A
LECTURE
ON
TRANSPORTATION MODEL

Outline

- Introduction
- Solution Procedure for Transportation Problem
- Finding an Initial Feasible Solution
- Finding the Optimal Solution
- Special Cases in Transportation Problems
- Maximisation in Transportation Problems
- Exercises

Operation Research

The main typical issues in OR :

- Formulate the problem
- Build a mathematical model
  - Decision Variable
  - Objective Function
  - Constraints
- Optimize the model

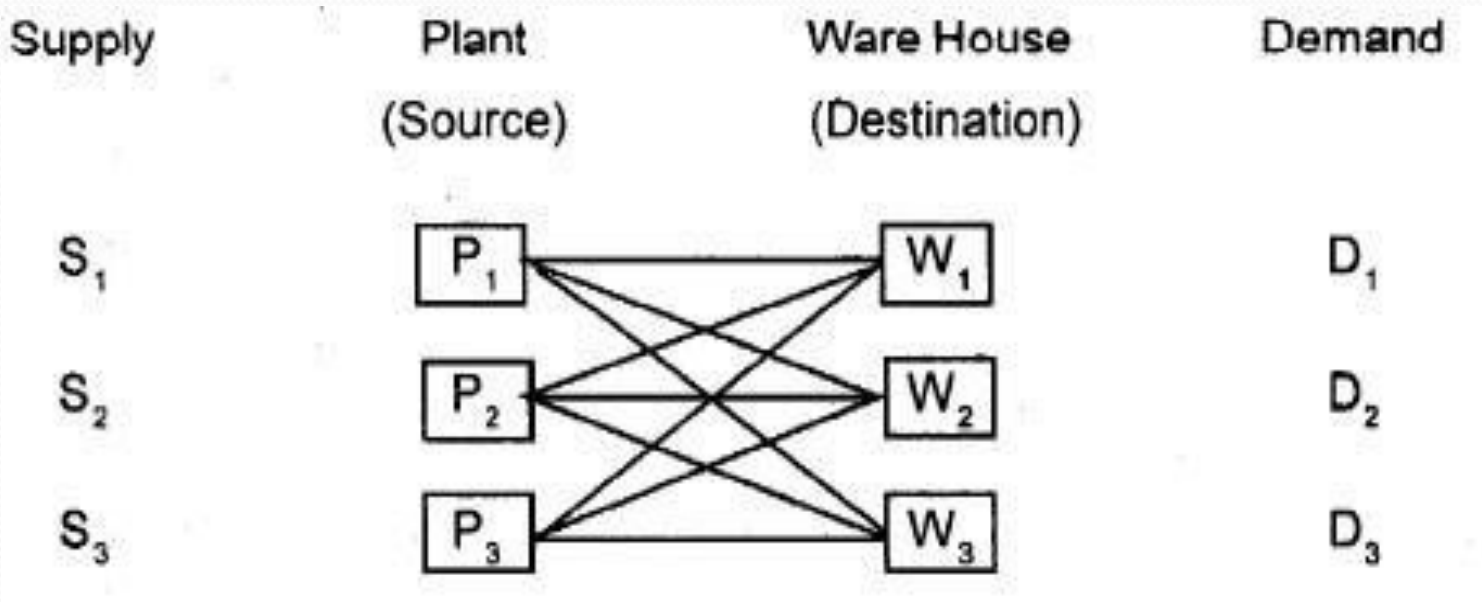
Introduction

- Transportation problem is one of the linear Programming Problem
- The objective is to minimize the cost of distribution a product from a no of sources or origin to a no of destination in such a manner to minimize the total transportation cost.

For example

- Manufacturer has three plants P_1, P_2, P_3 producing same products.
- From these plants, the product is transported to three warehouses W_1, W_2 and W_3.

- Each plant has a limited capacity, and each warehouse has specificdemand. Each planttransport to each warehouse, but transportation cost vary for different combinations.



Steps to solve a transportation problem

- Formulate the problem and setup in the matrix form.
- Obtain the initial basic feasible solution.
- Test the solution for optimality.
- Updating the solution if required

For example:

		Ware houses			
		W_1	W_2	W_3	Supply S_i
	P_1	7	6	9	20
Plants	P_2	5	7	3	28
	P_3	4	5	8	17
Demand	D_j	21	25	19	65

Finding an Initial Feasible Solution

There are a number of methods for generating an initial feasible solution for a transportation problem.

Consider three of the following

(i) North West Corner Method

(ii) Least Cost Method

(iii) Vogel's Approximation Method

North West Corner Method (NWCM)

The simplest of the procedures used to generate an initial feasible solution is NWCM. It is so called because we begin with the North West or upper left corner cell of our transportation table. Various steps are given

Step 1

Select the North West (upper left-hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand requirement i.e., min (S_1, D_1).

Step 2

Adjust the supply and demand numbers in the respective rows And columns allocation.

Step 3

(a) If the supply for the first row is exhausted, then move down to the first cell in t he second row and first column and go to step 2.

(b) If the demand for the first column is satisfied, then move horizontally to the next cell in the second column and first row and go to **step 2.**

Step 4

If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.

Step 5

Continue the procedure until the total available quantity id fully allocated to the cells as required.

Remark 1: The quantities so allocated are circled to indicated, the value of the corresponding variable.

Remark 2: Empty cells indicate the value of the corresponding variable as zero, I.e., no unit is shipped to this cell.

To illustrate the NWCM,

As stated in this method, we start with the cell ($P_1 W_1$) and Allocate the min (S_1, D_1) = min (20,21)=20. Therefore we allocate 20 Units this cell which completely exhausts the supply of Plant P_1 and leaves a balance of (21-20) =1 unit of demand at warehouse W_1

		WARE HOUSES			SUPPLY
		W_1	W_2	W_3	S_i
PLANTS	P_1	7 20	6	9	~~20~~ 0
	P_2	5 1	7 25	3 2	~~28~~ ~~27~~ ~~2~~ 0
	P_3	4	5	8 17	17
	D_j	~~21~~ ~~1~~ 0	~~25~~ 0	~~19~~ ~~17~~ 0	65

Least Cost Method

The allocation according to this method is very useful as it takes into consideration the lowest cost and therefore, **reduce the computation** as well as the amount of time necessary to arrive at the optimal solution.

Step 1

(a) Select the cell with the lowest transportation cost among all the rows or columns of the transportation table.

(b) If the minimum cost is not unique, then select arbitrarily any cell with this minimum cost.

Step 2

Allocate as many units as possible to the cell determined in Step 1 and eliminate that row (column) in which either supply is exhausted or demand is satisfied.

Repeat Steps 1 and 2 for the reduced table until the entire supply at different plants is exhausted to satisfied the demand at different warehouses.

	Ware House			Supply
	W_1	W_2	W_3	S_i
P_1	7	6 20	9	20 0
P_2	5 4	7 5	3 19	~~28~~ ~~9~~ ~~5~~ 0
P_3	4 17	5	8	~~17~~ 0
Demand D_j	~~21~~ ~~4~~ 0	~~25~~ ~~5~~ 0	~~19~~ 0	65

Total cost = 20 x 6 + x 5 + 5 x 7 + 19 x 3 + 17 x 4 = Rs. 300

This total cost is less than the total cost obtained by NWCM.

Vogel's Approximation Method (VAM)

This method is preferred over the other two methods because the initial feasible solution obtained is either optimal or very close to the optimal solution.

Step 1:

Compute a penalty for each row and column in the transportation table.

Step 2:

Identify the row or column with the largest penalty.

Step 3:

Repeat steps 1 and 2 for the reduced table until entire supply at plants are exhausted to satisfy the demand as different warehouses.

Warehouse

	W_1	W_2	W_3	S_i
P_1	$\boxed{7}$	$\boxed{6}$ 20	$\boxed{9}$	~~20~~ 0
P_2	$\boxed{5}$ 9	$\boxed{7}$	$\boxed{3}$ 19	~~28~~ ~~9~~ 0
P_3	$\boxed{4}$ 12	$\boxed{5}$ 5	$\boxed{8}$	~~17~~ ~~5~~ 0
D_j	~~21~~ ~~12~~ 0	~~25~~ ~~20~~ 0	~~19~~ 0	65

Column penalties

1	1	5
1	1	-
3	1	-

Row penalities

1	1	1
2	2	-
1	1	1

The total transportation cost associated with this method is

Total cost = 20 x 6 + 9 x 5 + 19 x 3 + 12 x 4 + 5 x 5 = Rs. 295.

Steps in Transportation Model :

i) Formulation of Model (IF UBTP -> BTP)

ii) Obtain IBFS (NWCRM or LCM or VAM)

iii) Test of Optimality

iv) Optimization by MODI method

a) Calculate Ri and Kj such that Cij = Ri + Kj for Stone cells

b) Determine Improvement Potential Cell values of Water

Cells: IP values = Cij –(Ri + Kj) >= 0

Finding the Optimal Solution

Once an initial solution has been found, the next step is to test that solution for optimality. The following two methods are widely used for testing the solutions:

- θ **Stepping Stone Method**
- θ **Modified Distribution Method**

Necessary condition

1. Make sure that the number of occupied cells is exactly equal to m+n-1, where m=number of rows and n=number of columns.
2. Each occupied cell will be at independent position.

Stepping-Stone Method

In this method we calculate the net cost change that can be obtained by introducing any of the unoccupied cells into the solution.

Steps

1. Check the optimality test necessary condition
2. Evaluate each unoccupied cells by following its closed path and determine its net cost change.
3. Determine the quality to be shipped to the selected unoccupied cell. Trace the closed path for the unoccupied cell and identify the minimum quality by considering the minus sign in the closed path.

Modified Distribution (MODI) Method

The MODI method is a more efficient procedure of evaluating the unoccupied cells. The modified transportation table of the initial solution is shown below

Steps

1. Determine the initial basic feasible solution by using any method.

2. Determine the value of dual variable u_i , v_j *by using* $c_{ij} = u_i + v_j$ for occupied cell . Associate a number, u_i, with each row and v_j with each column.

3. Compute opportunity cost for unoccupied cell by $d_{ij} = c_{ij} - u_i - v_j$.

Contd..

4. Check the sign for each opportunity cost. If the opportunity cost for each unoccupied cell is positive or zero then the solution is optimum. Otherwise
5. Select the unoccupied cell with largest negative opportunity cost draw a close loop.
6. Assign alternative positive and negative sign at corner points of the closed loop(start from unoccupied cell with positive sign)
7. Determine the maximum number of units that should be allocated to this unoccupied cell. This should be added to cells with positive sign and subtracted from negative sign
8. Repeat procedure till an optimal solution is obtained

Model to be optimized

Objective function

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

cij = variable cost

xij = number of unit transported from supply point i

to demand j

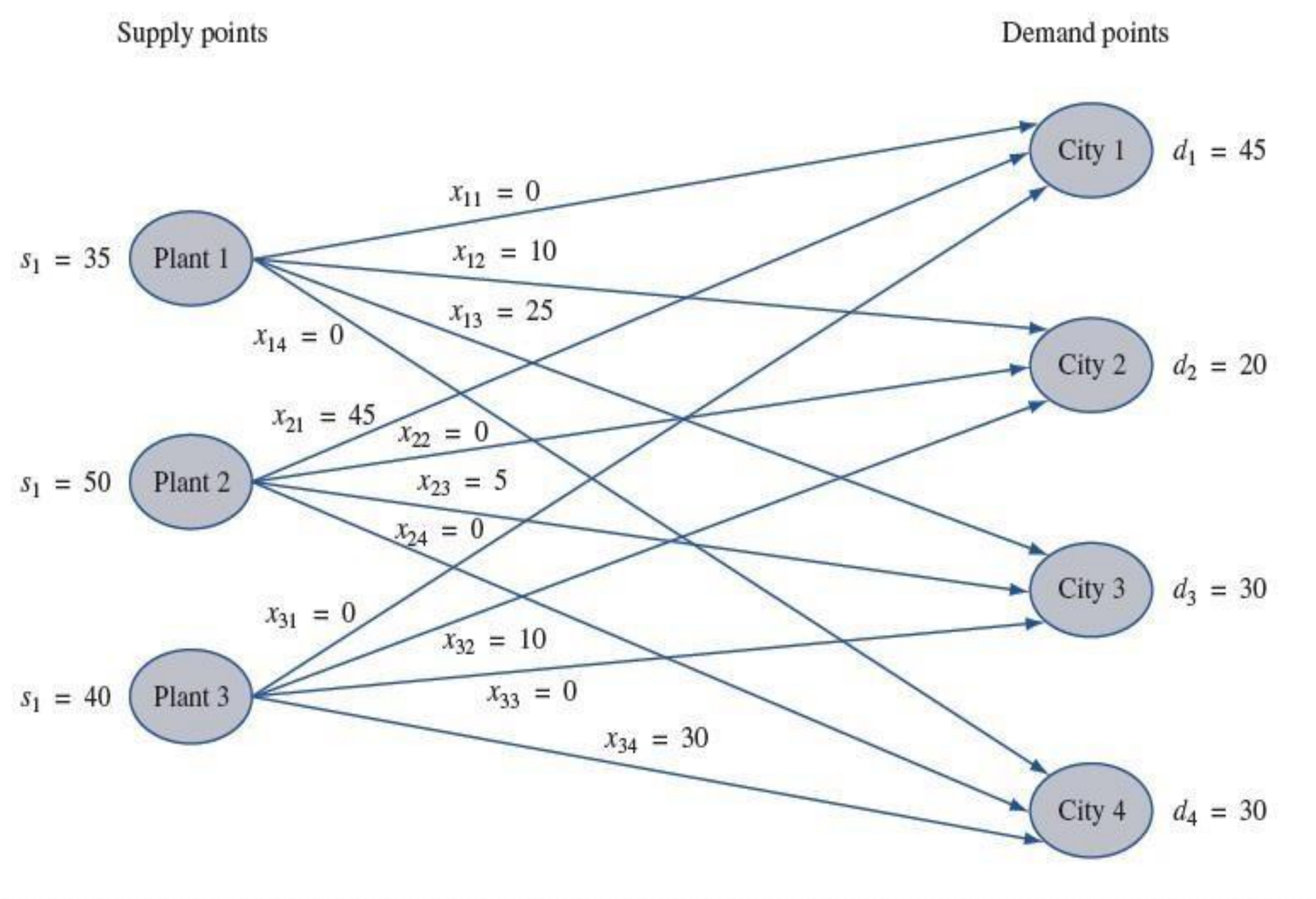
Example

Problem

Powerco has three electric power plants that supply the needs of four cities.[†] Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1—35 million; plant 2—50 million; plant 3—40 million (see Table 1). The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1—45 million; city 2—20 million; city 3—30 million; city 4—30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

Shipping Costs, Supply, and Demand for Powerco

	To				
From	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand (million kwh)	45	20	30	30	



i = number of sources **j = number of demands**

Decision variable

Number of millions energy (kWh) produced from sources and sent to demands

x (i,j) = number of sources

i = number of sources

j = number of demands

Objective function

- **Minimize cost**

cost =

$$
\begin{aligned}
& 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} && \text{(Cost of shipping power from plant 1)} \\
+ & 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} && \text{(Cost of shipping power from plant 2)} \\
+ & 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} && \text{(Cost of shipping power from plant 3)}
\end{aligned}
$$

Constraints

- **Constraints of number of Supply and demand**

  x (i,j) > 0 (i = 1,2,3) and j=1,2,3,4)

- **Constraints of Supply**

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \quad \text{(Plant 1 supply constraint)}$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50 \quad \text{(Plant 2 supply constraint)}$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \quad \text{(Plant 3 supply constraint)}$$

- **Constraints of Demands**

$$x_{11} + x_{21} + x_{31} \geq 45 \quad \text{(City 1 demand constraint)}$$
$$x_{12} + x_{22} + x_{32} \geq 20 \quad \text{(City 2 demand constraint)}$$
$$x_{13} + x_{23} + x_{33} \geq 30 \quad \text{(City 3 demand constraint)}$$
$$x_{14} + x_{24} + x_{34} \geq 30 \quad \text{(City 4 demand constraint)}$$

MODI Method (u-v method)

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	6	8 25	10 125	150
$u_2 =$	2	7	11	11 175	175
$u_3 =$	3	4 200	5 75	12	275
	Demand	200	100	300	600

$$u_i + v_j = c_{ij}$$

The value c_{ij} is the unit transportation cost for cell ij. For example, the formula for cell 1B is

$$u_1 + v_B = c_{1B}$$

and, since $c_{1B} = 8$,

$$u_1 + v_B = 8$$

The formulas for the remaining cells that presently contain allocations are

$$\begin{aligned}
x_{1C}&: \ u_1 + v_C = 10\\
x_{2C}&: \ u_2 + v_C = 11\\
x_{3A}&: \ u_3 + v_A = 4\\
x_{3B}&: \ u_3 + v_B = 5
\end{aligned}$$

Now there are five equations with six unknowns. To solve these equations, it is necessary to assign only one of the unknowns a value of zero. Thus, if we let $u_1 = 0$, we can solve for all remaining u_i and v_j values.

$$\begin{aligned}
x_{1B}&: \ u_1 + v_B = 8\\
&\quad 0 + v_B = 8\\
&\quad v_B = 8\\
x_{1C}&: \ u_1 + v_C = 10\\
&\quad 0 + v_C = 10\\
&\quad v_C = 10\\
x_{2C}&: \ u_2 + v_C = 11\\
&\quad u_2 + 10 = 11\\
&\quad u_2 = 1\\
x_{3B}&: \ u_3 + v_B = 5\\
&\quad u_3 + 8 = 5\\
&\quad u_3 = -3\\
x_{3A}&: \ u_3 + v_A = 4\\
&\quad -3 + v_A = 4\\
&\quad v_A = 7
\end{aligned}$$

	v_j	$v_A = 7$	$v_B = 8$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	$\boxed{6}$	$\boxed{8}$ 25	$\boxed{10}$ 125	150
$u_2 = 1$	2	$\boxed{7}$	$\boxed{11}$	$\boxed{11}$ 175	175
$u_3 = -3$	3	$\boxed{4}$ 200	$\boxed{5}$ 75	$\boxed{12}$	275
	Demand	200	100	300	600

IP values:

Next, we use the following formula to evaluate all *empty cells:*

$$c_{ij} - u_i - v_j = k_{ij}$$

where k_{ij} equals the cost increase or decrease that would occur by allocating to a cell.

For the *empty cells* in Table B-26, the formula yields the following values:

$$x_{1A}: \quad k_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1$$
$$x_{2A}: \quad k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1$$
$$x_{2B}: \quad k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2$$
$$x_{3C}: \quad k_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5$$

Table B-27
The Second Iteration of the MODI Solution Method

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	[6] 25	[8]	[10] 125	150
$u_2 =$	2	[7]	[11]	[11] 175	175
$u_3 =$	3	[4] 175	[5] 100	[12]	275
	Demand	200	100	300	600

$$
\begin{aligned}
x_{3A}:\quad u_3 + v_A &= 4\\
u_3 + 6 &= 4\\
u_3 &= -2\\
x_{3B}:\quad u_3 + v_B &= 5\\
-2 + v_B &= 5\\
v_B &= 7
\end{aligned}
$$

$$
\begin{aligned}
x_{1A}:\quad u_1 + v_A &= 6\\
0 + v_A &= 6\\
v_A &= 6\\
x_{1C}:\quad u_1 + v_C &= 10\\
0 + v_C &= 10\\
v_C &= 10\\
x_{2C}:\quad u_2 + v_C &= 11\\
u_2 + 10 &= 11\\
u_2 &= 1
\end{aligned}
$$

Table B-28
The New u_i and v_j Values for the Second Iteration

	v_j	$v_A = 6$	$v_B = 7$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	[6] 25	[8]	[10] 125	150
$u_2 = 1$	2	[7]	[11]	[11] 175	175
$u_3 = -2$	3	[4] 175	[5] 100	[12]	275
	Demand	200	100	300	600

The cost changes for the empty cells are now computed using the formula $c_{ij} - u_i - v_j = k_{ij}$.

$$x_{1A}: \quad k_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1$$
$$x_{2A}: \quad k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0$$
$$x_{2B}: \quad k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 7 = +3$$
$$x_{3C}: \quad k_{3C} = c_{3C} - u_3 - v_C = 12 - (-2) - 10 = +4$$

Find the optimal solution to the problem given in the following table.

Warehouse / Factory	W1	W2	W3	W4	Supply
F_1	14	25	45	5	6
F_2	65	25	35	55	8
F_3	35	3	65	15	16
Demand	4	7	6	13	30 / 30

1) Demand = suppl
2) I.B.F.S by VAM, N-W, LC
3) Test of optimal

Step 2 Initial basic feasible solution by VAM.

Warehouse / Factory	W_1	W_2	W_3	W_4	Supply	Column Penalty	R_i
F_1	14 / 4	25 / *	45 / *	5 / 2	6	9, 9, 40, 40 F	0
F_2	65 / *	25 / *	35 / 6	55 / 2	8	10, 20, 20, 20	50
F_3	35 / *	3 / 7	65 / *	15 / 9	16	12, 20, 50 F	10
Demand	4	7	6	13	30 / 30		
Row Penalty	21, 21 F	22 F	10, 10, 10, 10	10, 10, 10, 50			
K_j	+14	−7	−15	5			

Step 3 Test of optimality

Required no. of allocations to avoid degeneracy

$= m+n-1 = 3+4-1 = 6.$

Actual no. of allocations $= 6$.

Hence test of optimality.

Step 4: Optimization by MODI method.

a) Calculate R_i & K_j such that

$R_i + K_j = C_{ij}$ (stone square).

b) Improvement Potential = $C_{ij} - (R_i + K_j)$. (for water cells).

$$C_{ij} - (R_i + K_j).$$

$F_1W_2 \Rightarrow 25 - (0 - 7) = 32$

$F_1W_3 = 45 - (0 - 15) = 60.$

$F_2W_1 \Rightarrow 65 - (50 + 14) = 1$

$\boxed{F_2W_2 \Rightarrow 25 - (50 - 7) = -18}$

$F_3W_1 = 35 - (14 + 10) = 11$

$F_3W_3 \Rightarrow 65 - (10 - 15) = 70.$

As Improvement potential value is -ve for F_2W_2 location further improvement is possible by tracing a close loop and F_2W_2 is the starting point of close loop.

F \ W	W_1	W_2	W_3	W_4	R_i
F_1	14 / 4	25 / *	45 / *	5 / 2	0
F_2	65 / *	25 / 2	35 / 6	55 / *	32
F_3	35 / *	3 / 5	65 / *	15 / 11	10
K_j	14	−7	3	5	

IP cell.

$c_{ij} - (R_i + K_j)$ — water cells

$F_1W_2 \Rightarrow 25 - (0 - 7) = 32$

$F_1W_3 \Rightarrow 45 - (3 + 0) = 42$

$F_2W_1 = 65 - (32 + 14) = 19$

$F_2W_4 = 55 - (32 + 5) = 18$

$F_3W_1 = 35 - (10 - 14) = 11$

$F_3W_3 = 6.5 - (10 + 3) = 52$

As IP is +ve for each water cell, Above soln is optimal.

$\therefore$ Transportation Cost $= 14 \times 4 + 5 \times 2 + 25 \times 2 + 35 \times 6 + 3 \times 5 + 15 \times 11$

$= 506$.

Special Cases in Transportation Problem

- ♣Multiple Optimal Solutions
- ♣Unbalanced Transportation Problems
- ♣Degeneracy in Transportation Problem
- ♣Maximization In Transport Problems

Unbalanced Transportation Problems

Prob: A Cement manufacturing company wishes to transport cement from its 3 factories P, Q & R to 5 distribution centre located at A, B, C, D and E. The quantities produced at the factory i.e supply and quantities required by the distribution centre i.e demand and respective transportation cost in Rs from factories to distribution centre given in the following table.

Determine the least cost distribution problem for the company.

F \ D.	A	B	C	D	E	Supply
P	4	1	3	4	4	60
Q	2	3	2	2	3	35
R.	3	5	2	4	4	80
Demand	22	45	20	18	30	175 / 135.

As Demand ≠ Supply, ∴ Dummy D.C.

Ri	F \ D	A	B	C	D	E	Dummy D.C F	Supply	Column Penalty
0	P	4 *	1 45	3 *	4 *	4 15	0 *	60	1, 2, 1, 1 0
−1	Q	2 2	3 *	2 *	2 18	3 15	0 *	35	2, 1, 1, 1 1 F
0	R	3 20	5 *	2 20	4 *	4 *	0 40	80	2, 1, 1 1 F
	Demand	22	45	20	18	30	40	175 / 175	
	Row Penalty	1, 1, 1 1, 1, 2 F	2, 2 F	1, 1, 1 1, F	2, 2, 2 F	1, 1 1, 1	0 F		
	Kj	3	1	2	3	4	0		

3) Test of optimality.

Req. no. of allocations $= m+n-1$

$= 3+6-1 = 8$

Actual no. of allocations $= 8$

So test of optimality is satisfied.

4) MODI method calculate Ri & Kj.

a) $R_i + K_j = C_{ij}$ (for stone squares - such that).

b) IP for water cells such that

$c_{ij} - (R_j + K_j)$.

$PA \Rightarrow 4 - (3+0) = 1$
$PC \Rightarrow 3 - (2+0) = 1$
$PD \Rightarrow 4 - (3+0) = 1$
$PF \Rightarrow 0 - (0+0) = 0$
$QB \Rightarrow 3 - (-1+1) = 3$
$QC \Rightarrow 2 - (2-1) = 1$
$QF \Rightarrow 0 - (-1+0) = 1$
$RB \Rightarrow 5 - (0+1) = 4$
$RD \Rightarrow 4 - (0+3) = 1$
$RE \Rightarrow 4 - (0+4) = 0$.

As Improvement potentials values are +ve & zero. for each water cell, above solution is the optimal solution and therefore optimum transportation cost

$= 1\times45 + 4\times15 + 2\times2 + 2\times18 + 3\times15 + 3\times20 + 2\times20$
$+ 0\times40$
$=$ Rs. 290.

A company has three plants A, B, C and
three warehouses X, Y, Z. The no. of units
available at the plants is 60, 70, 80 resp.
The demand at X, Y, Z are 50, 80, 80 resp.
The unit costs of transportation are as follows:

	X	Y	Z	
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	

Find the allocation so that the total transportation cost is minimum.

Step 1 Supply = Dem

Soln:

	X	Y	Z	supp	C.P	Ri
A	8 *	7 *	3 60	60	4, 4 F	-6
B	3 50	8 *	9 20	70	5, 1, 1	0
C	11 *	3 80	5 ε	80	2, 2, 2	-4
Dem	50	80	80	210 / 210		
R.P	5 F	4, 4, 5	2, 2, 4			
Kj	3	7	9			

Step 2 :
I.B.F.S
by VAM.

Step 3.
Test of optimality
Req. no. of allocat.
$= m+n-1$
$= 3+3-1 = 5$

Actual no. of allocat. = 4

Step 4.
Optimised by MOD
a) Calcul. Ri & Kj such that
$K_{ij} = R_i + K_j$
for store squa

b) Calculat. I.P for water cells such that
$C_{ij} - (R_i + K_j)$ for W.C

I.P.

$AX = 8 - (3-6) = 11$

$AY = 7 - (7-6) = 6$

$BY = 8 - (7+0) = 1.$

$CX = 11 - (3-4) = 12$

As I.P value of W.C are +ve above

soln is optimum

$T.C = 3 \times 60 + 3 \times 50 + 9 \times 20 + 3 \times 80 + 5 \times \epsilon$

$= 750$ as $\epsilon \to 0.$

Maximization Transportation Problems

EXAMPLE 3.11 Consider the following profit matrix:

	A	B	C	D	E	*Supply*
1	19	21	16	15	15	150
2	9	13	11	19	11	200
3	18	19	20	24	14	125
Demand	80	100	75	45	125	

Maximize the profit.

Solution Since the given problem is a maximization problem, we have to convert it into a minimization problem by subtracting each element of profit matrix from the biggest element (24), producing the matrix as follows:

	A	B	C	D	E	*Supply*
1	5	3	8	9	9	150
2	15	11	13	5	13	200
3	6	5	4	0	10	125
Demand	80	100	75	45	125	

In the above table, the total demand (425) is less than the total supply (475), hence it is an unbalanced problem. So we need to introduce a dummy column.

Now by using Vogel's approximation method, the initial solution is as shown in the following table:

	A	B	C	D	E	*Dummy*	*Supply*
1	5 (50)	3 (100)	8	9	9	0	150
2	15 (25)	11	13	5	13 (125)	0 (50)	200
3	6 (5)	5	4 (75)	0 (45)	10	0	125
Demand	80	100	75	45	125	50	475

We have total eight allocations which is equal to $(m + n - 1)$, hence the problem is not degeneracy. Now, we will apply MODI method to test the optimality of this initial solution.

	A	B	C	D	E	*Dummy*	*Supply*	U_i
1	5 50	3 100	8 5	9 10	9	0 10	150	$U_1 = 5$
2	15 − 25	11 2	13 0	5 + −4	13 125	0 50	200	$U_2 = 15$
3	6 + 5	5 1	4 75	0 − 45	10 6	0 9	125	$U_3 = 6$
Demand	80	100	75	45	125	50	475	
V_j	$V_1 = 0$	$V_2 = -2$	$V_3 = -2$	$V_4 = -6$	$V_5 = -2$	$V_6 = -15$		

Since not all the opportunity costs are positive or zero, we will introduce cell (2, D) and drop cell (2,A), now the revised table will be as follows:

	A	B	C	D	E	*Dummy*	*Supply*	U_i
1	5 50	3 100	8 5	9 10	9 2	0 6	150	$U_1 = -6$
2	15 4	11 2	13 4	5 25	13 125	0 50	200	$U_2 = 0$
3	6 30	5 1	4 75	0 20	10 2	0 5	125	$U_3 = -5$
Demand	80	100	75	45	125	50	475	
V_j	$V_1 = 11$	$V_2 = 9$	$V_3 = 9$	$V_4 = 5$	$V_5 = 13$	$V_6 = 0$		

Since all the opportunity costs are positive or zero, therefore the above solution is an optimal solution.

$$\text{Total profit} = (50 \times 19) + (100 \times 21) + (25 \times 19) + (125 \times 11) + (30 \times 18) + (75 \times 20) + (20 \times 24) = \text{Rs. } 7420$$

Transshipment Problems

- A transportation problem allows only shipments that go directly from supply points to demand points. In many situations, shipments are allowed between supply points or between demand points.
- Sometimes there may also be points (called transshipment points) through which goods can be transshipped on their journey from a supply point to a demand point.
- Fortunately, the optimal solution to a transshipment problem can be found by solving a transportation problem.

- To solve such type of trans-shipment problem. If there are m sources and n destinations, the transportation table will be of size (m+n) (m+n).
- If the total number of units transported from all sources to all destinations in N, then the supply of each source and demand at each destinations is added to N and demand at source and the supply at each destination set is equal to N.
- Then, the problem can be solved usually by MODI method and also in the final solution, ignore or omit the values at diagonal cell as they depicts no transportation.

TRANSHIPMENT PROBLEMS

Definition. *A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a* ***transhipment problem.***

Main Characteristics of Transhipment Problems

Following are the main characteristics of transhipment problems :

1. The number of sources and destinations in the transportation problem are m and n respectively. But in transhipment problems, we have $m + n$ sources and destinations.
2. If S_i denotes the ith source and D_j denotes the jth destination, then commodity can move along the route $S_i \to D_i \to D_j$, $S_i \to S_j \to D_i \to D_j$, $S_i \to D_i \to S_j \to D_j$, or in various other ways. Clearly, transportation cost from S_i to S_i is zero and the transportation costs from S_i to S_j or S_i to D_i do not have to be symmetrical, *i.e.*, in general, $S_i \to S_j \neq S_j \to S_i$.
3. While solving the transhipment problem, we first obtain the optimum solution to the transportation problem, and then proceed in the same manner as in solving the transportation problems.
4. The basic feasible solution contains $2m + 2n - 1$ basic variables. If we omit the variables appearing in the $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

Computational Demonstration of Solution Procedure

The computational procedure for solving transhipment problems is best explained by the following example.

Example : *Consider the following transhipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule :*

	S_1	S_2	D_1	D_2	
S_1	0	1	3	4	5
S_2	1	0	2	4	25
D_1	3	2	0	1	
D_2	4	4	1	0	
			20	10	30

Solution. *Step 1. (To get modified transportation problem).* In the transhipment problem, each given source and destination can be considered a source or a destination. If we now take the quantity available at each of the sources D_1 and D_2 to be zero and also at each of the destinations S_1 and S_2 the requirement to be zero, then to have a supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as '*buffer stock*' is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to Σa_i or Σb_j. In our problem, the buffer-stock comes-out to be 30 units.

	S_1	S_2	D_1	D_2	Available
S_1	0	1	3	4	35
S_2	1	0	2	4	55
D_1	3	2	0	1	30
D_2	4	4	1	0	30
Required	30	30	50	40	

Step 2. *(To find initial solution of modified problem)*

By adding 30 units of commodity to each point of supply and demand, an initial basic feasible solution is obtained in *Table 2* by using *Vogel's Approximation method.*

Starting Table 2

				$a_i \downarrow$
30 • (0)	(1)	(3)	5 • (4)	35
(1)	30 • (0)	20 • (2)	5 • (4)	55
(3)	(2)	30 • (0)	(1)	30
(4)	(4)	(1)	30 • (0)	30
30	30	50	40	$b_j \rightarrow$

Step 3. *(To apply optimality test)*

The variables u_i $(i = 1, 2, 3, 4)$ and v_j $(j = 1, 2, 3, 4)$ have been determined by using successively the relations $u_i + v_j = c_{ij}$ for all the basic (occupied) cells. These values are then used to compute the net-evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic (empty) cells. Clearly d_{34} $(= -1)$ is the only negative quantity. Hence an unknown quantity θ is assigned to this cell (3, 4). After identifying the loop, we find that $\theta = 5$ and that the cell (2, 4) leaves the basis (*i.e.*, becomes empty).

					u_i
	30 • (0)	1 (1) $0 + 0$	1 (3) $0 + 2$	5 • (4)	0
	1 (1) $0 + 0$	30 • (0)	$20 + \theta$ (2)	$5 - \theta$ (4)	0
	5 (3) $-2 + 0$	4 (2) $-2 + 0$	$30 - \theta$ (0)	-1 $+\theta$ √ (1) $-2 + 4$	-2
	8 (4) $-4 + 0$	8 (4) $-4 + 0$	3 (1) $-4 + 2$	30 • (0)	-4
v_j	0	0	2	•	

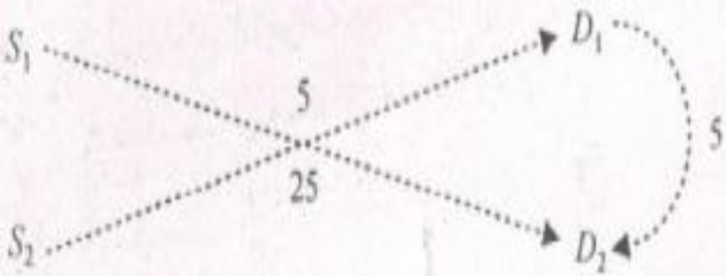
Table 3

Step 4. *Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then, again test the optimality of the revised solution.*

Since all the current net evaluations are non-negative, the current solution is an optimum one. It is shown in Table 48. The minimum transportation cost is :

$$z^* = 5 \times 4 + 25 \times 2 + 5 \times 1 = 75.$$

and the optimum transportation route is as shown below.



					u_i
	30 • (0)	0 (1) −1	0 (3) 3	5 • (4)	0
	2 (1) −1	30 • (0)	25 • (2)	1 (4) 3	−1
	6 (3) −3	4 (2) −2	25 • (0)	5 • (1)	−3
	8 (4) −4	7 (4) −3	2 (1) −1	30 • (0)	−4
v_j	0	1	3	4	

Table 4

Assignment Problems

i) By Hungarian Method
ii) By Branch and Bound Method /Technique (Little's method) (Travelling Salesman Problem)

Basic requirement of Hungarian method

i) Problem must be of Minimization. If the problem is of Maximization type convert it to minimization type before solving by Hungarian method.
ii) Problem must be of one to one basis of assignment i.e it must be of square matrix. If problem is of unbalanced assignment type introduce dummy row or dummy column as per the requirement.

Steps in Assignment Model

Step 1: Formulation of Problem statement

Step 2: Row Operation

Step 3 Column Operation

Step 4 Row Assignment

Step 5 Column Assignment

Step 6 Reduce the matrix
Cancelled once- As it is
Cancelled Twice- Add the minimum element
Not cancelled – Subtract minimum elelement

Step 7 Repeat step 4 to 6.

The Assignment Problem

- **In many business situations, management needs to assign - personnel to jobs, - jobs to machines, - machines to job locations, or - salespersons to territories.**
- **Consider the situation of assigning *n* jobs to *n* machines.**
- **When a job i (=1,2,....,n) is assigned to machine j (=1,2, ...n) that incurs a cost Cij.**
- **The objective is to assign the jobs to machines at the least possible total cost.**

The Assignment Problem

- **This situation is a special case of the Transportation Model And it is known as the *assignment problem*.**

- **Here, jobs represent "sources" and machines represent "destinations."**

- **The supply available at each source is 1 unit And demand at each destination is 1 unit.**

The Assignment Problem

	Machine					
		1	2	....	n	Source
Job	1	C11	C12	....	C1n	1
	2	C21	C22	....	C2n	1
	.	.	.		.	.
	.	.	.		.	.
	.	.	.		.	.
	n	Cn1	Cn2	....	Cnn	1
Destination		1	1	......	1	

The assignment model can be expressed mathematically as follows:

Xij= 0, if the job j is not assigned to machine i

1, if the job j is assigned to machine i

Assignment Problems

Example: Machineco has four jobs to be completed. Each machine must be assigned to complete one job. The time required to setup each machine for completingeach job is shown in the table below. Machinco wants to minimize the total setup time needed to complete the four jobs.

Setup times
(Also called the cost matrix)

	Time (Hours)			
	Job1	Job2	Job3	Job4
Machine 1	14	5	8	7
Machine 2	2	12	6	5
Machine 3	7	8	3	9
Machine 4	2	4	6	10

The Model

According to the setup table Machinco's problem can be formulated as follows (for *i*,*j*=1,2,3,4):

$$\min Z = 14X_{11} + 5X_{12} + 8X_{13} + 7X_{14} + 2X_{21} + 12X_{22} + 6X_{23} + 5X_{24}$$
$$+ 7X_{31} + 8X_{32} + 3X_{33} + 9X_{34} + 2X_{41} + X_{42} + 6X_{43} + 10X_{44}$$
$$s.t. X_{11} + X_{12} + X_{13} + X_{14} = 1$$
$$X_{21} + X_{22} + X_{23} + X_{24} = 1$$
$$X_{31} + X_{32} + X_{33} + X_{34} = 1$$
$$X_{41} + X_{42} + X_{43} + X_{44} = 1$$
$$X_{11} + X_{21} + X_{31} + X_{41} = 1$$
$$X_{12} + X_{22} + X_{32} + X_{42} = 1$$
$$X_{13} + X_{23} + X_{33} + X_{43} = 1$$
$$X_{14} + X_{24} + X_{34} + X_{44} = 1$$
$$X_{ij} = 0 or X_{ij} = 1$$

For the model on the previous page note that:

X_{ij}=1 if machine i is assigned to meet the demands of job j

X_{ij}=0 if machine i is assigned to meet the demands of job j

In general an assignment problem is balanced transportation problem in which all supplies and demands are equal to 1.

Assignment Problem

A bank officer has four subordinates and four task to be done. Subordinates differ in their efficiency, the task also differ in their intrinsic difficulty. His estimates of the time, each man/woman would take to perform each task is given in effectiveness matrix below. How should the task be allocated, one to one man so as to minimize the total man hours.

	Time (Hours)			
	Task1	Task2	Task3	Task4
Subordinate 1	18	26	17	11
Subordinate 2	13	28	14	26
Subordinate 3	38	19	18	15
Subordinate 4	19	26	24	10

Step 1

Formulation of Problem.

No need of formulation as given problem ~~is of~~ problem is in formulated form and basic requirements for solving problem by Hungarian

i) It is a minization type problem.

ii) One to ~~one~~ one basis of assignment.

step 1

Formulation of Problem.

No need of formulation as given problem ~~is~~ of problem is in formulated form and basic requirements for solving problem by Hungarian

step. 2 Row operation.

Row operation is performed to introduce at least one single zero in each row by subtracting the lowest element from the corresponding row. If already zero exist in particular row no need of row operation for that row

$$\begin{vmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{vmatrix} \qquad \begin{array}{r} 11 \\ 13 \\ 15 \\ 10 \\ \hline 49 \end{array}$$

Step 3.

Column operation

column operation is performed to introduce at least one zero in each column by subtracting the lowest element for the corresponding element for the correspond ing column. If already zero exists in particular column no need of column operation..

$$\begin{vmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{vmatrix}$$

$$\begin{array}{r} 49 \\ +\ 4 \\ 1 \\ \hline 54 \end{array}$$

Step 4

Row Assignment.

Assign one single zero i.e independent zero in each row by making square box around and cancel the

7	11	5	$\boxed{0}$
0	11	0	13
23	$\boxed{0}$	2	0
9	12	13	0

Step 5 Column Assignment.

Assign one single zero i.e independent zero in each Column by making square box around and cancel the corresp. row.

7	11	5	$\boxed{0}$
$\boxed{0}$	11	0	13
23	$\boxed{0}$	2	0
9	12	13	0

Step 6 . Reduce Matrix further.

For reducing matrix further follow the following step.

a) cancelled ones - As it is.

b) cancelled twice - Add minimum not cancelled element

c) Not cancelled - subtract minimum not ~~subtracted 2~~ cancelled element.

Reduce Matrix I

$$\begin{vmatrix} 5 & 11 & 3 & \boxed{0} \\ \boxed{0} & 13 & 0 & 15 \\ 21 & \boxed{0} & 0 & 0 \\ 7 & 12 & 11 & 0 \end{vmatrix}$$

$$\begin{array}{r} 54 \\ +\ \ 2 \\ \hline 56 \end{array}$$

Repeat step 4 to step 6 till required number of allocations are obtained

Reduce Matrix II

$$\begin{array}{c|cccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} & \boxed{\mathrm{IV}} \\ \hline A & 2 & 8 & \boxed{0} & 0 \\ B & \boxed{0} & 13 & 0 & 18 \\ C & 21 & \boxed{0} & 0 & 3 \\ D & 4 & 9 & 8 & \boxed{0} \end{array}$$

$$\begin{array}{r} 56 \\ +\ 3 \\ \hline 59 \end{array}$$

Subordinate	Job	time
A	III	17
B	I	13
C	II	19
D	IV	10
		59.

Beta Corporation has four plants A, B, C, D each of which can manufacture any of the products P_1, P_2, P_3 and P_4. Production cost differ from one plant to another as do sales revenues. From the data given below, find out which product each plant should produce to maximise profit.

Plant	Productwise Sales Revenue (Rs. 000's)			
	P_1	P_2	P_3	P_4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

Plant	Productwise Production Cost (Rs 000's)			
	P_1	P_2	P_3	P_4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

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Sol^n:

1) Formulat of Problem

Profit matrix (Max).

Plant	P_1	P_2	P_3	P_4
A	1	8	4	1
B	5	7	6	5
C	3	5	4	2
D	3	1	6	3

$\Rightarrow$

Loss matrix (Minimi).

$$\begin{vmatrix} 7 & 0 & 4 & 7 \\ 3 & 1 & 2 & 3 \\ 5 & 3 & 4 & 6 \\ 5 & 7 & 2 & 5 \end{vmatrix}$$

2) Row operati..

$$\begin{vmatrix} 7 & 0 & 4 & 7 \\ 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 \\ 3 & 5 & 0 & 3 \end{vmatrix}$$

3) Column operati

$$\begin{vmatrix} 5 & \boxed{0} & 4 & 5 \\ 0 & 0 & 1 & \boxed{0} \\ \boxed{0} & 0 & 1 & 1 \\ 1 & 5 & \boxed{0} & 1 \end{vmatrix}$$

$A \longrightarrow P_2$ 8

$B \longrightarrow P_4$ 5

$C \longrightarrow P_1$ 3

$D \longrightarrow P_3$ 6

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