Chapter 2 - Independence

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Exercise Solutions

Theorem 2.5

The only implication we will be proving is *(ii)* $\implies$ *(iii)*, the rest are obvious. Let $\{\beta_i\}_{i\in I}$ be an arbitrary binary sequence and let $\{\alpha_i\}_{i\in I}$ be the binary sequence which makes the class of events independent, let $J \subset I$ be finite (in particular suppose that $J = \{1, ..., n\}$) and let $L \subset J$ be the set such that $i \in L \iff \alpha_i \neq \beta_i$. We define the following algorithm

$\mathcal{B} = \{C_i\}_{1\leq i\leq n}$
for $1 \leq i \leq n$ **do**
    $C_i = B_i^{\alpha_i}$
end for
for $1 \leq i \leq n$ **do**
    if $i \in L$ **then**
        $C_i = C_i^c$
    end if
end for
return $\mathcal{B}$

We claim that the follwing two assertions are invariants of the algorithm

- The family of events $\mathcal{B}$ is independent
- At the end of the i-th iteration we have that $\{C_j\}_{1\leq j\leq i} = \left\{B_j^{\beta_j}\right\}_{1\leq j\leq i}$

Before the first iteration, $\mathcal{B}$ is independent by hypothesis and $1 - 1 = 0$, so both assertions hold.

Assume that both assertions hold at the beginning of iteration i, we prove that they hold at it's end. If $i \notin L$, $\mathcal{B}$ is not modified and $C_i = B_i^{\beta_i}$, thus both properties hold. If $i \in L$, clearly the second assertin holds as we now have $C_i = B_i^{\beta_i}$, we prove that the first assertion holds through the following derivation

$$\mathbb{P}\left[\bigcap_{j=1}^{n} C_j\right] = \mathbb{P}\left[C_i\right] \prod_{1\leq j\neq i\leq n} \mathbb{P}\left[C_j\right] = \mathbb{P}\left[C_i\right]\mathbb{P}\left[\bigcap_{1\leq j\neq i\leq n} C_j\right]$$

Using the observation stated above the theorem, we have that the event C_i^c and the intersection of all the other events in $\mathcal{B}$ are independent which proves that the first assertion holds after the i-th iteration is completed.

Looking at the last iteration of the algorithm and putting together these two invariant we deduce that $\left\{B_i^{\beta_i}\right\}_{i\in J}$ is independent, as we wanted.

Useful Properties