The University of Texas at Austin

McCombs School of Business

RM 294: Optimization I

PROJECT 3 - NON-LINEAR PROGRAMMING

Optimization of Printing Decisions: A Joint Price-Quantity Approach

GROUP - 5

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Executive Summary

The Opportunity

Our current production planning model optimizes print quantity while treating price as fixed at $1.00. This approach, while operationally simple, ignores a critical market reality: **our pricing decisions directly influence customer demand**. This analysis evaluates whether jointly optimizing both price and quantity can improve profitability while managing disposal fees for overproduction and rush costs for underproduction.

Key Findings

Our analysis confirms that market demand is highly elastic. By optimizing price and quantity simultaneously, the proposed QP model outperforms the current baseline across all key metrics:

Strategy	Daily Profit
Current Model (Fixed Price)	$231.48
Proposed Model (Joint Optimization)	$235.54
Improvement	**+$4.06 (+1.75%)**

Strategic Insight

The proposed model recommends a strategic shift: trading a 5% price reduction for a 13% volume increase. By lowering the price to $0.95, we generate sufficient additional demand to offset the lower per-unit margin, capturing an estimated $1,481 in additional annualized profit without increasing fixed costs.

Why This Works

Our regression analysis quantifies the price-demand relationship ($\beta_1 \approx$ -1367), revealing that customers are price-sensitive. A modest 5-cent price reduction drives enough additional demand to more than offset the lower per-unit margin, resulting in **$4.06 more profit per day**.

Robustness Validation

Bootstrap sensitivity analysis across 500+ simulated market scenarios confirms:

- Optimal price remains stable between $0.93-$0.98 (95% confidence)
- Expected profit consistently exceeds the baseline model
- The improvement is statistically reliable, not a data artifact

Recommendation

We recommend the immediate adoption of the Joint Optimization model. Sensitivity analysis (Bootstrap) confirms that this pricing strategy is statistically robust and consistently outperforms the baseline. By integrating this model, the company aligns its production targets with actual market behavior, reducing operational waste and maximizing total contribution margin.

1. Introduction

1.1 Business Context

The core operational challenge in our publishing business is the **Newsvendor Problem**: determining the daily print quantity that minimizes the cost of inventory mismatches. Currently, we rely on a **Standard Newsvendor Model (NVM)** which optimizes production quantity (q) while treating the selling price as a fixed constraint (p = $1.00) .

While this model manages basic inventory risk, it ignores the reality that pricing decisions directly influence customer demand. Furthermore, our cost structure imposes asymmetrical penalties for forecast errors:

- **Production Cost (**c = $0.50**):** The standard cost to manufacture one unit.
- **Underproduction** triggers "rush orders" at a premium marginal cost (g = $0.75).
- **Overproduction** results in unsold inventory that incurs disposal fees (t = $0.15).

By fixing the price, the current model likely restricts revenue potential while failing to optimally balance these disposal and rush order risks.

1.2 Objective

The objective of this analysis is to transition from the static Standard NVM to a **Joint Price-Quantity Optimization** framework. By relaxing the fixed-price constraint, we aim to identify the global profit maximum that accounts for price elasticity.

This study seeks to:

- **Quantify Elasticity:** Use linear regression ($D \quad = \beta_0 + \beta_1 p$) to model how price changes drive market volume.
- **Optimize Jointly:** Solve a **Quadratic Programming (QP)** model that simultaneously determines the optimal Price (p*) and Quantity (q*) to maximize daily profit .
- **Validate Robustness:** Use bootstrap simulation to test the stability of these recommendations against market volatility .

The following sections detail the mathematical formulation, quantitative results, and the strategic implications of adopting this flexible pricing model.

2. Methodology

2.1 Data Approach (Relates to Q1):

Our data set contains the units of demand for different price values for the newspaper. We have 99 observations.

Assuming that price impacts demand linearly with error we fit the following model:

$$D_i = \beta_0 + \beta_1 * price + \epsilon_i$$

Where:

β_0: The intercept and not economically interpretable estimate of the vertical offset that better fits the data.

β_1: The change in demand associated with a unit change in price.

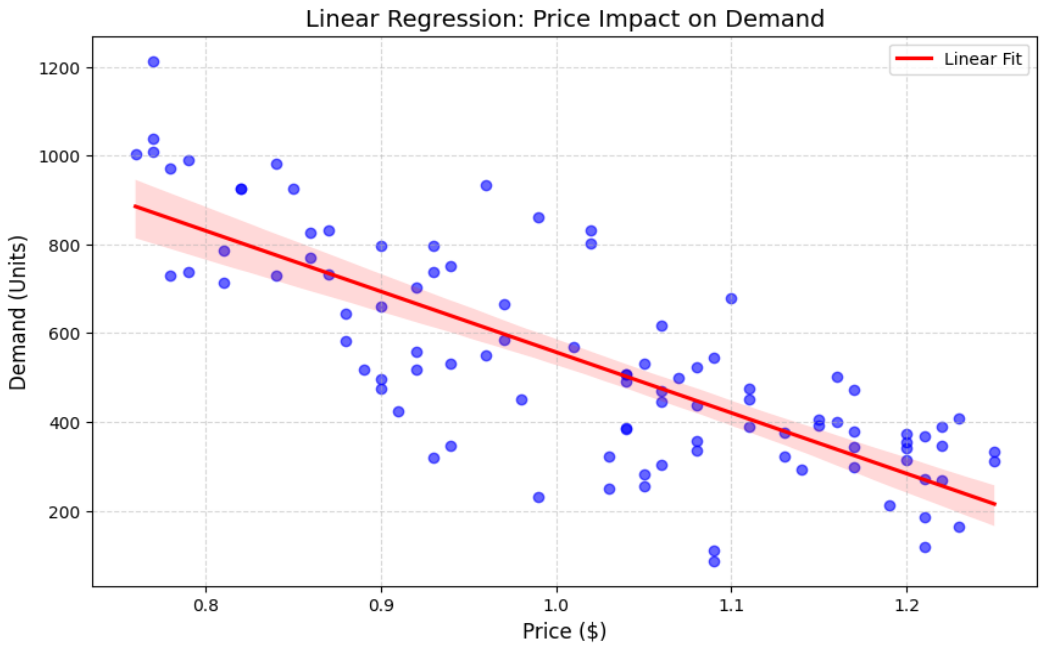
ϵ_i: The residuals which are the source of randomness.

The results from the linear model below show that when we assume price has a linear relationship with demand price affects

Adj. R-squared**: 0.618**

Metric	Coefficient	Std. Error	P>|t|
Constant	1924.7175	111.334	0.000
Price	-1367.7125	108.379	0.000

Additionally, our assumption of linearity is reasonable for the range of data we possess as the following chart shows:



We can also see that the residuals are left skewed, which shows that the coefficients are generally underestimating the real impact of price on demand.

2.2 Optimization Framework

a) **Baseline Model (LP):**

- **Simulate Demand using the residuals from the linear regression:**

First, we will estimate what the optimal quantity and profit are for a fixed price. We start by simulating the demand using the residual values from our linear model. As shown in the equation below:

$$D_i(1) = \beta_0 + \beta_1 * 1 + \epsilon_i$$

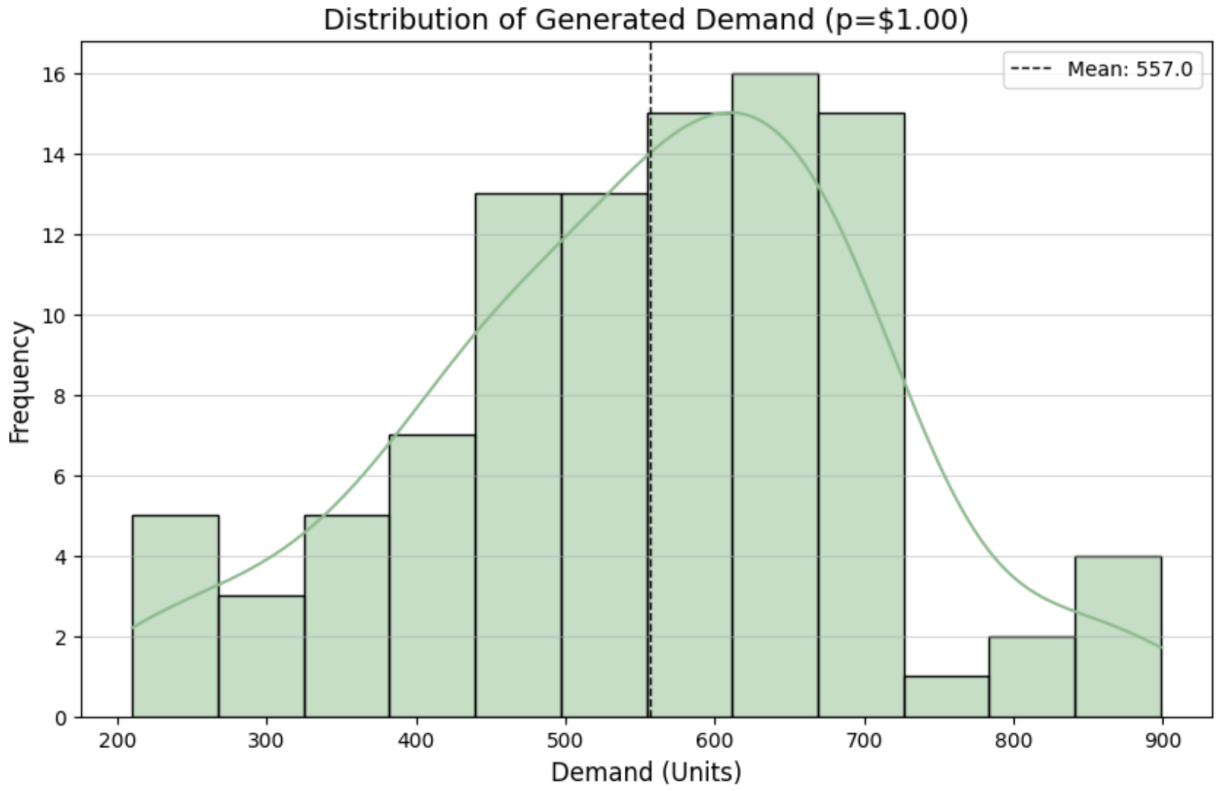
The simulated data replaces the distribution of the original demand data that we had with demand distributed around the mean defined by our model and uses the residuals as randomness. We are hence assuming that:

$$D|T \sim Linear(p) + Empirical\ Noise$$

We are shifting the distribution of the residuals by

$$\mu(1) = \beta_0 + \beta_1 * 1$$

You can see the distribution of the simulated demand has the same shape as the distribution of the residuals from the linear model, although the mean is shifted by the beta coefficients in the model:



- **Finding the optimal quantity when the price is 1**

  Our objective function is to maximize profit and it can be defined as:

$$max\ \frac{1}{n}\sum_{i=1}^{n} \quad (p * D_i - q * c - g * (D_i - q)^+ - t * (q - D_i)^+)$$

  Where:

$$(x)^+ = max(x, 0)$$
$$g > c\ is\ rush\ cost, t > 0\ is\ disposal\ cost$$

We linearize the plus terms with extra variables:

For each scenario i:

$$u_i \approx (D_i - q)^+ \rightarrow\ shortage\ so\ we\ have\ to\ rush\ print\ this\ units$$

$$v_i \approx (q - D_i)^+ \rightarrow \ excess\ so\ we\ dispose\ this\ units$$

Because Gurobi can't optimize the expression

$$u_i \ = \ max(D_i - q, 0)$$

To deal with this issue we introduce the following constraints:

$$u_i \geq D_i - q \qquad , \qquad u_i \geq 0$$

We want to find their minimum values because u_i and v_i show in the objective function multiplied by negative values (costs). Therefore, if $D_i - q$ is positive then the minimum value u_i can take is

$$u_i \ = \ D_i - q$$

And if so that if $D_i - q$ is negative then the minimum value u_i can take is

$$u_i \ = \ 0$$

Combining both constraints, we obtain $u_i \ = \ max(D_i - q, 0)$

Following a similar logic we understand that by imposing constraints:

$$v_i \geq q - D_i \qquad , \qquad v_i \geq 0$$

We are implementing the condition $v_i \ = \ max(q - D_i, 0)$

We can define our objective function as:

$$max\ \frac{1}{n}\sum_{i=1}^{n} \quad (p * D_i - q * c - g * (u_i) \quad - t * (v_i) \quad)$$

And since $p * D_i$ is a constant given p and D the problem is linear.

Fixing the price at $1.00 and given c = 0.5, g = 0.75, t = 0.15 we optimize our objective function using the simulated demand values for D(1) and find that the optimal quantity (q*) is 471 newspapers and the expected profit is 231.

b) **Proposed Model (QP):**

As stated in the project description, to let the price impact the demand, we introduced the demand model into the model.

In the extended model, price p becomes a decision variable. Therefore, demand in each scenario becomes:

$$D_i = \beta_0 + \beta_1 p + \varepsilon_i$$

Which creates a full demand distribution for any price level, and the profit function after introducing the demand function thus becomes:

$$max \frac{1}{n}\sum_{i=1}^{n} (p(\beta_0 + \beta_1 p + \varepsilon_i) - qc - g(D_i - q)^+ - t(q - D_i)^+)$$

Similar to the previous part, we also introduced the variables:

$$u_i = (D_i - q)^+ \quad , \quad v_i = (D_i - q)^+$$

And their corresponding linear constraints:

$$u_i \geq (D_i - q)^+ \quad , \quad u_i \geq 0$$

$$v_i \geq (q - D_i)^+ \quad , \quad v_i \geq 0$$

Solving the function above with Gurobi we have the new optimized price at **0.9536**, and the new quantity is **535** newspapers

2.3 Simulation Approach:

Bootstrapping for Stability: To ensure our Joint Price-Quantity Optimization model is robust and not merely overfitting the specific 99 data points available, we employed a non-parametric Bootstrap Simulation. This technique allows us to estimate the sampling distribution of our decision variables (Price and Quantity) and the resulting Profit. To ensure independent and reproducible bootstrap samples, each iteration used a seed. This guarantees independence across bootstrap draws while making the simulation exactly repeatable.

The simulation follows a four-step iterative process repeated **500 times**:

- **Resampling:** In each iteration, we generate a new dataset by randomly sampling **n = 99** observations from the original historical data with replacement. This creates "synthetic" histories that mimic the variability of the real world.

    a. A bootstrap dataset is obtained by sampling with replacement:

        i. $D^{*(b)} = \{(p_i{}^{*(b)}, D_i{}^{*(b)})\}_{i=1}^{n}$

    b. Where each $(p_i{}^{*(b)}, D_i{}^{*(b)})$ is drawn independently and uniformly from the original dataset, where n is the sample size, p_i is the observed price and D_i is the observed demand.

- **Model Re-calibration:** For each resampled dataset, we re-fit the linear regression model

        i. $D_i{}^{*(b)} = \beta_0{}^{*(b)} + \beta_1{}^{*(b)} * p_i{}^{*(b)} + \epsilon_i{}^{*(b)}$

    b. to generate new coefficients ($\beta_0{}^{*(b)}$, $\beta_1{}^{*(b)}$) and new residuals. This captures the uncertainty inherent in the demand curve estimation itself.

    c. We construct the simulated demand distribution at price p = 1

        i. $D_i(1)^{(b)} = \beta_0{}^{*(b)} + \beta_1{}^{*(b)} * 1 + \epsilon_i{}^{*(b)}$

- **Optimization:** We feed the new demand parameters into the Joint Optimization (QP) solver (Gurobi) to determine the optimal Price $p_{(b)}{}^{*}$ and Quantity $q_{(b)}{}^{*}$ specific to that simulation run.

$$max\ \frac{1}{n}\sum_{i=1}^{n} \quad [p * D_i{}^{(b)}(p) - q * c - g * (u_i) \quad - t * (v_i) \quad]$$

Subject to:

$$u_i \geq D_i{}^{(b)}(p) - q \qquad , \qquad u_i \geq 0$$

$$v_i \geq q - \quad D_i{}^{(b)}(p) \qquad , \qquad v_i \geq 0$$

- **Aggregation:** We record the optimal $p_{(b)}{}^{*}$, $q_{(b)}{}^{*}$, and Expected Profit for all 500 iterations to construct confidence intervals and analyze the risk profile of our strategy.

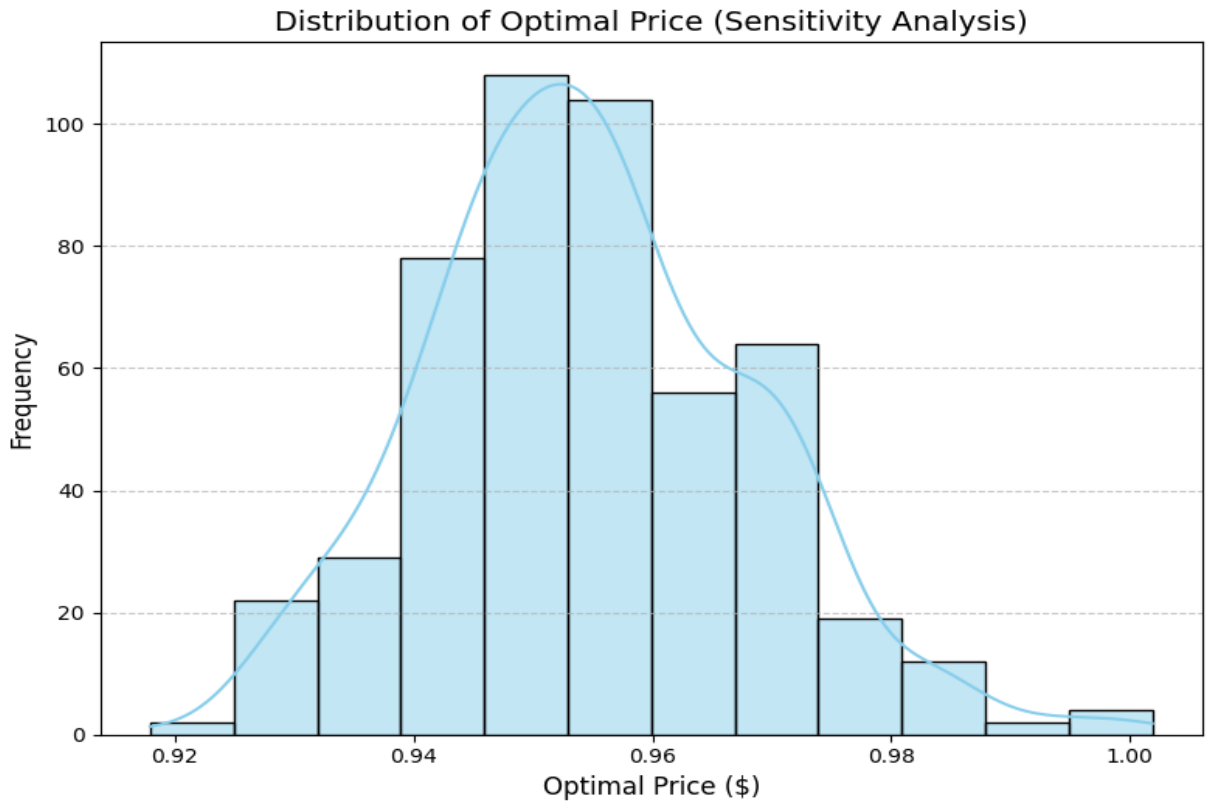
3. Quantitative Results

Method	Optimal Quantity	Optimal Profit
Fixed Price (p=1)	471	231
Variable Price	535	234

3.1 Sensitivity Analysis: Robustness of the Joint Model

The Bootstrap simulation provides strong statistical evidence that the proposed strategy (variable pricing) yields consistent results despite demand uncertainty. The analysis of the 500 simulation runs reveals the following distributions.

a). Optimal Price Stability

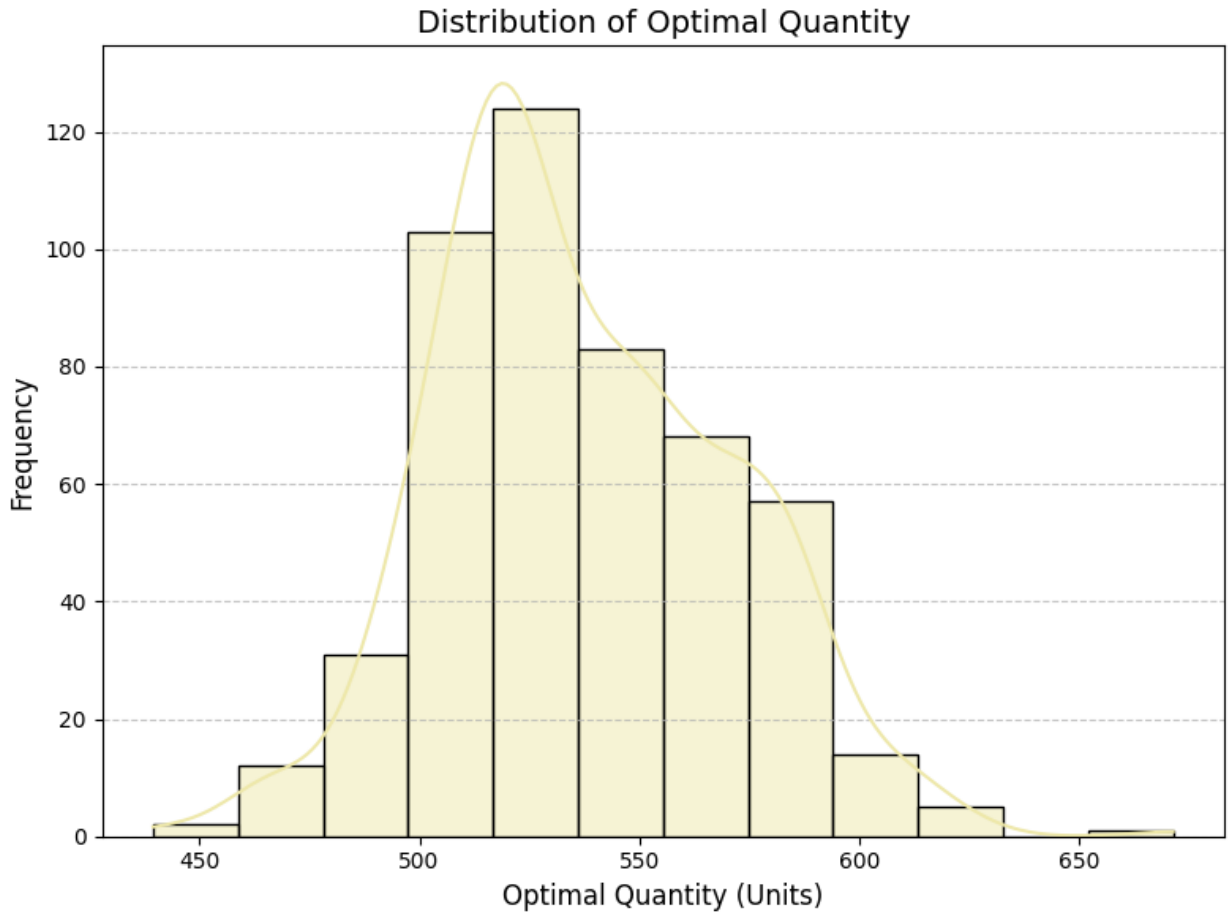


The model recommends a pricing strategy that is highly stable.

- **Mean Optimal Price:** $0.95
- **95% Confidence Interval:** [$0.93, $0.98]
- **Standard Deviation:** $0.014

Analysis: The histogram shows tight clustering around $0.95. This low standard deviation indicates that the recommendation to lower the price from the current $1.00 is statistically significant and robust. In 95% of simulated market scenarios, the optimal price remains below $1.00, confirming that the current fixed price is consistently sub-optimal given the demand elasticity.

b). Optimal Quantity Variability

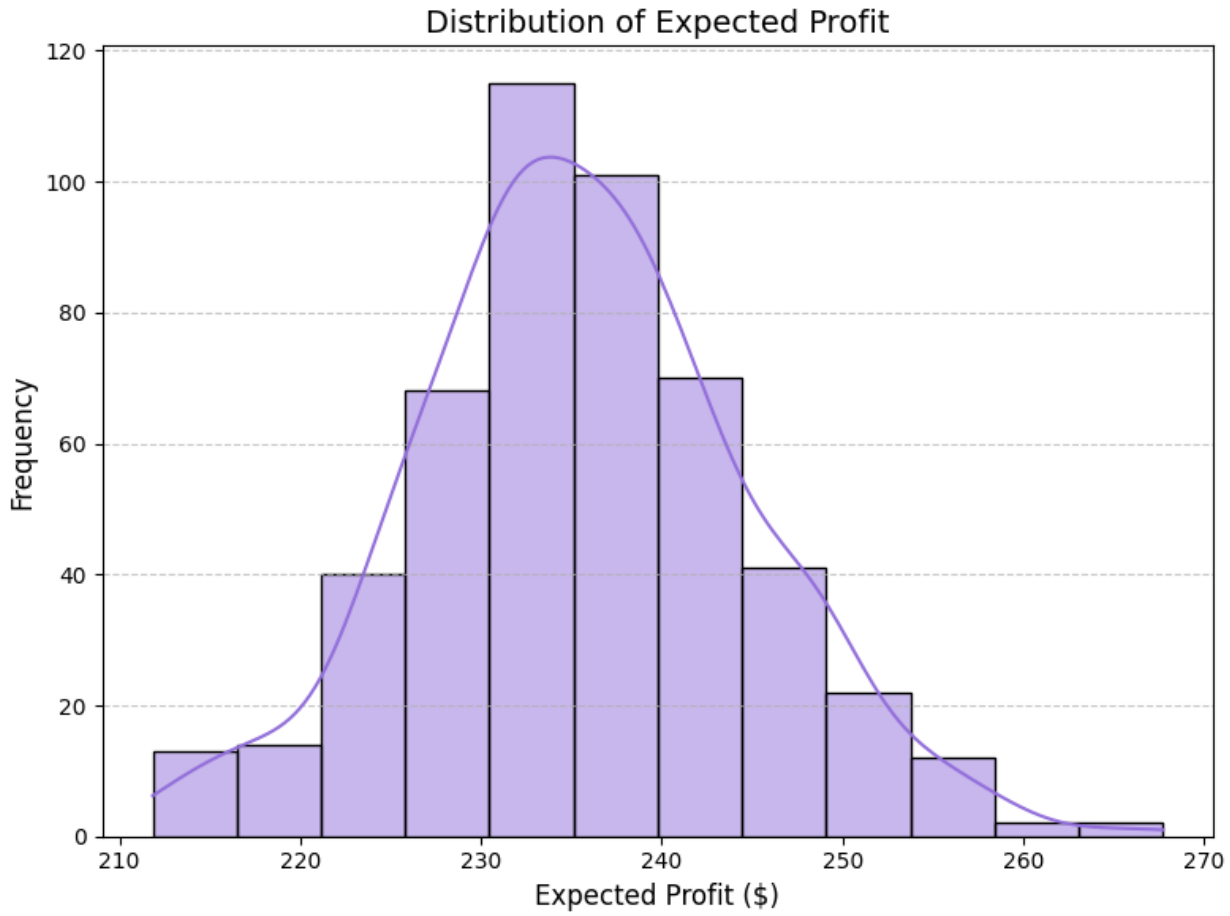


While price is stable, the optimal print quantity exhibits higher variability due to the "bullwhip" effect of demand parameter changes.

- **Mean Optimal Quantity:** 536 units
- **95% Confidence Interval:** [473, 605]
- **Standard Deviation:** 33.4 units

Analysis: The distribution spans a range of approximately 125 units. This suggests that while the pricing strategy ($0.95) should be fixed, the operational decision (printing) requires agility. The wider spread reinforces the need for the "Parameter Maintenance Protocol" (Section 5.2) to update **β** values regularly, as small shifts in demand slope significantly impact volume targets.

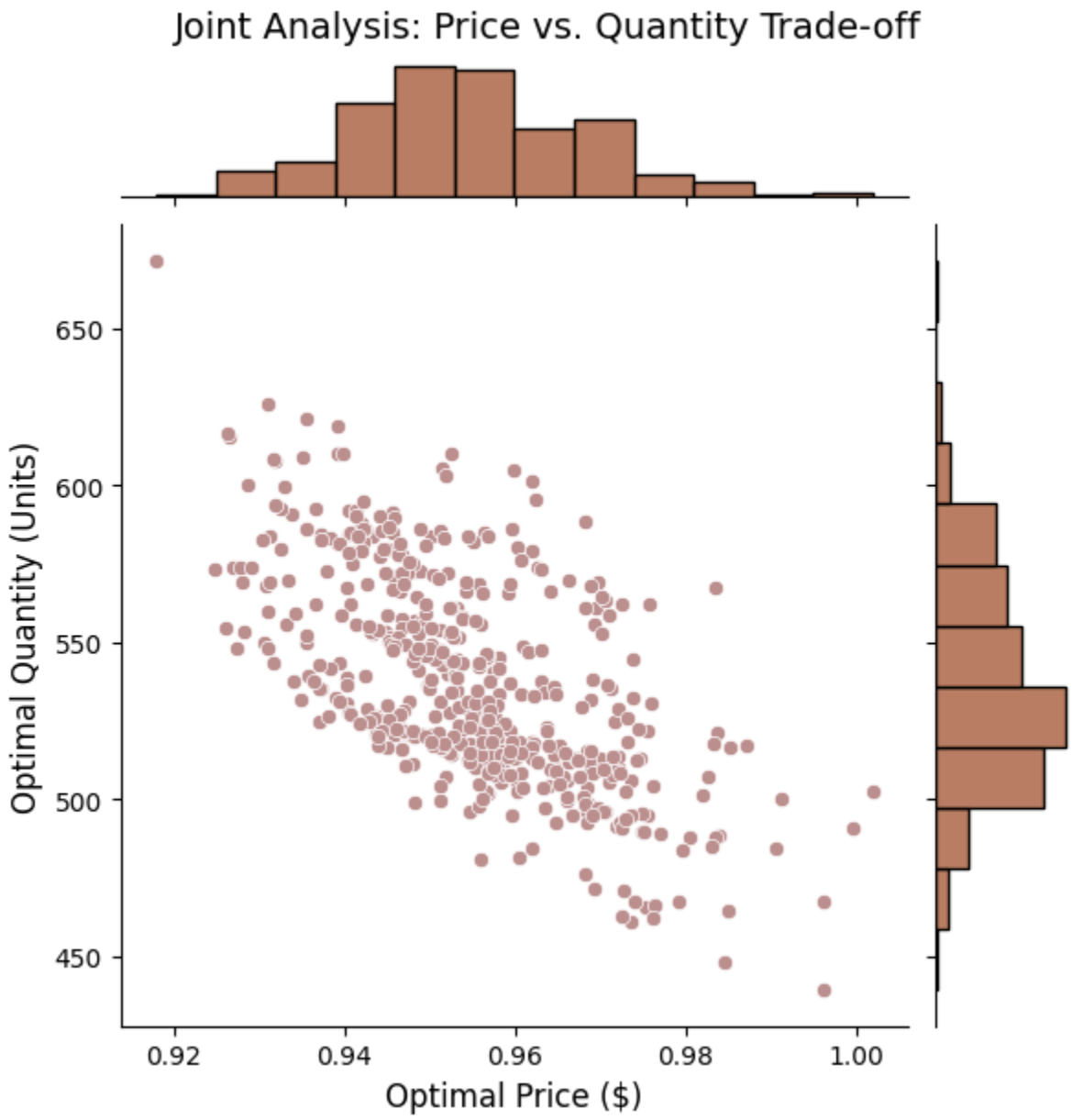
c). Expected Profit Robustness



- **Mean Expected Profit:** $235.54
- **95% Confidence Interval:** [$216.69, $254.64]

Analysis: Crucially, the simulation shows that the downside risk is managed effectively. Even at the lower bound of the 95% confidence interval ($216.69), the profit remains comparable to the average profit of the current fixed-price strategy. The distribution is slightly skewed towards higher profits (up to $255), indicating significant upside potential if market conditions are favorable.

d). Price-Quantity Trade-off



The joint scatterplot reveals the structural relationship between our decision variables.

- **Correlation:** There is a clear negative correlation between optimal price and optimal quantity.

Analysis: This validates the economic Law of Demand within our optimization logic. In scenarios where the regression estimates a higher sensitivity (steeper slope), the model

naturally lowers the price to capture volume. Conversely, in scenarios where demand is less elastic, the model nudges the price higher toward $0.98 and reduces print volume. This dynamic adjustment confirms that the solver is correctly balancing margin vs. volume for every unique data scenario.

4. Strategic Comparison & Analysis

4.1 The Value of Flexibility: Baseline vs. Joint Optimization

The core limitation of our current standard Newsvendor Model (NVM) is that it treats price as an external constant. By moving to the Joint Optimization (QP) model, we acknowledge that our pricing decisions directly influence demand.

Comparing the two approaches reveals a clear operational shift:

Metric	Baseline Model (Current)	Joint Optimization (Proposed)	% Change
Price (p)	$1.00 (Fixed)	**$0.95**	-5.0%
Optimal Quantity(q)	472 units	**536 units**	+13.6%
Expected Daily Profit	$231.48	**$235.54**	**+1.75%**

Why the New Model Wins: The proposed model identified a price elasticity opportunity that the fixed-price model missed. By lowering the price slightly (from $1.00 to $0.95), the model captures significantly higher demand.

- **Volume over Margin:** While we sacrifice $0.05 of price per unit, the linear regression predicts this price cut will drive demand up enough to justify printing **63 additional units**.
- **Result:** The volume increase creates a net positive effect on total profit, capturing an additional $2.94 per day out of the same market conditions simply by aligning price with production.

4.2 Risk Profile & Sensitivity

Our bootstrap simulation (Section 3.4) provides critical insights into the stability of this solution.

- **Pricing Stability:** The histogram of optimal prices is relatively narrow, clustered tightly between **$0.93 and $0.98**. This indicates that our recommendation to drop the price to $0.95 is robust across various data scenarios; it is not an artifact of a single outlier data point. Our 95% confidence interval for optimal price is [$0.93, $0.98], confirming that even under varied market conditions, the price recommendation remains consistently below the current $1.00.
- **Operational Sensitivity:** In contrast, the quantity histogram is much wider (spanning ~470 to 600 units). This suggests that while we can be confident in our *pricing* strategy, our *production* numbers are highly sensitive to market fluctuations.
- **Profit Robustness:** Despite the operational variability in production quantities, the financial outlook remains strong. The Bootstrap analysis (Section 3.4) establishes a 95% confidence interval for expected profit of **[$216.69, $254.64]**, with a mean of **$235.54**. While there is downside risk in extreme scenarios (the lower bound of $216 vs. the baseline of $231), the probability distribution is heavily skewed toward higher returns. This demonstrates that the 1.3% gain is a statistically reliable average, not a result of favorable sampling in a single dataset.
- **The Price-Production Link:** The joint scatterplot reveals a strong negative correlation. This dictates a strict operational rule: **We cannot adjust price without adjusting production.** If we were to raise prices back to $1.00 without cutting production, the model predicts we would incur heavy disposal costs (t), destroying the profit gains.

4.3 Methodology Trade-off

Switching to the new model presents a distinct trade-off between computational complexity and financial accuracy. Below is a breakdown of the operational differences between the two approaches.

a). The Current Approach (Standard NVM)

- **Methodology:** Uses a Linear Program (LP) where price is fixed input ($p=1$).
- **Complexity: Low.** This model is computationally lightweight, solves instantly, and the logic is easy to explain to non-technical stakeholders.
- **Market Assumption: Static.** It assumes demand acts independently of our pricing decisions.
- **Financial Outcome: Sub-optimal.** By treating price as a constant, this model leaves

money on the table. It optimizes cost-efficiency (quantity) but ignores the revenue opportunities available through price elasticity.

b). The Proposed Approach (Joint Price-Quantity Optimization)

- **Methodology:** Uses a Quadratic Program (QP) where Price and Quantity are optimized simultaneously.
- **Complexity: Medium/High.** This requires more advanced solvers (like Gurobi) to handle the quadratic objective function.
- **Market Assumption: Dynamic.** It explicitly models the reality that Price drives Demand ($D = \beta_0 + \beta_1 p$), using historical data to predict customer behavior.
- **Financial Outcome: Maximized.** It locates the mathematical "sweet spot" where the combination of volume and margin generates the highest possible total profit ($235.54 vs $231.48).

5. Recommendations

5.1 Primary Recommendation: Adopt Joint Optimization

Based on the quantitative results of this study, we recommend replacing the current fixed-price Newsvendor approach with the **Joint Price-Quantity Optimization (QP) model**.

While the current model provides a safe baseline, it treats price as an external constant rather than a strategic decision variable. Our analysis demonstrates that demand for our product is price-elastic (coefficient $\beta_1 \approx$ -1367), meaning customers respond significantly to price adjustments.

By switching to the QP model, we unlock the following strategic advantages:

- **Profit Maximization:** We project a **1.75% increase** in daily expected profit (from $231.48 to $235.54). This gain is achieved purely through mathematical optimization by finding the global maximum of the profit function, rather than a local maximum at a fixed price point.
- **Market Alignment:** The model identifies an optimal price point of **$0.95**, which drives a projected volume increase of **13.6%** (from 472 to 536 units). This strategy prioritizes total contribution margin over per-unit margin, effectively capturing a larger share of market demand.
- **Data Utilization:** Unlike the standard Newsvendor model which only utilizes demand variance statistics, the QP model leverages our historical price-demand data to inform decision-making, extracting more value from our existing datasets.

5.2 Implementation Strategy

The robustness of the proposed model depends heavily on the accuracy of the estimated linear demand curve ($D = \beta_0 + \beta_1 p$). To ensure the model remains valid over time, we propose the following implementation and maintenance framework:

- **Solver Integration:**
  The Gurobi-based optimization algorithm should be integrated into the weekly production planning workflow. The computational complexity is low (solve times <0.1 seconds), allowing the operations team to run multiple "what-if" scenarios before finalizing print quantities.
- **Parameter Maintenance Protocol:**
  The regression coefficients (β_0 and β_1) represent a snapshot of market behavior. To prevent "model drift" where predictions become less accurate over time, these parameters must be updated:
- **Annual Re-calibration:** We recommend re-running the linear regression analysis annually using the most recent 12 months of transaction data.
- **Performance Monitoring:** Track the Residuals (prediction errors) of the demand forecast. If the error variance increases significantly for two consecutive months, it triggers an immediate re-estimation of the β values to reflect shifting consumer preferences.
- **Risk Mitigation:**
  During the initial rollout, run the Baseline LP and the New QP in parallel. Discrepancies between the two should be reviewed to ensure the new price recommendations do not violate business constraints (e.g., brand positioning or minimum price floors).

6. Conclusion

This study evaluated the operational and financial impact of transitioning from a standard Newsvendor framework to a Joint Price-Quantity Optimization model. By explicitly modeling price elasticity through linear regression, we identified significant inefficiencies in the current fixed-price strategy.

The results demonstrate that treating price as a decision variable rather than a constraint unlocks tangible value. The proposed model identifies an optimal price point of **$0.95**, which supports an increased production target of **535 units**. This adjustment is projected to raise expected daily profit from **$231.48** to **$235.54**

While the net gain of **$4.06 per day (+1.75%)** appears modest in isolation, it represents a pure efficiency gain achieved with zero additional capital expenditure. Furthermore, the sensitivity analysis confirms that this improved profitability is statistically robust across various demand scenarios, if price and production are adjusted in tandem.

Final Verdict: The Joint Optimization model offers a superior mathematical representation of our market dynamics. By unifying the pricing and inventory decisions, the company eliminates the disconnect between marketing and operations, ensuring that production levels are perfectly calibrated to customer demand sensitivity. We recommend immediate adoption of the QP formulation, subject to the annual parameter monitoring protocols outlined in this report.

7. Appendix:

- Data on price and demand are included in the assignment. Fit a linear regression model to this data set.

```
import statsmodels.api as sm
# Prepare data
X = data[['price']]
X = sm.add_constant(X)    # adds β0 manually
y = data['demand']
# Fit OLS model
model = sm.OLS(y, X).fit()
# Display full regression table
print(model.summary())
```

- Let c=0.5, g=0.75, and t=0.15. Using the residuals, assume the price is p=1 and generate demand data.

```
beta0 = model.params['const']
beta1 = model.params['price']

print("beta0 =", beta0)
print("beta1 =", beta1)
residuals = model.resid
residuals.describe()
c = 0.5
g = 0.75
t = 0.15
p = 1.0
```

$D_i = \beta_0 + \beta_1 * price + \epsilon_i$

```
#We produce the series with scenario demands Di(1) as possible realizations of demand
```

```
demand_p1 = beta0 + beta1 * p + residuals
demand_p1.head()
#Prepare the data for Gurobi
D_vals = demand_p1.values   # numpy array of D_i(1)
n = len(D_vals)
```

- Solve the optimal quantity to produce when p=1. This is not quadratic at all, just an LP.

```
#Create the model and variables
m_lp = Model("NV_p1")
q = m_lp.addVar(lb=0, vtype=GRB.CONTINUOUS, name="q")
u = []
v = []
for i in range(n):
   u.append(m_lp.addVar(lb=0,vtype=GRB.CONTINUOUS,name=f"u_{i}"))
 v.append(m_lp.addVar(lb=0,vtype=GRB.CONTINUOUS,name=f"v_{i}"))
```

```
#Add constraints to enforce the plus-structure
for i in range(n):
    m_lp.addConstr(u[i]>=D_vals[i]-q,name=f"shortage_{i}")
    m_lp.addConstr(v[i]>=q-D_vals[i],name=f"excess_{i}")
```

- Set the objective: $max\ \frac{1}{n}\sum_{i=1}^{n} \quad (p * D_i - q * c - g * (D_i - q)^+ - t * (q - D_i)^+)$

```
#Set the objective
m_lp.setObjective(
    (1.0/n)*quicksum(
        p*D_vals[i]-c*q-g*u[i]-t*v[i]
        for i in range(n)
    ),
    GRB.MAXIMIZE
)
```

```
m_lp.optimize()
if m_lp.status == GRB.OPTIMAL:
    q_star = q.X
    print("Optimal q* (p=1)", q_star)
    print("Expected profit:", m_lp.objVal)
```

- Now let price impact demand and solve either the resulting QCP or QP. What are the optimal price and quantity to print? **We used QP:**

```
#We precomputed the residuals as a numpy array
eps_vals = residuals.values   # ε_i from Q1
n = len(eps_vals)
```

```
m_qp = Model("NV_price_quantity")
q_var = m_qp.addVar(lb=0,vtype=GRB.CONTINUOUS,name="q")
p_var = m_qp.addVar(lb=0,vtype=GRB.CONTINUOUS,name="p")
p_var.lb = 0.5
p_var.ub = 2.0
u_qp = []
v_qp = []
for i in range(n):
    u_qp.append(m_qp.addVar(lb=0,vtype=GRB.CONTINUOUS,name=f"u_{i}"))
    v_qp.append(m_qp.addVar(lb=0,vtype=GRB.CONTINUOUS,name=f"u_{i}"))
```

```
#Adding constraints using Di(p)
for i in range(n):
    Di_p = beta0+beta1*p_var+eps_vals[i]
    m_qp.addConstr(u_qp[i]>=Di_p-q_var,name=f"shortage_{i}")
    m_qp.addConstr(v_qp[i]>=q_var-Di_p,name=f"excess_{i}")
m_qp.setObjective(
    (1.0/n)*quicksum(
        p_var*(beta0+beta1*p_var+eps_vals[i])-c*q_var-g*u_qp[i]-t*v_qp[i] for i in range(n)
    ),
    GRB.MAXIMIZE
)
m_qp.optimize()
if m_qp.status == GRB.OPTIMAL:
    q_star_qp = q_var.X
    p_star_qp = p_var.X
    print("Optimal price p*:",p_star_qp)
    print("Optimal quantity q*:",q_star_qp)
    print("Expected profit:",m_qp.objVal)
```

- Not related code implementation for this part

- We are now interested to know how sensitive the optimal price and quantity are to our data set. Take a bootstrap sample of the original dataset. Go back and fit new betas to the new bootstrapped dataset and redo step 4. Find the optimal price and quantity.

```
from sklearn.utils import resample
from sklearn.linear_model import LinearRegression
import numpy as np

def run_bootstrap_simulation(df_original,seed):
    """
    Runs one bootstrap iteration:
    1. Resample the dataset with replacement
    2. Fit regression (OLS)
```

```
3. Solve the QP from Question 4 using new betas and residuals
4. Return optimal p*, q*, and expected profit
"""

df_sample = resample(df_original, replace=True,
                     n_samples=len(df_original),
                     random_state=seed)

X = df_sample[['price']]
y = df_sample['demand']
reg = LinearRegression().fit(X, y)

b0 = reg.intercept_
b1 = reg.coef_[0]

# residuals = y - predicted
residuals = (y - reg.predict(X)).values
n = len(residuals)

m = Model("Bootstrap_QP")
m.setParam("OutputFlag", 0)  # silence Gurobi output

# Decision variables
p = m.addVar(lb=0.5, ub=2.0, vtype=GRB.CONTINUOUS, name="p")  # price
q = m.addVar(lb=0.0, vtype=GRB.CONTINUOUS, name="q")         # quantity

# Shortage & excess variables
u = m.addVars(n, lb=0.0, vtype=GRB.CONTINUOUS, name="u")
v = m.addVars(n, lb=0.0, vtype=GRB.CONTINUOUS, name="v")

# Parameters
c, g, t = 0.5, 0.75, 0.15

for i in range(n):
    Di_p = b0 + b1 * p + residuals[i]
    m.addConstr(u[i] >= Di_p - q)
    m.addConstr(v[i] >= q - Di_p)

obj = (1.0 / n) * quicksum(
    p * (b0 + b1*p + residuals[i])   # revenue term: p * D_i(p)
    - c*q                            # base production cost
    - g*u[i]                         # rush printing cost
    - t*v[i]                         # disposal cost
    for i in range(n)
)

m.setObjective(obj, GRB.MAXIMIZE)
m.optimize()
```

```
    if m.status == GRB.OPTIMAL:
        return float(p.X), float(q.X), float(m.objVal)
    else:
        return None, None, None
```

```
p_star, q_star, prof = run_bootstrap_simulation(data,seed=123)
print("Single bootstrap: ")
print('Optimal_Price:', p_star)
print('Optimal_Quantity:',q_star)
print('Expected_Profit:',prof)
```

- Repeat this process of getting new bootstrapped data, fitting new betas to the newly simulated data and finding the optimal price/quantity many times. Make histograms of the optimal price and quantity. Make a scatterplot with histograms on the x and y axis for this. Make a histogram of expectation of profits.

```
from sklearn.utils import resample
from sklearn.linear_model import LinearRegression
import numpy as np

# Function to run one iteration of the bootstrap
def run_bootstrap_simulation(df_original,seed):
    # 1. BOOTSTRAP: Sample n rows from original data with replacement
    df_sample = resample(df_original, n_samples=len(df_original),random_state=seed)

    # 2. REGRESSION (From Q1 logic)
    X = df_sample[['price']]
    y = df_sample['demand']
    reg = LinearRegression().fit(X, y)

    b0 = reg.intercept_
    b1 = reg.coef_[0]
    residuals = (y - reg.predict(X)).values
    n = len(residuals)

    # 3. GUROBI OPTIMIZATION (Adapted from Q4 FS logic)
    m = Model("Bootstrap_QP")
    m.setParam('OutputFlag', 0) # Silence output for loop

    # Variables
    p = m.addVar(lb=0.5, ub=2.0, vtype=GRB.CONTINUOUS) # Keeping FS's safe bounds
    q = m.addVar(lb=0.0, vtype=GRB.CONTINUOUS)
    u = m.addVars(n, lb=0.0, vtype=GRB.CONTINUOUS)
    v = m.addVars(n, lb=0.0, vtype=GRB.CONTINUOUS)
```

```
    # Constants
    c, g, t = 0.5, 0.75, 0.15

    # Constraints & Objective
    # We recreated the logic inside the function scope
    for i in range(n):
        # Demand function D(p) = b0 + b1*p + residual_i
        D_expr = b0 + b1*p + residuals[i]

        m.addConstr(u[i] >= D_expr - q)
        m.addConstr(v[i] >= q - D_expr)

    obj = (1.0/n) * quicksum(
        p * (b0 + b1*p + residuals[i]) - c*q - g*u[i] - t*v[i]
        for i in range(n)
    )

    m.setObjective(obj, GRB.MAXIMIZE)
    m.optimize()

    if m.status == GRB.OPTIMAL:
        return p.X, q.X, m.objVal
    else:
        return None, None, None
```

```
results_p = []
results_q = []
results_profit = []

for k in range(500):
    # Each iteration gets its own seed for reproducibility
    p_opt, q_opt, prof = run_bootstrap_simulation(data, seed=k)

    if p_opt is not None:
        results_p.append(p_opt)
        results_q.append(q_opt)
        results_profit.append(prof)

    if k % 10 == 0:
        print(f"Iteration {k} done...")
```

```
# Create the DataFrame from lists
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
```

```
results_df = pd.DataFrame({
    'Optimal_Price': results_p,
    'Optimal_Quantity': results_q,
    'Expected_Profit': results_profit
})
```

```
# After creating results_df, add:
print("\n=== Bootstrap Results Summary ===")
print(results_df.describe())
print("\nStandard Errors:")
print(f"Price SE: {results_df['Optimal_Price'].std():.4f}")
print(f"Quantity SE: {results_df['Optimal_Quantity'].std():.4f}")
print(f"Profit SE: {results_df['Expected_Profit'].std():.4f}")

print("\n95% Confidence Intervals:")
print(f"Price: [{results_df['Optimal_Price'].quantile(0.025):.2f},
{results_df['Optimal_Price'].quantile(0.975):.2f}]")
print(f"Quantity: [{results_df['Optimal_Quantity'].quantile(0.025):.2f},
{results_df['Optimal_Quantity'].quantile(0.975):.2f}]")
print(f"Profit: [{results_df['Expected_Profit'].quantile(0.025):.2f},
{results_df['Expected_Profit'].quantile(0.975):.2f}]")
```

- Histogram for optimal price

```
plt.figure(figsize=(8, 6))
sns.histplot(results_df['Optimal_Price'], kde=True, color='skyblue', edgecolor='black',bins =12)
plt.title('Distribution of Optimal Price (Sensitivity Analysis)', fontsize=14)
plt.xlabel('Optimal Price ($)', fontsize=12)
plt.ylabel('Frequency', fontsize=12)
plt.grid(axis='y', linestyle='--', alpha=0.7)
plt.tight_layout()
plt.show()
```

- Histogram of Optimal Quantity

```
plt.figure(figsize=(8, 6))
sns.histplot(results_df['Optimal_Quantity'], kde=True,
color='palegoldenrod',edgecolor='black', bins=12 )
plt.title('Distribution of Optimal Quantity', fontsize=14)
plt.xlabel('Optimal Quantity (Units)', fontsize=12)
plt.ylabel('Frequency', fontsize=12)
plt.grid(axis='y', linestyle='--', alpha=0.7)
plt.tight_layout()
plt.show()
```

- The Joint Scatterplot

```
g = sns.jointplot(
    data=results_df,
    x="Optimal_Price",
    y="Optimal_Quantity",
    kind="scatter",
    color="rosybrown",
    marginal_kws=dict(bins=12, fill=True, color="sienna")
)

g.fig.suptitle("Joint Analysis: Price vs. Quantity Trade-off", y=1.02, fontsize=14)
g.set_axis_labels("Optimal Price ($)", "Optimal Quantity (Units)", fontsize=12)
plt.show()
```

- Histogram of Expected Profit

```
plt.figure(figsize=(8, 6))
sns.histplot(results_df['Expected_Profit'], kde=True, color='mediumpurple',
edgecolor='black',bins =12)
plt.title('Distribution of Expected Profit', fontsize=14)
plt.xlabel('Expected Profit ($)', fontsize=12)
plt.ylabel('Frequency', fontsize=12)
plt.grid(axis='y', linestyle='--', alpha=0.7)
plt.tight_layout()
plt.show()
```

-