Linear Programming: Model Formulation and Graphical Solution

Chapter 2

Chapter Topics

- Model Formulation
- A Maximization Model Example
- Graphical Solutions of Linear Programming Models
- A Minimization Model Example
- Irregular Types of Linear Programming Models
- Characteristics of Linear Programming Problems

Linear Programming: An Overview

- Objectives of business decisions frequently involve ***maximizing profit*** or ***minimizing costs***.
- Linear programming uses ***linear algebraic relationships*** to represent a firm's decisions, given a business ***objective***, and resource ***constraints***.
- Steps in application:
  1. Identify problem as solvable by linear programming.
  2. Formulate a mathematical model of the unstructured problem.
  3. Solve the model.
  4. Implementation

Model Components

- **Decision variables** - mathematical symbols representing levels of activity of a firm.
- **Objective function** - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.
- **Constraints** – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.
- **Parameters** - numerical coefficients and constants used in the objective function and constraints.

Summary of Model Formulation Steps

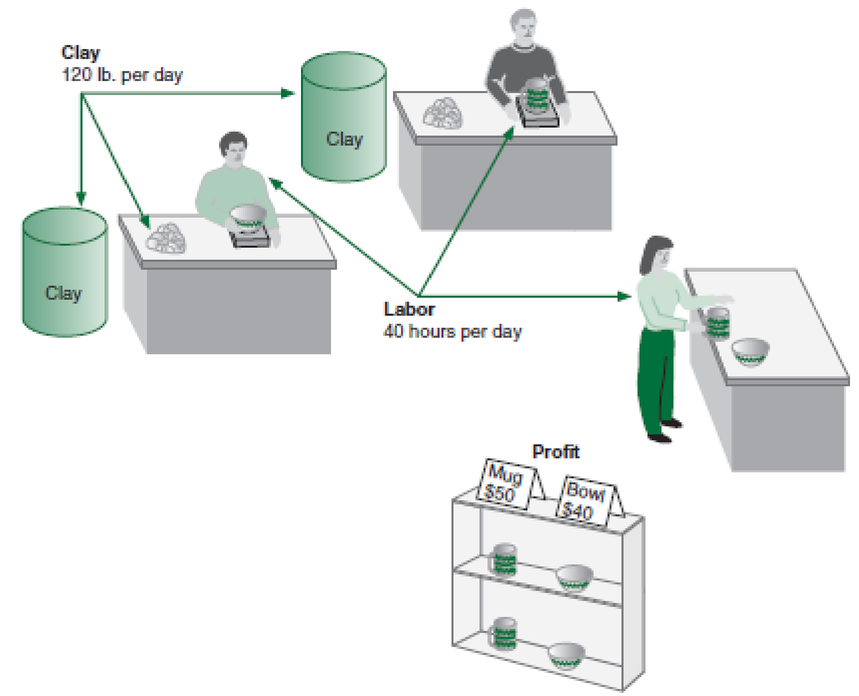
Step 1 : Clearly define the decision variables

Step 2 : Construct the objective function

Step 3 : Formulate the constraints

LP Model Formulation
A Maximization Example (1 of 3)

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50



- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

LP Model Formulation
A Maximization Example (2 of 3)

Resource Availability:	40 hrs of labor per day 120 lbs of clay
Decision Variables:	x_1 = number of bowls to produce per day x_2 = number of mugs to produce per day
Objective Function:	Maximize $Z = \$40x_1 + \$50x_2$ Where Z = profit per day
Resource Constraints:	$1x_1 + 2x_2 \leq 40$ hours of labor $4x_1 + 3x_2 \leq 120$ pounds of clay
Non-Negativity Constraints:	$x_1 \geq 0$; $x_2 \geq 0$

LP Model Formulation
A Maximization Example (3 of 3)

Complete Linear Programming Model:

Maximize $Z = \$40x_1 + \$50x_2$

subject to:

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$ bowls
$x_2 = 10$ mugs
$Z = \$40x_1 + \$50x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 \leq 40$ hours
Clay constraint check: $4(5) + 3(10) = 70 \leq 120$ pounds

Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: $x_1 = 10$ bowls
$x_2 = 20$ mugs
$Z = \$40x_1 + \$50x_2 = \$1400$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).

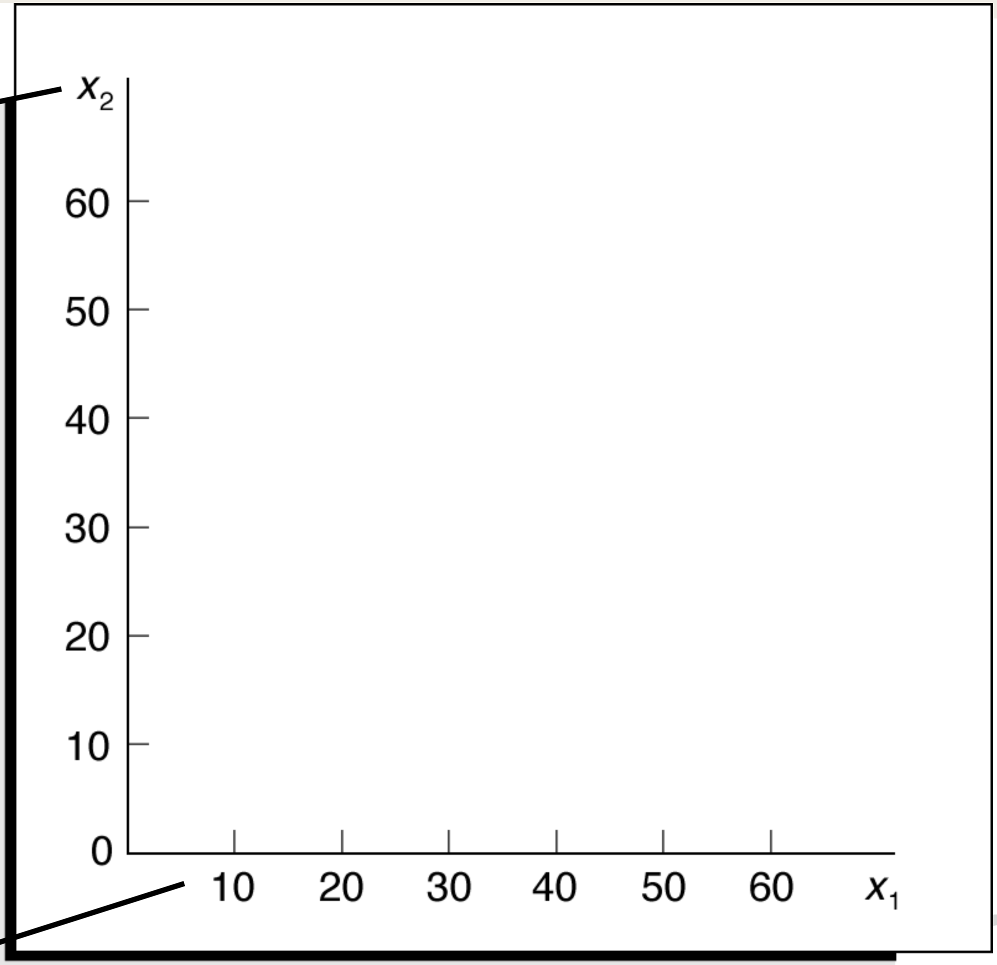
- Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.

Coordinate Axes
Graphical Solution of Maximization Model (1 of 12)

X_2 is mugs

Maximize $Z = \$40x_1 + \$50x_2$

subject to:
$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$



X_1 is bowls

Figure 2.2 Coordinates for graphical analysis

Labor Constraint
Graphical Solution of Maximization Model (2 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

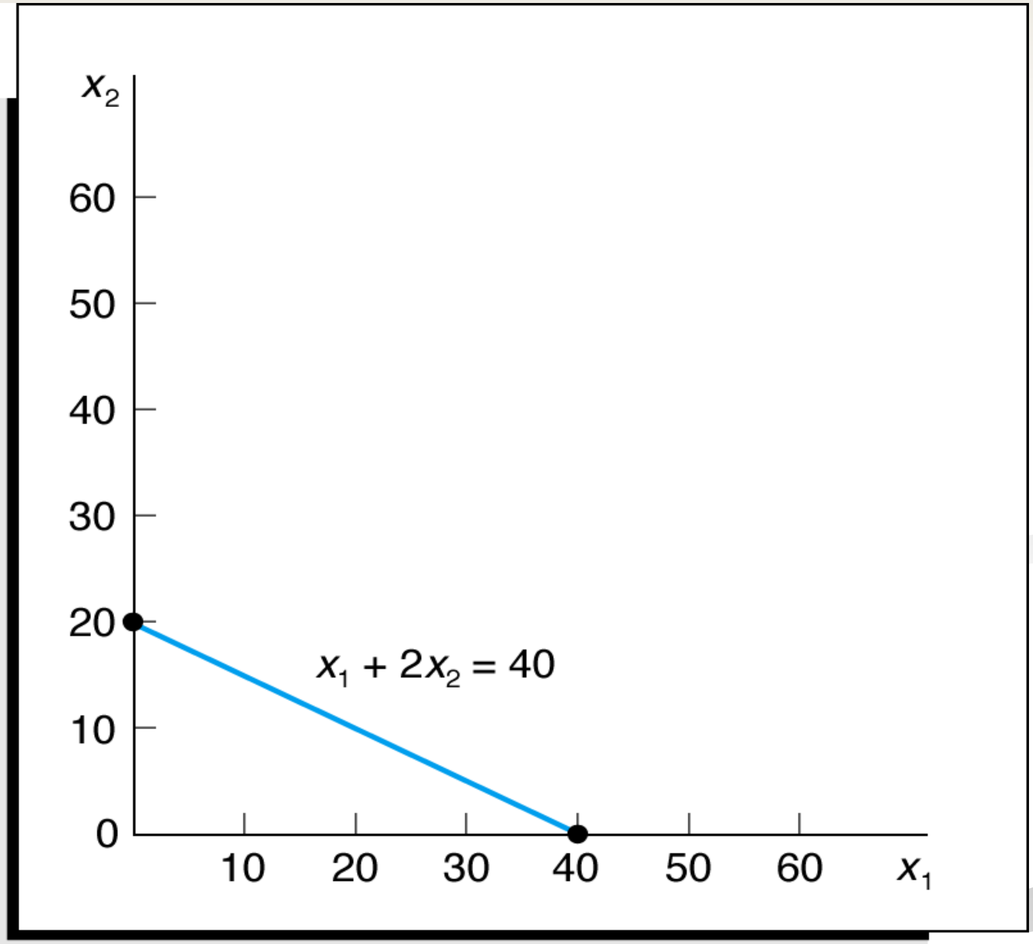


Figure 2.3 Graph of labor constraint

Labor Constraint Area
Graphical Solution of Maximization Model (3 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

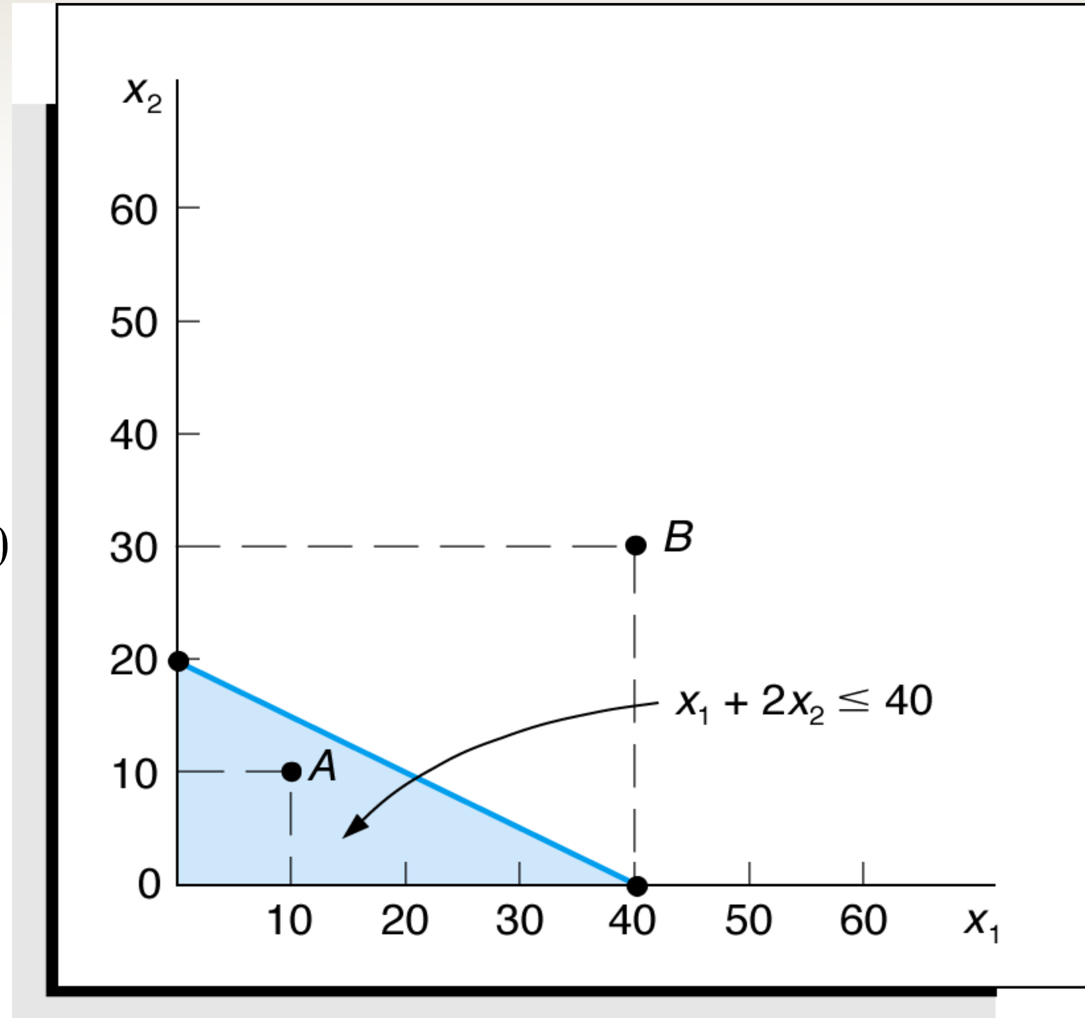


Figure 2.4 Labor constraint area

Clay Constraint Area
Graphical Solution of Maximization Model (4 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

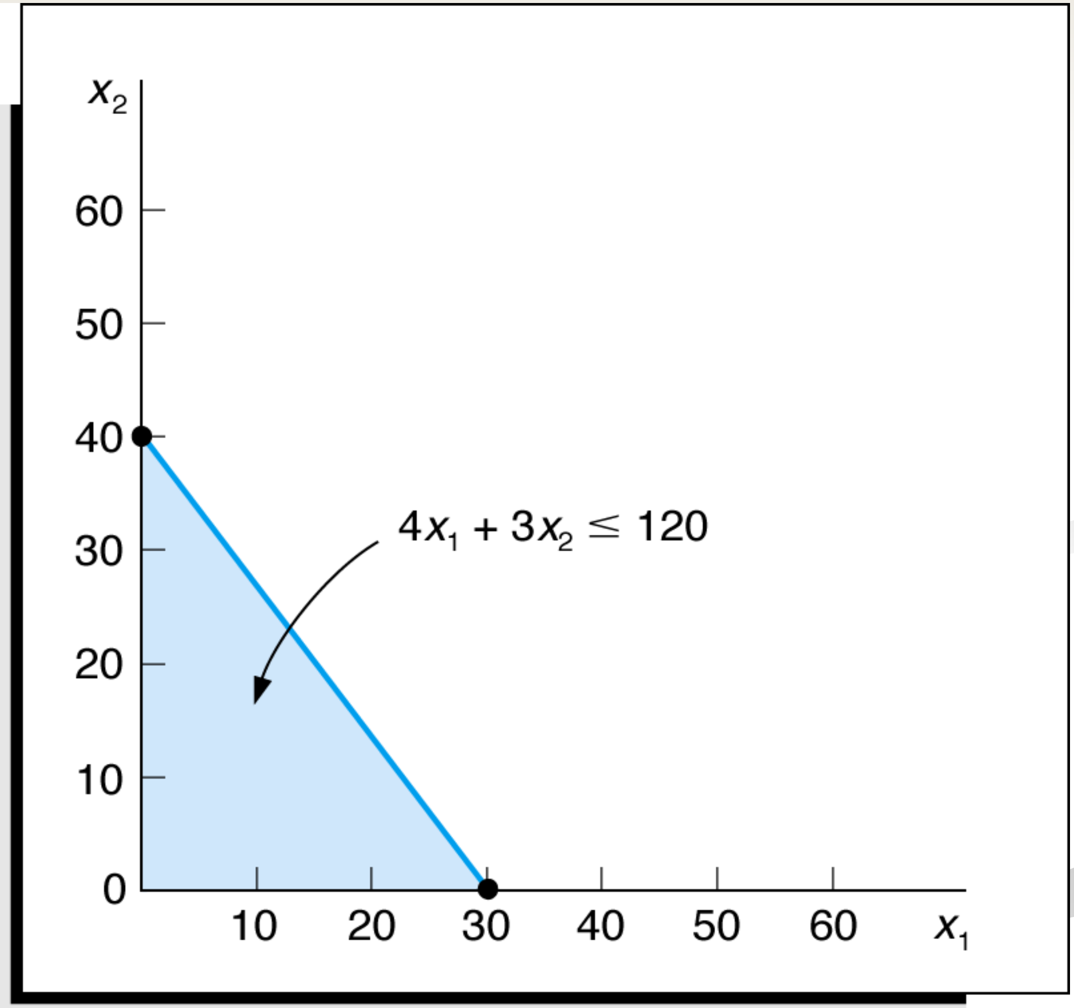


Figure 2.5 The constraint area for clay

Both Constraints
Graphical Solution of Maximization Model (5 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

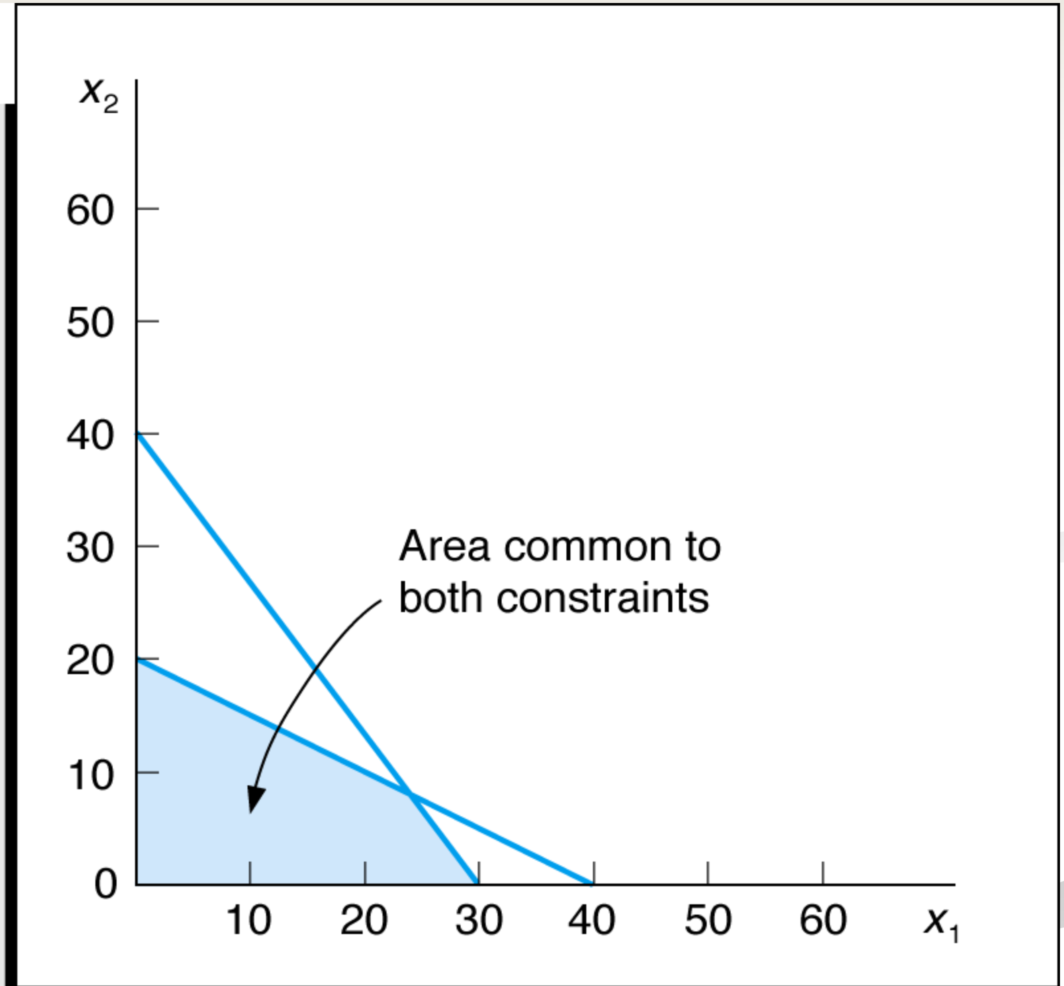


Figure 2.6 Graph of both model constraints

Feasible Solution Area
Graphical Solution of Maximization Model (6 of 12)

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

subject to:

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

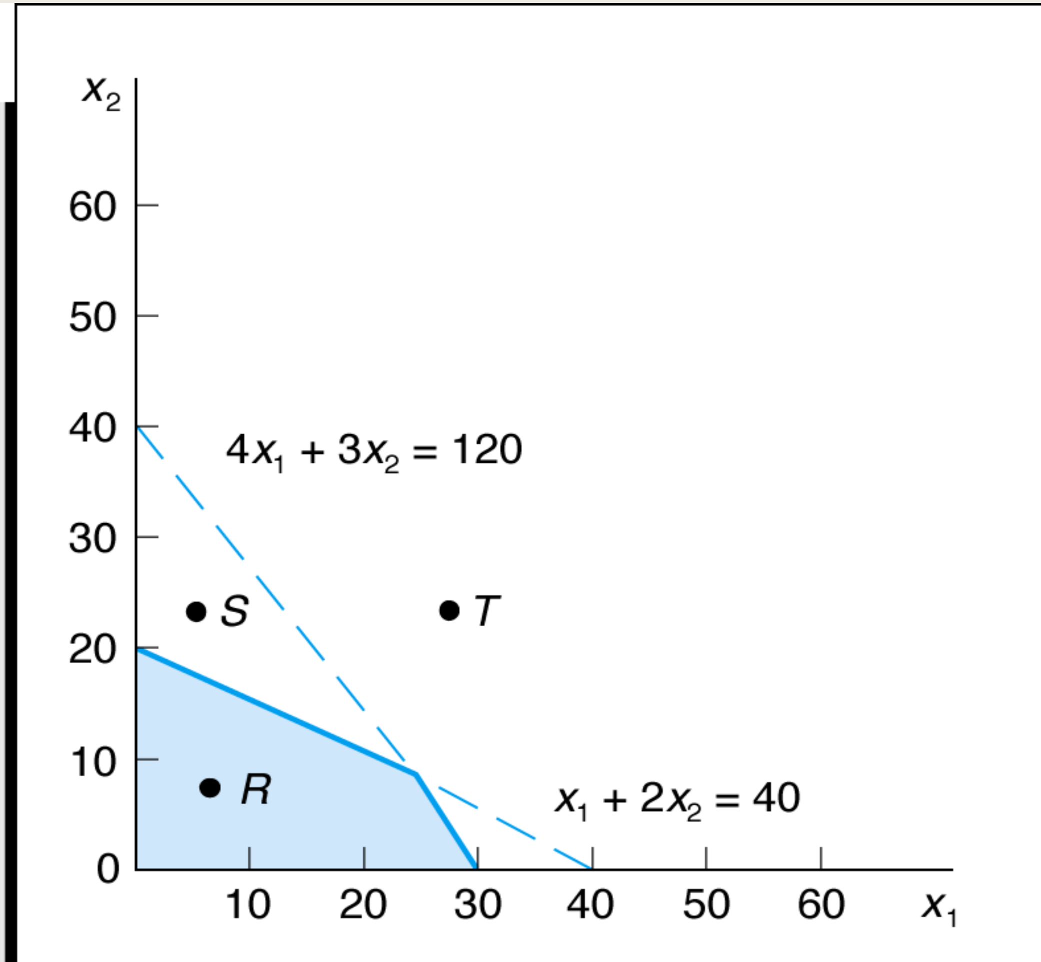


Figure 2.7 The feasible solution area

Objective Function Solution = \$800
Graphical Solution of Maximization Model (7 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

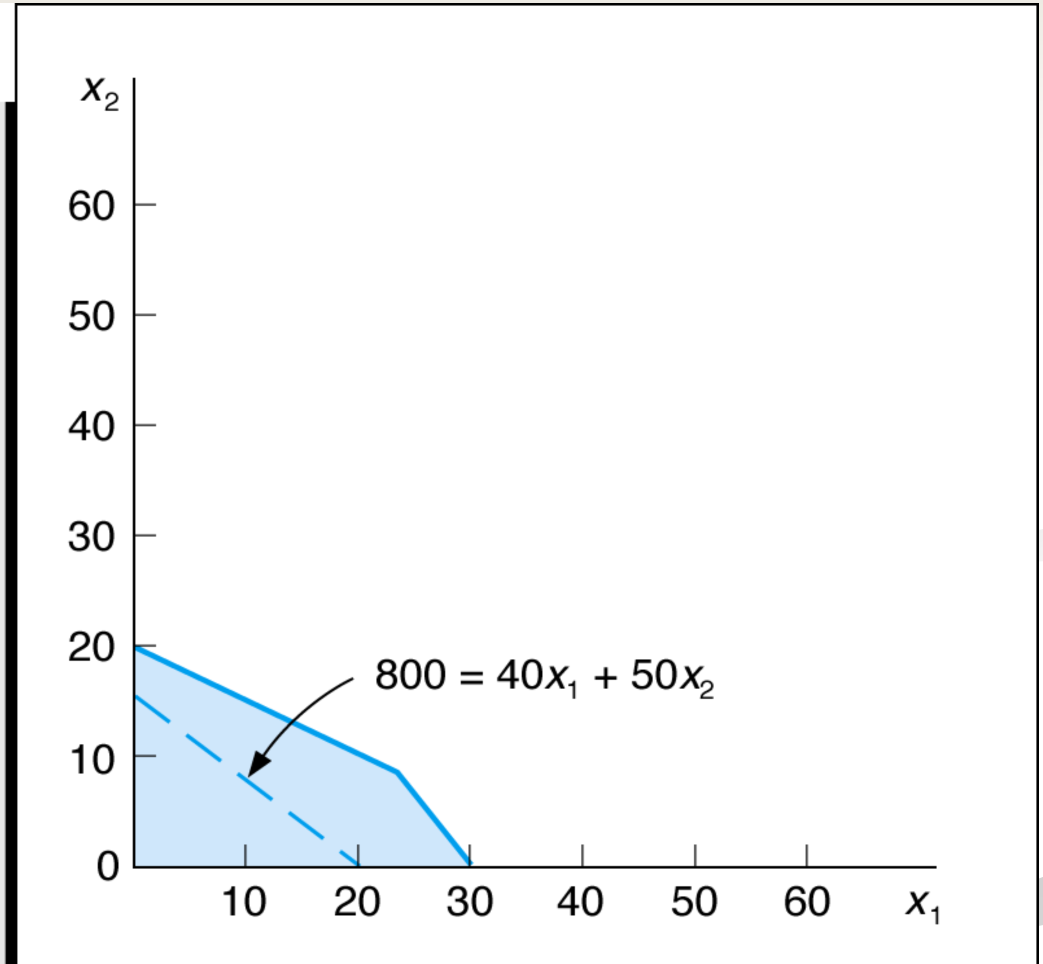


Figure 2.8 Objective function line for Z = \$800

Alternative Objective Function Solution Lines Graphical Solution of Maximization Model (8 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

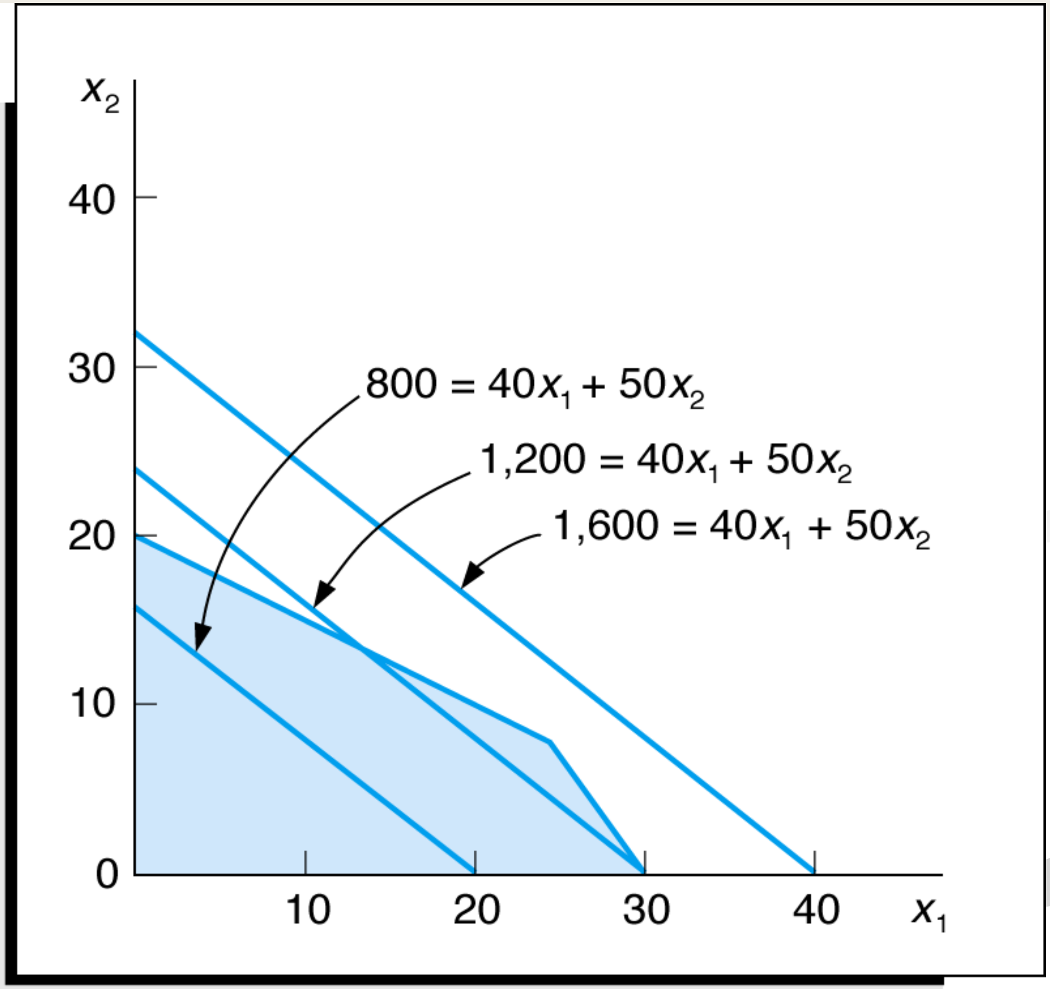


Figure 2.9 Alternative objective function lines for profits Z of \$800, \$1,200, and \$1,600

Optimal Solution
Graphical Solution of Maximization Model (9 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

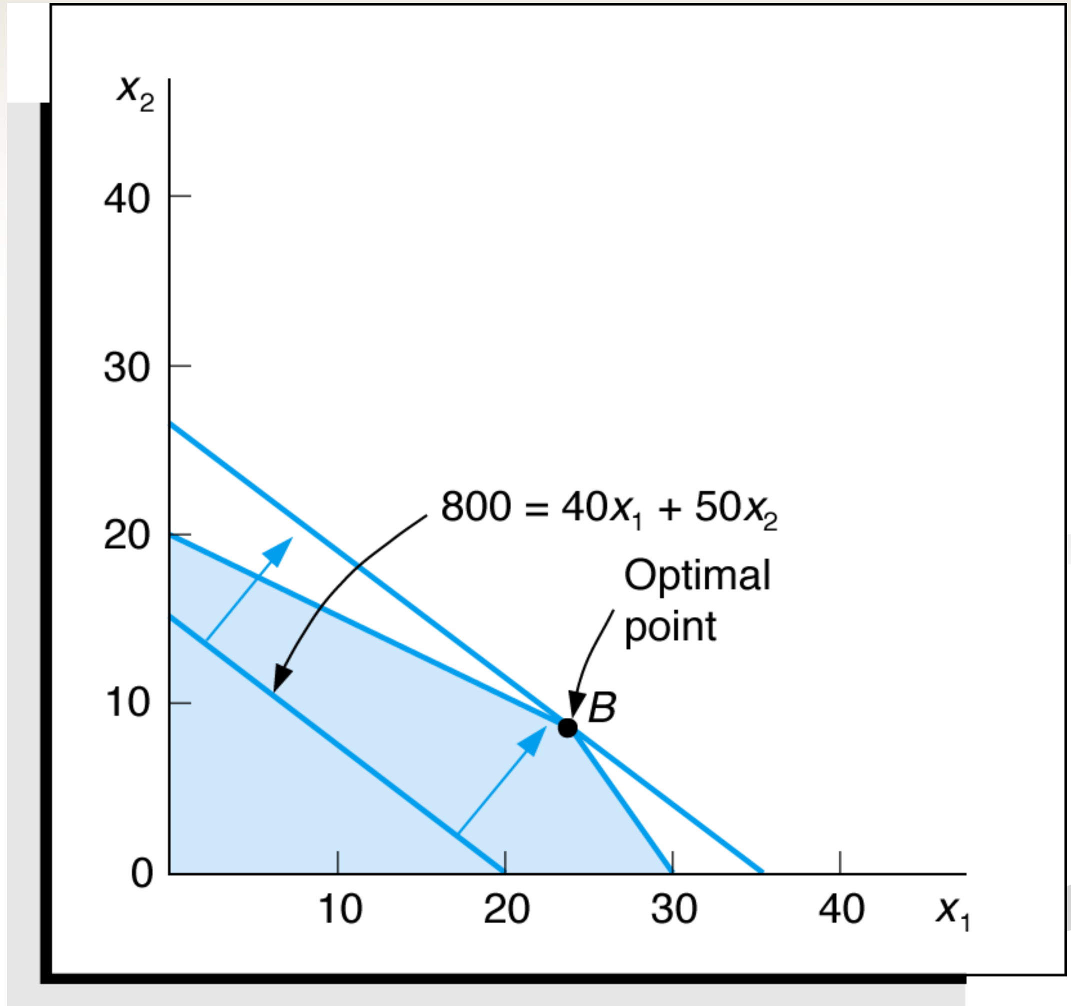


Figure 2.10 Identification of optimal solution point

Optimal Solution Coordinates
Graphical Solution of Maximization Model (10 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

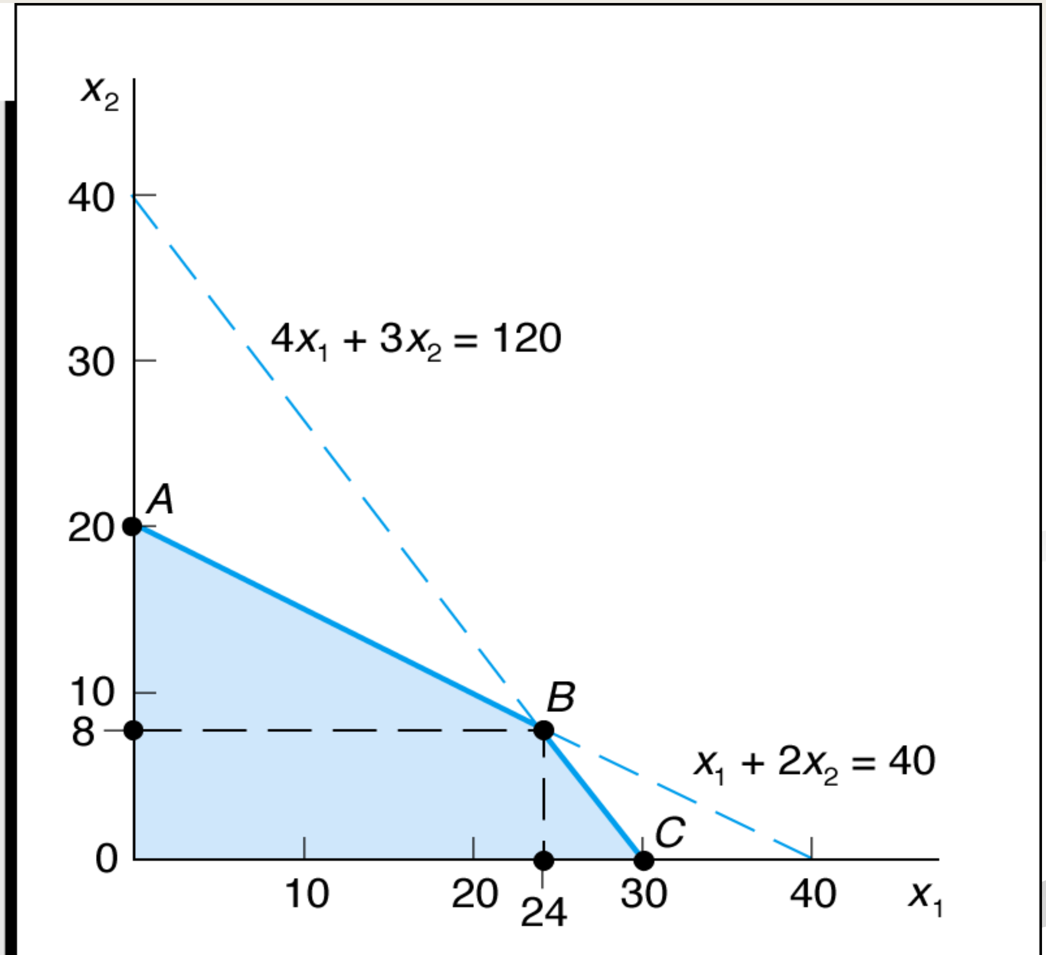


Figure 2.11 Optimal solution coordinates

Extreme (Corner) Point Solutions
Graphical Solution of Maximization Model (11 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

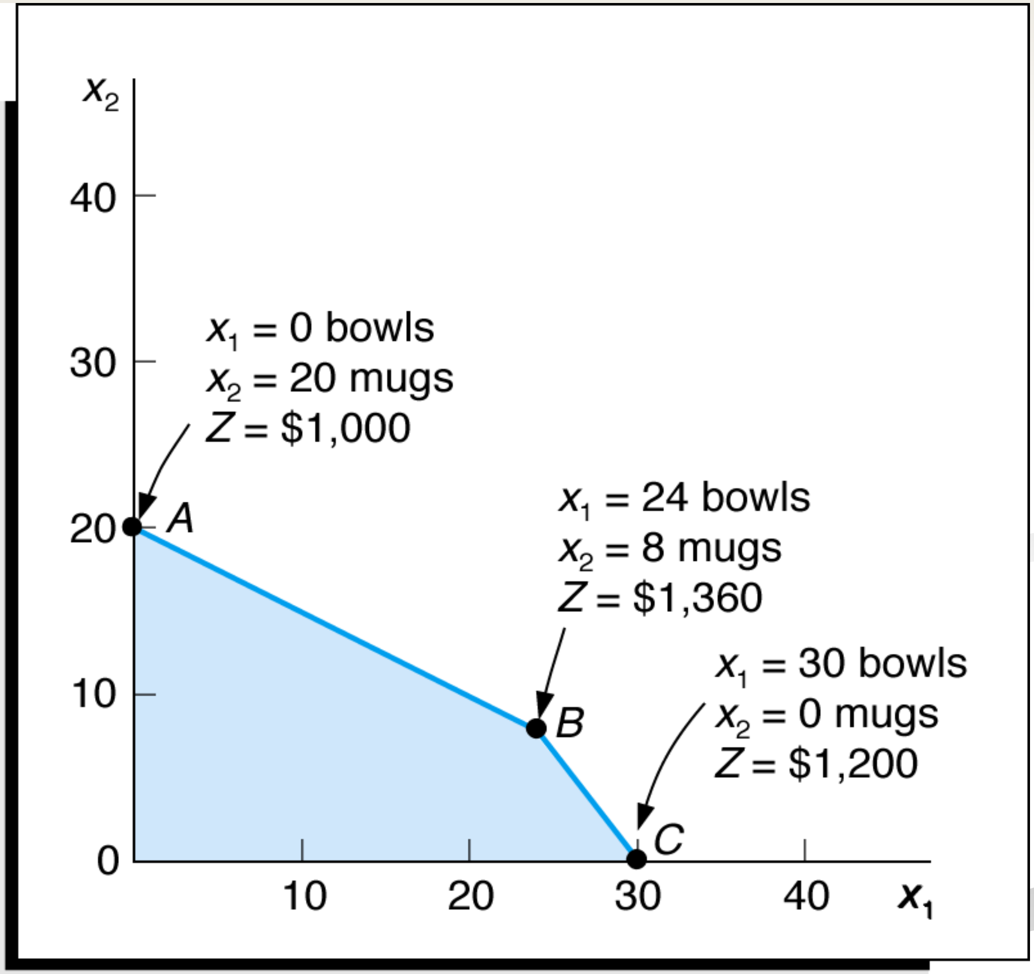


Figure 2.12 Solutions at all corner points

Optimal Solution for New Objective Function Graphical Solution of Maximization Model (12 of 12)

Maximize $Z = \$70x_1 + \$20x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

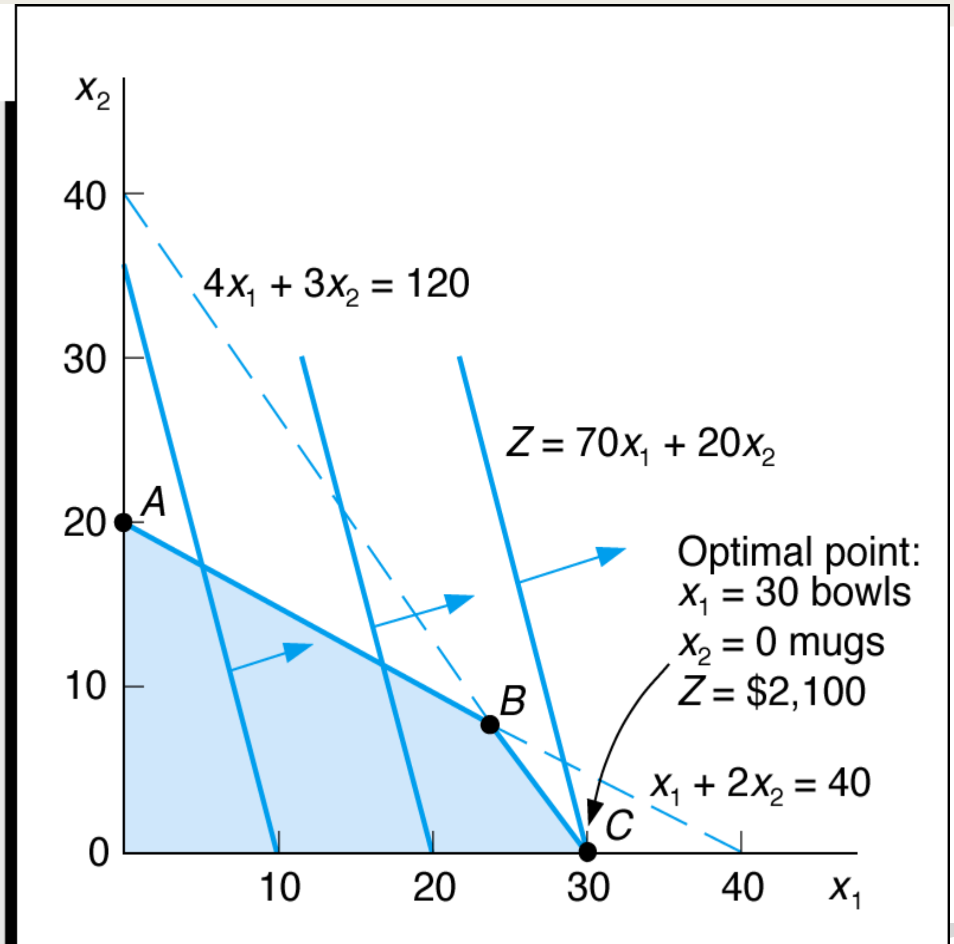


Figure 2.13 Optimal solution with $Z = 70x_1 + 20x_2$

Slack Variables

- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is ***added to a ≤ constraint*** (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an ***unused resource***.
- A slack variable ***contributes nothing*** to the objective function value.

Linear Programming Model: Standard Form

Max $Z = 40x_1 + 50x_2 + s_1 + s_2$

subject to: $1x_1 + 2x_2 + s_1 = 40$

$4x_1 + 3x_2 + s_2 = 120$

$x_1, x_2, s_1, s_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

s_1, s_2 are slack variables

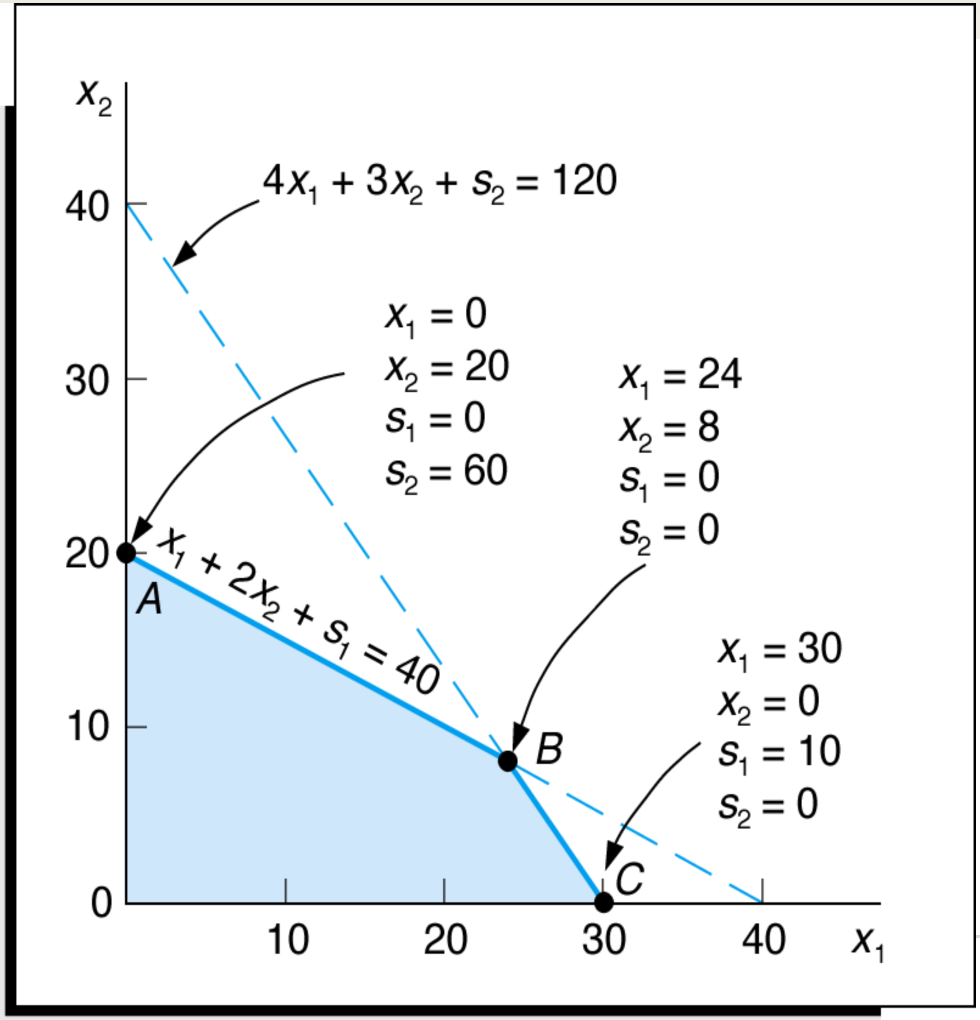
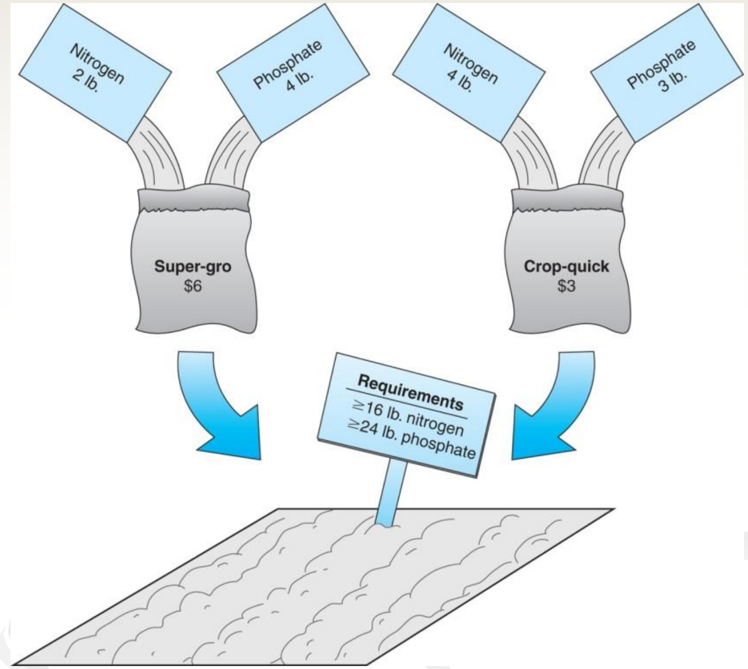


Figure 2.14 Solutions at points A, B, and C with slack

LP Model Formulation – Minimization (1 of 7)

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs $6 per bag, Crop-quick $3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?



Brand	Chemical Contribution Nitrogen (lb/bag)	 Phosphate (lb/bag)
Super-gro	2	4
Crop-quick	4	3

LP Model Formulation – Minimization (2 of 7)

Decision Variables:

x_1 = bags of Super-gro
x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$
Where: $\$6x_1$ = cost of bags of Super-Gro
$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)
$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)
$x_1, x_2 \geq 0$ (non-negativity constraint)

Constraint Graph – Minimization (3 of 7)

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

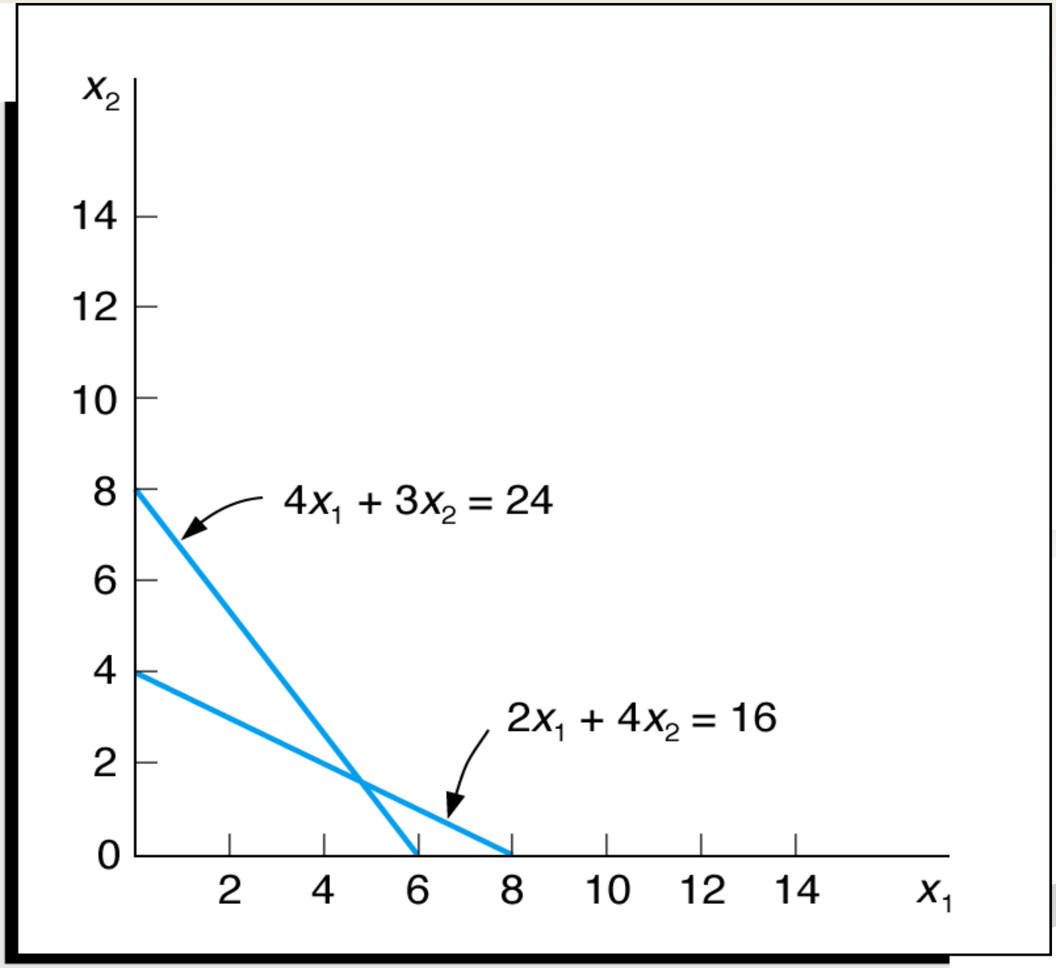


Figure 2.16 Constraint lines for fertilizer model

Feasible Region– Minimization (4 of 7)

$$\text{Minimize } Z = \$6x_1 + \$3x_2$$

subject to:

$$2x_1 + 4x_2 \geq 16$$
$$4x_1 + 3x_2 \geq 24$$
$$x_1, x_2 \geq 0$$

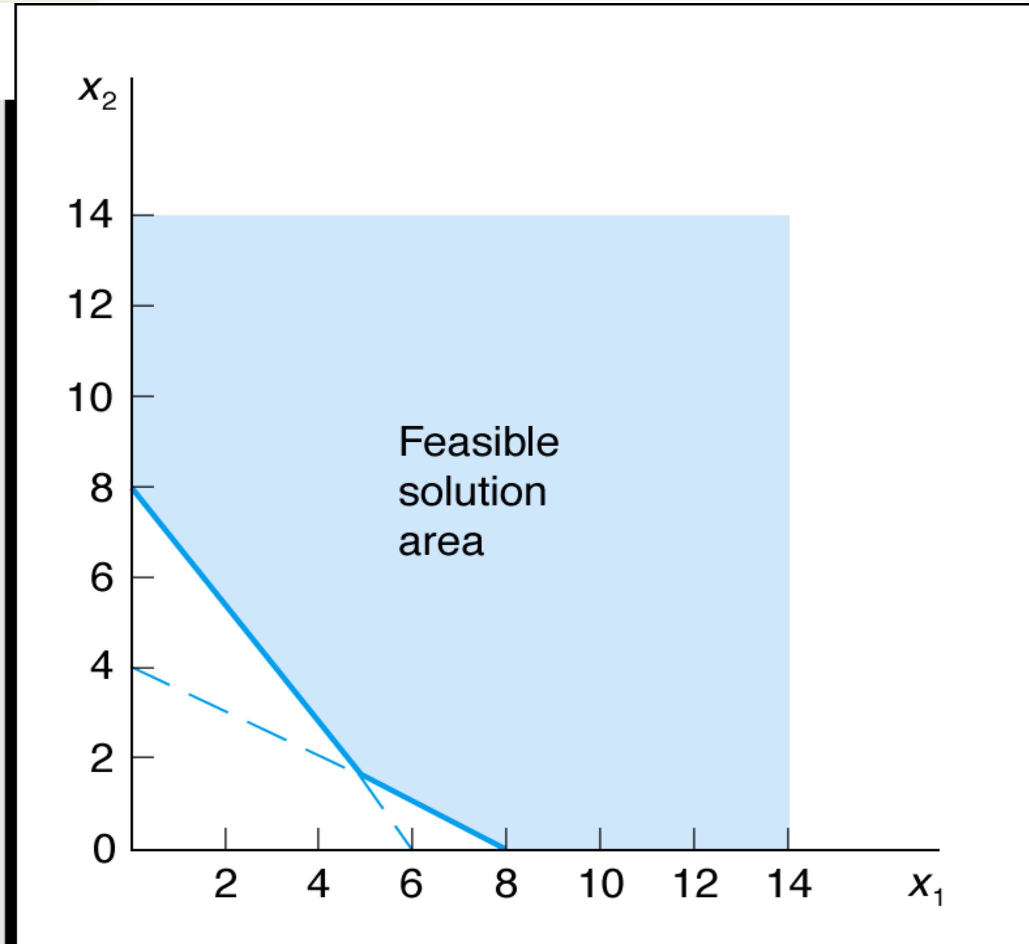


Figure 2.17 Feasible solution area

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_2 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

The optimal solution of a minimization problem is at the extreme point closest to the origin.

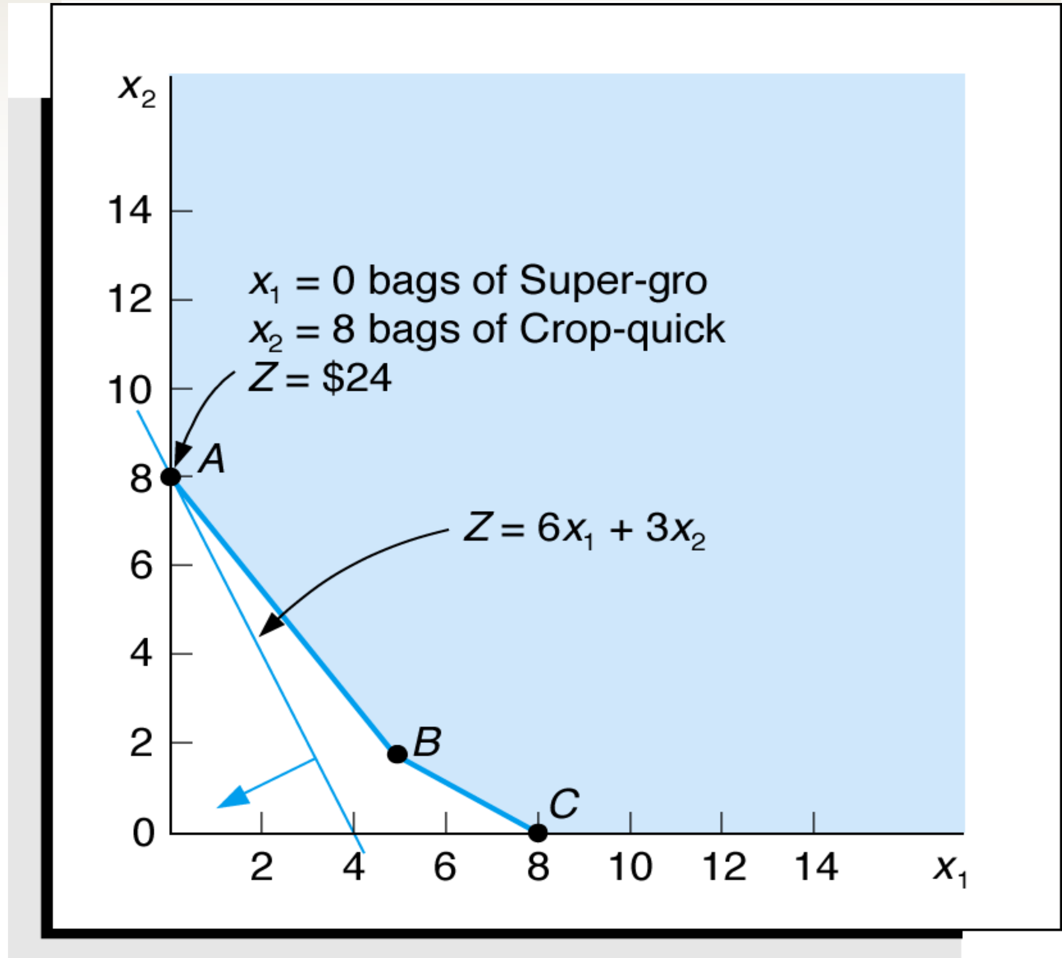


Figure 2.18 The optimal solution point

Surplus Variables – Minimization (6 of 7)

- A surplus variable is ***subtracted from a ≥ constraint*** to convert it to an equation (=).
- A surplus variable ***represents an excess*** above a constraint requirement level.
- A surplus variable ***contributes nothing*** to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}$$
$$4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}$$

Graphical Solutions – Minimization (7 of 7)

Minimize $Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$

subject to:
$$2x_1 + 4x_2 - s_1 = 16$$
$$4x_1 + 3x_2 - s_2 = 24$$
$$x_1, x_2, s_1, s_2 \geq 0$$

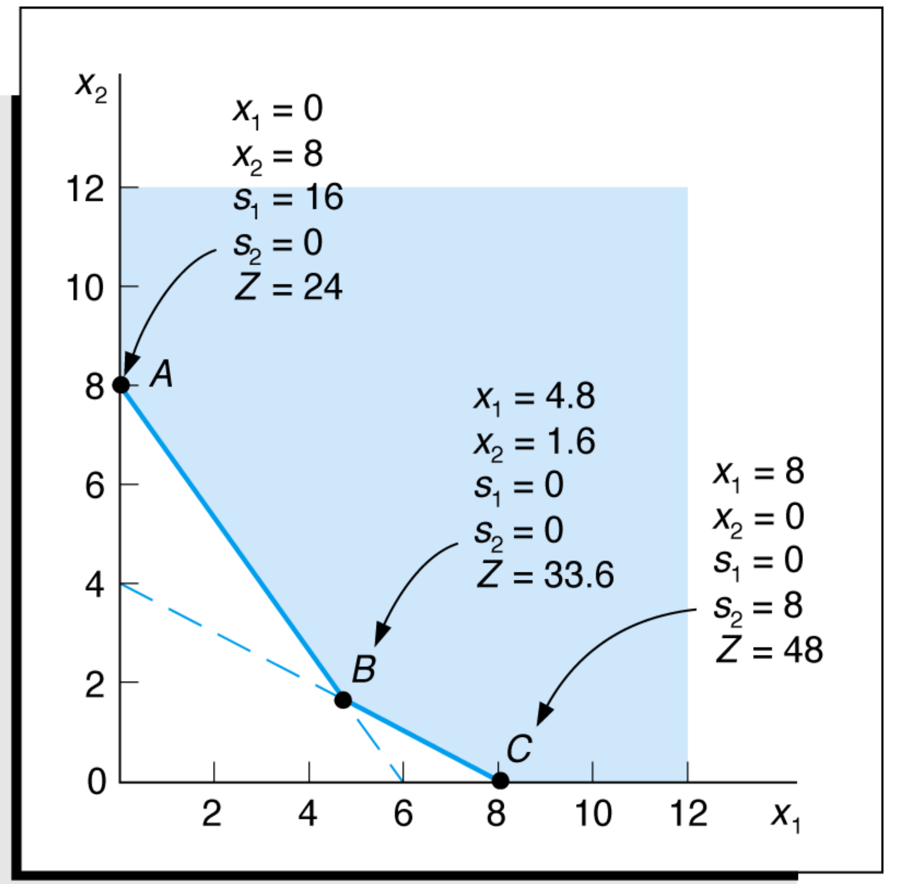


Figure 2.19 Graph of the fertilizer example

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

Special types of problems include those with:

- Multiple optimal solutions
- Infeasible solutions
- Unbounded solutions

Multiple Optimal Solutions Beaver Creek Pottery

The objective function is **parallel** to a constraint line.

Maximize $Z = \$40x_1 + 30x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

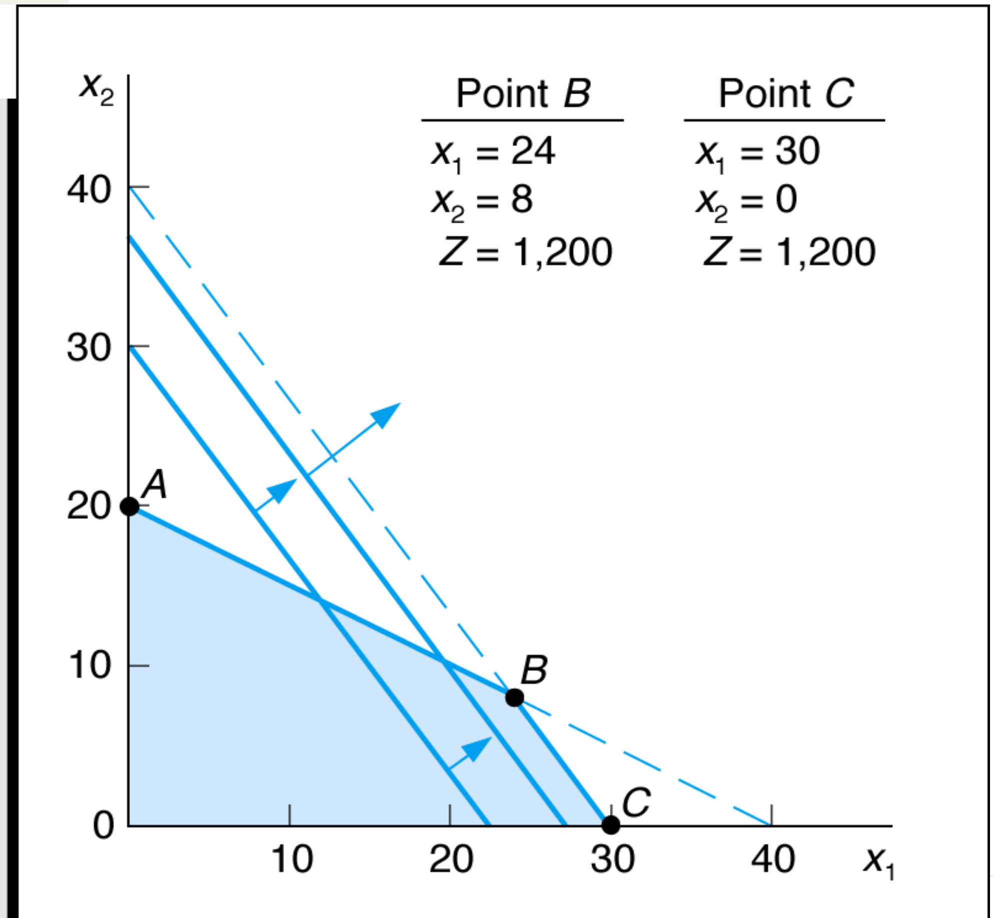


Figure 2.20 Example with multiple optimal solutions

An Infeasible Problem

Every possible solution **violates** at least one constraint:

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$

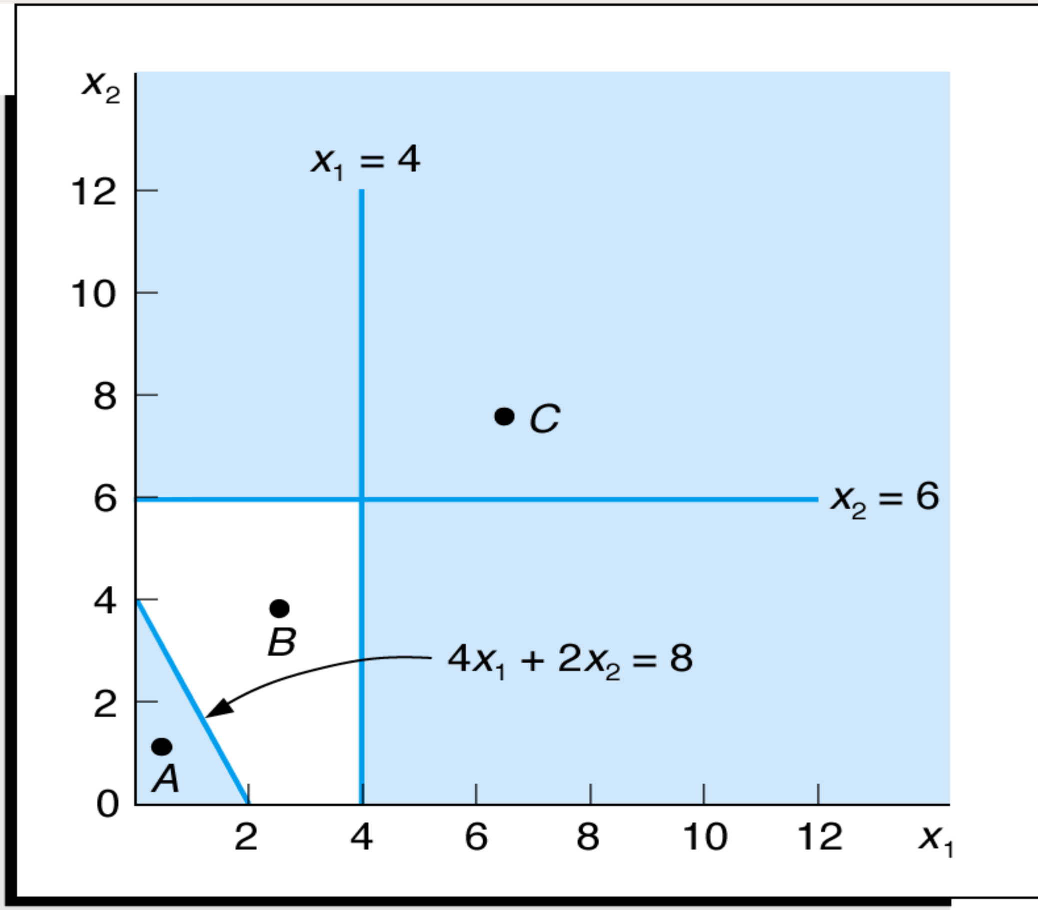


Figure 2.21 Graph of an infeasible problem

An Unbounded Problem

Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

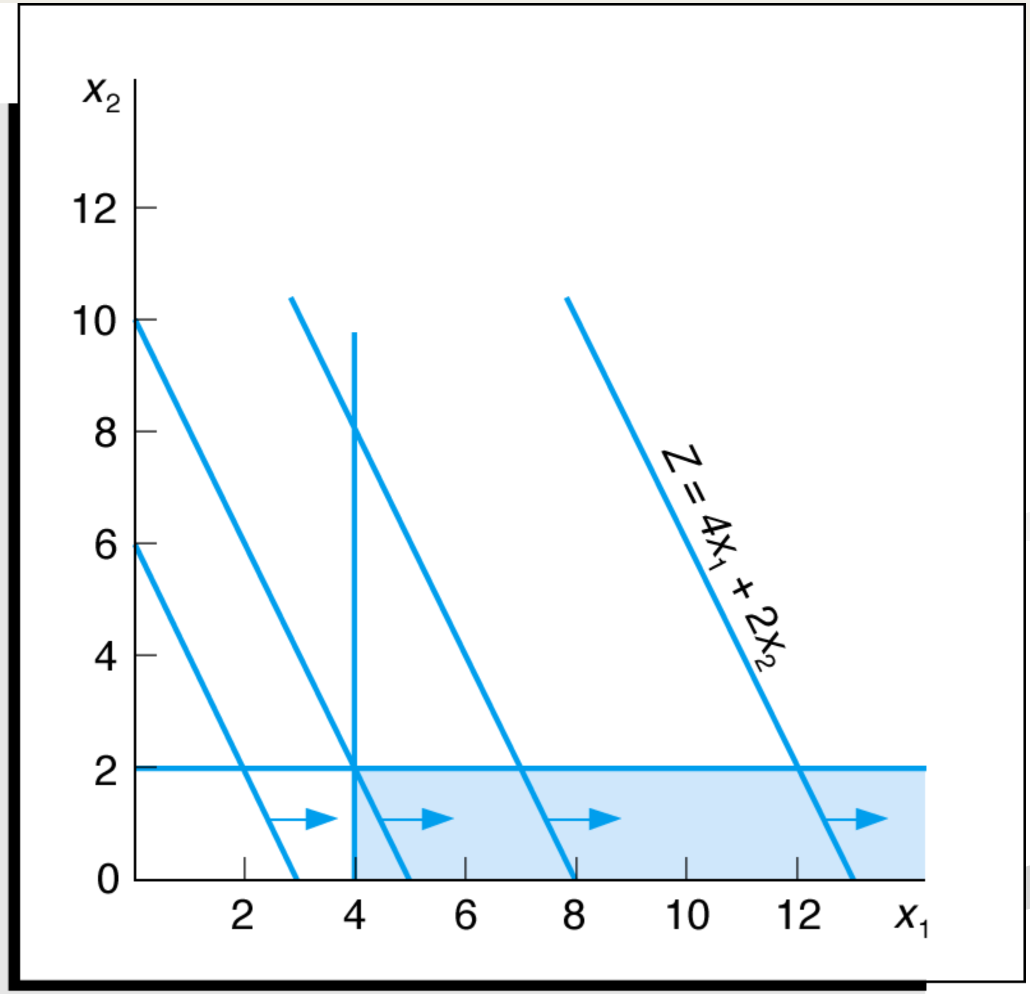


Figure 2.22 Graph of an unbounded problem

Characteristics of Linear Programming Problems

- A decision amongst alternative courses of action is required.
- The decision is represented in the model by **decision variables**.
- The problem encompasses a goal, expressed as an **objective function**, that the decision maker wants to achieve.
- Restrictions (represented by **constraints**) exist that limit the extent of achievement of the objective.
- The objective and constraints must be definable by **linear** mathematical functional relationships.

Properties of Linear Programming Models

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.
- **Divisibility** - Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

Problem Statement
Example Problem No. 1 (1 of 3)

- Hot mixture in 1000-pound batches.
- Two ingredients, a (\$3/lb) and b (\$5/lb).
- Recipe requirements:

  at least 500 pounds of "a"

  at least 200 pounds of "b"
- Ratio of a to b must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

Solution
Example Problem No. 1 (2 of 3)

Step 1:

Identify decision variables.

x_1 = lb of a in mixture

x_2 = lb of b in mixture

Step 2:

Formulate the objective function.

Minimize $Z = \$3x_1 + \$5x_2$

where Z = cost per 1,000-lb batch

$\$3x_1$ = cost of a

$\$5x_2$ = cost of b

Solution
Example Problem No. 1 (3 of 3)

Step 3:

Establish Model Constraints

$$x_1 + x_2 = 1{,}000 \text{ lb}$$
$$x_1 \geq 500 \text{ lb of a}$$
$$x_2 \geq 200 \text{ lb of b}$$
$$x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

The Model: Minimize $Z = \$3x_1 + 5x_2$

subject to: $x_1 + x_2 = 1{,}000$ lb

$$x_1 \geq 50$$
$$x_2 \geq 200$$
$$x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

Example Problem No. 2 (1 of 3)

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations

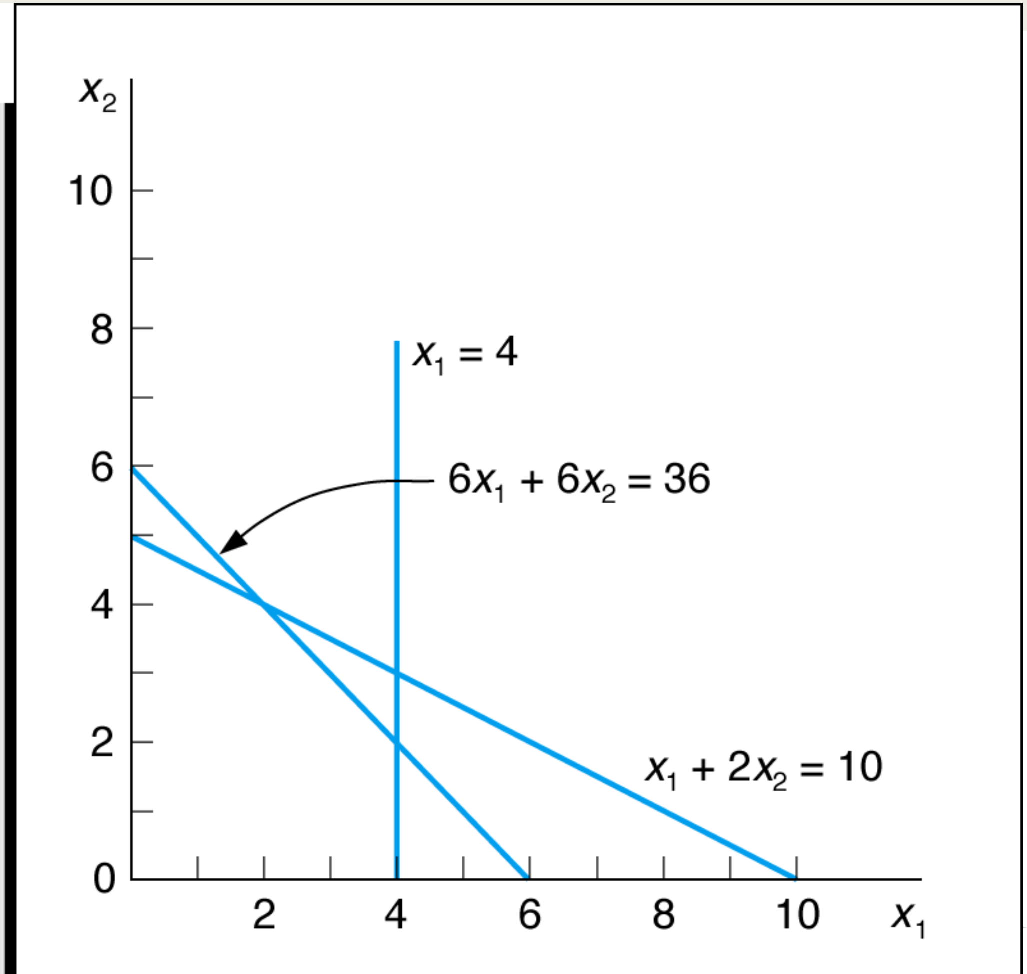


Figure 2.23 Constraint equations

Example Problem No. 2 (2 of 3)

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space

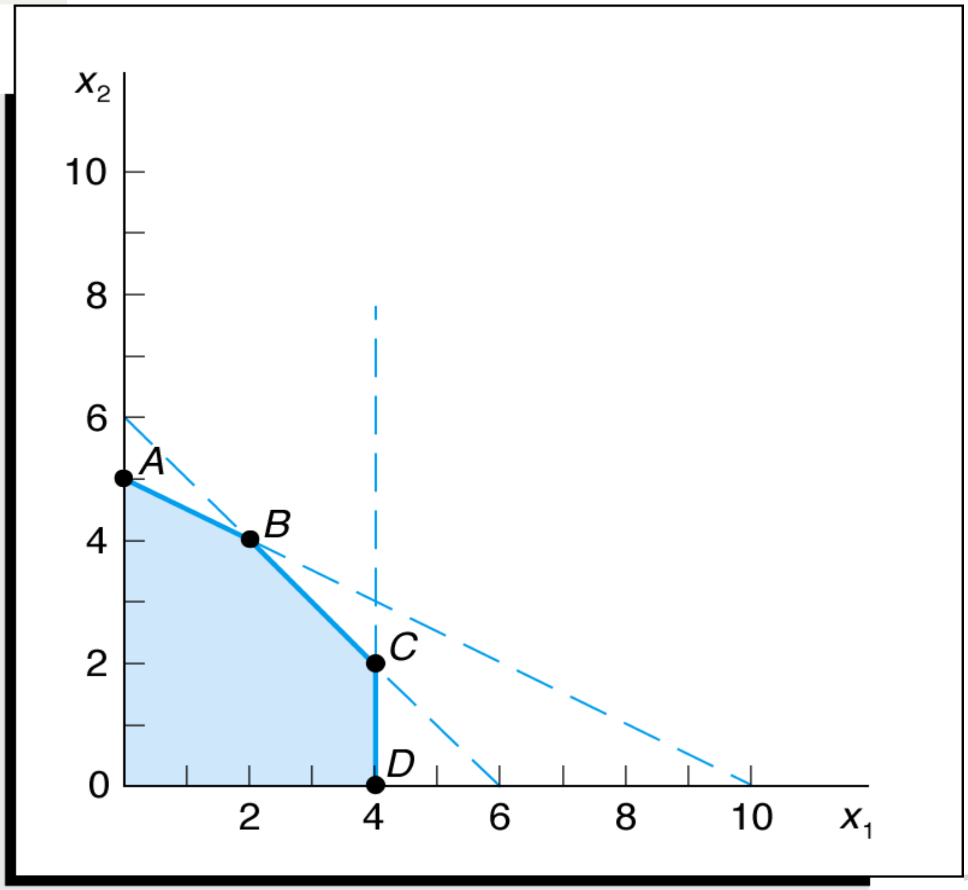


Figure 2.24 Feasible solution space and extreme points

Example Problem No. 2 (3 of 3)

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution

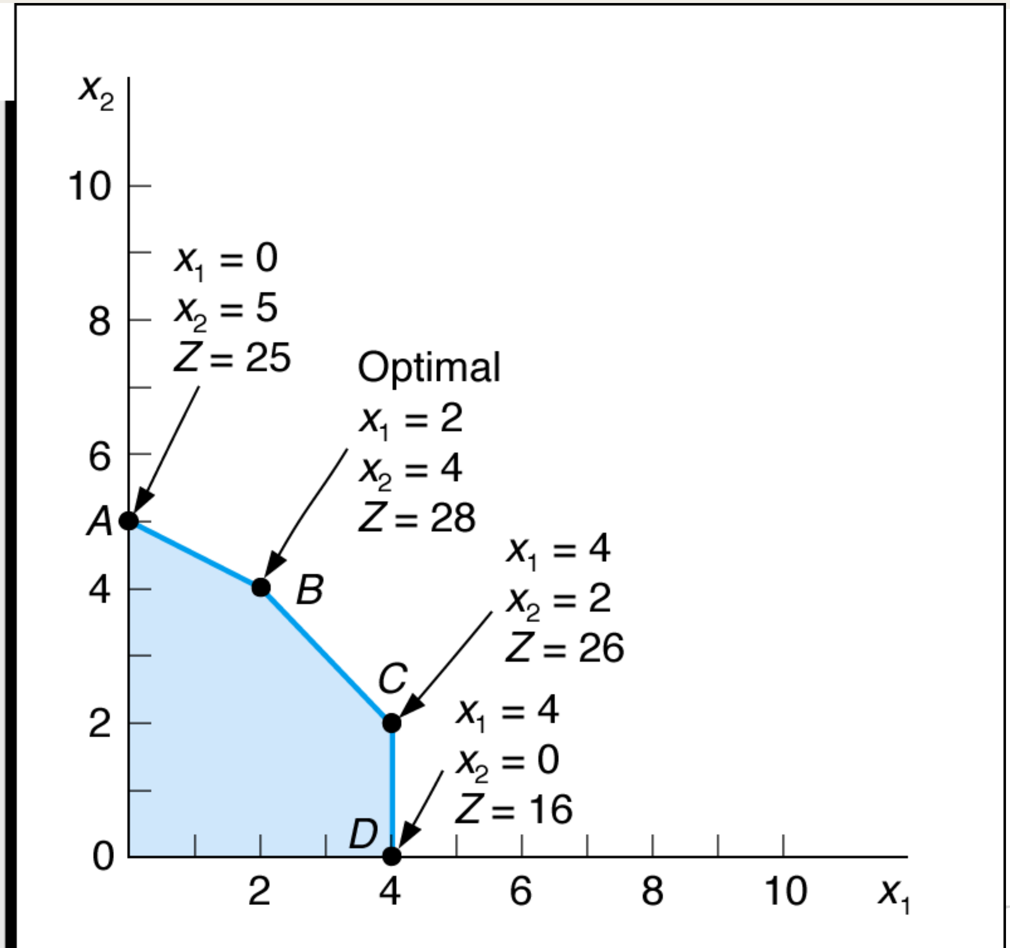


Figure 2.25 Optimal solution point