# A Rigorous Formalization of the Semantic Compression Lattice

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**Abstract**

This document provides a mathematically rigorous formalization of the *Semantic Compression Lattice* (SCL), a structure that models semantic compression, invariant preservation, and constrained geometric optimization within latent spaces. We refine the curvature functional, formalize admissible projections, define explicit meet and join operations on SCLs, and introduce well-posed Riemannian gradient flows respecting invariant shells. The resulting structure is a complete lattice equipped with a compression-monotone semantics.

## 1 Semantic Compression Lattice

**Definition 1** (Semantic Compression Lattice)**.** A *Semantic Compression Lattice* (SCL) is a tuple

$$\mathcal{L} = (V, \mathcal{E}, \kappa, \mathcal{I}, \nabla_{\mathrm{SAL}})$$

with components defined below.

### 1.1 Meaning Atoms and Curvature-Like Functional

Let $V \subset \mathbb{R}^d$ be a finite set. Let $\mathcal{L}_{\mathrm{world}} : \mathbb{R}^d \to \mathbb{R}$ be $C^2$.

**Definition 2** (Curvature-Like Functional)**.** For $\lambda > 0$, define

$$\kappa(v) = \|\nabla \mathcal{L}_{\mathrm{world}}(v)\|_2 \;+\; \lambda \|v\|_2, \quad v \in V.$$

The functional is monotone, smooth, coercive, and invertible up to radial symmetry. It is not geometric curvature but a proxy measuring "semantic energy" plus position regularity.

### 1.2 Directed Hypergraph Structure

**Definition 3** (Directed Hyperedge)**.** A directed hyperedge is a pair $e = (T_e \to h_e)$ with $T_e \subseteq V$ and $h_e \in V$.

**Definition 4** (Admissibility)**.** Fix $\delta > 0$. A hyperedge $e = (T_e \to h_e)$ is *admissible* if

$$\kappa(h_e) + \delta \;\leq\; \min_{u \in T_e} \kappa(u).$$

**Definition 5** (Hyperedge Weight)**.** For inverse temperature $\beta > 0$, define

$$w_e = \exp\bigl(-\beta(\kappa(h_e) - \min_{u \in T_e} \kappa(u))\bigr) \in (0, 1].$$

## 2 Invariant Shells

**Definition 6** (Invariant Shell)**.** An invariant shell is a triple $S = (V_S, \varphi_S, \varepsilon_S)$ where:

(i) $V_S \subseteq V$,

(ii) $\varphi_S : V_S \to \{0, 1\}$ decidable,

(iii) $\varepsilon_S \geq 0$ the curvature tolerance.

### 2.1 Shell Constraint Manifolds and Projections

For each shell $S$ define its constraint set

$$\mathcal{M}_S = \{v \in V_S : |\kappa(v) - \kappa(v_0)| \leq \varepsilon_S,\ \varphi_S(v) = \varphi_S(v_0)\},$$

where $v_0$ is the original untransformed element.

The global constraint set is

$$\mathcal{M}_\mathcal{I} = \bigcap_{S \in \mathcal{I}} \mathcal{M}_S,$$

which is nonempty by assumption of shell compatibility.

**Definition 7** (Admissible Projection)**.** For any $v \in \mathbb{R}^d$, define

$$\Pi_\mathcal{I}(v) = \operatorname*{argmin}_{u \in \mathcal{M}_\mathcal{I}} \|u - v\|_2.$$

Existence and uniqueness follow from convexity of constraints.

## 3 Riemannian Teleological Gradient Flow

Define the objective with corrected inner product:

**Definition 8** (Objective Functional)**.** For $\alpha > 0$,

$$J(V) = \sum_{v \in V} \kappa(v) - \alpha \sum_{v \in V} \langle \nabla_{\mathrm{SAL}}(v), v \rangle.$$

**Definition 9** (Constrained Riemannian Gradient Flow)**.** A differentiable path $\Phi_t : V \to V$ satisfies

$$\frac{d}{dt}\Phi_t(v) = -\operatorname{Proj}_{T_{\Phi_t(v)}\mathcal{M}_\mathcal{I}}\big(\nabla_v J\big)$$

and is projected via

$$\Phi_{t+\eta}(v) = \Pi_\mathcal{I}\left(\Phi_t(v) - \eta \nabla_v J\right).$$

Under standard Lipschitz assumptions, existence and uniqueness hold.

## 4 Refinement Order and Lattice Operations

Let

$$\mathcal{L}_i = (V_i, \mathcal{E}_i, \kappa_i, \mathcal{I}_i, \nabla_{\mathrm{SAL},i}), \quad i = 1, 2.$$

**Definition 10** (Refinement Order). $\mathcal{L}_1 \sqsubseteq \mathcal{L}_2$ iff there exists $\varphi : V_1 \to V_2$ with:

(i) $\kappa_2(\varphi(v)) \le \kappa_1(v)$ for all $v$,

(ii) Every shell in $\mathcal{I}_1$ refines to one in $\mathcal{I}_2$:

$$\varphi(V_{S_1}) \subseteq V_{S_2},\ \varepsilon_{S_2} \le \varepsilon_{S_1},\ \varphi_{S_2} \circ \varphi = \varphi_{S_1}.$$

**Theorem 1** (Completeness of the SCL Lattice)**.** Every pair of SCLs has:

1. a greatest lower bound (meet), defined by intersection of vertex sets, admissible edges, and shell intersections,
2. a least upper bound (join), defined by disjoint union completion, pushforward of curvature, and shell union with tolerances minimized.

Hence SCLs form a complete lattice.

## 5 Theorems

**Theorem 2** (Monotonicity of Constrained Curvature Flow)**.** If all hyperedges are admissible, then along the constrained flow

$$\frac{d}{dt} \sum_{v \in V} \kappa(\Phi_t(v)) \le 0.$$

Any nontrivial step strictly improves any functional decreasing in $\kappa$.

**Theorem 3** (Kolmogorov Correspondence (Precise Form))**.** Assume:

1. $\mathcal{L}_{\text{world}}$ is the negative log-likelihood of a universal generative family,
2. $\lambda \to 0$,
3. model capacity $\to \infty$,
4. $\kappa$ approximates MDL in the limit,
5. the constrained flow converges to fixed points.

Then minimal fixed points of the curvature flow correspond, up to additive constants, to shortest programs computing the same total deterministic function. In particular,

$$\min_{\text{fixed points}} \sum_v \kappa(v) \ \sim\ K(f) \quad \text{up to } O(1).$$

**Theorem 4** (Shell Soundness)**.** If $S$ has $\varepsilon_S = 0$, then for all $v \in V_S$,

$$\kappa(\tau(v)) = \kappa(v) \quad \text{and} \quad \varphi_S(\tau(v)) = \varphi_S(v).$$

**Theorem 5** (Logarithmic Memory Scaling)**.** Let $B : V^{\mathbb{N}} \to \Sigma^*$ be a measure-preserving discretization with pointer reuse. If the number of distinct meaning-atoms grows $o(n)$ along a reasoning trace of length $n$, then

$$\mathrm{mem}(n) \in O(\log n).$$