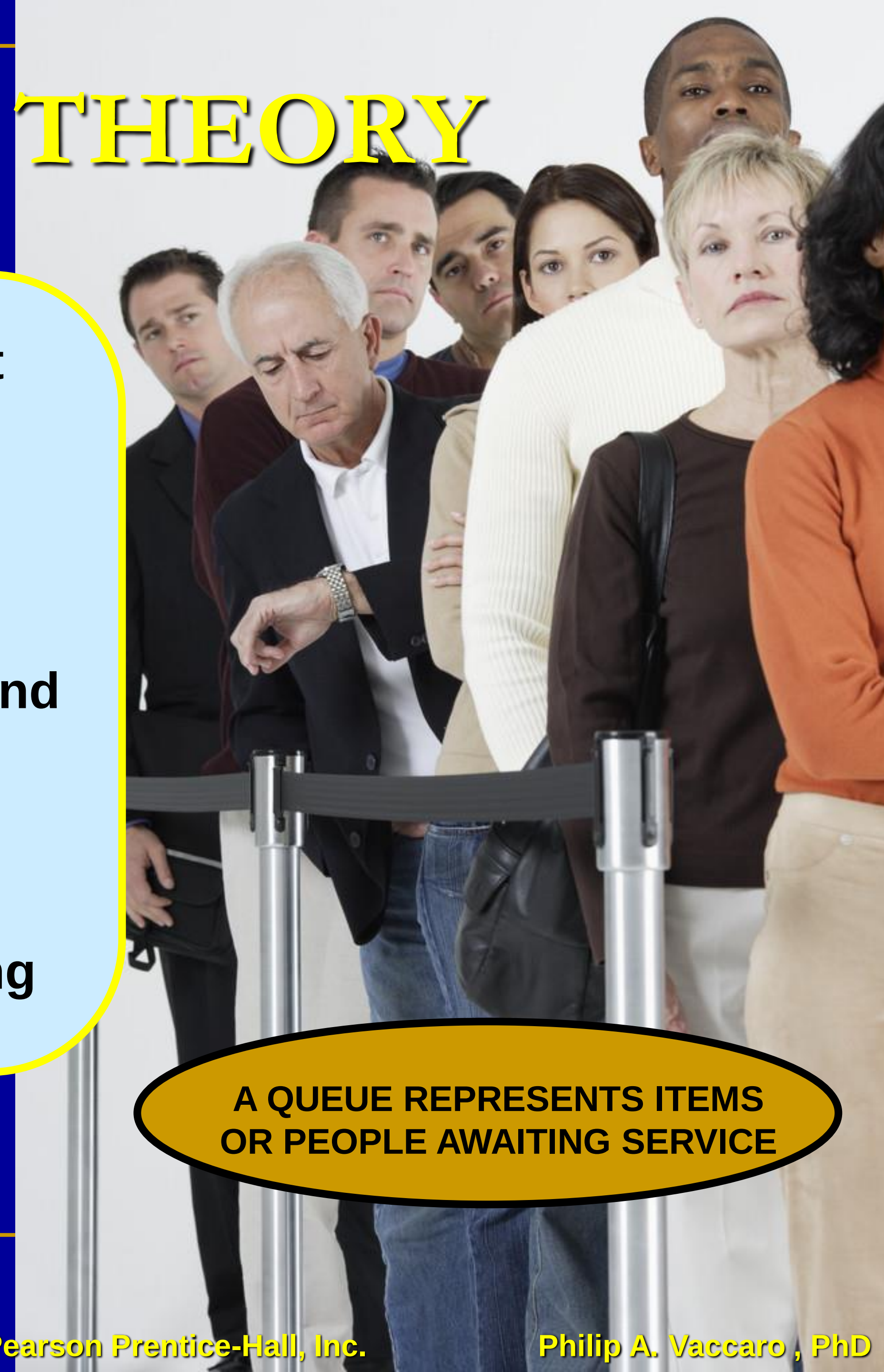
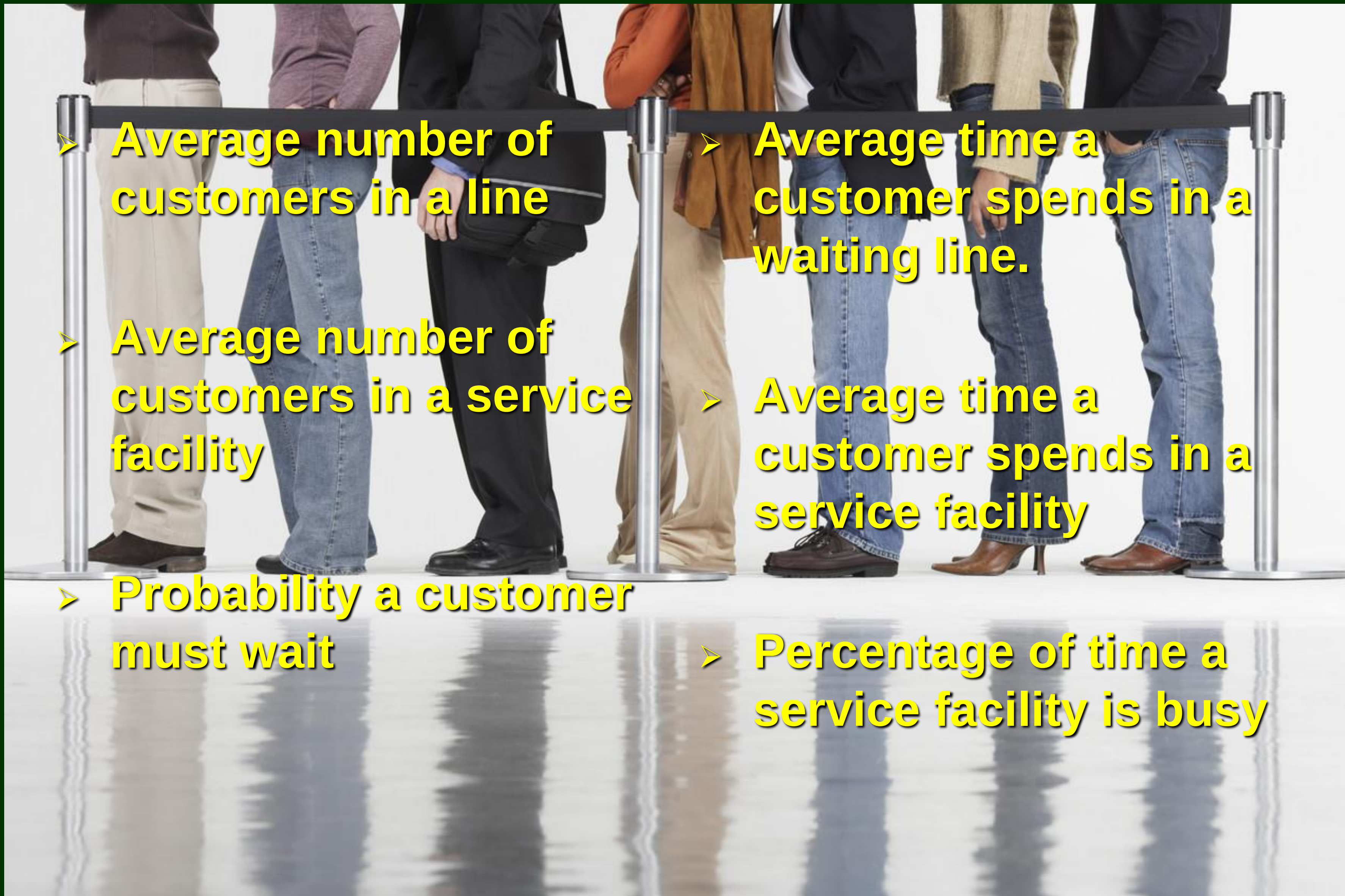
QUEUING THEORY

- Body of knowledge about waiting lines
- Helps managers to better understand systems in manufacturing, service, and maintenance
- Provides competitive advantage and cost saving



Queue Characteristics



- Average number of customers in a line
- Average number of customers in a service facility
- Probability a customer must wait
- Average time a customer spends in a waiting line.
- Average time a customer spends in a service facility
- Percentage of time a service facility is busy

Queuing System Examples

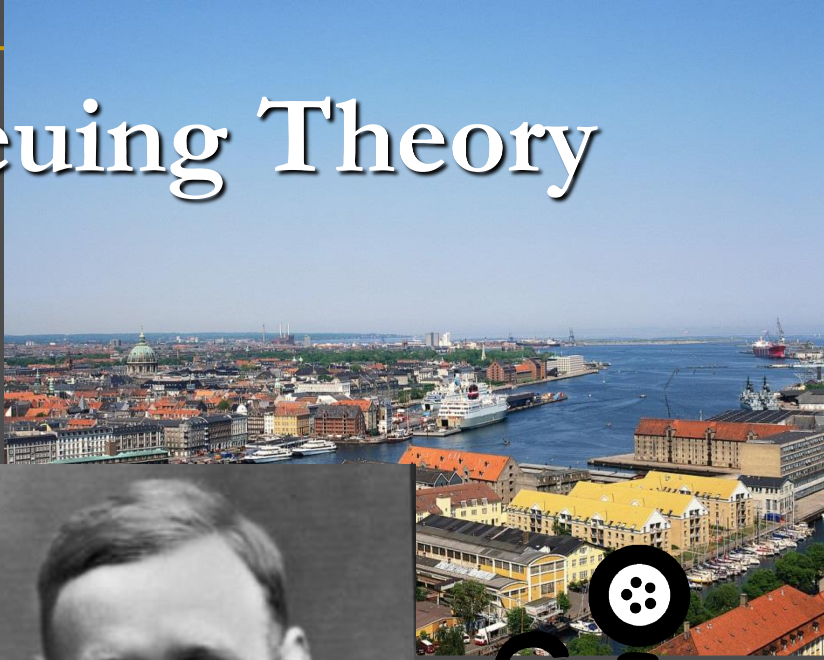
System	Customers	Servers
Grocery Store	Shoppers	Checkout Clerks
Phone System	Phone Calls	Switching Equipment
Toll Highway	Vehicles	Tollgate
Restaurant	Parties of Diners	Tables & Waitstaff
Factory	Products	Workers

The Father of Queuing Theory

Danish engineer, who, in 1909 experimented with fluctuating demand in telephone traffic in Copenhagen.

In 1917, he published a report addressing the delays in automatic telephone dialing equipment.

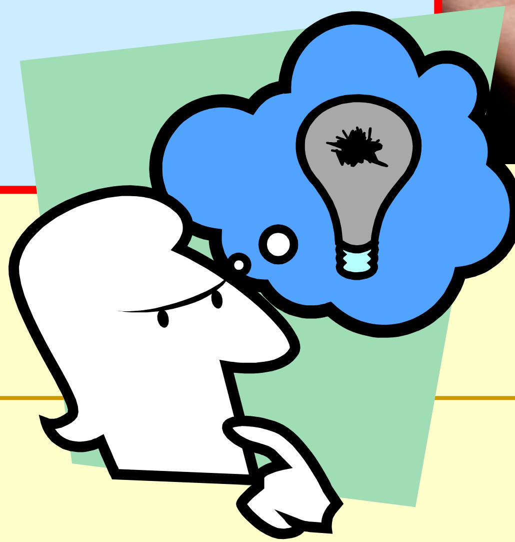
At the end of World War II, his work was extended to more general problems, including waiting lines in business.



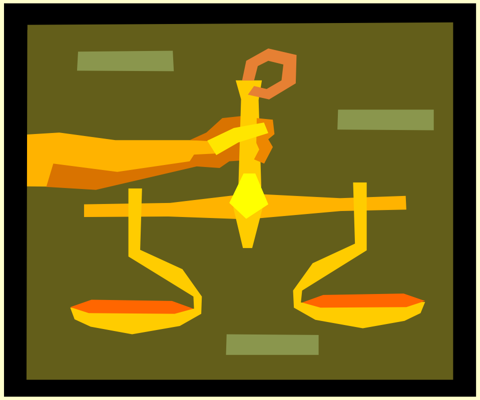
AGNER K. ERLANG

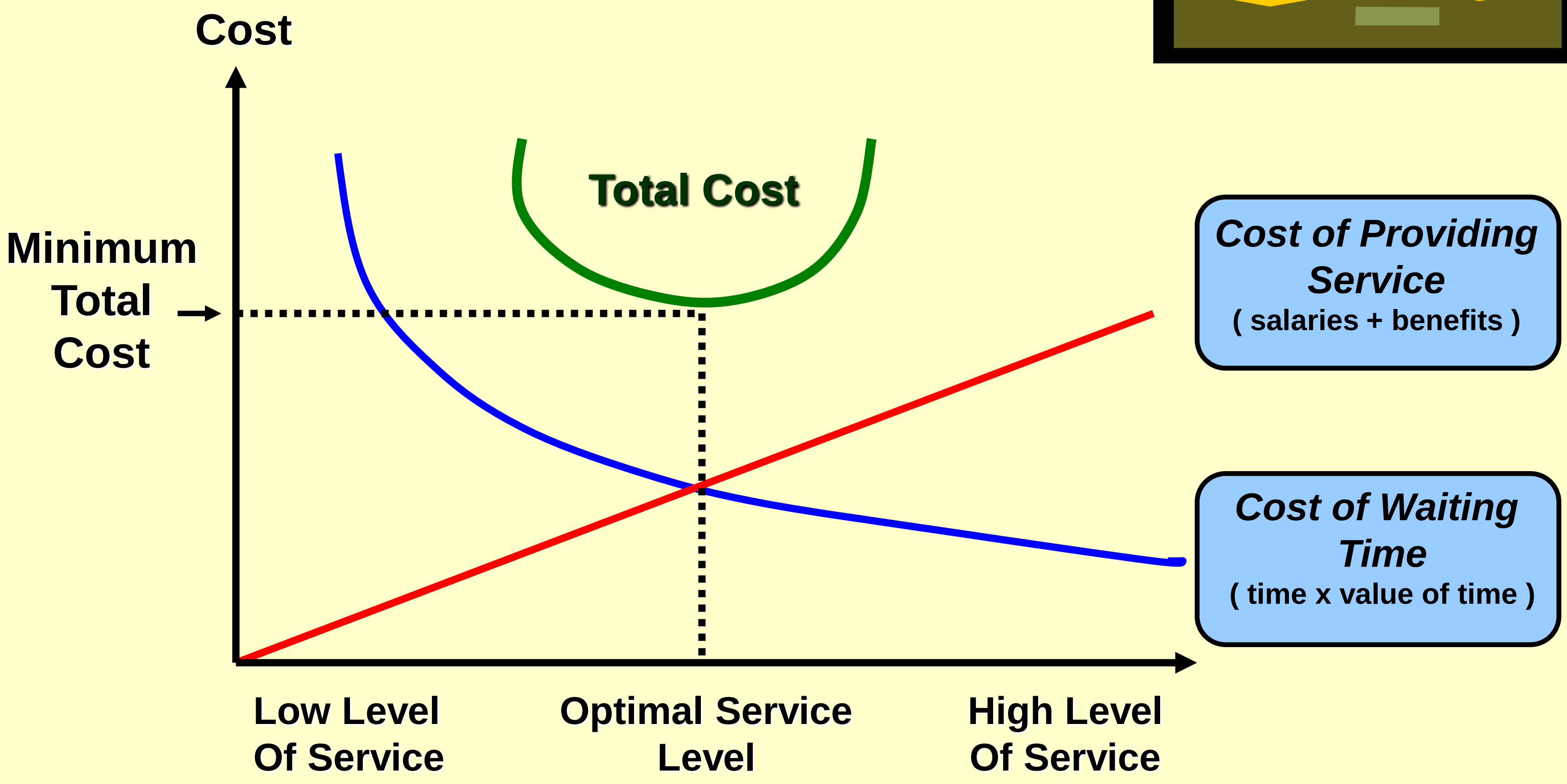
Lack of Managerial Intuition Surrounding Waiting Lines

Queuing theory is not a matter of common sense. It is one of those applications where diligent, intelligent managers will arrive at drastically wrong solutions if they fail to thoroughly appreciate and understand the mathematics involved.



THE QUEUING COST TRADE-OFF





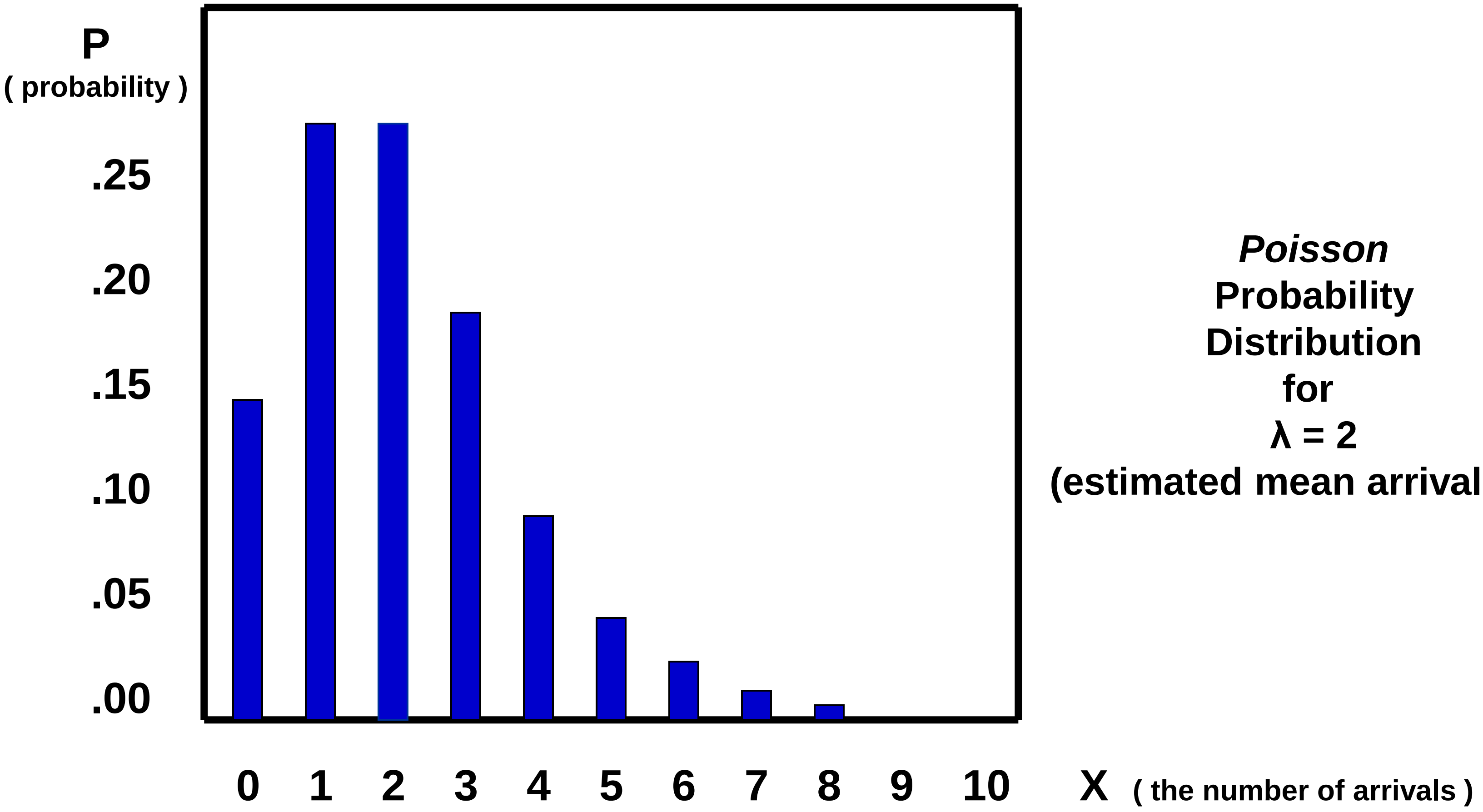
Aspects of a Queuing Process

- SYSTEM ARRIVALS
- THE QUEUE ITSELF
- THE SERVICE FACILITY

The Calling Population

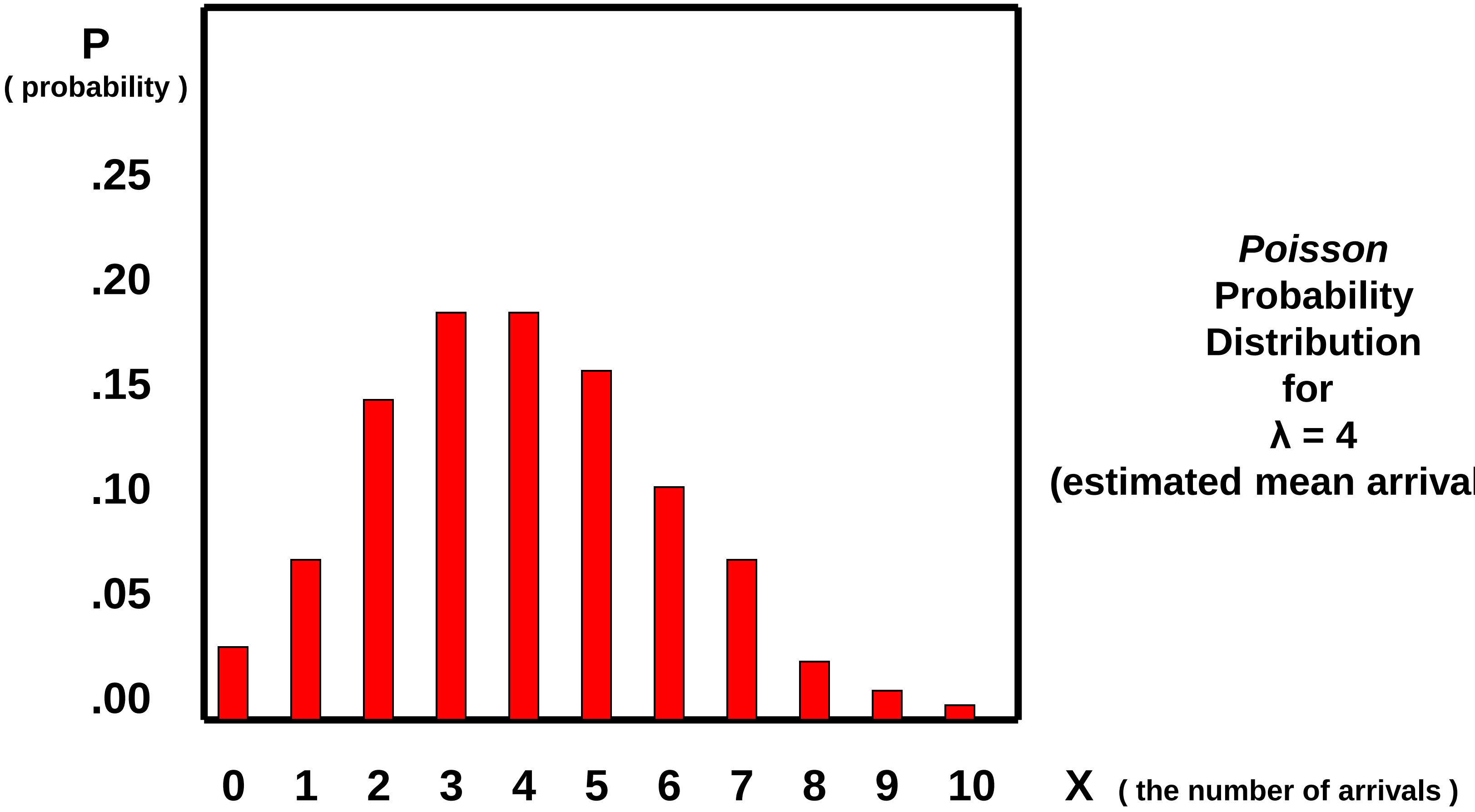
- The source of all system arrivals
- It is usually of infinite size
- Theoretically, any person or object can enter the service facility during operating hours

Poisson Arrival Distribution



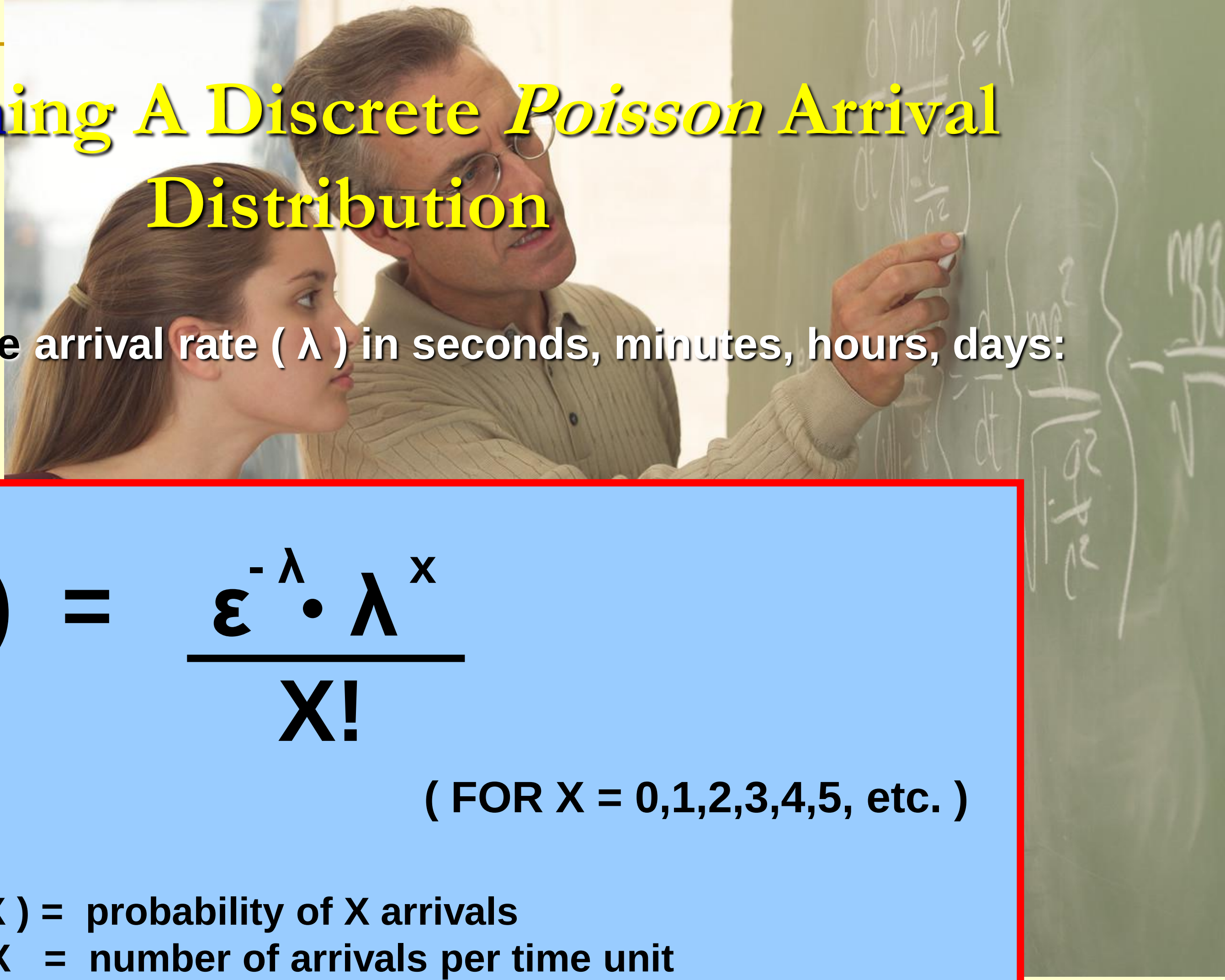
Poisson
Probability
Distribution
for
$\lambda = 2$
(estimated mean arrival rate)

Poisson Arrival Distribution



Poisson
Probability
Distribution
for
$\lambda = 4$
(estimated mean arrival rate)

Establishing A Discrete *Poisson* Arrival Distribution



Given any average arrival rate (λ) in seconds, minutes, hours, days:

$$P (X) = \frac{\varepsilon^{-\lambda} \cdot \lambda^{x}}{X!}$$

(FOR X = 0,1,2,3,4,5, etc.)

Where : P (X) = probability of X arrivals
X = number of arrivals per time unit
λ = the average arrival rate
ε = 2.7183 (base of the natural logarithm)

EXAMPLE

If the average arrival rate per hour is two people ($\lambda = 2$) , what is the probability of three (3) arrivals per hour?

Solution

$$P(X) = \frac{\varepsilon^{-\lambda}\ \lambda^{X}}{X\,!}$$

Given λ = 2 :

$$P(3) = \frac{2.7183^{-2} \bullet 2^{3}}{3\,!}$$

$$= \frac{[\,1 / 7.389\,] \times 8}{(3)(2)(1)}$$

$$= \frac{.1353 \times 8}{6} = .1804 \approx 18\%$$

The Remaining Probabilities

GIVEN THAT λ = 2

P (0 arrivals) = 14%
P (1 arrival) = 28%
P (2 arrivals) = 28%
P (3 arrivals) = 18%
P (4 arrivals) = 9%
P (5 arrivals) = 4%
P (6 arrivals) = 2%
P (7 arrivals) = 1%
P (8 arrivals) = .8%
P (9 arrivals) = .6%
P (=> 10 ") = 0%

Poisson Probability Table

For a given value of λ , entry indicates the probability of obtaining a specified value of ‘X’

X	λ = 1.8	λ = 1.9	λ = 2.0
0	.1653	.1496	.1353
1	.2975	.2842	.2707
2	.2678	.2700	.2707
3	.1607	.1710	**.1804**
4	.0723	.0812	.0902
5	.0260	.0309	.0361
6	.0078	.0098	.0120
7	.0020	.0027	.0034
8	.0005	.0006	.0009
9	.0001	.0001	.0002

EXAMPLE

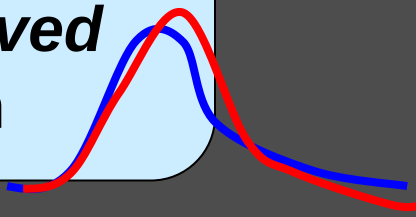
Precise Terminology

Theoretical Distribution	Observed Distribution
The discrete arrival probability distribution, based on the average arrival rate (λ) which was computed from the actual system observations	The actual discrete arrival probability distribution that was constructed from the actual system observations

THIS DISTRIBUTION MAY OR MAY NOT BE POISSON DISTRIBUTED.

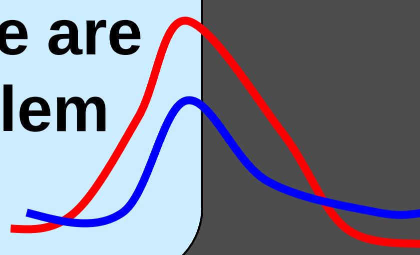
CAUTION

The *theoretical poisson* arrival probability distribution must be statistically identical to the *observed* arrival probability distribution

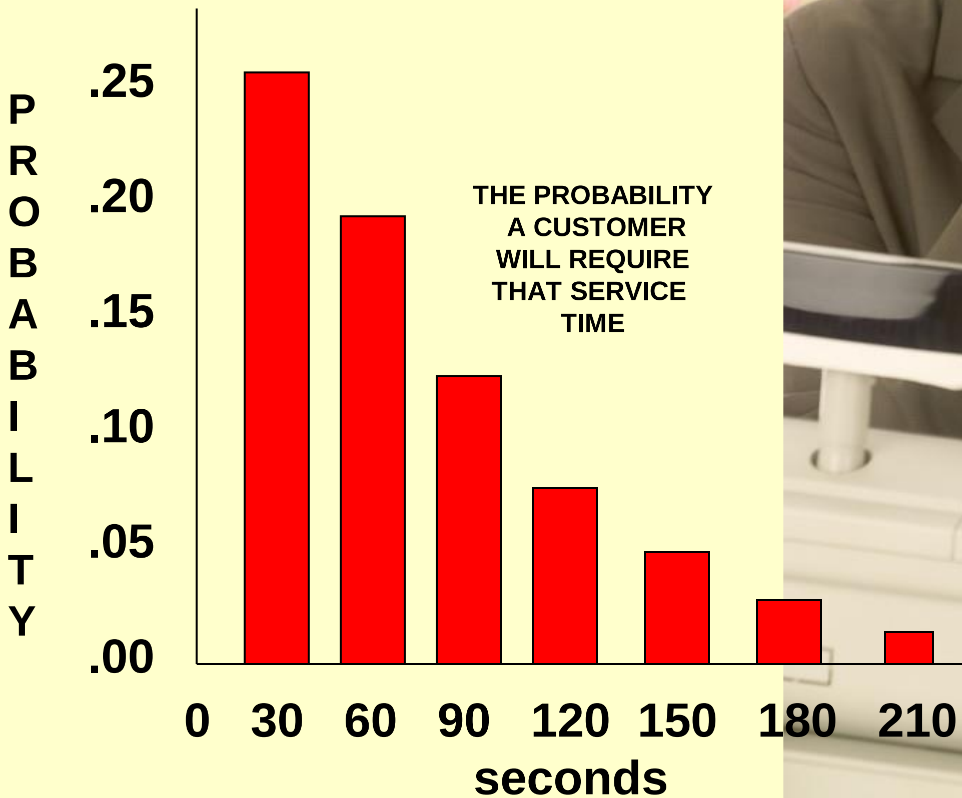


THIS CAN BE ESTABLISHED BY A *GOODNESS – OF - FIT* HYPOTHESIS TEST

If the two probability distributions are not found to be statistically identical, we are forced to study and solve the problem via simulation modeling



Service Times



Service times normally follow a negative exponential probability distribution

Queue Discipline

- ***Balking***, ***reneging***, **and** ***jockeying*** **are not permitted in the service system.**

- ***Jockeying*** **is the switching from one waiting line to another.**

JOCKEYING CAN BE DISCOURAGED BY PLACING BARRICADES SUCH AS MAGAZINE RACKS AND IMPULSE ITEM DISPLAYS BETWEEN WAITING LINES

Queuing Theory Variables

- *Lambda* (λ) is the average arrival rate of people or items into the service system.
- It can be expressed in seconds, minutes, hours, or days.
- From the Greek small letter “ L “.

Queuing Theory Variables

- *Mu* (μ) is the average service rate of the service system.
- It can be expressed as the number of people or items processed per second, minute, hour, or day.
- From the Greek small letter “ M “.

μ

Queuing Theory Variables

- *Rho* (ρ) is the % of time that the service facility is busy on the average.
- It is also known as the utilization rate.
- From the Greek small letter " R ".

"BUSY" IS DEFINED AS AT LEAST ONE PERSON OR ITEM IN THE SYSTEM

ρ

Queuing Theory Variables

- *Mu* (M) is a channel or service point in the service system.
- Examples are gasoline pumps, checkout counters, vending machines, bank teller windows.
- From the Greek large letter “ M “.

M

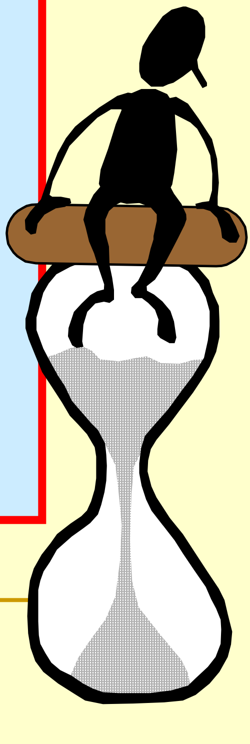
Queuing Theory Variables

- *Phases* are the number of service points that must be negotiated by a customer or item before leaving the service system.
- They have no symbol.

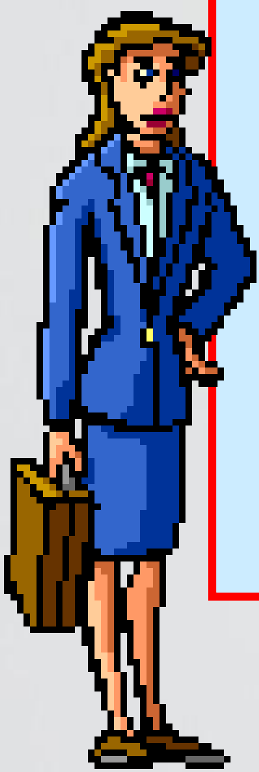
A CARWASH TAKES A VEHICLE THROUGH SEVERAL PHASES: PRE-WASH, WASH, WAX, AND DRY BEFORE IT IS ALLOWED TO LEAVE THE FACILITY.

Queuing Theory Variables

- *Po* or (*1 – ρ*) is the percentage of time that the service facility is idle.
- *L* is the average number of people or items in the service system both waiting to be served and currently being served.
- *Lq* is the average number of people or items in the waiting line (queue) only !



Queuing Theory Variables



- *W* is the average time a customer or item spends in the service system, both waiting and receiving service.

- *Wq* is the average time a customer or item spends in the waiting line (queue) only.

- *Pw* is the probability that a customer or item must wait to be served.

Queuing Theory Variables

“Mμ” is the effective service rate.*

The average number of customers or items processed by the entire service system

It can be expressed in seconds, minutes, hours, or days

*** [NUMBER OF SERVERS] x [AVERAGE SERVICE RATE PER SERVER]**

IMPORTANT CONSIDERATION

- The average service rate must always exceed the average arrival rate.

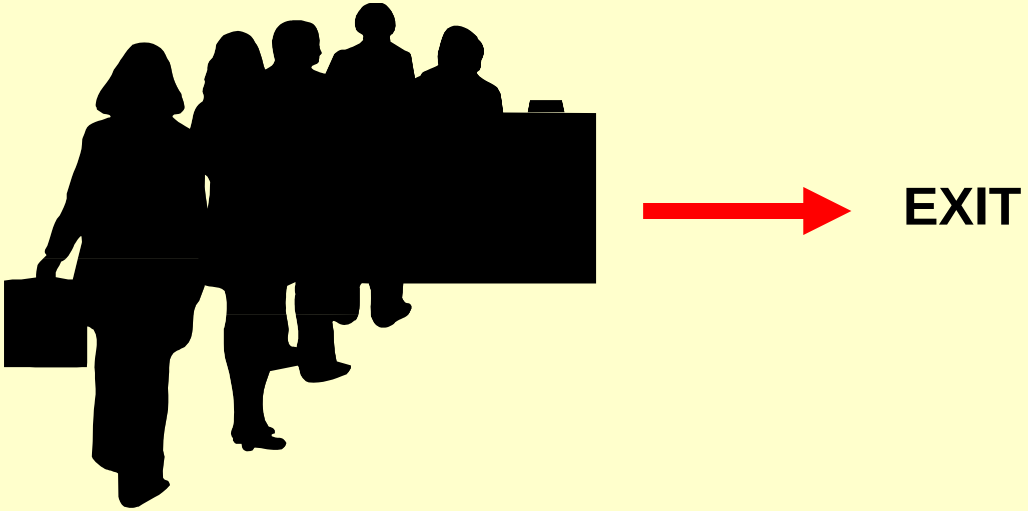
$$\mu > \lambda$$

- Otherwise, the queue will grow to infinity.

$$\infty$$

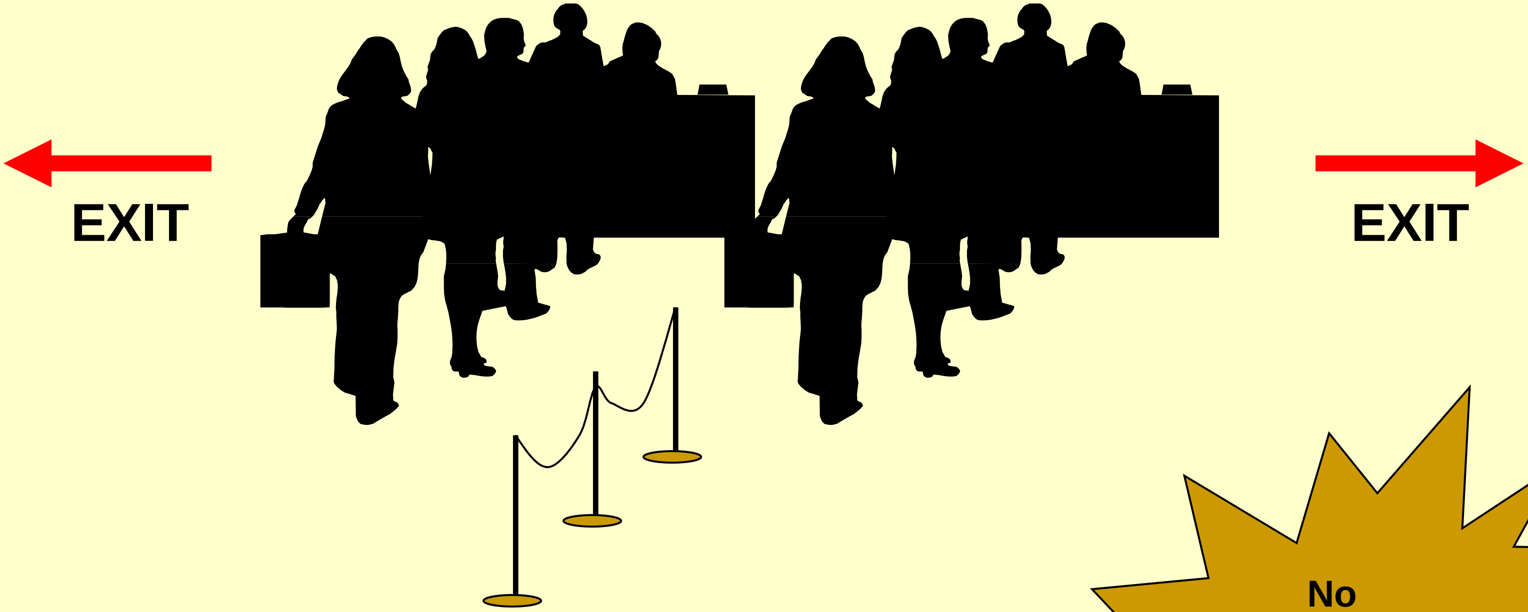
THERE WOULD BE NO SOLUTION !

Single-Channel / Single-Phase System



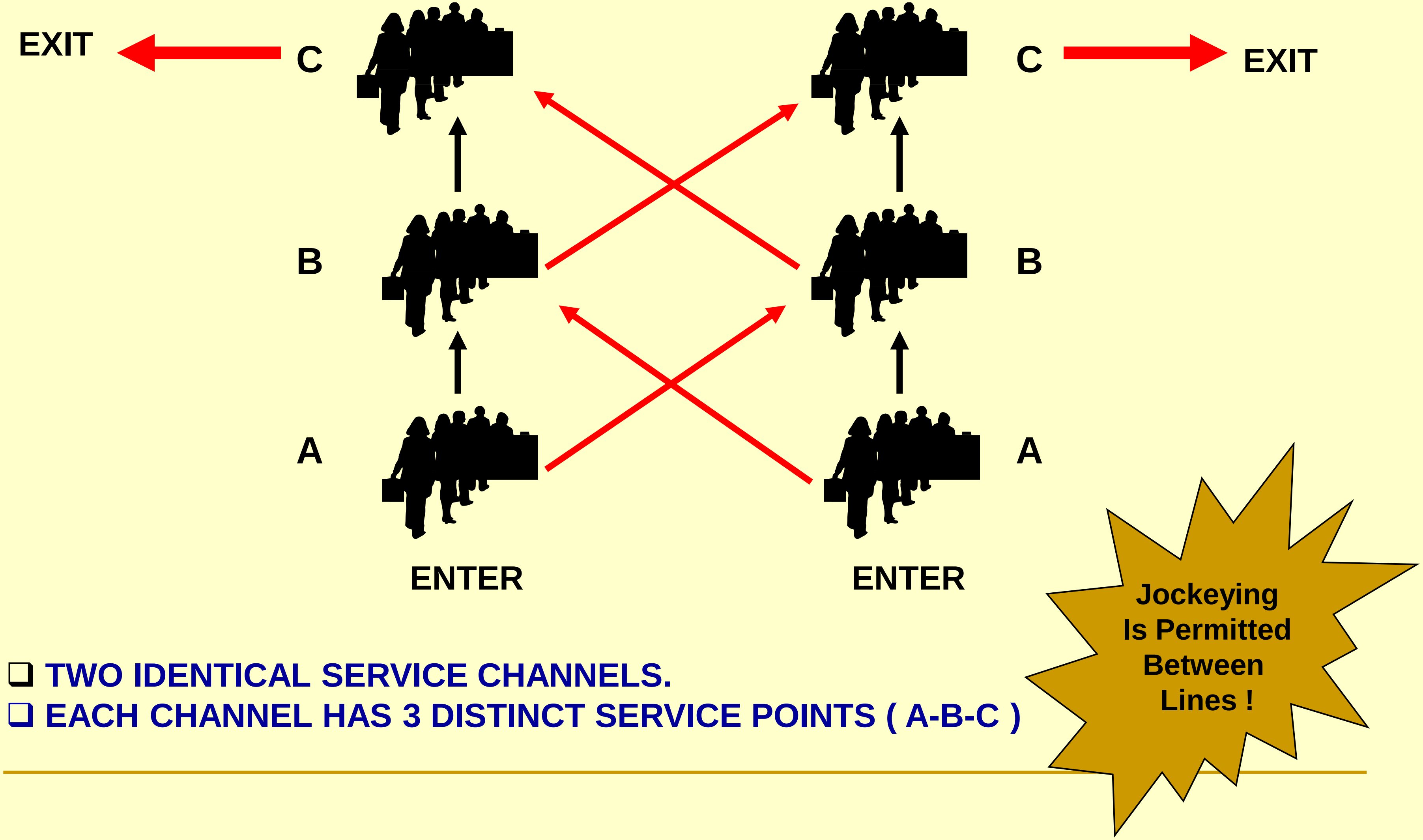
- ONE WAITING LINE or QUEUE
- ONE SERVICE POINT or CHANNEL

Dual-Channel / Single-Phase System



- ONE OR TWO WAITING LINES
- TWO DUPLICATE SERVICE POINTS

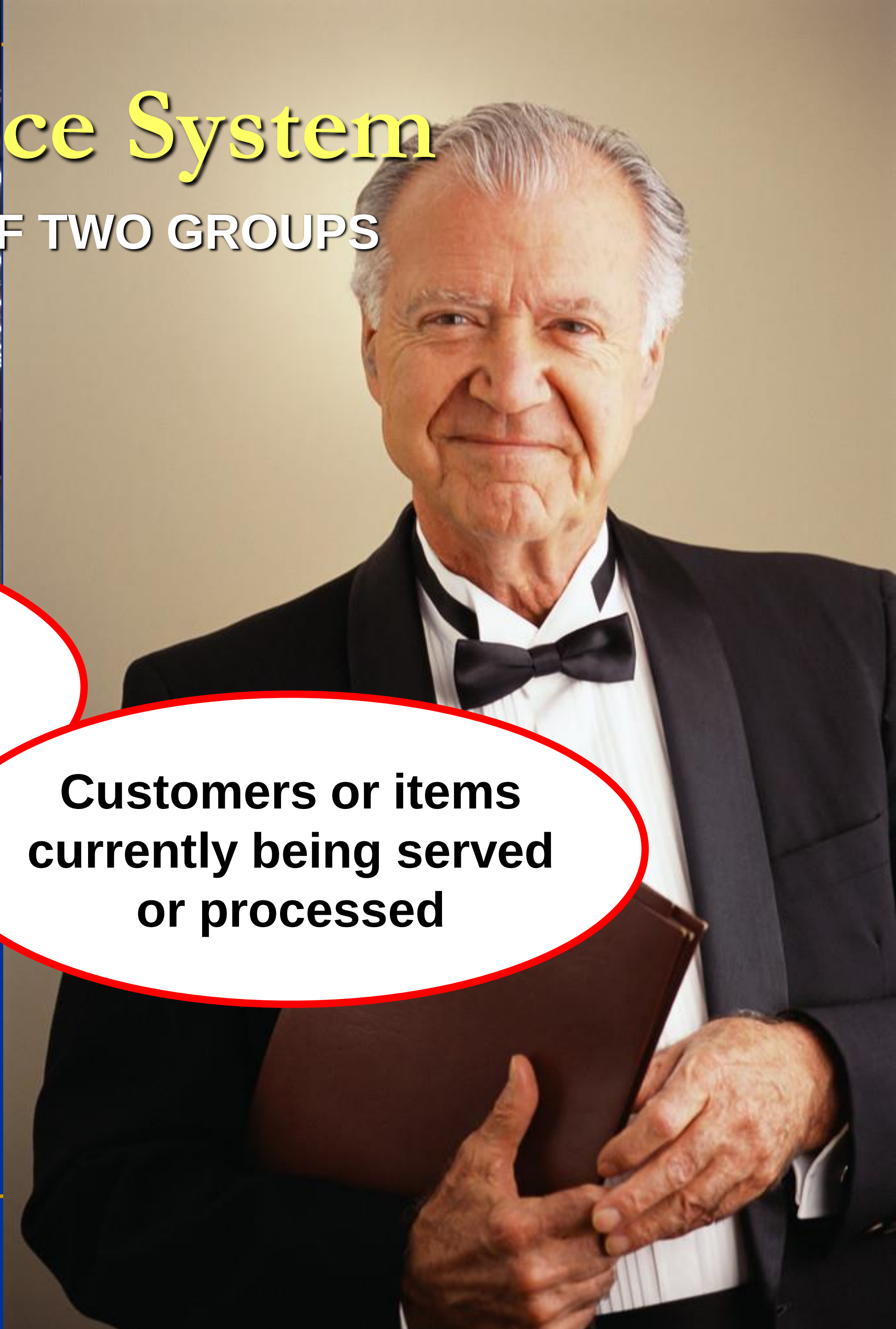
Dual-Channel / Triple-Phase System



- TWO IDENTICAL SERVICE CHANNELS.
- EACH CHANNEL HAS 3 DISTINCT SERVICE POINTS (A-B-C)

The Service System

COMPRISED OF TWO GROUPS



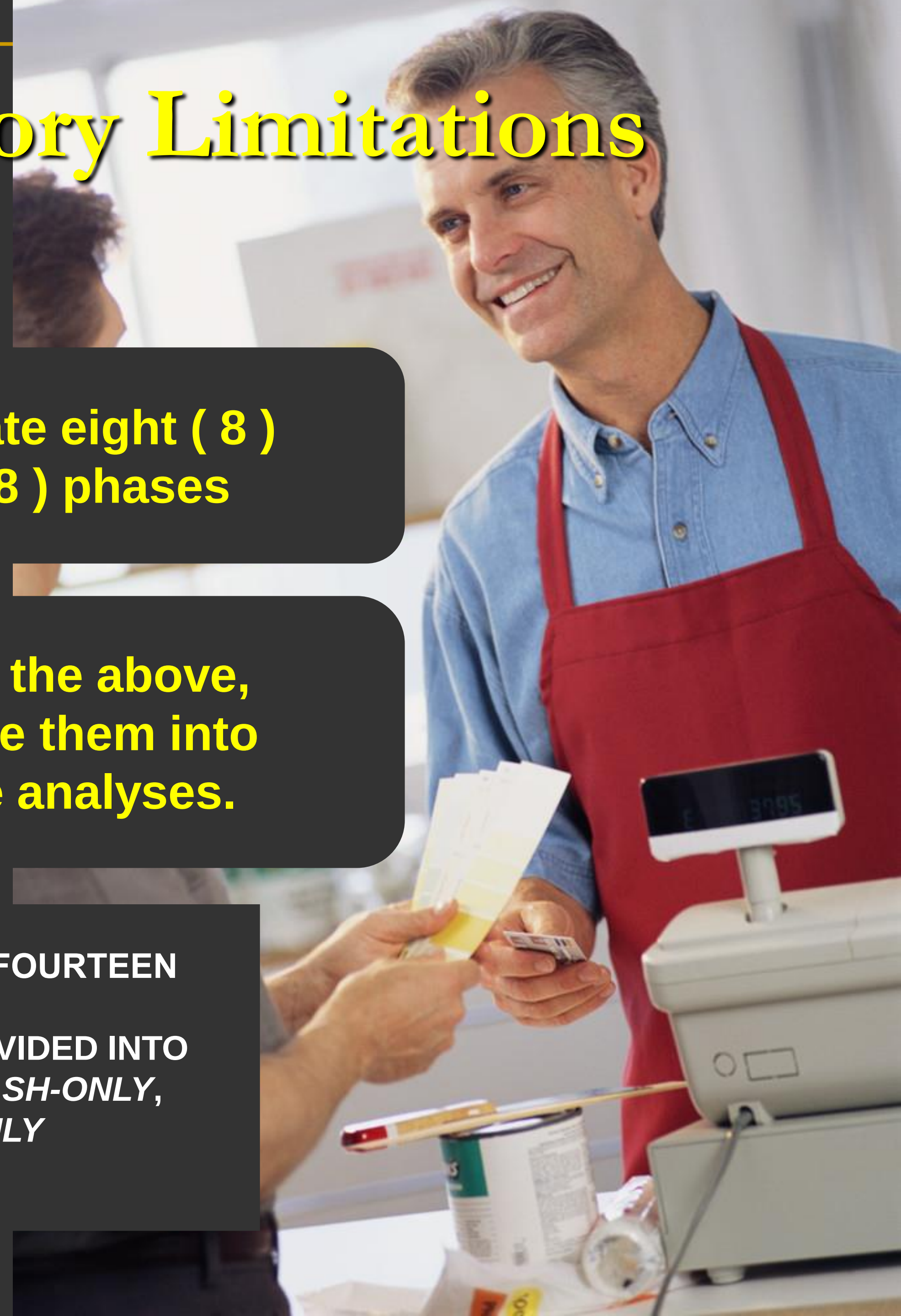
Customers or items waiting to be served or processed

Customers or items currently being served or processed

SUPERMARKET SHOPPERS ARE NOT IN THE SERVICE SYSTEM UNTIL THEY MOVE TO THE CHECKOUT AREA

RESTAURANT PATRONS ENTER THE SERVICE SYSTEM AS SOON AS THEY ARRIVE

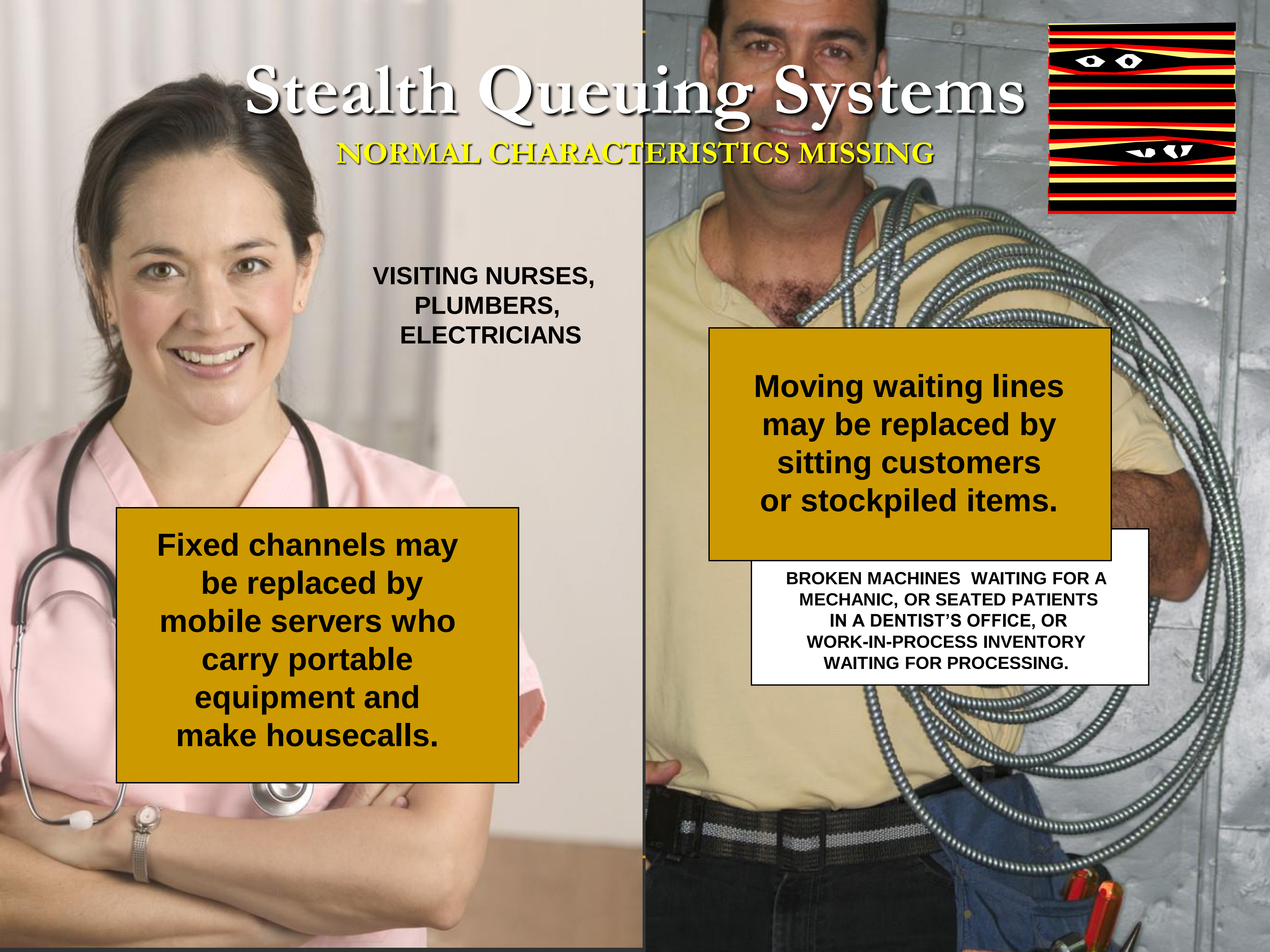
Queuing Theory Limitations



Formulae only accommodate eight (8) channels and / or eight (8) phases

If service systems exceed the above, it may be possible to divide them into sub-systems for separate analyses.

BJ's WHOLESALE CLUB HAS FOURTEEN (14) CHECKOUTS.
HOWEVER,THEY COULD BE DIVIDED INTO *CONTRACTOR*, *EXPRESS*, *CASH-ONLY*, AND *CREDIT-CARD-ONLY* SUBSYSTEMS.


Stealth Queuing Systems
NORMAL CHARACTERISTICS MISSING
VISITING NURSES,
PLUMBERS,
ELECTRICIANS
Fixed channels may be replaced by mobile servers who carry portable equipment and make housecalls.
Moving waiting lines may be replaced by sitting customers or stockpiled items.
BROKEN MACHINES WAITING FOR A MECHANIC, OR SEATED PATIENTS IN A DENTIST'S OFFICE, OR WORK-IN-PROCESS INVENTORY WAITING FOR PROCESSING.

Behavioral Considerations

QUEUING THEORY

- Customer willingness to wait depends on what is perceived as reasonable.
- Waiting lines that are always moving are perceived as less painful.
- Customer willingness to wait is higher if they know that others are also waiting their turn.
- Customers should be permitted to perform the services that they can easily provide for themselves.

Behavioral Considerations

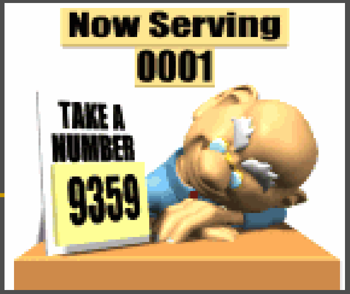
QUEUING THEORY

- Well projected waiting times allow customers to adjust their expectations and therefore their aggravation.
- If customers are kept busy, their waiting time may not be construed as wasted time.

FILLING OUT SURVEYS AND FORMS, BEING ENTERTAINED

- Customers should be rewarded with price discounts or gifts if they must wait beyond a certain period of time.

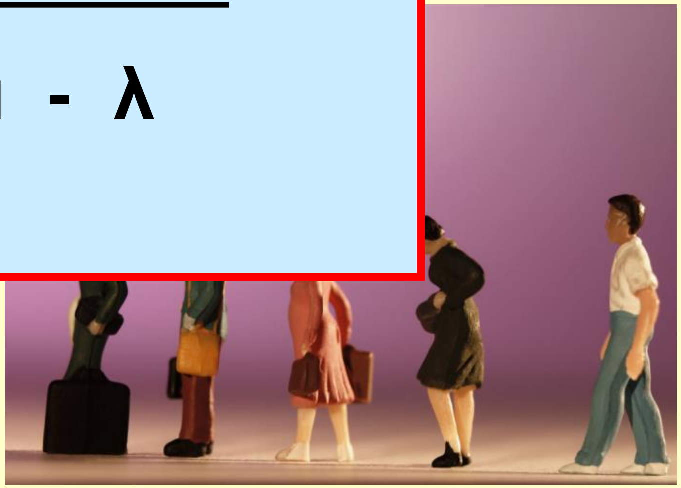
IT SHOWS THAT THE FIRM VALUES THEIR TIME AND IS WILLING TO PAY THEM FOR IT IF THE WAIT IS TOO LONG



Single-Channel / Single-Phase Model

The Average Number of Customers in the System

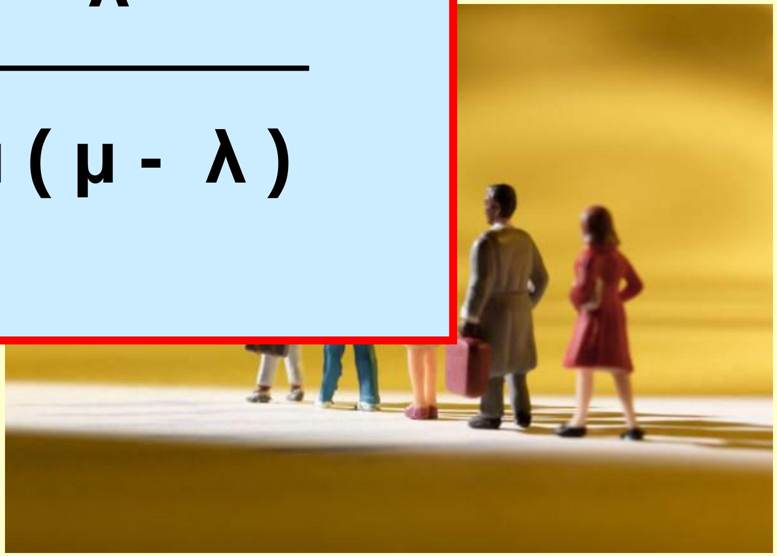
$$L = \frac{\lambda}{\mu - \lambda}$$



Single-Channel / Single-Phase Model

The Average Number Just Waiting in Line

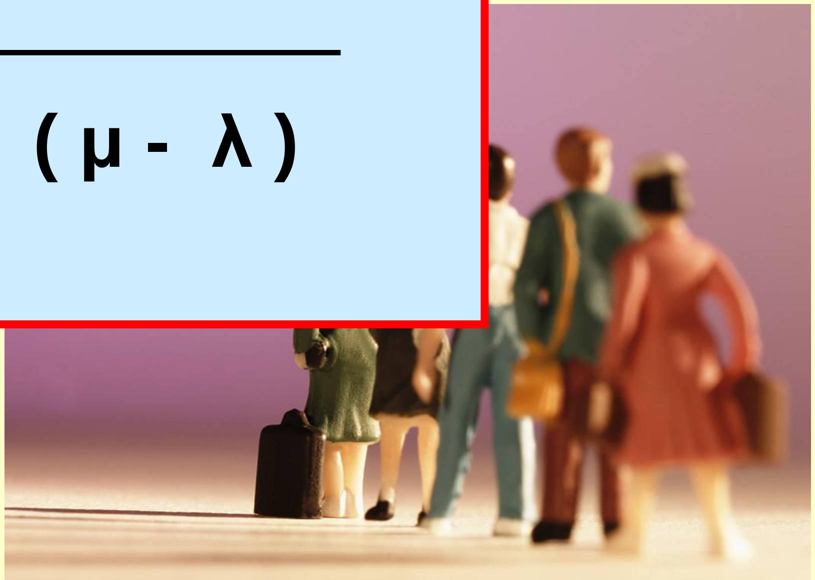
$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$



Single-Channel / Single-Phase Model

Average Customer Time Spent in the System

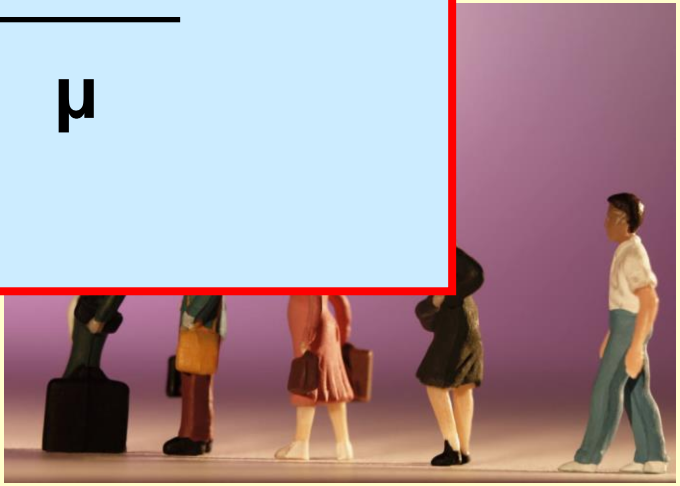
$$W = \frac{1}{(\mu - \lambda)}$$



Single-Channel / Single-Phase Model

Percentage of Time the System is Busy

$$\rho = \frac{\lambda}{\mu}$$



Single-Channel / Single-Phase Model

APPLICATION

- **A clerk can serve thirty customers per hour on average.**
- **Twenty customers arrive each hour on average.**

Therefore:

$\mu = 30$

$\lambda = 20$

M = 1

Single-Channel / Single-Phase Model

APPLICATION

The Average Number of Customers in the System

$$L = \frac{20}{(30 - 20)} = 2$$

Single-Channel / Single-Phase Model

APPLICATION

The Average Number Just Waiting in Line

$$Lq = \frac{(20)^2}{30(30 - 20)} = 1.33$$

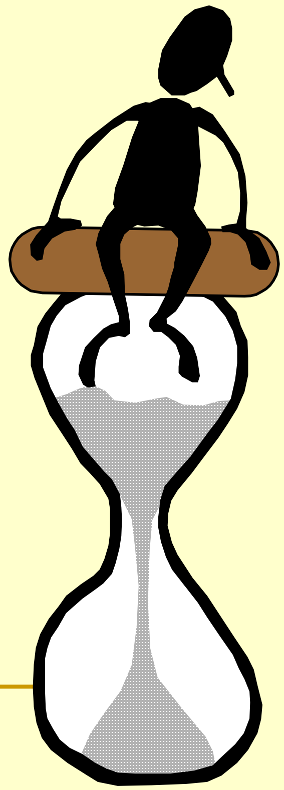
Single-Channel / Single-Phase Model

APPLICATION

The Average Customer Time Spent in the System

$$W = \frac{1}{(30 - 20)} = .10 \text{ hrs}$$

(6 minutes)

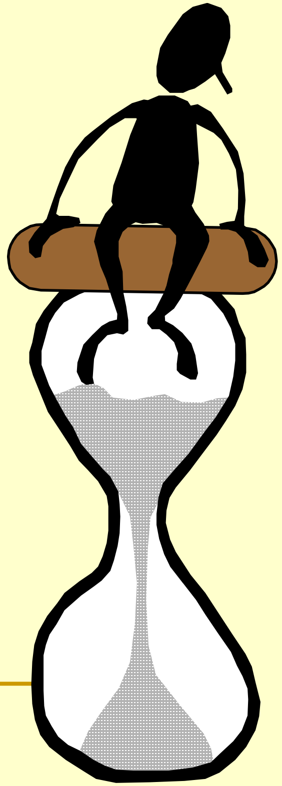


Single-Channel / Single-Phase Model

APPLICATION

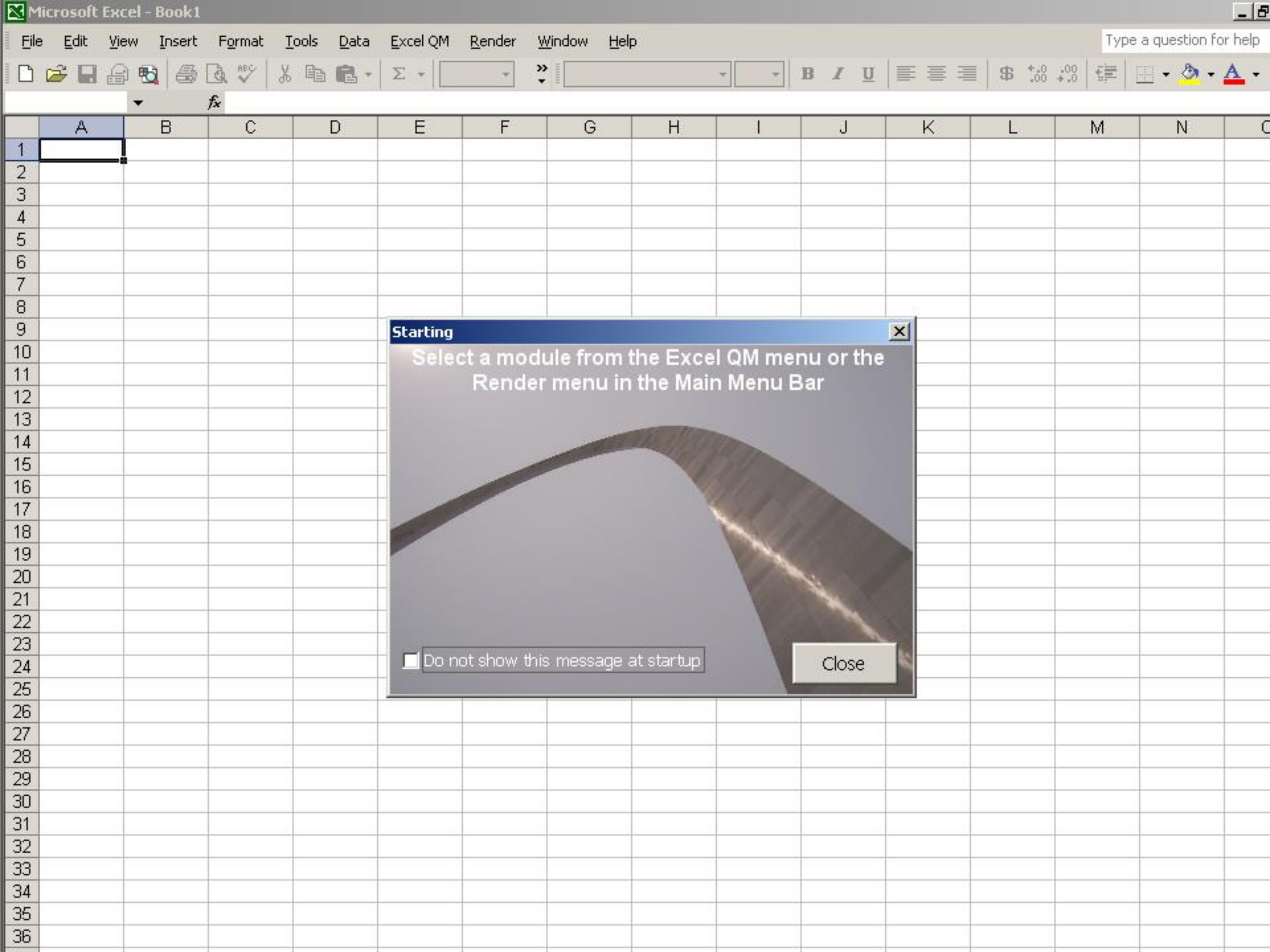
The Percentage of Time the System is Busy

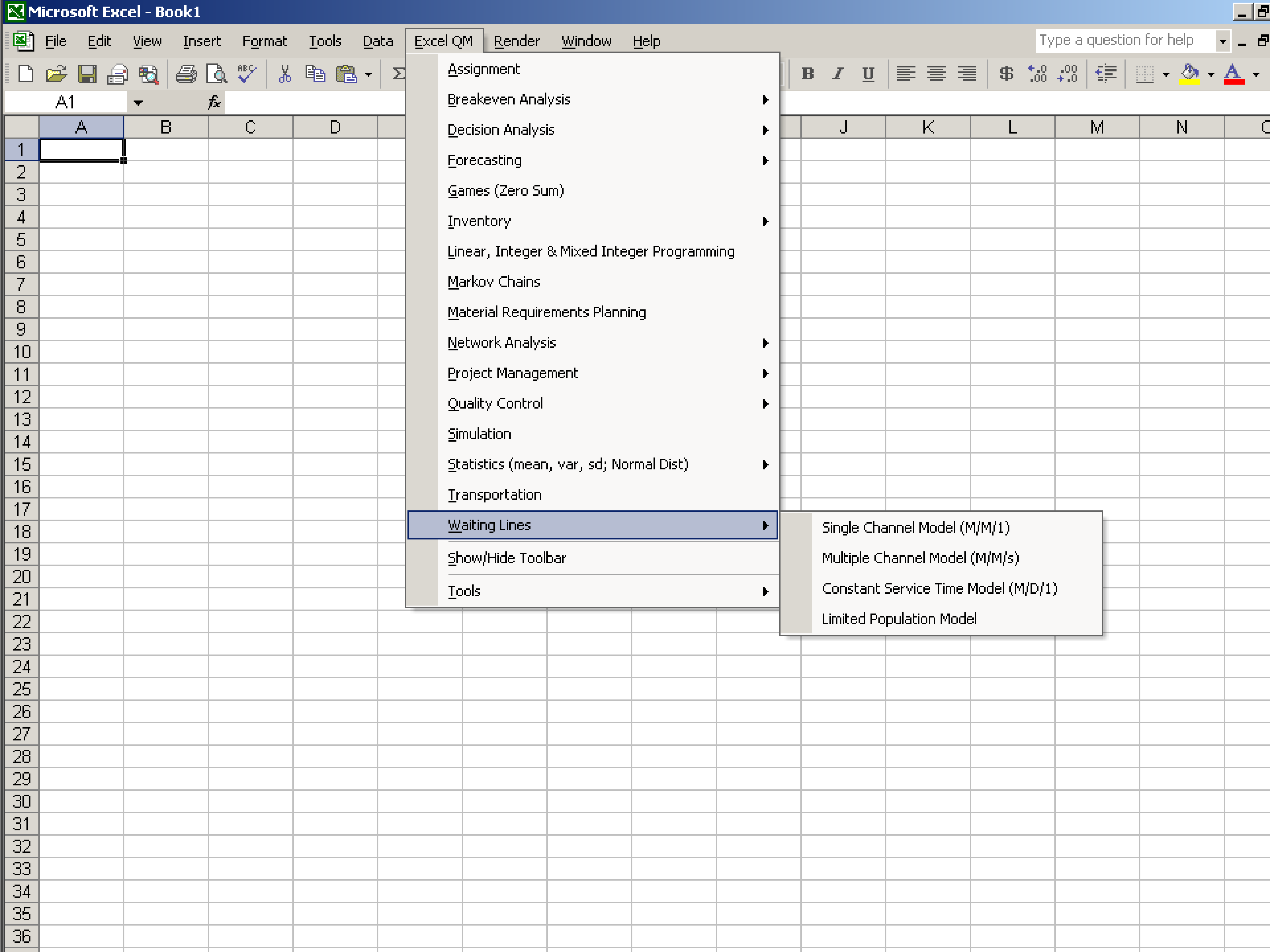
$$\rho = \frac{20}{30} = 67\%$$



Queuing Theory Modeling with *Excel Solver* Software

Single-Channel Single-Phase **Model**


Microsoft Excel - Book1
File Edit View Insert Format Tools Data Excel QM Render Window Help
Type a question for help
Starting
Select a module from the Excel QM menu or the Render menu in the Main Menu Bar
Do not show this message at startup
Close


Microsoft Excel - Book1
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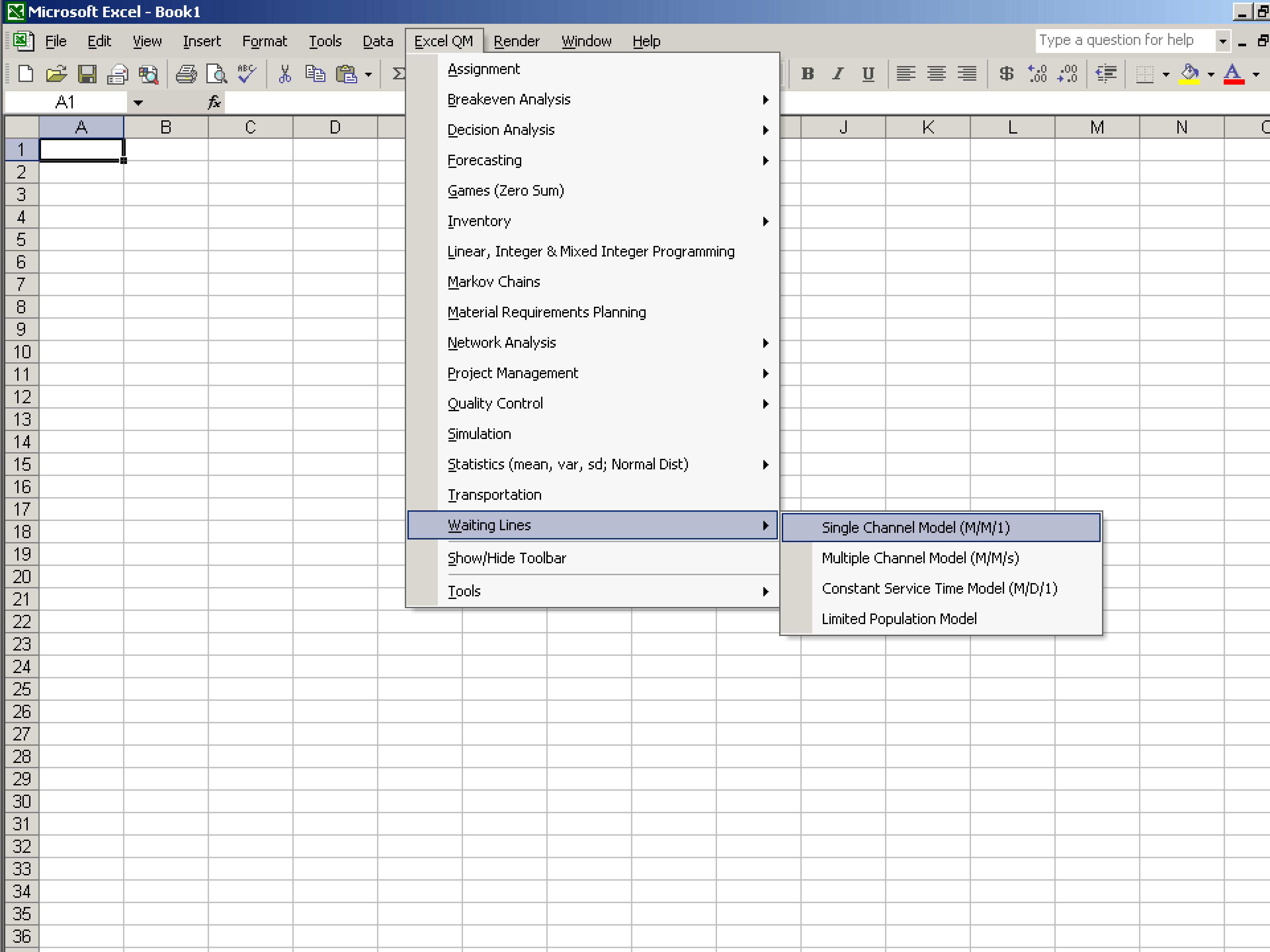
Type a question for help
A1
Assignment
Breakeven Analysis
Decision Analysis
Forecasting
Games (Zero Sum)
Inventory
Linear, Integer & Mixed Integer Programming
Markov Chains
Material Requirements Planning
Network Analysis
Project Management
Quality Control
Simulation
Statistics (mean, var, sd; Normal Dist)
Transportation
Waiting Lines
Show/Hide Toolbar
Tools
Single Channel Model (M/M/1)
Multiple Channel Model (M/M/s)
Constant Service Time Model (M/D/1)
Limited Population Model


Microsoft Excel - Book1
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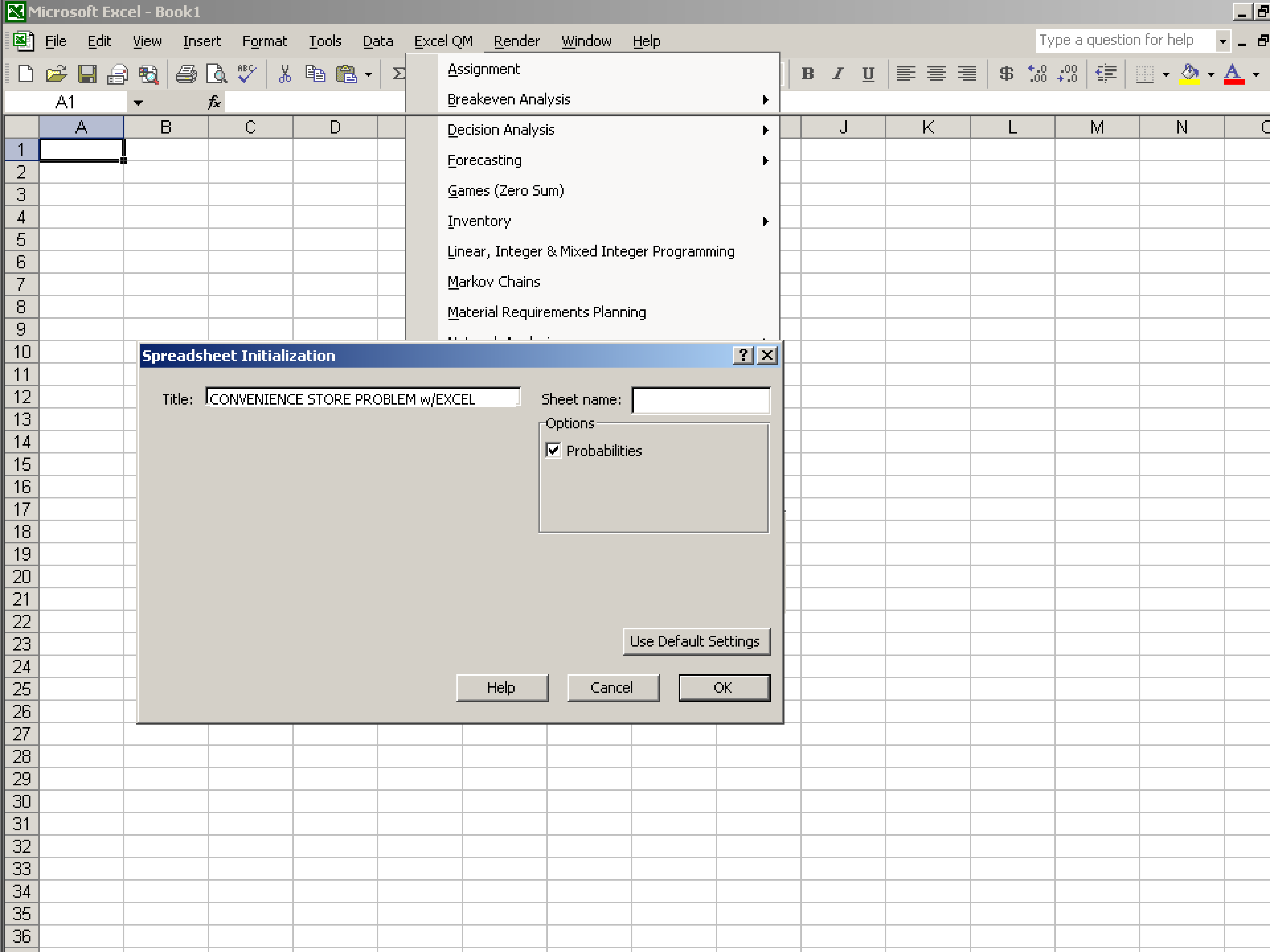
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Markov Chains
Material Requirements Planning
Spreadsheet Initialization
Title:
CONVENIENCE STORE PROBLEM w/EXCEL
Sheet name:
Options
Probabilities
Use Default Settings
Help
Cancel
OK

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File Edit View Insert Format Tools Data Excel QM Render Window Help

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Dr. Philip A. Vaccaro

CONVENIENCE STORE PROBLEM w/EXCEL

Waiting Lines — M/M/1 (Single Server Model)

The arrival RATE and service RATE both must be rates and use the same time unit. Given a time such as 10 minutes, convert it to a rate such as 6 per hour.

Data

Arrival rate (λ)	3
Service rate (μ)	4

Results

Average server utilization(ρ)	0.75
Average number of customers in the queue(L_q)	2.25
Average number of customers in the system(L_s)	3
Average waiting time in the queue(W_q)	0.75
Average time in the system(W_s)	1
Probability (% of time) system is empty (P_0)	0.25

Probabilities

Number	Probability	Cumulative Probability
0	0.250000	0.250000
1	0.187500	0.437500
2	0.140625	0.578125
3	0.105469	0.683594
4	0.079102	0.762695
5	0.059326	0.822021
6	0.044495	0.866516
7	0.033371	0.899887
8	0.025028	0.924915
9	0.018771	0.943686
10	0.014078	0.957765
11	0.010559	0.968324
12	0.007919	0.976243
13	0.005939	0.982182

Template and Sample Data

Microsoft Excel - Book1

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CONVENIENCE STORE PROBLEM w/EXCEL

Waiting Lines — M/M/1 (Single Server Model)

The arrival RATE and service RATE both must be rates and use the same time unit. Given a time such as 10 minutes, convert it to a rate such as 6 per hour.

Data

Arrival rate (λ)	20
Service rate (μ)	30

Results

Average server utilization(ρ)	0.666667
Average number of customers in the queue(L_q)	1.333333
Average number of customers in the system(L_s)	2
Average waiting time in the queue(W_q)	0.066667
Average time in the system(W_s)	0.1
Probability (% of time) system is empty (P_0)	0.333333

Probabilities

Number	Probability	Cumulative Probability
0	0.333333	0.333333
1	0.222222	0.555556
2	0.148148	0.703704
3	0.098765	0.802469
4	0.065844	0.868313
5	0.043896	0.912209
6	0.029264	0.941472
7	0.019509	0.960982
8	0.013006	0.973988
9	0.008671	0.982658
10	0.005781	0.988439
11	0.003854	0.992293
12	0.002569	0.994862
13	0.001713	0.996575

Microsoft Excel - Book1

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Data

Arrival rate (λ)	20
Service rate (μ)	30

Results

Average server utilization(ρ)	0.666667
Average number of customers in the queue(L_q)	1.333333
Average number of customers in the system(L_s)	2
Average waiting time in the queue(W_q)	0.066667
Average time in the system(W_s)	0.1
Probability (% of time) system is empty (P_0)	0.333333

Probabilities

Number	Probability	Cumulative Probability
0	0.333333	0.333333
1	0.222222	0.555556
2	0.148148	0.703704
3	0.098765	0.802469
4	0.065844	0.868313
5	0.043896	0.912209
6	0.029264	0.941472
7	0.019509	0.960982
8	0.013006	0.973988
9	0.008671	0.982658
10	0.005781	0.988439
11	0.003854	0.992293
12	0.002569	0.994862
13	0.001713	0.996575
14	0.001142	0.997716
15	0.000761	0.998478
16	0.000507	0.998985
17	0.000338	0.999323
18	0.000226	0.999549
19	0.000150	0.999699
20	0.000100	0.999800

Kendall-Lee Convention

- Widely accepted classification system for queuing models.
- Indicates the *pattern of arrivals,* the *service time distribution,* and the *number of channels* in a model.
- Often encountered in queuing software.
- Known also as the *Kendall Notation.*