

Chapter 2 - Independence

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Exercise Solutions

Theorem 2.5

The only implication we will be proving is $(ii) \implies (iii)$, the rest are obvious. Let $\{\beta_i\}_{i \in I}$ be an arbitrary binary sequence and let $\{\alpha_i\}_{i \in I}$ be the binary sequence which makes the class of events independent, let $J \subset I$ be finite (in particular suppose that $J = \{1, \dots, n\}$) and let $L \subset J$ be the set such that $i \in L \iff \alpha_i \neq \beta_i$. We define the following algorithm

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 $\mathcal{B} = \{C_i\}_{1 \leq i \leq n}$ 
for  $1 \leq i \leq n$  do
   $C_i = B_i^{\alpha_i}$ 
end for
for  $1 \leq i \leq n$  do
  if  $i \in L$  then
     $C_i = C_i^c$ 
  end if
end for
return  $\mathcal{B}$ 
```

We claim that the following two assertions are invariants of the second loop

- The family of events \mathcal{B} is independent
- At the end of the i -th iteration we have that $\{C_j\}_{1 \leq j \leq i} = \{B_j^{\beta_j}\}_{1 \leq j \leq i}$

Before the first iteration, \mathcal{B} is independent by hypothesis and $1 - 1 = 0$, so both assertions hold.

Assume that both assertions hold at the beginning of iteration i , we prove that they hold at its end. If $i \notin L$, \mathcal{B} is not modified and $C_i = B_i^{\beta_i}$, thus both properties hold. If $i \in L$, clearly the second assertion holds as we now have $C_i = B_i^{\beta_i}$. Assume that C_i has not been modified this iteration, we then have

$$\mathbb{P} \left[\bigcap_{j=1}^n C_j \right] = \mathbb{P}[C_i] \prod_{1 \leq j \neq i \leq n} \mathbb{P}[C_j] = \mathbb{P}[C_i] \mathbb{P} \left[\bigcap_{1 \leq j \neq i \leq n} C_j \right]$$

Using the observation stated above the theorem, we have that the event C_i^c and the intersection of all the other events in \mathcal{B} are independent which proves that the first assertion holds after the i -th iteration is completed.

Looking at the last iteration of the algorithm and putting together these two invariant we deduce that $\left\{B_i^{\beta_i}\right\}_{i \in J}$ is independent, as we wanted.

Useful Properties