CS2040S: Data Structures and Algorithms

Discussion Group Problems for Week 8

For: March 6–March 10

Problem 1. (Priority queue)

There are situations where, given a data set containing n unique elements, we want to know the top k highest value elements. A possible solution is to store all n elements first, sort the data set in $O(n \log n)$, then report the right-most k elements. This works, but we can do better.

(a) Design a data structure that supports the following operation better than $O(n \log n)$:

- `getKLargest()`: returns the top k highest value elements in the data set.

(b) Instead of having a static data set, you could have the data streaming in. However, your data structure must still be ready to answer queries for the top k elements efficiently. Expand on your data structure to support the following two operations better than $O(n \log n)$:

- `insertNext(x)`: adds a new item x into the data set.
- `getKLargest()`: returns the current top k highest value elements in the data set.

For example, if the data set contains {1, 13, 7, 9, 8, 4} initially and we want to know the top 3 highest value elements, calling `getKLargest()` should return the values {13, 9, 8}.

Suppose we then add the number 11 into the data set by calling `insertNext(11)`. The data set now contains {1, 13, 7, 9, 8, 4, 11} and calling `getKLargest()` should return {13, 11, 9}.

Since Priority Queues are tested for the midterms, we have decided to include the solution to this problem on the next page. However, you should still give the problem a try before looking at the solution.

Solution: For part (a), we can quick-select the k^{th} largest element in expected $O(n)$ time. The elements that are required will be found between this element and the end of the array.

For part (b), a key thing to realise is that we only need to store the largest k elements at any time. We can maintain a min priority queue of no more than k elements. For every element that is given to us, add it into the priority queue. If the priority queue contains more than k elements, keep removing the smallest element until the priority queue size is k. Both insert and remove-min operations run in $O(\log k)$ time since the priority queue contains at most k elements.

For `insertNext(x)`, we can add it into the priority queue. If the priority queue contains more than k elements, keep removing the smallest element until the priority queue size is k. Both insert and remove-min operations run in $O(\log k)$ time since the priority queue contains at most k elements. For `getKLargest()`, we can simply return all the elements in the priority queue.

The overall complexity over n calls of `insertNext` is $O(n \log k)$. Certain languages (such as C++) use this idea to implement `partial_sort`. Priority queue must be used here (instead of waiting for all elements before we do a quick select) so that we can always have access to k largest elements throughout all stages of the stream instead of only at the end.

Problem 2. (Union-Find Review)

Problem 2.a. What is the worst-case running time of the find operation in Union-Find with path compression, but no weighted union (i.e Quick-Union with path compression)?

Problem 2.b. Here's another algorithm for Union-Find based on a linked list. Each set is represented by a linked list of objects, and each object is labelled (e.g., in a hash table) with a set identifier that identifies which set it is in. Also, keep track of the size of each set (e.g., using a hash table). Whenever two sets are merged, relabel the objects in the smaller set and merge the linked lists. What is the running time for performing m Union and Find operations, if there are initially n objects each in their own set?

More precisely, there is: (i) an array id where $id[j]$ is the set identifier for object j; (ii) an array $size$ where $size[k]$ is the size of the set with identifier k; (iii) an array $list$ where $list[k]$ is a linked list containing all the objects in set k.

```
Find(i, j):
    return (id[i] == id[j])

Union(i, j):
    if size[i] < size[j] then Union(j,i)
    else: // size[i] >= size[j]
        k1 = id[i]
        k2 = id[j]
```

```
for every item m in list[k2]:
    set id[m] = k1
append list[k2] on the end of list[k1] and set list[k2] to null
size[k1] = size[k1] + size[k2]
size[k2] = 0
```

Assume for the purpose of this problem that you can append one linked list on to another in $O(1)$ time. (How would you do that?)

Problem 2.c. Imagine we have a set of n corporations, each of which has a (string) name. In order to make a good profit, each corporation has a set of jobs it needs to do, e.g., corporation j has tasks $T^j[1 \ldots m]$. (Each corporation has at most m tasks.) Each task has a priority, i.e., an integer, and tasks must be done in priority order: corporation j must complete higher priority tasks before lower priority tasks.

Since we live in a capitalist society, every so often corporations decide to merge. Whenever that happens, two corporations merge into a new (larger) corporation. Whenever that happens, their tasks merge as well.

Design a data structure that supports three operations:

- `getNextTask(name)` that returns the next task for the corporation with the specified name.
- `executeNextTask(name)` that returns the next task for the corporation with the specified name and removes it from the set of tasks that corporation does.
- `merge(name1, name2, name3)` that merges corporation with names `name1` and `name2` into a new corporation with `name3`.

Give an efficient algorithm for solving this problem.

Problem 3. **Optional. No need to prepare!** TAs will pick some puzzles and solve as a class!

Problem 3.a. [*Warm-up*] **Heaven or Hell**

You have two doors in front of you. One door leads to heaven, and the other to hell. There are two male guards, one by each door. One guard always tells the truth, and the other always lies, but you don't know who is who.

You can only ask **one** question to **one** guard to find the door to heaven. What question would you ask?

Problem 3.b. *What..?* **The Housewife and the Bartender**
Source: AlphaLab

A housewife finally has time for a break. She goes to a bar and challenges a bartender to guess her kids' ages:

"I have three children and the product of their ages is 72."

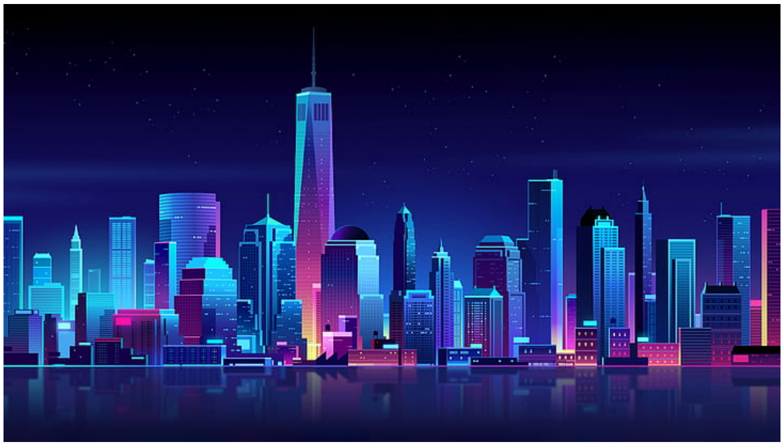
The bartender claims he doesn't know.

"The sum of their ages is the number of this building"

The bartender does some calculation but claims he still doesn't know.

"Well, guess i'm getting a free drink. It's nice to relax. I've been playing chess with my youngest every night, she's improving too quickly."

Now he knows! So what are the ages?



Problem 3.c. *'Shared' Information* - **100 green-eyed Captives**
Source: TED-Ed

100 green-eyed logicians are held captive by a (un)reasonable dictator, Tdolf Ailer, on a secluded island. There is no escape but there is in place, one peculiar rule: *Any prisoner can approach the guard at night and ask to leave. If the prisoners have green eyes, they'll be released. Otherwise, they are tossed into the volcano.*

As it happens, all 100 prisoners have green eyes! But they've lived there since birth. And Tdolf has ensured they can't learn their own eye colour. There are no reflective surfaces (even water is in opaque containers), and most importantly, they cannot communicate among themselves. However, they do see each other during each morning headcount. Nevertheless, they all know no one would ever risk trying to leave without absolute certainty of success.

After much pressure from human rights groups. Tdolf reluctantly agrees to let you visit the island and speak to the prisoners under the following conditions:

- You may only make **1 statement**
- You cannot tell them any new information

You thought long and hard on what to say to avoid incurring the wrath of the dictator. Finally, you tell the crowd: *"At least one of you has green eyes!"*

The dictator is suspicious but reassures himself that your statement has no consequential impact. You leave, and life seemingly goes on as before. But on the 100th morning after your visit, all 100 prisoners are gone, each having asked to leave the previous night.

So how did you outsmart the dictator?

Hint: *Start with a much smaller group! What's the minimum group size you can consider?*

Problem 3.d. *An optimization problem* - **Cross The Bridge**
Source: TED-Ed

Quick! Your team needs to get away from a bunch of zombies ***fast***. With you, there are the young biologist, the middle-aged physicist, and the old mathematician.

There is only one way to safety - across an old rope bridge, spanning a massive gorge. Here's how fast each of your team can dash across:

- You: 1 minute
- Biologist: 2 minutes

- Physicist: 5 minutes
- Mathematician: 10 minutes

The mathematician calculated that you have just **17 minutes** before the zombies catch up. So, you only have that much time to get everyone across and cut the ropes. Sadly, the bridge can only hold **two people** at a time and they need to **walk at the pace of the slower one**. Worse, the place is pitch black! And you only have **1 lantern** that illuminates a tiny area. You have no other tools, and must make sure everyone is safely across before the first zombie steps onto the bridge.

Can you figure out a way to have everyone successfully escape?

(Bonus, optional) If you would like a greater challenge, try tackling the more general version of this problem here: https://open.kattis.com/problems/bridge