

Chapter 1 - Basic Measure Theory

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Exercise Solutions

Exercise 1.1.3

Exercise 1.3.1

Exercise 1.3.3

Exercise 1.4.1 The *only if* assertion is more than clear, we proceed with the *if* one. We base the proof on the observation that if h is an even function then $x \in h^{-1}(A)$ implies $-x \in h^{-1}(A)$ for any $x \in \mathbb{R}$ and any $A \subset \mathbb{R}$. With this in mind, it suffices to show that $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}) : x \in A \implies -x \in A\} \subset \sigma(f)$, this is what we will do.

More specific, we will show that for all $B \in \mathcal{B}(\mathbb{R})$ with $B \subset \mathbb{R}_{\geq 0}$, $B \cup -B \in \sigma(f)$. To do this observe that $\mathcal{C} = \{(-\infty, -a] \cup [a, \infty) : a \in \mathbb{R}_{\geq 0}\} \subset \sigma(f)$ and that by exchanging the generator with the trace we get

$$\begin{aligned}\mathcal{B}(\mathbb{R})|_{\mathbb{R}_{\geq 0}} &= \sigma(\{[a, \infty) : a \in \mathbb{R}\})|_{\mathbb{R}_{\geq 0}} \\ &= \sigma(\{[a, \infty) : a \in \mathbb{R}_{\geq 0}\}) \\ &= \sigma(\mathcal{C})|_{\mathbb{R}_{\geq 0}}\end{aligned}$$

The remaining steps are simple and left to the reader.

Exercise 1.4.2 Let f be as in **Ex 1.4.1** and let $(\mathbb{R}, \sigma(f), \lambda)$ be our measure space, we define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(x) = \begin{cases} 2 & \text{if } x = 1 \\ -2 & \text{if } x = -1 \\ |x| & \text{otherwise} \end{cases}$$

Now, f and g only differ in $\{1, -1\}$ which clearly has null measure. However, we have that $1 \in g^{-1}(2)$ and $-1 \notin g^{-1}(2)$ which implies that the set $g^{-1}(2)$ cannot be measurable (this assertion can be deduced from the solution of **Ex 1.4.1**) which concludes the proof.

Exercise 1.4.3 The differentiability of f implies it's continuity which in turn implies it's measurability. Now define the sequence of functions

$$f_n(c) = \frac{f(c + \frac{1}{n}) - f(c)}{(c + \frac{1}{n}) - c}$$

It is easy to verify that these are measurable functions and given the existence of the limit of the difference quotient of f we have $f'(c) = \limsup_{n \rightarrow \infty} f_n(c)$ for all $c \in \mathbb{R}$ which implies the measurability of f' and concludes the proof.

Exercise 1.4.5

Useful Properties