Games theory - 2

Mixed strategies (2×2) game

- Arithmetic method
- Algebric method

Arithmetic method

Prob 1

In a game of matching coins, player A wins Rs 2 if there are two heads, wins nothing if there are two tails and loses Re 1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of the game to player A.

Solution

The payoff matrix for player A is

		Player B	
		H	T
Player A	H	2	-1
	T	-1	0

Check for saddle point

		H	T	max(min)
Pl. A	H	2	-1	(-1)
	T	-1	0	-1
	min(max)	2	(0)	

As min (max) ≠ max(min)

There is no saddle point

By applying Arithmetic method

		Player B			
		H	T		
Player A	H	2	-1	1	1/4
	T	-1	0	3	3/4
		1	3		
		1/4	3/4		

Value of the game can be obtained by

Using A's oddments

(i) B plays | selects H

$$\text{game value} = v = \left(\frac{2\times1 + (-1)\times3}{1+3}\right)$$

$$= \frac{2-3}{4} = -\frac{1}{4}$$

(ii) B plays | selects T

$$\text{game value} = v = \left(\frac{(-1)\times1 + 0\times3}{1+3}\right) = -\frac{1}{4}$$

Using B's oddments

(iii) A plays with H

$$\text{game value} = v = \left(\frac{2\times1 + (-1)\times3}{1+3}\right) = -\frac{1}{4}$$

(iv) A plays with T

$$\text{game value} = v = \left(\frac{(-1)\times1 + 0\times3}{1+3}\right) = -\frac{1}{4}$$

∴ value of game = $v = -1/4$

Optimum strategies for player A = $\{1/4, 3/4\}$

Optimum strategies for player B = $\{1/4, 3/4\}$

Prob 2

Reduce the game and solve it.

Player B

		I	II	III	IV	max (min)
Player A	I	3	2	4	0	0
	II	3	4	2	4	(2)
	III	4	2	4	0	0
	IV	0	4	0	8	0
min (max)		4	(4)	4	8	

As min (max) ≠ max (min)

There is no saddle point

By applying dominance

Row III dominates Row I

B

		I	II	III	IV
A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Coloumn III dominates Col. I

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B

		II	III	IV
	II	4	2	4
A	III	2	4	0
	IV	4	0	8

Average of col. III & IV dominates col II.

B

		III	IV
	II	2	4
A	III	4	0
	IV	0	8

Average of row III & IV dominates row II

B

		III	IV		
A	III	4	0	8	$8/12 = 2/3$
	IV	0	8	4	$4/12 = 1/3$
		8	4		
		$8/12$	$4/12$		
		$2/3$	$1/3$		

$$\text{game } v = \left(\frac{4\times 8 + 0\times 4}{8+4}\right) = \frac{32}{12} = 8/3$$

Optimum strategies for player A = $\{0, 0, 2/3, 1/3\}$

Optimum strategies for player B = $\{0, 0, 2/3, 1/3\}$

Prob5

Solve the game.

P_2

P_1		1	2	3	4	max (min)
	1	8	15	-4	-2	-4
	2	19	16	17	16	(16)
	3	20	20	15	5	5
min (max)		20	20	17	(16)	

Row 3 dominates Row 1.

	1	2	3	4
2	19	16	17	16
3	20	20	15	5

Col. 4 dominates Col. 3

	1	2	4
2	19	16	16
3	20	20	5

Col. 2 dominates Col 1.

	2	4	max (min)
2	16	16	(16)
3	20	15	15
min (max)	20	(16)	

game value $= v = 16$

$P_1 = \{0, 1, 0\}$

Prob 4

Two firms A and B are competing for business. Whatever business gained by A is lost by B. Advertising strategies available to both the firms along with utilities to firm A in terms of various market share %age are shown in the table below.

firm B

	Press	Radio	TV
Press	65	50	45
Radio	80	80	65
TV	85	65	75

Solution

B

A		P	R	TV	max (min)
	P	65	50	45	45
A	R	80	80	65	65
	TV	85	65	75	(65)
	min (max)	85	80	(75)	

As min(max) $\neq$ max(min)

There is no saddle point

Col. T.V. dominates col. P.

	R	TV
P	50	45
R	80	65
TV	65	75

Row T.V. dominates row P.

	R	T.V.	max (min)
R	80	65	65
T.V.	65	75	(65)
min (max)	80	(75)	

As min (max) $\neq$ max (min)
There is no saddle point.

Let's solve by Arithmetic method

	R	T.V.			
R	80	65	10	2	2/5
T.V.	65	75	15	3	3/5
	10	15			
	2	3			
	2/5	3/5			

Game value $= v = \left(\frac{2 \times 80 + 65 \times 3}{5}\right) = \underline{71}$

Optimal strategy for player A $= \{0, 2/5, 3/5\}$

Optimal strategy for player B $= \{0, 2/5, 3/5\}$

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$\boxed{Prob5}$ In a game of matching coins with two coins, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails. and loses ½ unit of value when there are one head and one tail. Determine payoff matrix, optimum strategies and game value to player A.

Player B

No saddle point.

		H	T		
	H	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
player A	T	$-\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{3}{4}$
		$\frac{1}{2}$	$\frac{3}{2}$		
		$\frac{1}{4}$	$\frac{3}{4}$		

Optimum strategies for player A = $\{\frac{1}{4}, \frac{3}{4}\}$

Optimum strategies for player B = $\{\frac{1}{4}, \frac{3}{4}\}$

$$\text{Game value} = v = \left(\frac{1 \times \frac{1}{2} + (-\frac{1}{2}) \times \frac{3}{2}}{\frac{1}{2} + \frac{3}{2}}\right)$$

$$= \frac{\frac{1}{2} - \frac{3}{4}}{2} = \frac{-\frac{1}{4}}{2}$$

$$= \underline{-\frac{1}{8}}$$

Algebric method

Probl

The two armies at war. Army A has two airbases one of which is thrice as valueable as other. Army B can destroy undefended air base, but it can destroy only one of them. Army A can also defend only one of them at a time. find the best strategies for A to minimize it's loss.

		Army B: Attack small Air base	Army B: Attack large air base	
Army A	Defend small Air base	0	-3	x
	Defend large air base	-1	0	$(1-x)$
		y	$(1-y)$	

No saddle point.

Let x and $(1-x)$ be the probabilities with which Army A selects first & second strategies respectively

$$\underbrace{0 \times x}_{0} + (-1)(1-x) = (-3) \times x + \underbrace{0(1-x)}_{0}$$

$$-1+x = -3x$$

$$-4x = -1$$

$$\therefore \ x = 1/4 \quad \text{and} \quad (1-x) = 3/4$$

Similarly let y and $(1-y)$ be the probabilities with which Army B selects first & second strategies respectively

$$\underbrace{0 \times y}_{0} + (-3)(1-y) = (-1)y + \underbrace{0(1-y)}_{0}$$

$$-3+3y = -y \quad \therefore \ -4y = -3$$

Army B

$$\begin{array}{cc|cc|c} & & S & L & \\ \hline \text{Army A} & S & 0 & -3 & 1/4 \\ & L & -1 & 0 & 3/4 \\ \hline & & 3/4 & 1/4 & \end{array}$$

game value (using Army A)

$$v = 3/4 \left\{ \underbrace{0 \times 1/4}_{0} + (-1) \times 3/4 \right\} + 1/4 \left\{ -3 \times 1/4 + \underbrace{0 \times 3/4}_{0} \right\}$$

$$= 3/4 \left(-3/4\right) + \frac{1}{4}\left(-3/4\right)$$

$$= -9/16 - \frac{3}{16} = -\frac{12}{16} = -3/4$$

or

game value (using Army B)

$$v = 1/4 \left\{ \underbrace{0 \times 3/4}_{0} + (-3) \times 1/4 \right\} + 3/4 \left\{ -1 \times 3/4 + \underbrace{0 \times 1/4}_{0} \right\}$$

$$= \frac{1}{4}\left(-3/4\right) + 3/4 \left(-3/4\right)$$

$$= -\frac{3}{16} - \frac{9}{16} = -\frac{12}{16} = -3/4$$

Optimal strategies for Army A = $\{1/4, 3/4\}$

Optimal strategies for Army B = $\{3/4, 1/4\}$

Method of subgames (2×n or m×2 game)

Prob 7 Solve the game

Player II

		A	B	C
Player I	A	275	−50	−75
	B	125	130	150

Solution check for saddle point – No saddle point

Subgame 1 (ignoring col. C) – No saddle point

		A	B			
I	A	275	−50	5	1	1/66
	B	125	130	325	65	65/66
		180	150			
		36	30			
		36/66	30/66			

$$\text{game value } v_1 = \left(\frac{275\times 1 + (125\times 65)}{66}\right) = \text{Rs } 127.30$$

Subgame 2 (ignoring col. B) – No saddle point

	A	C			
A	275	−75	25	1	1/15
B	125	150	350	14	14/15
	225	150			
	9	6			
	9/15	6/15			

$$\text{game value} = v_2 = \left(\frac{275\times 1 + 125\times 14}{1+14}\right) = \text{Rs } 135$$

Subgame 3 (ignoring col. A)

	B	C	max(min)
A	-50	-75	-75
B	130	150	(130)
min(max)	(130)	150	

As min(max) = max(min)

Saddle point exists.

∴ game v_3 = 130

As the least game value is with sub game 1

∴ Optimum strategy for player A = $\{ \frac{1}{66}, \frac{65}{66} \}$

Optimum strategy for player B = $\{ \frac{36}{66}, \frac{30}{66}, 0 \}$

For 2×m – Select the least subgame value

For m×2 – Select the largest game value

Method of matrices (3×3 game)

$$
\begin{array}{cc|ccc|c}
 & & & B & & \\
 & & 1 & 2 & 3 & \text{max (min)} \\
\hline
 & 1 & 7 & 1 & 7 & 1 \\
A & 2 & 9 & -1 & 1 & -1 \\
 & 3 & 5 & 7 & 6 & (5) \\
\hline
\text{min(max)} & & 9 & (7) & 7 &
\end{array}
$$

As min (max) $\neq$ max (min) — No saddle point.

Not possible to reduce by dominance.

By applying method of matrices.

$$
\begin{array}{cc|ccc|cc}
 & & & B & & & \\
 & & 1 & 2 & 3 & & \\
\hline
 & 1 & 7 & 1 & 7 & 6 & -6 \\
A & 2 & 9 & -1 & 1 & 10 & -2 \\
 & 3 & 5 & 7 & 6 & -2 & 1 \\
\hline
 & & -2 & 2 & 6 & & \\
 & & 4 & -8 & -5 & &
\end{array}
$$

Neglect sign

oddment for $A_1 = \begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix} = (10-4) = 6$

oddment for $A_2 = \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = (6-12) = -6 = 6$

oddment for $A_3 = \begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix} = (-12+60) = \underline{48}$

oddment for $B_1 = \begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix} = (-10+48) = 38$

oddment for $B_2 = \begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix} = (10-24) = 14$

oddment for $B_3 = \begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix} = (16-8) = \underline{8}$

		B 1	2	3				
A	1	7	1	7	6	1	1/10	3/30
	2	9	-1	1	6	1	1/10	3/30
	3	5	7	6	48	8	8/10	24/30
		38	14	8				
		19	7	4				
		19/30	7/30	4/30.				

$$v = \frac{(7\times1)+(9\times1)+(5\times8)}{10} = \frac{56}{10} = \frac{28}{5}$$

Thus optimum strategies are

$$A = \left\{ \frac{3}{30}, \frac{3}{30}, \frac{24}{30} \right\}$$

$$B = \left\{ \frac{19}{30}, \frac{7}{30}, \frac{4}{30} \right\}$$

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