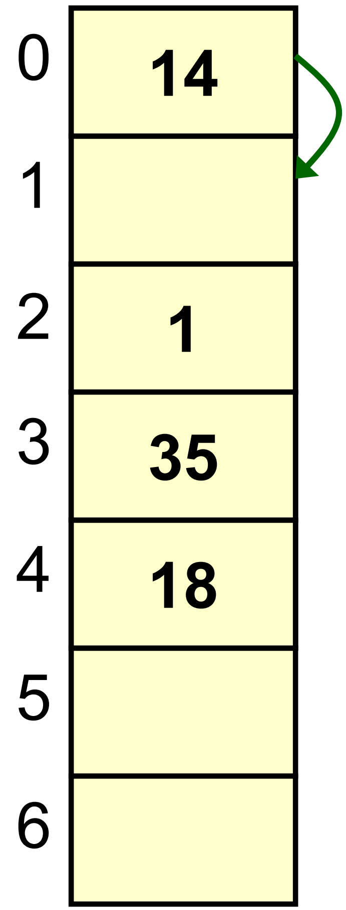
0	**14**
1	**21**
2	**1**
3	**35**
4	**18**
5	
6	

We cannot simply remove a value, because it can affect find()!

5.2 Linear Probing: Find 35

hash(*k*) = *k* mod 7

hash(35) = 0



We cannot simply remove a value, because it can affect find()!

35 NOT found!

Incorrect!

Hence for deletion, cannot simply remove the key value!

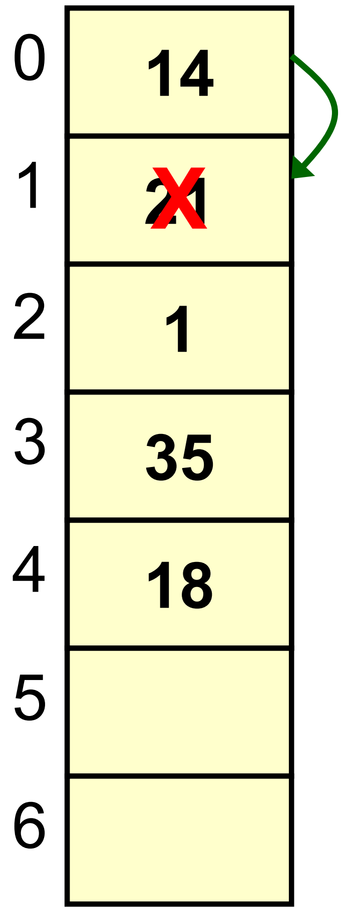
5.2 How to delete?

- Lazy Deletion
- Use three different states of a slot
  - Occupied
  - Occupied but mark as deleted
  - Empty
- When a value is removed from linear probed hash table, we just mark the status of the slot as “deleted”, instead of emptying the slot.

5.2 Linear Probing: Delete 21

hash(*k*) = *k* mod 7

hash(21) = 0

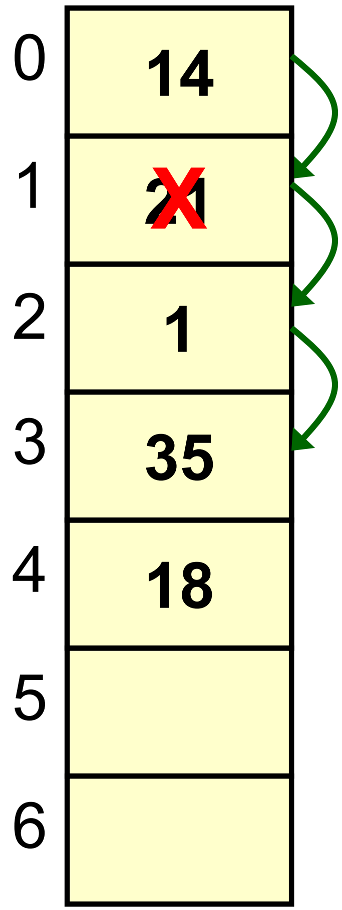


Slot 1 is occupied but now marked as deleted.

5.2 Linear Probing: Find 35

hash(*k*) = *k* mod 7

hash(35) = 0



Found 35

Now we can find 35

5.2 Linear Probing: Insert 15 (1/2)

hash(k) = k mod 7

hash(15) = 1

0	**14**
1	~~**21**~~
2	**1**
3	**35**
4	**18**
5	
6	

Slot 1 is marked as deleted.

We continue to search for 15, and found that 15 is not in the hash table (total 5 probes).

So, we insert this new value 15 into the slot that has been marked as deleted (i.e. slot 1).

5.2 Linear Probing: Insert 15 (2/2)

hash(*k*) = *k* mod 7

hash(15) = 1

0	**14**
1	**15**
2	**1**
3	**35**
4	**18**
5	
6	

So, 15 is inserted into slot 1, which was marked as deleted.

Note: We should insert a new value in first available slot so that the find operation for this value will be the fastest.

5.2 Linear Probing

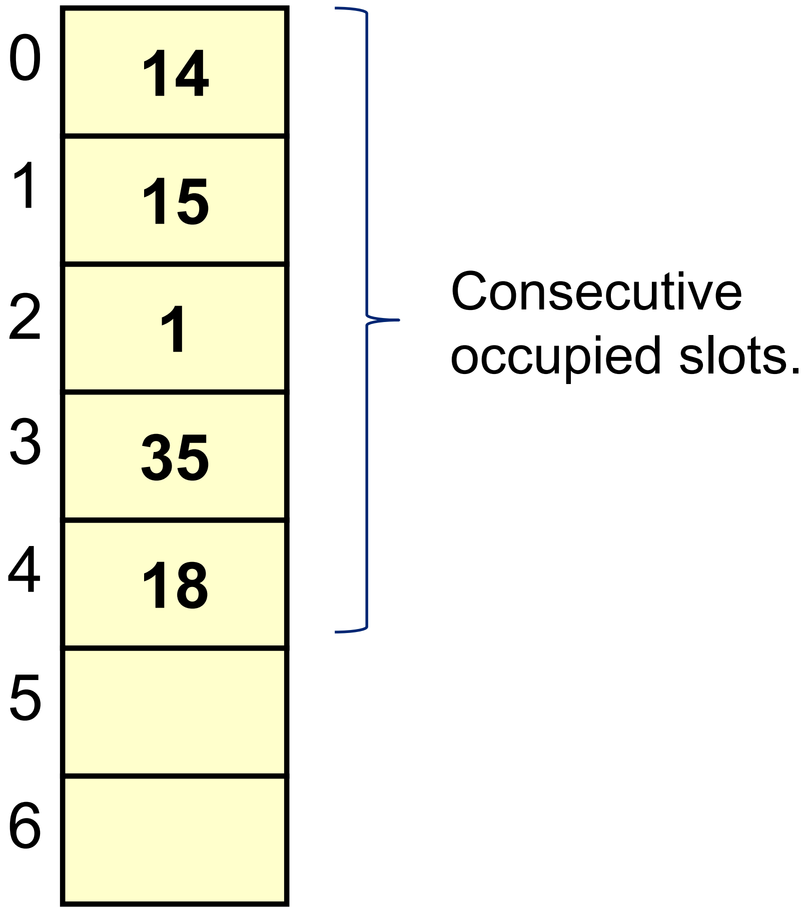
The probe sequence of this linear probing is:

$$\text{hash(key)}$$
$$(\ \text{hash(key)} + \mathbf{1}\ )\ \%\ m$$
$$(\ \text{hash(key)} + \mathbf{2}\ )\ \%\ m$$
$$(\ \text{hash(key)} + \mathbf{3}\ )\ \%\ m$$
$$\vdots$$

5.2 Problem of Linear Probing

It can create many **consecutive occupied slots**, increasing the running time of find/insert/delete.

This is called Primary Clustering



5.2 Modified Linear Probing

Q: How to modify linear probing to avoid primary clustering?

We can modify the probe sequence as follows:

$$\begin{gathered}
\text{hash(key)} \\
(\ \text{hash(key)} + \mathbf{1} * \boldsymbol{d}\ )\ \%\ m \\
(\ \text{hash(key)} + \mathbf{2} * \boldsymbol{d}\ )\ \%\ m \\
(\ \text{hash(key)} + \mathbf{3} * \boldsymbol{d}\ )\ \%\ m \\
\vdots
\end{gathered}$$

where *d* is some constant integer >1 and is co-prime to *m*.

Note: Since *d* and *m* are co-primes, the probe sequence covers all the slots in the hash table.

5.3 Quadratic Probing

For quadratic probing, the probe sequence is:

$$\text{hash(key)}$$
$$(\ \text{hash(key)} + \mathbf{1}\ )\ \%\ m$$
$$(\ \text{hash(key)} + \mathbf{4}\ )\ \%\ m$$
$$(\ \text{hash(key)} + \mathbf{9}\ )\ \%\ m$$
$$\vdots$$
$$(\ \text{hash(key)} + \boldsymbol{k^2}\ )\ \%\ m$$

5.3 Quadratic Probing: Insert 3

hash(*k*) = *k* mod 7

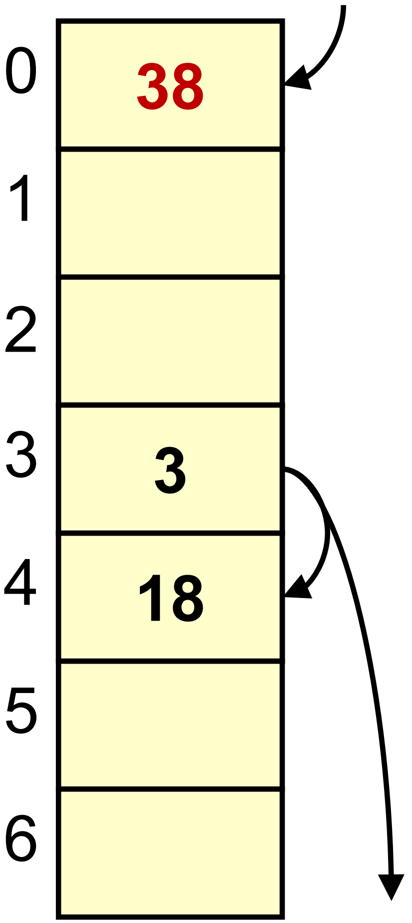
hash(3) = 3

0	
1	
2	
3	**3**
4	**18**
5	
6	

5.3 Quadratic Probing: Insert 38

hash(*k*) = *k* mod 7

hash(38) = 3



5.3 Theorem of Quadratic Probing

- If $\alpha < 0.5$, and m is prime, then we can always find an empty slot.
  (m is the table size and α is the load factor)
- Note: $\alpha < 0.5$ means the hash table is less than half full.
- Q: How can we be sure that quadratic probing always terminates?
- Insert 12 into the previous example, followed by 10. See what happen?

5.3 Problem of Quadratic Probing

- If two keys have the same initial position, their probe sequences are the same.
- This is called secondary clustering.
- But it is not as bad as linear probing.

5.4 Double Hashing

Use 2 hash functions:

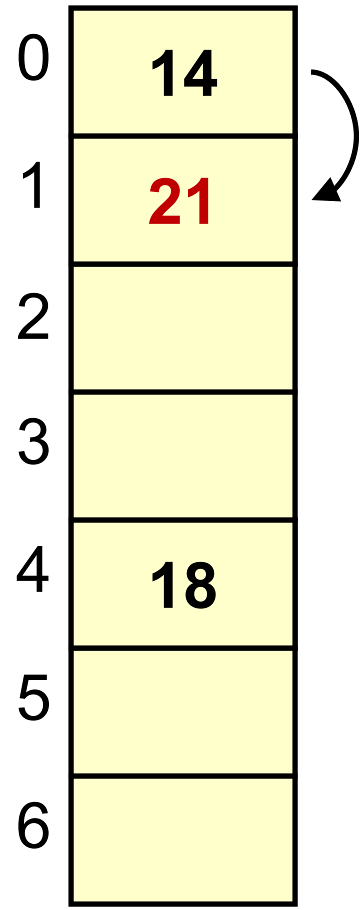
$$\text{hash(key)}$$
$$(\ \text{hash(key)} + 1 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$(\ \text{hash(key)} + 2 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$(\ \text{hash(key)} + 3 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$\vdots$$

$hash_2$ is called the secondary hash function, the number of slots to jump each time a collision occurs.

5.4 Double Hashing: Insert 21

hash(k) = k mod 7
hash$_2$(k) = k mod 5

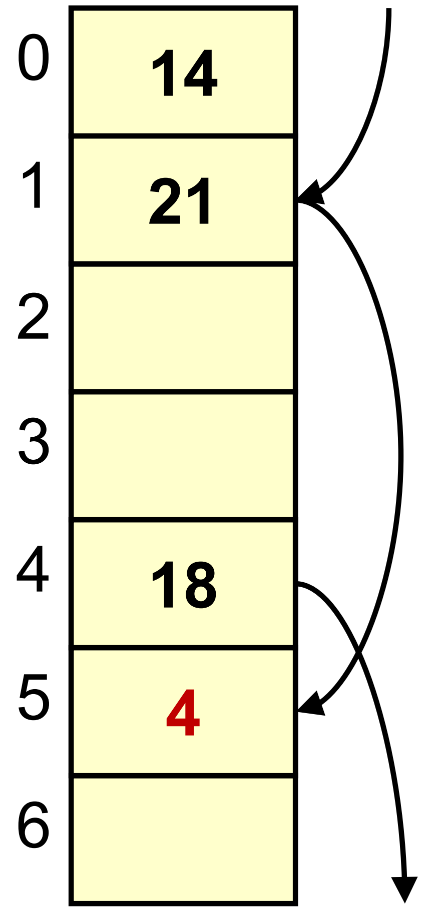
hash(21) = 0
hash$_2$(21) = 1



5.4 Double Hashing: Insert 4

hash(k) = k mod 7
hash$_2$(k) = k mod 5

hash(4) = 4
hash$_2$(4) = 4

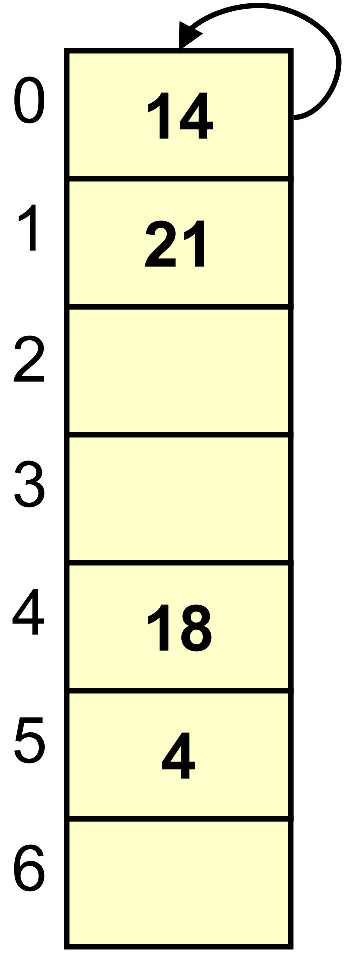


If we insert 4, the probe sequence is 4, 8, 12, …

5.4 Double Hashing: Insert 35

hash(*k*) = *k* mod 7
$hash_2$(*k*) = *k* mod 5

hash(35) = 0
$hash_2$(35) = 0



But if we insert 35, the probe sequence is **0, 0, 0,** …

What is wrong?
Since $hash_2$(35)=**0**.
Not acceptable!

5.4 Warning

- **Secondary hash function must not evaluate to 0!**
- To solve this problem, simply change $hash_2(key)$ in the above example to:

$$hash_2(key) = 5 - (key \% 5)$$

Note:

- If $hash_2(k) = 1$, then it is the same as linear probing.
- If $hash_2(k) = d$, where d is a constant integer > 1, then it is the same as modified linear probing.

5.5 Criteria of Good Collision Resolution Method

- Minimize clustering
- Always find an empty slot if it exists
- Give different probe sequences when 2 initial probes are the same (i.e. no secondary clustering)
- Fast

5.6 Worst case performance of hashing

- For separate chaining
  - O(n) time for find/insert/delete
  - Such a case is when all items hash to one index so you have a linked list of size n

- For open addressing
  - O(n) time for find/insert/delete
  - Such a case is when you have to probe every slot in the table to determine you cannot insert or what is to be found/deleted is not in the table

5.6 Average case performance of hashing: Using Separate Chaining

- α (load factor) is the average size of the linked list for each slot
- Thus if α is bounded, find/insert/delete will take O(1) time on average

5.6 Average case performance of hashing: Using Open Addressing

- Average case = average number of probes

- Assuming each probe location is generated randomly and independently *(such collision resolution cannot be used in practice !)* so that clustering does not happen

- For each probe

$$\text{Probability of finding empty slot} = 1 - \propto$$

$$\text{Probablity of finding non-empty slot} = \propto$$

5.6 Average case performance of hashing: Using Open Addressing

- For unsuccessful find, unsuccessful delete and successful insert (need to hit an empty slot)

$$\text{Average number of probes} = \frac{1}{1-\alpha}$$

- For successful find and successful delete

$$\text{Average number of probes} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha}$$

- Again if α is bounded, find/insert/delete will take O(1) time on average

6 Set ADT

- A set as you have learned in high school is simply a unordered collection of items with no duplicates (with duplicates it's a multi-set)
  - E.g {1,2,3} is a set of 3 integers and this set is the same set as {3,1,2}, since order does not matter
- Some simple Set operations include the following
  - find(x) – retrieve x from the set if it exist in the set
  - insert(x) – insert x into the set if it doesn't already exist in set
  - remove(x) – remove x from the set if it exist in the set
  - union(s) – return union of this set with another set s
  - intersect(s) – return intersection of this set with another set s

6 Using Hashtable for simple Set

- HashTable can easily and efficiently implement a Set ADT If we do not need complex operations like set intersection and union

Set Operations	HashTable implementation	Time complexity
find(x)	find(x)	Average O(1)
Insert(x)	insert(x,x) – make <key,value> pair the same	Average O(1)
remove(x)	remove(x)	Average O(1)

7 Summary

- How to hash? Criteria for good hash functions?
- How to resolve collision?

  Collision resolution techniques:
  - separate chaining
  - linear probing
  - quadratic probing
  - double hashing
- Problem on deletions
- Primary clustering and secondary clustering.

8 Java HashMap Class

8 Class HashMap <K, V>

public class **HashMap<K,V>**
extends AbstractMap<K,V>
implements Map<K,V>, Cloneable, Serializable

- This class implements a hash map, which maps keys to values. Any non-null object can be used as a key or as a value.
  e.g. We can create a hash map that maps people names to their ages. It uses the names as keys, and the ages as the values.
- The AbstractMap is an abstract class that provides a skeletal implementation of the Map interface.
- Generally, the default load factor (0.75) offers a good tradeoff between time and space costs.
- The default HashMap capacity is 16.

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- **Constructors** summary
  - `HashMap()`

    Constructs an empty HashMap with a default initial capacity (16) and the default load factor of 0.75.

  - `HashMap(int initialCapacity)`

    Constructs an empty HashMap with the specified initial capacity and the default load factor of 0.75.

  - `HashMap(int initialCapacity, float loadFactor)`

    Constructs an empty HashMap with the specified initial capacity and load factor.

  - `HashMap(Map<? extends K, ? extends V> m)`

    Constructs a new HashMap with the same mappings as the specified Map.

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Some methods

- `void clear()`
  Removes all of the mappings from this map.
- `boolean containsKey(Object key)`
  Returns true if this map contains a mapping for the specified key.
- `boolean containsValue(Object value)`
  Returns true if this map maps one or more keys to the specified value.
- `V get(Object key)`
  Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.
- `V put(K key, V value)`
  Associates the specified value with the specified key in this map.
- ...

8 Example

- Example: Create a hashmap that maps people names to their ages. It uses names as key, and the ages as their values.

```
HashMap<String, Integer> hm = new HashMap<String, Integer>();
// placing items into the hashmap
hm.put("Mike", 52);
hm.put("Janet", 46);
hm.put("Jack", 46);
// retrieving item from the hashmap
System.out.println("Janet => " + hm.get("Janet"));
```

TestHash.java

The output of the above code is:

```
Janet => 46
```

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