Waiting line theory/Queuing theory/ Queuing model

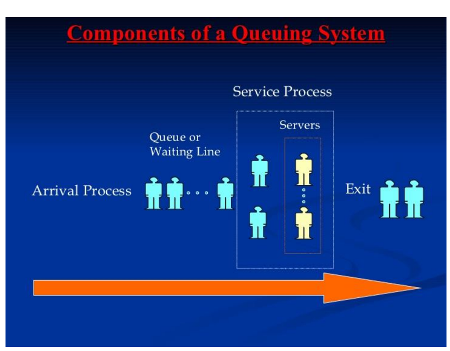
We experience number of waiting lines in our day to day to life. Waiting lines builds up for getting some service. That is there are two essential elements of a queue

i. customer or element who is in need of service and

ii. Service facility or service

For example:

1. Person going to hospital to get medical advice from the doctor is an element or a customer,
   Doctor is a service facility and medical care is a service
2. A person going to railway station or a bus station to purchase a ticket for the journey is a customer or an element,
   Ticket counter is a service facility and issue of ticket is service.
3. A person at ticket counter of a cinema hall is an element or a customer,
   Ticket counter is a service facility and issue of ticket is service.
4. A person at a grocery shop to purchase consumables is an element or a customer,
   Shop owner is a service facility and issue of items is service.
5. A bank pass book tendered to a bank clerk for withdrawal of money is an element or a customer,
   Bank clerk is a service facility and passing the cheque is service.
6. A machine break down and waiting for the attention of a maintenance crew is an element or a customer.
   Maintenance crew is service facility and repairing the machine is service.
7. Vehicles waiting at traffic signal are elements or customers,
   Traffic signals are service facility and control of traffic is service.



Queue and system or queuing system

Customers excluding the customer who is in process of availing service forms a queue.

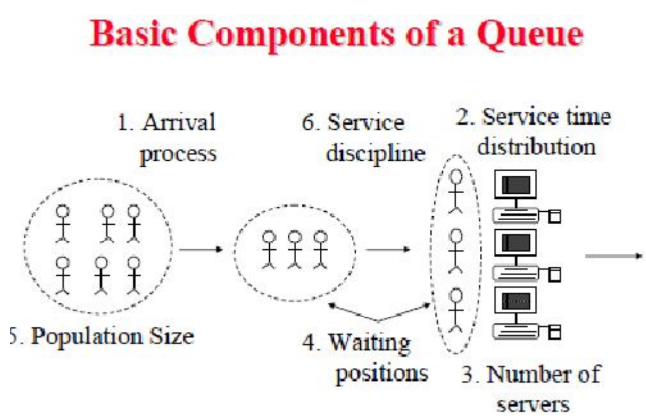
Customers including the customer who is in process of availing service forms a system or queuing system.

Hence, the length of a system is greater than the length of a queue. Moreover, the waiting time per customer in a system is greater than waiting time per customer in a queue.

QUEUING SYSTEM OR PROCESS

A queuing system is described by the following essential elements

1. Arrival distribution
2. Service distribution
3. Number of service channels
4. Service discipline
5. Maximum number of customers in the system and
6. Calling population



1. Arrival distribution

Customers arrive and join in the queue according to a probability distribution. The arrival may be single or bulk. In this chapter arrival distribution is considered as **Poisson distribution**.

2. Service distribution

The service offered by- the server also follows a probability distribution. In this chapter service distribution is considered as **exponential distribution**.

3. **Number of service channels**

The services can be offered by the servers in a series, parallel or network stations. A facility comprise a number of series stations through which the customer many pass for service is called 'Tandem queues'. Waiting lines may or may not be allowed between the stations. Similarly parallel queue and network queue are defined.

4. **Service discipline**

Service discipline indicates the process of providing service to the customers.

For example:

FIFO - First In First Out,

LIFO - Last In First Out

SIRO - Service in Random Order

5. **Maximum number of customers in the system or Queue size**

Generally it is referred as length of the queue or line length. Queue size may be finite or infinite (i.e., a very large queue). Queue size along with the server(s) form the capacity of the system.

6. **Calling population or calling source**

Customers join in the queue from a source is known as calling population which may be finite or infinite (i.e., a very large number). To reserve a ticket in a railway reservation counter, customers may come from anywhere of a city. Then the population of the city forms the calling population which can be considered as infinite.

Let us consider an example of purchasing a ticket at cinema hall

The arrival and service patterns are considered as **Poisson and exponential distributions** respectively. If the ticket is issued at a single counter then, it **single server queue**. The ticket is issued with queue discipline first in first out **(FIFO).** Any number of customers can be a part of queue system so the maximum number of customers in the queue will be **infinite**. Similarly, anyone can be the customer and hence the calling population will be **infinite**.

Queuing system can be different types according to changes in the above six elements.

Note: For our syllabus, only queues with Poisson and exponential distributions for arrival and service pattern, single server with FIFO queue discipline and with maximum number of customers in the system and calling population as infinity are considered. i.e. $(M/M/1:FCFS/\infty/\infty)$

Kendall Notation for queuing system

Kendall-Lee designates a queue as (a/b/c: d/e/f)

Where

a- Arrival distribution

b- Service time distribution

c- Number of service channels

d- Service discipline

e- Maximum number of customers in the system and

f- Size of calling population

Note: While solving a problem

1. **Identify arrival rate and service rate. Ensure that units for the arrival and service rate are the same.**
2. **Calculate utilization factor or traffic intensity.**
3. **Calculate the various values by using the appropriate formula.**
4. **Remember the values of system such as length or waiting time per customer are greater than the corresponding values of queue.**
5. **The derivation of the various formulae is not expected. These are the results of Little's Law and the distributions i.e. Poisson and exponential.**

LIST OF FORMULAE

1. **Average number of arrivals/customers per unit time = λ**

2. **Average number of customers/units served per unit time = μ**

3. **Traffic intensity/utility ratio/utilization factor = ρ**

$$\rho = \frac{\lambda}{\mu}$$ **the condition is (μ > λ)**

4. **Probability that the system is empty/idle**

$$P_0 = (1 - \rho)$$

5. **Probability that there are 'n' units in the system**

$$P_n = \rho^n P_0$$
$$P_n = \rho^n (1 - \rho)$$

6. **Average number of units or customers in the system/length of system**

$$L_S = \frac{\rho}{(1-\rho)} = \frac{\lambda}{(\mu - \lambda)}$$

7. **Average number of units or customers in the quque/length of queue**

$$L_q = \frac{\rho^2}{(1-\rho)} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

8. **Waiting time per customer in the system**

$$W_S = \frac{1}{(\mu - \lambda)}$$

9. **Waiting time per customer in the queue**

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

10. **The length of the queue that forms from time to time**

$$L_n = \frac{1}{(1-\rho)} = \frac{\mu}{(\mu - \lambda)}$$

11. **Probability that system remains busy = ρ**

12. **Idle time proportion = (1 – ρ)**

13. **Probability that a customer has to wait = ρ**

14. **Probability that an arrival has to wait for more than 't' minutes in a queue =** $\rho e^{-(\mu-\lambda)t}$

15. **Probability that an arrival has to wait for more than 't' minutes in a system =** $e^{-(\mu-\lambda)t}$

Problem 1

A T. V. repairman finds that the time spent on his jobs have exponential distribution with mean of 30 minutes. If he repairs sets in the order in which they come in and if the arrival of sets is approximated as Poisson with an average rate of 10 per 8 hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution

Data:

Arrival rate = λ = 10 sets per 8 hour day = 10/8 = **5/4** sets per hour

Service time spent on each radio set = 30 minutes

Therefore, per hour 2 radio sets can be repaired

Service rate = μ = **2** sets per hour

Traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{5}{4}}{2} = \frac{5}{8} = 0.625$$

Repairman's expected idle time

As the shop remains open for 8 hours

Repairman's expected idle time = (1 – ρ) X 8

= (1 – 0.625) X 8 = 3 hours

Number of jobs ahead of the average set just brought in

This will be equal to the length of the system

$$L_S = \frac{\rho}{(1-\rho)} = \frac{0.625}{(1-0.625)} = 1.67 sets$$

or

$$L_S = \frac{\lambda}{(\mu-\lambda)} = \frac{\frac{5}{4}}{\left(2-\frac{5}{4}\right)} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3} = 1.67 sets$$

Problem 2

The arrivals at a telephone booth are considered to be following Poisson law of distribution with an average time of 10 minutes between one arrival and the next. Length of the phone call is assumed to be distributed exponentially with a mean of 3 minutes.

i. **What is the probability that a person arriving at the booth will have to wait?**
ii. **What is the average length of queue that forms time to time?**
iii. **The telephone department will install a second booth when convinced that an arrival would expect to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?**

Solution

Data

Time between two arrivals = 10 min

No. of arrivals per minute i.e. arrival rate = λ = 1/10 customers/minutes

A phone call takes 3 minutes

Service rate = μ = 1/3 customers/min

Traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10} = 0.33$$

i. Probability that a person arriving at the booth will have to wait

Any person who is coming to booth has to wait when there is somebody in the queue. He need not wait when there is nobody in the queue i.e. the queue is empty.

Hence the probability that an arrival does not wait

$$P_0 = (1-0.33) = 0.67$$

Hence the probability that an arrival has to wait = 1- probability that an arrival does not wait

= 1-P_0 = 1 – 0.67 = 0.33

That is the system is busy for 0.33 equal to traffic intensity for which an arrival has to wait.

ii. The average length of queue that forms time to time

$$L_n = \frac{1}{(1-\rho)} = \frac{1}{1-0.33} = \frac{1}{0.67} = 1.49 customers$$

or

$$L_n = \frac{\mu}{(\mu-\lambda)} = \frac{\frac{1}{3}}{\left(\frac{1}{3}-\frac{1}{10}\right)} = \frac{30}{21} = 1.43 customers$$

iii. Increase in flow to justify second booth when an arrival has to wait for 3 minutes for call

The customer is waiting for 3 minutes for a call i.e. waiting time for customer in queue is 3 minutes

Let λ_1 = New arrival rate which is required for waiting time in queue

$$W_q = \frac{\lambda_1}{\mu(\mu - \lambda_1)}$$

$$3 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)}$$

$$\boldsymbol{\lambda_1 = 1/4}$$

Hence, increase in arrival flow rate= $\lambda_1 - \lambda$ = ¼ - 1/10 = 3/20

Problem 3

In a departmental store one cashier is there to serve the customers, and the customers pick up their needs by themselves. The arrival rate is 9 customers for every 5 minutes and the cashier can serve 10 customers in 5 minutes. Assuming Poisson's arrival rate and exponential distribution for service rate find

i. **Average number of customers in the system**
ii. **Average number of customers in the queue or average length of the queue**
iii. **Average time a customer spends in the system and**
iv. **Average time a customer waits before being served**

Solution

As 9 customers arrive for every 5 minutes

Arrival rate = λ = 9/5 customers/min

10 customers are served in 5 minutes

Service rate = μ = 10/5 = 2 customers/min

i. **Average number of customers in the system**

$$L_S = \frac{\lambda}{(\mu - \lambda)} = \frac{\frac{9}{5}}{\left(2 - \frac{9}{5}\right)} = \frac{\frac{9}{5}}{\frac{1}{5}} = 9 customers$$

ii. **Average number of customers in the queue or average length of the queue**

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\left(\frac{9}{5}\right)^2}{2\left(2-\frac{9}{5}\right)} = \frac{81}{10} = 8.1 customers$$

iii. **Average time a customer spends in the system**

$$W_S = \frac{1}{(\mu-\lambda)} = \frac{1}{\left(2-\frac{9}{5}\right)} = 5\min$$

iv. **Average time a customer waits before being served**

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\frac{9}{5}}{2\left(2-\frac{9}{5}\right)} = \frac{9}{2} = 4.5\min$$

Problem 4

A branch of a nationalized bank has only one typist. Since typing work varies in length (number of pages to be typed), the typing rate is randomly distributed approximating a Poisson distribution with a mean service rate of 8 letters per hour. The letter arrives at a rate of 5 per hour during the entire 8-hour workday. If the typist is valued at Rs. 150 per hour, determine

i. **Equipment utilization**
ii. **The percent time an arriving letter has to wait**
iii. **Average waiting time in the system and**
iv. **Average idle time cost of the typewriter per day**

Solution

Data

Arrival rate = λ = 5 letters/hour

Service rate = μ = 8 letters/hour

i. **Equipment utilization**

$$\rho = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625$$

ii. **The percent time an arriving letter has to wait**

The percent time an arriving letter has to wait = ρ = 0.625 i.e. 62.5%

iii. **Average waiting time in the system**

$$W_S = \frac{1}{(8-5)} = 1/3\, hrs = 20 \min$$

iv. **Average idle time cost of the typewriter per day**

= Idle time X 8 X Cost of hour

= (1 – ρ) X 8 X 150

= (1 – 0.625) X 8 X 150 = Rs 450.00

Problem 5

A product manufacturing plant at a city distributes its products by trucks, loaded at the factory warehouse. It has its own fleet of trucks plus trucks of a private transport company. This transport company has complained that sometimes its trucks have to wait in line and thus the company loses money paid for a truck and driver of waiting truck. The company has asked the plant manager either to go in for a second warehouse or discount prices equivalent to the waiting time. The data available is

Average arrival rate of all trucks = 3 per hour

Average service rate = 4 per hour

The transport company has provided 40% of the total number of trucks. Assuming that these rates are random according to Poisson distribution, determine

i. **The probability that a truck has to wait**
ii. **The waiting time of a truck that has to wait and**
iii. **The expected waiting time of company trucks per day**

Solution

Arrival rate = λ = 3 trucks/hr

Service rate = μ = 4 trucks/hr

i. **The probability that a truck has to wait**

Probability that a truck has to wait = $\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

Hence a truck has to wait for **75% of the time.**

ii. **The waiting time of a truck that has to wait**

This waiting time of a truck in the system

$$W_S = \frac{1}{(\mu - \lambda)} = \frac{1}{(4-3)} = 1hr$$

iii. **The expected waiting time of company trucks per day**

= [(Trucks per day) X (% company trucks) X expected waiting time per truck] X ρ

$$= \left[(3X8)X\,0.40X\,\frac{\lambda}{\mu(\mu-\lambda)}\right]X\rho$$

$$= \left[24X\,0.40X\,\frac{3}{4(4-3)}\right]X\,0.75$$

= 7.2 hrs/day

Problem 6

Arrival rate of telephone calls at a telephone booth is according to Poisson distribution with an average time of 9 minutes between consecutive arrivals. The length of telephone call is exponentially distributed with a mean of 3 minutes. Find

i. **Determine the probability that a person arriving at the booth will have to wait**
ii. **Find the average queue length that forms from time to time**
iii. **The telephone company will install a second booth when conveniences that an arrival would expect to have to wait at least four minutes for the phone. Find the increase in flow of arrivals, which will justify a second booth.**
iv. **What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free?**
v. **What is the probability that they will have to wait for more than 10 min before the phone is available and the call is also complete?**
vi. **Find the fraction of a day that the phone will be in use.**

Solution

Data

Time between two arrivals = 9 min

No. of arrivals per minute i.e. arrival rate = λ = 1/9 customers/minutes

A phone call takes 3 minutes

Service rate = μ = 1/3 customers/min

Traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{3}{9} = 0.33$$

i. Probability that a person arriving at the booth will have to wait

Any person who is coming to booth has to wait when there is somebody in the queue. He need not wait when there is nobody in the queue i.e. the queue is empty.

Hence the probability that an arrival does not wait

$$P_0 = (1 - 0.33) = 0.67$$

Hence the probability that an arrival has to wait = 1- probability that an arrival does not wait

= 1-P_0 = 1 – 0.67 = 0.33

That is the system is busy for 0.33 equal to traffic intensity for which an arrival has to wait.

ii. Find the average queue length that forms from time to time

$$L_n = \frac{\mu}{(\mu - \lambda)} = \frac{\frac{1}{3}}{\left(\frac{1}{3} - \frac{1}{9}\right)} = \frac{9}{6} = 1.5 customers$$

iii. Increase in flow to justify second booth when an arrival has to wait for 4 minutes for call

The customer is waiting for 4 minutes for a call i.e. waiting time for customer in queue is 4 minutes

Let λ_1 = New arrival rate which is required for waiting time in queue

$$W_q = \frac{\lambda_1}{\mu(\mu - \lambda_1)}$$

$$4 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)}$$

λ_1 = 4/21

Hence, increase in arrival flow rate= $\lambda_1 - \lambda$ =4/21 - 1/9 = 5/63

iv. The probability that an arrival will have to wait for more than 10 minutes before the phone is free

$$= \rho e^{-(\mu - \lambda)t}$$

$$= \frac{1}{3} e^{-(\frac{1}{3}-\frac{1}{9})10} = \frac{1}{30}$$

v. The fraction of a day that the phone will be in use.

The fraction of a day that the phone will be in use = ρ= 0.33