LPP-Artificial Variable Techniques

Before we start

- While converting given LPP to standard form

Type of constraint	Variables to added
Less than or equal to (≤)	Add slack variable
Greater than or equal to (≥)	Subtract surplus variable and add artificial variable
Equal to (=)	Add artificial variable

- While adding variables to constraints to convert to standard form first add slack variables and subtract surplus variables from first to last constraint as applicable and add artificial variables from first to last constraint for (≥) and (=) type of constraints
- Only slack and artificial variables are considered in the basis during starting simplex table no surplus variable

Artificial variable technique

1. Big M method or M-technique or Method of penalties and
2. Two phase method

Big M method

- **Assign huge cost (-M) to the artificial variable added and solve.**
- **In case the artificial variable leaves the basis, then further calculations for the artificial variable may or may not be continued.**

PROBLEM 1

Solve

Max $z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \leq 2$

$x_1 + 3x_2 \geq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Introducing slack variables S_1, S_2, surplus variable S_3 and artificial variable A_1, the problem can be changed to standard form

Max $z = 3x_1 - x_2 + 0S_1 + 0S_2 - 0S_3 – MA_1$

$2x_1 + x_2 + S_1 = 2$

$x_1 + 3x_2 - S_2 + A_1 = 3$

$x_2 + S_3 = 4$

x_1, x_2, S_1, S_2, S_3 and $A_1 \geq 0$

IBFS

Basis		**C_j**	**3**	**-1**	**0**	**0**	**0**	**-M**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_4	
S_1	0	2	2	1	1	0	0	0	2
A_4	-M	3	1	$\boxed{3}$	0	-1	0	1	→ 1
S_3	0	4	0	1	0	0	1	0	4
	Z= -3M		-M+3	↑ -3M+1	0	M	0	0	

SBFS

Basis		**C_j**	**3**	**-1**	**0**	**0**	**0**	**-M**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_4	
S_1	0	2	2	1	1	0	0	0	2
A_4	-M	3	1	$\boxed{3}$	0	-1	0	1	→ 1
S_3	0	4	0	1	0	0	1	0	4
	Z= -3M		-M+3	↑ -3M+1	0	M	0	0	
S_1	0	**1**	$\boxed{5/3}$	0	1	1/3	0	-1/3	→ 3/5
x_2	-1	**1**	1/3	1	0	-1/3	0	1/3	3
S_3	0	**3**	-1/3	0	0	1/3	1	-1/3	---
	Z= -1		-10/3 ↑	0	0	1/3	0	2/3M	

TBFS

Basis		C_j	**3**	**-1**	**0**	**0**	**0**	**-M**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_4	
S_1	0	2	2	1	1	0	0	0	2
A_4	-M	3	1	3	0	-1	0	1	1
S_3	0	4	0	1	0	0	1	0	4 →
	Z= -3M		-M+3	↑-3M+1	0	M	0	0	
S_1	0	1	$\boxed{5/3}$	0	1	1/3	0	-1/3	→ 3/5
x_2	-1	1	1/3	1	0	-1/3	0	1/3	3
S_3	0	3	-1/3	0	0	1/3	1	-1/3	---
	Z= -1		↑-10/3	0	0	1/3	0	2/3M	
x_1	3	3/5	1	0	3/5	1/5	0	-1/5	
x_2	-1	4/5	0	1	-1/5	-2/5	0	-2/5	
S_3	0	16/5	0	0	1/5	2/5	1	-2/5	
	Z= 1		0	0	2	1	0	-1/5 +M	

Optimal Solution is Z_{max}= 1 when x_1= 3/5 and x_2= 4/5

PROBLEM 2

An air force is experimenting with three types of bombs P, Q and R in which three kinds of explosives viz. A, B and C will be used. Taking the various factors into account it has been decided to use at the maximum 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. Bomb P requires 3, 2, and 2 kg, bomb Q requires 1, 4, and 3 kg, and bomb R requires 4, 2, and 3 kg of explosives A, B and C respectively. Bomb P is estimated to give the equivalent of 2 ton explosion, bomb Q 3 ton explosion and bomb R 4 ton explosion respectively. Under that production schedule can the air force makes the biggest bang?

Let x_1, x_2 and x_3 be the number of bombs of type P, Q and R respectively.

Max $z = 2x_1 + 3x_2 + 4x_3$

Subject to

$3x_1 + x_2 + 4x_3 \leq 600$

$2x_1 + 4x_2 + 2x_3 \geq 480$

$2x_1 + 3x_2 + 3x_3 = 540$

$x_1, x_2, x_3 \geq 0$

Introducing slack variable S_1 surplus variable S_3 and artificial variables A_1 and A_2 the problem can be changed to standard form

Max $z = 2x_1 + 3x_2 + 4x_3 + 0S_1 + 0S_2 – MA_1 – MA_2$

$3x_1 + x_2 + 4x_3 + S_1 = 600$

$2x_1 + 4x_2 + 2x_3 - S_2 + A_1 = 480$

$2x_1 + 3x_2 + 3x_3 + A_2 = 540$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

≥ 0

Starting table

Basis		C_j	**2**	**3**	**4**	**0**	**0**	**-M**	**-M**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	
S_1	0	600	3	1	4	1	0	0	0	600
A_1	-M	480	2	$\boxed{4}$	2	0	-1	1	0	120
A_2	-M	540	2	3	3	0	0	0	1	180
	Z= -940M		-4M-2	-7M-3	-5M-4	0	M	0	0	

SBFS

Basis		C_j	**2**	**3**	**4**	**0**	**0**	**-M**	**-M**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	
S_1	0	600	3	1	4	1	0	0	0	600
A_1	-M	480	2	$\boxed{4}$	2	0	-1	1	0	120
A_2	-M	540	2	3	3	0	0	0	1	180
	Z= -940M		-4M-2	-7M-3	-5M-4	0	M	0	0	
S_1	0	480	5/2	0	7/2	1	¼	-1/4	0	960/7
x_2	3	120	½	1	½	0	-1/4	1/4	0	240
A_2	-M	180	½	0	$\boxed{3/2}$	0	3/4	-3/4	1	120
	Z= -180M+360		-M/2-1/2	0	-3M/2-5/2	0	-3M/4-1/4	7M/4+1/4	0	

TBFS

Basis		C_j	2	3	4	0	0	-M	-M	$\text{Min}\{x_B/x_k\}>0$
	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	
S_1	0	600	3	1	4	1	0	0	0	600
A_1	-M	480	2	$\boxed{4}$	2	0	-1	1	0	120
A_2	-M	540	2	3	3	0	0	0	1	180
	Z= -940M		-4M-2	-7M-3	-5M-4	0	M	0	0	
S_1	0	480	5/2	0	7/2	1	¼	-1/4	0	960/7
x_2	3	120	½	1	½	0	-1/4	1/4	0	240
A_2	-M	180	½	0	$\boxed{3/2}$	0	3/4	-3/4	1	120
	Z= -180M+360		-M/2-1/2	0	-3M/2-5/2	0	-3M/4-1/4	7M/4 +1/4	0	
S_1	0	60	4/3	0	0	1	-3/2	3/2	-7/3	
x_2	3	60	1/3	1	0	0	-1/2	1/2	-1/3	
x_3	4	120	1/3	0	1	0	1/2	-1/2	2/3	
	Z= 660		1/3	0	0	0	1/2	½+M	5/3+M	

Optimal Solution is $Z_{max}= 660$ when $x_1= 0$, $x_2= 60$ and $x_3= 120$

Two phase method

Phase I

Convert given LPP into auxiliary LPP by assigning 0 cost to all the variables and (-1) to artificial variables.

If

i. Max Z > 0 and at least one artificial variable appears in the basis, then the given problem has no feasible solution and the procedure terminates.

ii. Max Z = 0 and no artificial variable appears in the basis, then proceed to phase II

Phase II

Consider the final table of Phase I as starting table of Phase II, assign real costs and solve.

PROBLEM 3

Use the two-phase simplex method to

Max $z = 5x_1 - 4x_2 + 3x_3$

Subject to

$2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \leq 76$

$8x_1 - 3x_2 + 6x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Phase I

Converting it to standard form by adding slack and artificial variables and converting it to Auxiliary LPP we get

Max $z = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1$

Subject to

$2x_1 + x_2 - 6x_3 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$

$8x_1 - 3x_2 + 6x_3 + S_2 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

IBFS

Basis		**C_j**	**0**	**0**	**0**	**0**	**0**	**-1**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	X_3	S_1	S_2	A_1	
A_1	-1	20	2	1	-6	0	0	1	10
S_1	0	76	6	5	10	1	0	0	76/6
S_2	0	50	$\boxed{8}$	-3	6	0	1	0	25/4 →
	Z= -20		↑ -2	-1	6	0	0	0	

SBFS

Basis		C_j	**0**	**0**	**0**	**0**	**0**	**-1**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	X_3	S_1	S_2	A_1	
A_1	-1	20	2	1	-6	0	0	1	10
S_1	0	76	6	5	10	1	0	0	76/6
S_2	0	50	$\boxed{8}$	-3	6	0	1	0	25/4 →
	Z= -20		↑ -2	-1	6	0	0	0	
A_1	-1	15/2	0	7/4	-15/2	0	-1/4	1	30/7 →
S_1	0	77/2	0	29/4	11/2	1	-3/4	0	154/29
x_1	0	25/4	1	-3/8	3/4	0	1/8	0	--
	Z= -15/2		0	↑ -7/4	15/2	0	1/4	0	

TBFS

Basis		C_j	**0**	**0**	**0**	**0**	**0**	**-1**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	X_3	S_1	S_2	A_1	
A_1	-1	20	2	1	-6	0	0	1	10
S_1	0	76	6	5	10	1	0	0	76/6
S_2	0	50	$\boxed{8}$	-3	6	0	1	0	25/4 →
	Z= -20		↑ -2	-1	6	0	0	0	
A_1	-1	15/2	0	$\boxed{7/4}$	-15/2	0	-1/4	1	30/7 →
S_1	0	77/2	0	29/4	11/2	1	-3/4	0	154/29
x_1	0	25/4	1	-3/8	3/4	0	1/8	0	--
	Z= -15/2		0	↑ -7/4	15/2	0	1/4	0	
x_2	0	30/7	0	1	-30/7	0	-1/7	4/7	
S_1	0	52/7	0	0	256/7	1	2/7	-29/7	
x_1	0	55/7	1	0	-6/7	0	1/14	3/14	
	Z= 0		0	0	0	0	0	1	

Phase II

By assigning real costs and deleting artificial variables we get

Basis		C_j	**0**	**0**	**0**	**0**	**0**	**Min{x_B/x_k}>0**
	C_B	X_B	x_1	x_2	X_3	S_1	S_2	
x_2	-4	30/7	0	1	-30/7	0	-1/7	
S_1	0	52/7	0	0	256/7	1	2/7	
x_1	5	55/7	1	0	-6/7	0	1/14	
	Z= 155/7		0	0	69/7	0	13/14	

Optimal Solution is Z_{max}= **155/7** when x_1= 55/7, x_2= 30/7 and x_3= 0