Chapter 1 - Basic Measure Theory

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Exercise Solutions

Exercise 1.1.3 Let $\epsilon, \delta > 0$ we define

$$U_{\epsilon,\delta} = \{x \in \Omega_1 : \exists\ y, z \in B(x, \delta) \text{ with } d_2(f(y), f(z)) > \epsilon\}$$

Let's see that these sets are open for all ϵ and δ and that U_f can be generated from them.

Let $\epsilon, \delta > 0$ we take $x \in U_{\epsilon,\delta}$ and define $m = \delta - \max\{d_1(x, y), d_1(x, z)\}$, we want to see that $B(x, m) \subset U_{\epsilon,\delta}$. If $w \in B(x, m)$ then

$$d_1(w, y) \leq d_1(w, x) + d_1(x, y) < m + d_1(x, y) \leq \delta$$

This proves that $y \in B(w, \delta)$ and the same reasoning can be applied to prove that $z \in B(w, \delta)$. Combining both statementes we get that $w \in U_{\epsilon,\delta}$.

Now, observe that $x \in \Omega_1$ is a point of discontinuity of f if and only if there exists an $\epsilon > 0$ such that for all $\delta > 0$ we have that $f(B(x, \delta)) \not\subset B(f(x), \epsilon)$. Expressing this with set operations we get

$$U_f = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} U_{\frac{1}{n},\frac{1}{m}}$$

This proves what we wanted.

Exercise 1.3.1

Exercise 1.3.3

Exercise 1.4.1 The *only if* assertion is more than clear, we procede with the *if* one. We base the proof on the observation that if h is an even function then $x \in h^{-1}(A)$ implies $-x \in h^{-1}(A)$ for any $x \in \mathbb{R}$ and any $A \subset \mathbb{R}$. With this in mind, it suffices to show that $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}) : x \in A \implies -x \in A\} \subset \sigma(f)$, this is what we will do.

More specific, we will show that for all $B \in \mathcal{B}(\mathbb{R})$ with $B \subset \mathbb{R}_{\geq 0}$, $B \cup -B \in \sigma(f)$. To do this observe that $\mathcal{C} = \{(-\infty, -a] \cup [a, \infty) : a \in \mathbb{R}_{\geq 0}\} \subset \sigma(f)$ and that by exchanging the generator with the trace we get

$$\begin{aligned}\mathcal{B}(\mathbb{R})|_{\mathbb{R}_{\geq 0}} &= \sigma(\{[a,\infty) : a \in \mathbb{R}\})|_{\mathbb{R}_{\geq 0}} \\ &= \sigma(\{[a,\infty) : a \in \mathbb{R}_{\geq 0}\}) \\ &= \sigma(\mathcal{C})|_{\mathbb{R}_{\geq 0}}\end{aligned}$$

The remaining steps are simple and left to the reader.

Exercise 1.4.2 Let f be as in **Ex 1.4.1** and let $(\mathbb{R}, \sigma(f), \lambda)$ be our measure space, we define the function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \begin{cases} 2 & \text{if } x = 1 \\ -2 & \text{if } x = -1 \\ |x| & \text{otherwise} \end{cases}$$

Now, f and g only differ in $\{1, -1\}$ which clearly has null measure. However, we have that $1 \in g^{-1}(2)$ and $-1 \notin g^{-1}(2)$ which implies that the set $g^{-1}(2)$ cannot be measurable (this assertion can be deduced from the solution of **Ex 1.4.1**) which conclues the proof.

Exercise 1.4.3 The differentiability of f implies it's continuity which in turn implies it's measurability. Now define the sequence of functions

$$f_n(c) = \frac{f(c + \frac{1}{n}) - f(c)}{(c + \frac{1}{n}) - c}$$

It is easy to verify that these are measurable functions and given the existance of the limit of the difference quotient of f we have $f'(c) = \limsup_{n\to\infty} f_n(c)$ for all $c \in \mathbb{R}$ which implies the measurability of f' and concludes the proof.

Exercise 1.4.5

Useful Properties