LINEAR PROGRAMMING PROBLEM (LPP)

Progamming techniques are widely used for decision making.

Programming techniques can be

Linear Programming and

Non linear programming

A model, which is used for optimum allocation of scare or limited resource to competing products or activities under such assumptions as certainty, linearity, fixed technology and constant profit per unit, is linear programming.

LPP is one of the **most versatile, powerful and useful technique** for making **managerial decision making**. It can be used for solving broad range of problems arising in industries, business, government, hospitals etc.

For solving the problem it is necessary to

Formulate the problem (Problem formulation) and

Solving the formulated by using suitable method such as graphical method, Simplex method

Properties of LPP Model/Problem

Any LPP model or problem must have the following properties

i. The relationship between variables and constraints must be **linear.**
ii. The model must have an **objective function.**
iii. The model must have structural **constraints.**
iv. The model must have **non-negativity constraints.**

While formulating the problem as LPP

i. LPP will be will be essentially having an objective function and set of constraints
ii. Set of constraints will be of two types-linear structural constraints and non-negativity constraints
iii. Define the decision variables x_1, x_2, x_3... which are contributing to the decision or objective function such as number of products of each type A and B or X and Y to manufacture so as to maximize the profit.
iv. Objective function may be of either minimization or maximization type. It will be of minimization type when the decision is concerned with time and cost, where as it will be of maximization when the decision is concerned with profit, efficiency and effectiveness.

v. **Express the given information/content in the problem in the tabular form.**

vi. **State the objective function using the defined variables such as Minimum $z = 3x_1 + 4x_2$ or Maximum $z = 2x_1 + 3x_2$.**

vii. **The constraints can be of three types less than equal to (≤), greater than equal to (≥) or equal to (=). Constraints describe the availability of resources or requirement of decision variables.**

- ✓ **When the resources are limited or restricted for example available resources are limited to 200 units, the capacity is limited to 50 hours etc; then the relationship among the decision variables will be in the form of less than equal to (≤).**
- ✓ **When there is minimum requirement or demand for the particular decision variable; then the relationship among the decision variables will be in the form of greater than equal to (≥).**
- ✓ **If conditions such as minimum requirement or capacities are not described in the problem then the constraint may be equal to (=) type.**

viii. **Finally, end up the formulation by introducing non-negativity constraints for the decision variables defined for example- $x_1, x_2 \geq 0$, it means that it is not possible to have negative values of decision variables defined.**

Let us consider the following problem

PROBLEM 1

A company manufactures two products X and Y, which require the following resources. The resources are the capacities of machines M_1, M_2 and M_3. The available capacities are 50, 25 and 15 hours respectively. Product X requires 1 hour of machine M_2 and 1 hour of machine M_3. Product Y requires 2 hours of machine $M_1$and 1 hour of machine M_3. The profit contribution of products X and Y are Rs 5 and Rs 4 respectively. Find out the number of products of X and Y to be manufactured so as to maximize the total profit. Formulate the problem as LPP.

Solution

Here the decision is concerned with maximization of profit; hence the objective function will be of maximization type.

Let x_1 and x_2 be the number of units of product X and Y respectively. (Defining the decision variables)

The total profit is calculated by the number of products of X and Y and their contribution (profit/unit). As profit contribution of products X and Y are Rs 5 and Rs 4 respectively. Therefore, the **objective function** will be

$$\text{Maximum } z = 5x_1 + 4x_2$$

The content in the problem can be expressed in the tabular form as below

Machines	*Products*		*Availability in hours*
	X	*Y*	
M_1	0	2	50
M_2	1	2	25
M_3	1	1	15
Profit in Rs. Per unit	5	4	

As information is concerned with availability of the resources (hours of each machine), all the structural constraints will be of less than equal to type (≤) and can be written as below.

$$0x_1 + 2x_2 \leq 50$$

$$x_1 + 2x_2 \leq 25$$

$$x_1 + x_2 \leq 15$$

The non-negativity constraint can be written as

$$x_1 , x_2 \geq 0$$

Hence the problem formulation can be

$$\textbf{Maximum z} = 5x_1 + 4x_2$$

Subjected to the constraints

$$0x_1 + 2x_2 \leq 50$$

$$x_1 + 2x_2 \leq 25$$

$$x_1 + x_2 \leq 15$$

$$x_1 , x_2 \geq 0$$

PROBLEM 2

A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity however is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs 2 per unit and type B a profit of Rs 5 per unit. How many of each type the store should stock per week to maximize the total profit? Formulate the problem as LPP**.**

Solution

Let x_1 and x_2 be the number of units of type A and B shirts respectively.

As type A shirt fetches a profit of Rs 2 per unit and type B a profit of Rs 5 per unit, the objective function will be

$$\textbf{Maximum } \mathbf{z} = 2x_1 + 5x_2$$

Subjected to constraints

$x_1 \leq 400$ Sales constraint

$x_2 \leq 300$ Sales constraint

$x_1 + x_2 \leq 600$ Storage capacity constraint

$x_1 , x_2 \geq 0$ Non negativity constraints

PROBLEM 3

A patient consulted a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and Y, which are having vitamin A and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin daily. The cost of tonics X and Y and proportion of vitamin A and D that present in X and Y are given in the table below.

Vitamins	Tonics		Daily requirement In units
	X	Y	
A	2	4	40
D	3	2	50
Cost (Rs/unit)	5	3	

Formulate LPP to minimize the cost of tonics.

Solution

Let x_1 and x_2 be the number of units of tonic X and tonic Y respectively patient purchases

$$\textbf{Minimize } \mathbf{z} = 5x_1 + 3x_2$$

Subjected to constraints

$$x_1 + x_2 \geq 40$$

$$x_1 + x_2 \geq 50$$

$$x_1 , x_2 \geq 0$$

PROBLEM 4

A company owns two flour mills viz. A and B which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs 2000 and Rs 1500 per day to run mill A and B respectively. On a day, mill A produces 6, 2 and 4 quintals of high, medium and low quality flour, mill B produces 2, 4 and 12 quintals of high, medium and low flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

Solution

	Mill		Requirement
	Mill A	Mill B	
High	6	2	8
Medium	2	4	12
Low	4	12	24
Operating cost/day	2000	1500	

Let x_1, and x_2 be the number of days per month mill A and Mill B is operated Y respectively

Minimize z = $2000x_1 + 1500x_2$

Subjected to constraints

$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

PROBLEM 5

A farmer has Rs 80 and space for 20 hens. The cost of feeding per hen is Re 1 per week. The purchasing of young hen is Rs 5 and it lays 5 eggs per week, whereas the old hen can be bought for Rs 2 and it lays 3 eggs per week. If an egg is worth for 30 paise, how many hens of each type a farmer should purchase to maximize his profit?

Solution

Let x_1, and x_2 be the number of hens of young and old type respectively a farmer purchases

Profit= (0.30 X number of eggs –cost of feeding) X No. of hens

Maximize z = $(0.3 \text{ X } 5 - 1)\, x_1 + (0.3 \text{ X } 3 - 1)\, x_2$

Maximize z = $0.5\, x_1 - 0.1x_2$

Subjected to constraints

$$x_1 + x_2 \leq 20$$

$$5x_1 + 2x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

PROBLEM 6

A company produces two types of belts- Belt A and Belt B. Belt A is superior to belt B and requires twice time as much belt B. If all the belts of type B are produced then a company can produce 1500 belts. The leather supply is limited to produce 1000 belts and each type of belt requires same amount of leather. Belt A requires a fancy buckle for which supply is limited to 400 buckles per day. The profit per unit for belt A and belt B is Rs 5 and Rs 4 respectively. How many belts of each type a company can manufacture so as to maximize the total profit. Formulate the problem as LPP.

Solution

Let x_1 and x_2 be the number of belts of type A and type B manufactured respectively

Maximize $z = 5x_1 + 4x_2$

Subjected to constraints

$$2x_1 + x_2 \leq 1500$$

$$x_1 + x_2 \leq 1000$$

$$x_1 \leq 400$$

$$x_1, x_2 \geq 0$$

PROBLEM 7

A ship has three cargo holds, forward, aft and centre. The capacity limits are

Forward- 2000 tons, 1,00,000 cubic meters

Centre - 3000 tons, 1,35,000 cubic meters

Aft - 1500 tons, 30,000 cubic meters

The following cargoes are offered, the ship owners may accept all or any part of each commodity:

Commodity	Amount (tons)	Volume/ton (cubic meters)	Profit/ton (Rs)
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate the problem as LPP.

Let x_1, x_2 and x_3 be the tons of commodities of type A, B and C respectively.

Maximum z = $60x_1 + 80x_2 + 50\ x_3$

The weight constraint is

$$6000x_1 + 4000x_2 + 2000x_3 \le 6500$$

The tonnage of commodity A is 6000 and each tone occupies of 60 cubic meters, hence there are 100 cubic meters capacity is available. Similarly cubic capacity for commodity B and C will be 80 cubic meters.

Hence capacity constraint will be

$$100x_1 + 80x_2 + 80x_3 \le 6500$$

Hence the formulation will be

Maximum z = $60x_1 + 80x_2 + 50\ x_3$

Subjected to constraints

$$6000x_1 + 4000x_2 + 2000x_3 \le 6500$$

$$100x_1 + 80x_2 + 80x_3 \le 6500$$

$$x_1\ ,\ x_2\ ,x_3 \ge 0$$

General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for $x_1, x_2 \dots x_n$ so as to maximize or minimize

$$Z = c_1x_1 + c_2x_2 + \dots\dots\dots\dots + c_nx_n$$

subject to restrictions

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots + a_1nx_n\ (\le \text{ or } \ge)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots + a_2nx_n\ (\le \text{ or } \ge)\ b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n\ (\le \text{ or } \ge)\ b_m$$

and

$$x_1 \ge 0,\ x_2 \ge 0, \dots,\ x_n \ge 0$$

Where

Z = value of overall measure of performance
x_j = level of activity (for j = 1, 2, ..., n)
c_j = increase in Z that would result from each unit increase in level of activity j
b_i = amount of resource i that is available for allocation to activities (for i = 1,2, …, m)
a_{ij} = amount of resource i consumed by each unit of activity j

Standard form of LPP

In the standard form of LPP

I. The objective function is of maximization or minimization type
II. All the constraints are expressed in the form of equations using slack or surplus variables
III. The right hand side of each constraint is non-negative.

- It is the LP model with the specific form of the **constraints**:

$$\begin{aligned} \text{max (or min)} \quad & z = c_1x_1 + c_2x_2 + \cdots c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \end{aligned}$$

- m equalities and n nonnegativity constraints with $m \leq n$

Assumptions in Linear programming model

I. **Proportionality-** The rate of change (slope) fo the objective function and constraints equations is constant.

II. **Additivity-** Terms in the objective function and constraint equations must be additive.

III. **Divisibility-** Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.

IV. **Certainty-** Values of all the model parameters are assumed to be known with certainty (non-probabilistic)

Limitations of Linear Programming model

I. Linear programming is applicable only to problems where the constraints and objective function are linear i. e. where they can be expressed as equations which represent straight lines. In real life situations, when constraints or objective functions are not linear, this technique cannot be used.

II. Factors such as uncertainty and time are not taken into consideration.

III. Parameters in the model are assumed to be constant but in real life situations they are not constants.

IV. Linear programming deals with only single objective, whereas in real life situation may have multiple and conflicting objectives.

V. In solving a LPP there is no guarantee that we get an integer value. In some cases of no of men/machine a non integer value is meaningless.

Theory questions

1. **State the general form of LPP.**
2. **State the standard form of LPP.**
3. **State assumptions in LPP model.**
4. **State limitations of LPP model.**