Linear Programming

Introduction

- Problems of this kind are called "linear programming problems" or "LP problems" for short; linear programming is the branch of applied mathematics concerned with these problems.
- A *linear programming problem* is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints.
- Standard form:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

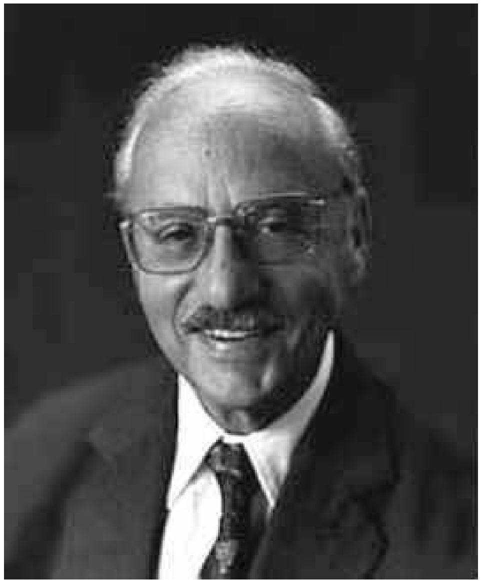
$$\begin{aligned} \text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad (i = 1, 2, \ldots, m) \\ x_j &\geq 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

History of Linear Programming

- It started in 1947 when G.B.Dantzig design the “simplex method” for solving linear programming formulations of U.S. Air Force planning problems.

- It soon became clear that a surprisingly wide range of apparently unrelated problems in production management could be stated in linear programming terms and solved by the simplex method.

- Later, it was used to solve problems of management. It’s algorithm can also used to network flow problems.



History of Linear Programming

- On Oct.14th,1975, the Royal Sweden Academy of Science awarded the Nobel Prize in economic science to L.V.Kantorovich and T.C.Koopmans ”for their contributions to the theory of optimum allocation of resources”

- The breakthrough in looking for a theoretically satisfactory algorithm to solve LP problems came in 1979 when L.G.Khachian published a description of such an algorithm.

Computer Solution of Linear Programs With Any Number of Decision Variables

- Linear programming software packages solve large linear models.
- Most of the software packages use the algebraic technique called the Simplex algorithm.
- The input to any package includes:
  - The objective function criterion (Max or Min).
  - The type of each constraint: $\leq, =, \geq$.
  - The actual coefficients for the problem.

Simplex Today

- A large variety of Simplex-based algorithms exist to solve LP problems.
- Other (polynomial time) algorithms have been developed for solving LP problems:
  - Khachian algorithm (1979)
  - Kamarkar algorithm (AT&T Bell Labs, mid 80s)
  - See Section 4.10

BUT,

none of these algorithms have been able to beat Simplex in actual practical applications.

HENCE,

Simplex (in its various forms) is and will most likely remain the most dominant LP algorithm for at least the near future

Fundamental Theorem

Extreme point (or Simplex filter) theorem:

If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the region.

Convex set:

A set (or region) is convex if, for any two points (say, x1 and x2) in that set, the line segment joining these points lies entirely within the set.

A point is by definition convex.

What does the extreme point theorem imply?

- A finite number of extreme points implies a finite number of solutions!
- Hence, search is reduced to a finite set of points
- However, a finite set can still be too large for practical purposes
- Simplex method provides an efficient systematic search guaranteed to converge in a finite number of steps.

Basic Steps of Simplex

1. Begin the search at an extreme point (i.e., a basic feasible solution).

2. Determine if the movement to an adjacent extreme can improve on the optimization of the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.

3. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.

4. Continue steps 2 and 3 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Step 0 – Obtain Canonical Form

IMPORTANT: Simplex only deals with equalities

General Simplex LP model:

min (or max) $z = \Sigma c_i x_i$

s.t.

$A x = b$

$x \geq 0$

In order to get and maintain this form, use

- *slack,* if $x \leq b$, then $x + \text{slack} = b$
- *surplus,* if $x \geq b$, then $x - \text{surplus} = b$
- *artificial variables* (sometimes need to be added to ensure all variables ≥ 0, see page 101)

Compare constraint conversion with goal conversions using deviation variables

Different "components" of a LP model

- LP model can always be split into a <u>basic</u> and a <u>non-basic</u> part.
- “Transformed” or “reduced” model is another good way to show this.
- This can be represented in mathematical terms as well as in a LP or simplex tableau.

Movement to Adjacent Extreme Point

Given any basis we move to an adjacent extreme point (another basic feasible solution) of the solution space by **exchanging one of the columns that is in the basis for a column that is not in the basis.**

Two things to determine:

1) which (nonbasic) column of A should be brought into the basis so that the solution improves?
2) which column can be removed from the basis such that the solution stays feasible?

Entering and Departing Vector (Variable) Rules

General rules:

- The one non-basic variable to come in is the one which provides the highest reduction in the objective function.

- The one basic variable to leave is the one which is expected to go infeasible first.

NOTE: THESE ARE HEURISTICS!!

Variations on these rules exist, but are rare.

Simplex Variations

Various variations on the simplex method exist:

- "regular" simplex
- two-phase method: Phase I for feasibility and Phase II for optimality
- condensed/reduced/revised method: only use the non-basic columns to work with
- (revised) dual simplex

Computational Considerations

- Unrestricted variables (unboundedness)
- Redundancy (linear dependency, modeling errors)
- Degeneracy (some basic variables = 0)
- Round-off errors

Limitations of Simplex

1. Inability to deal with multiple objectives
2. Inability to handle problems with integer variables

Problem 1 is solved using Multiplex

Problem 2 has resulted in:

- Cutting plane algorithms (Gomory, 1958)
- Branch and Bound (Land and Doig, 1960)

However,

solution methods to LP problems with integer or Boolean variables are still far less efficient than those which include continuous variables only

Solve Problem by simplex Method.

Max. $Z = 120x + 100y$.

Subj. to

$$2x + 2.5y \leq 1000$$

$$3x + 1.5y \leq 1200$$

$$1.5x + 4y \leq 1200$$

$$x, y \geq 0$$

Converting cannonical form to standard form.

$$\text{Max. } Z = 120x + 100y + 0.S_1 + 0.S_2 + 0.S_3$$

Subj to

$$2x + 2.5y + S_1 = 1000$$

$$3x + 1.5y + S_2 = 1200$$

$$1.5x + 4y + S_3 = 1200$$

$$x, y, S_1, S_2, S_3 \geq 0$$

tion

1st iteration

	C_j	120	100	0	0	0		
e_i	CVS	x	y	S_1	S_2	S_3	b	$\theta = b_i / a_{ik}$
0	S_1	2	2.5	1	0	0	1000	1000/2 = 500
0	S_2	3 key elem	1.5	0	1	0	1200	1200/3 = 400 ← K_R (outgoing)
0	S_3	1.5	4	0	0	1	1200	1200/1.5 = 800
	$E_j = \sum e_i a_{ij}$	0	0	0	0	0		
	$C_j - E_j$	120	100	0	0	0		

↑ K_C

(incoming variable)

II nd Iteration

	C_j	120	150	0	0	0			
e_i	CVS	x	y	s_1	s_2	s_3	b	$\theta_i = b_i/a_{ik}$	KR
0	s_1	0	1.5	1	−0.66	0	200	$\frac{200}{1.5} = 133.33$	outgoing variable
120	x	1	0.5	0	0.33	0	400	$\frac{400}{0.5} = 800$	
0	s_3	0	3.25	0	−0.495	1	600	$\frac{600}{3.25} = 184.6$	
$E_j = \sum e_i a_{ij}$		120	60	0	39.6	0			
$C_j - E_j$		0	40	0	−39.6	0			

Pivot element

KC

Incoming variable

New Row Element = Element of the Row to be replaced – [(Intersection of old row & key column)x (Corresponding element in new pivot row)].

e.g. a_{11}=2-[2(1)]=0; a_{12} = 2.5-[2(0.5)]=1.5 and so on…

a_{31} = 1.5- [1.5(1)]=0; a_{32} = 4 – [1.5(0.5)] =3.25 and so on…

Condition for Max. Cj –Ej ≤ 0; Min. Cj –Ej ≥ 0

III rd Iteration

	C_j	120	100	0	0	0		
C_i	CVS	x	y	S_1	S_2	S_3	b	θ
100	y	0	1	0.66	-0.44	0	133.33	
120	x	1	0.	-0.33	0.55	0	333.335	
0	S_3	0	0	-2.145	0.925	1	166.67	
$E_j = \Sigma c_i a_{ij}$		120	100	26.4	22	0		
$C_j - E_j$		0	0	-26.4	-22	0		

As $C_j - E_j$ row contains 0 and −ve value solution is the optimal solution

$y = 133.33$ Units

$x = 333.33$ Units

$\therefore Z = 120x + 100y$

$= 120 \times 333.33 + 100 \times 133.33$

$$\boxed{Z = 53,332.6}$$

BIG M method

$$\text{Min } Z = 4x_1 + 3x_2$$

Subject to

$$2x_1 + x_2 \geq 40$$

$$x_1 + 2x_2 \geq 50$$

$$x_1 + x_2 \geq 35$$

$$x_1, x_2 \geq 0$$

$$\text{Min } Z = 4x_1 + 3x_2$$

Subject to

$$2x_1 + x_2 \geq 40$$
$$x_1 + 2x_2 \geq 50$$
$$x_1 + x_2 \geq 35$$
$$x_1, x_2 \geq 0$$

$$\text{Min } Z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2 + 0.S_3 + M.A_1 + M A_2 + M.A_3$$

$$2x_1 + x_2 - S_1 + A_1 = 40$$
$$x_1 + 2x_2 - S_2 + A_2 = 50$$
$$x_1 + x_2 - S_3 + A_3 = 35$$
$$x_1, x_2, S_1, A_1, S_2, A_2, S_3, A_3 \geq 0$$

first Iteration

Ci	C_j	4	3	0	0	0	M	M	M		
c_i	CVS	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	b	$\theta = b/a$
M	A_1	2	1	−1	0	0	1	0	0	40	$40/1 = 40$
M	A_2	1	2 key element	0	−1	0	0	1	0	50	$50/2 = 25$
M	A_3	1	1	0	0	−1	0	0	1	35	$35/1 = 35$
	$E_j = \sum a_{ij} c_i$	4M	4M	−M	−M	−M	M	M	M		outgoing variable
	$C_j - E_j$	4−4M	3−4M	M	M	M	0	0	0		

(max −ve)

incoming Key (key column)

Condition for Min. problem $C_j - E_j \geq 0$

IInd Iteration

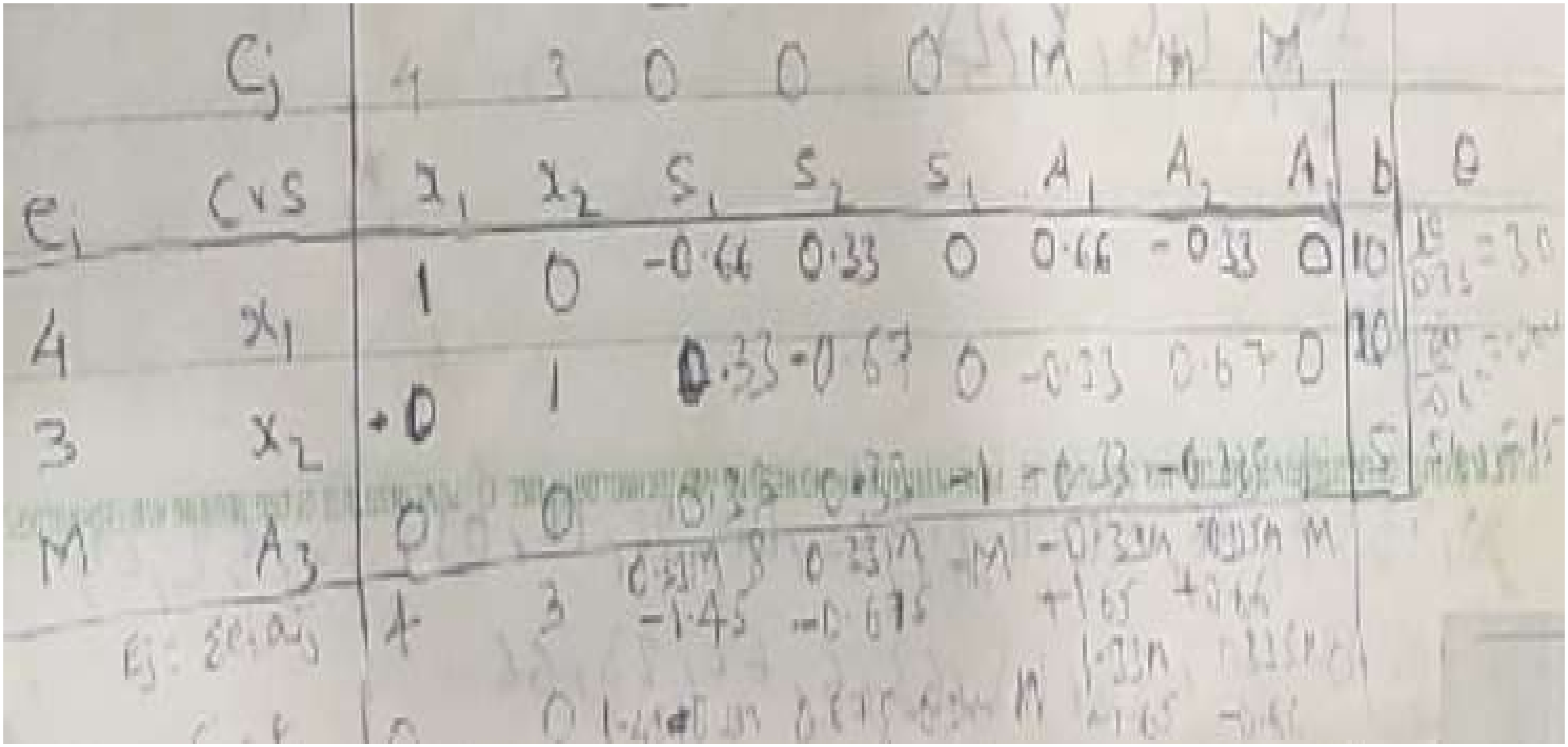
	C_j	4	3	0	0	0	M	M	M		
C_i	CVS	x_1 (key element)	x_2	S_1	S_2	S_3	A_1	A_2	A_3	b	$\theta = \frac{b_i}{a_{i1}}$
M	A_1	1.5	0	-1	0.5	0	1	-0.5	0	15	$\frac{15}{1.5} = 10$
3	x_2	0.5	1	0	-0.5	0	0	0.5	0	25	$\frac{25}{0.5} = 50$
M	A_3	0.5	0	0	0.5	-1	0	-0.5	1	10	$\frac{10}{0.5} = 20$
$E_j = \sum a_{ij} \cdot C_i$		2M+1.5	3	-M	M-1.5	-M	M	1.5-M	M		
$C_j - E_j$		2.5-2M	0	M	1.5-M	M	0	2M-1.5	0		

↑ (most -ve)
incoming variable k_c (key column)

New Row Element = Element of the Row to be replaced − [(Intersection of old Row & key column) × (corresponding elements in new pivot row)]

Condition for Min. problem Cj –Ej ≥ 0

III rd Iteration



	C_j	4	3	0	0	0	M	M	M		
C_i	CVS	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	b	θ
4	x_1	1	0	−0.66	0.33	0	0.66	−0.33	0	10	$\frac{10}{0.33}=30$
3	x_2	0	1	0.33	−0.67	0	−0.33	0.67	0	10	[illegible]
M	A_3	0	0	0.33	0.33	−1	−0.33	−0.33	1	5	[illegible]
$E_j=\sum C_i a_{ij}$		4	3	0.33M −1.45	0.33M −0.675	−M	−0.33M +1.65	−0.33M +0.66	M		
C_j-E_j		0	0	1.45 − 0.33M	0.675 − 0.33M	M	1.33M −1.65	1.33M −0.66	[illegible]		

Condition for Min. problem Cj –Ej ≥ 0

IV th Iteration

C_j		4	3	0	0	0	M	M	M		
C_i	CVS	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	b	θ
4	x_1	1	0	−0.99	0	0.99	0.99	0	−0.99	5.05	
3	x_2	0	1	1	0	−2.01	−1	0	2.01	30.05	
0	S_2	0	0	1	1	−3	−1	−1	3	15	
$E_j = \Sigma a_{ij} C_i$		4	3	−0.96	0	−2.07	0.96	0	2.07		
$C_j - E_j$		0	0	0.96	0	2.07	M − 0.96	M	M − 2.07		

As $C_j - E_j$ row contains 0 and +ve value above solution is the optimal solution and

$x_1 = 5.05$ & $x_2 = 30.05$

$\therefore Z = 4x_1 + 3x_2$

$= 4 \times 5.05 + 3 \times 30.05$

$\boxed{Z = 110.35}$

Condition for Min. problem Cj –Ej ≥ 0

Example Problem

Maximize **Z =** $5x_1 + 2x_2 + x_3$

subject to

$$x_1 + 3x_2 - x_3 \leq 6,$$

$$x_2 + x_3 \leq 4,$$

$$3x_1 + x_2 \leq 7,$$

$$x_1, x_2, x_3 \geq 0.$$

Simplex and Example Problem

Step 1. Convert to Standard Form

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &\le b_1, \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &\ge b_2, \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &\le b_m,
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + x_{n+1} &= b_1, \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n - x_{n+2} &= b_2, \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n + x_{n+k} &= b_m,
\end{aligned}
$$

In our example problem:

$$
\begin{aligned}
x_1 + 3x_2 - x_3 &\le 6, \\
x_2 + x_3 &\le 4, \\
3x_1 + x_2 &\le 7, \\
x_1, x_2, x_3 &\ge 0.
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
x_1 + 3x_2 - x_3 + x_4 &= 6, \\
x_2 + x_3 + x_5 &= 4, \\
3x_1 + x_2 + x_6 &= 7, \\
x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0.
\end{aligned}
$$

Simplex: Step 2

Step 2. Start with an initial basic feasible solution (b.f.s.) and set up the initial tableau.

In our example

Maximize $\mathbf{Z} = 5x_1 + 2x_2 + x_3$

$$x_1 + 3x_2 - x_3 + x_4 = 6,$$

$$x_2 + x_3 + x_5 = 4,$$

$$3x_1 + x_2 + x_6 = 7,$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

Step 2: Explanation

Adjacent Basic Feasible Solution

If we bring a nonbasic variable x_s into the basis, our system changes from the basis, x_b, to the following (same notation as the book):

$$
\begin{aligned}
x_1 \qquad\qquad\qquad &+ \bar{a}_{1s}x_s = \bar{b}_1 \\
&\ \vdots \\
x_r \qquad &+ \bar{a}_{rs}x_r = \bar{b}_r \\
&\ \vdots \\
x_m &+ \bar{a}_{ms}x_s = \bar{b}_s
\end{aligned}
$$

$x_i = \bar{b}_i - \bar{a}_{is}$ for i =1, …, m

$x_s = 1$

$x_j = 0$ for $j=m+1, \dots, n$ and $j \neq s$

The new value of the objective function becomes:

$$Z = \sum_{i=1}^{m} c_i (\bar{b}_i - \bar{a}_{is}) + c_s$$

Thus the change in the value of Z per unit increase in x_s is

$$
\begin{aligned}
\bar{c}_s &= \text{new value of Z - old value of Z} \\
&= \sum_{i=1}^{m} c_i (\bar{b}_i - \bar{a}_{is}) + c_s - \sum_{i=1}^{m} c_i \bar{b}_i \\
&= c_s - \sum_{i=1}^{m} c_i \bar{a}_{is}
\end{aligned}
$$

This is the Inner Product rule

Simplex: Step 3

Use the inner product rule to find the relative profit coefficients

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

$$\bar{c}_j = c_j - c_B \overline{P}_j$$

$\underline{c}_1 = 5 - 0(1) - 0(0) - 0(3) = 5$ -> largest positive

$\underline{c}_2 = \ldots.$

$\underline{c}_3 = \ldots.$

Step 4: Is this an optimal basic feasible solution?

Simplex: Step 5

Apply the minimum ratio rule to determine the basic variable to leave the basis.

The new values of the basis variables:

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \qquad \text{for } i = 1, \ldots, m$$

$$\max x_s = \min_{\bar{a}_{is} > 0} \left[\frac{\bar{b}_i}{\bar{a}_{is}} \right]$$

In our example:

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

Row	Basic Variable	Ratio
1	x_4	6
2	x_5	-
3	x_6	7/3

Simplex: Step 6

Perform the pivot operation to get the new tableau and the b.f.s.

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

New iteration: find entering variable:

$$\bar{c}_j = c_j - c_B \bar{P}_j$$

$c_B = (0\ 0\ 5)$

$c_2 = 2 - (0)\ 8/3 - (0)\ 1 - (5)\ 1/3 = 1/3$

$c_3 = 1 - (0)\ (-1) - (0)\ 1 - (5)\ 0 = 1$

$c_6 = 0 - (0)\ 0 - (0)\ 0 - (5)\ 1/3 = -5/3$

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	8/3	-1	1	0	0	11/3
0	x_5	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	1/3	1	0	0	-5/3	Z=35/3

Final Tableau

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	8/3	-1	1	0	0	11/3
0	x_5	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	1/3	1	0	0	-5/3	Z=35/3

x_3 enters basis,
x_5 leaves basis

Wrong value!
4 should be 11/3

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	4	0	1	1	0	23/3
1	x_3	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	-2/3	0	0	-1	-5/3	Z=47/3