Supplementary Materials to: Election Methods and Political Polarization

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Abstract

The Supplementary Materials serve three primary objectives: First, they demonstrate that the main findings remain valid in scenarios involving more than three candidates. Second, they provide detailed simulation results. Third, they offer additional explanations regarding the R and Python scripts available on GitHub.

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1. Ballot cardinality and vote splitting

A fourth candidate does not affect the number of votes favoring the polarizing candidate. In the best case, the fourth candidate gets no vote, and the plurality winner depends on whether γ_1 exceeds γ_2 or vice versa. Apart from the best case, the despisers split off their votes.

Let, in line with our previous notations, be candidate B the most-favored candidate by all despisers. Further, let $\gamma_2 > \gamma_1$. Candidate B is the plurality winner, defeating the (unique) PC. Now, consider vote-splitting as a result of the candidacy of the fourth candidate. Some of the despisers now cast a vote favoring the fourth candidate, which can alter the electoral outcome.

Again, under Borda's rule the electoral effect differs from that under PR, as can be illustrated by the following example: Consider 100 voters and three candidates; among them both a PC and a CC. All voters rank the CC medium, 51 rank the PC top. For the sake of exposition and to keep things simple, let us consider a Borda scheme where the top-ranked gets one Borda score, the second-ranked two and so on. The Borda winner is then the candidate with the fewest scores. The PC becomes the Borda winner with 198 digit points (200 for the CC). Now, consider a fourth candidate on the ballot. 51 voters still rank the PC top, 49 still bottom and all voters rank the CC second. Now, the PC is defeated with $247 (> 200)$ points.

We will generalize this illustration as follows: We calculate the expected Borda scores for a (unique) PC (denoted B_{pc}) and the expected Borda scores for the despisers' top-choice given that s/he is a consensus candidate (B_{cc}). Then, we analyze the effect of the expected scores if the number of candidates increases.

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$$B_{pc} = \gamma_1 + k \cdot \gamma_2 + \frac{(1-\gamma_1-\gamma_2)}{k} \cdot \sum_{j=1}^{k} j \tag{1}$$

$$B_{cc} = \frac{\gamma_1}{k-2} \cdot \sum_{j=2}^{k-1} j + \gamma_2 + \frac{(1-\gamma_1-\gamma_2)}{k-1} \cdot \sum_{j=1}^{k-1} j. \tag{2}$$

Lemma 1
Under Borda's rule, and on domain $R \in \mathscr{R}$, more contestants harm a polarizing candidate more than the despisers' top-choice, given the latter is a CC.

Proof (Proof of Lemma 1). By using the Gaussian sum, we find after some rearrangements

$$\Delta_k\left(B_{pc}\right) = \frac{1}{2}(1-\gamma_1+\gamma_2), \text{and} \tag{3}$$

$$\Delta_k\left(B_{cc}\right) = \frac{1-\gamma_2}{2}. \tag{4}$$

$\Delta_k\left(B_{pc}\right)$ exceeds $\Delta_k\left(B_{cc}\right)$ (viz., higher k harm the PC more than the CC) whenever $2{\cdot}\gamma_2 > \gamma_1$, a condition always fulfilled given Assumption **??**.

Secondly, we will argue that the PC is also negatively affected by more candidacies even when some restrictions are relaxed. In particular, let us assume that the despisers have no favored candidate in common. Instead, they rank all candidates but the PC randomly with equiprobability on ranks 1 to $k-1$. Aside from A, the supporters rank any candidates randomly on ranking positions $2, \ldots, k$. Finally, the moderates rank all candidates randomly over the entire scale. The assumption is that all candidates (except the PC) are sorted according to the so-called impartial culture approach (ICA). It denotes a uniform distribution over linear preference orders, such that all orderings are equally likely for every voter (Sen, 2017, Ch. 10.2). It is rightly claimed that such preferences are hardly conceivable (Tsetlin et al., 2003; Lehtinen and Kuorikoski, 2007). Miller (2019) reasonably stated that the ICA "is in effect to assume the absence of any 'culture' at all—that is, the absence of any characteristic structuring of voter preferences." However, without attempting to defend the ICA, we use it to demonstrate that Lemma 1 can be generalized even by assuming no consensus candidate to run. It can be argued that the ICA depicts the best case for a PC. In this best case, the expected Borda score for the PC remain unaltered, and those for the CC becomes

$$B_{cc}^{(2)} = \frac{\gamma_1}{k-1} \cdot \sum_{j=2}^{k} j + \frac{\gamma_2}{k-1} \cdot \sum_{j=1}^{k-1} j + \frac{(1-\gamma_1-\gamma_2)}{k} \cdot \sum_{j=1}^{k} j. \tag{5}$$

The effect of an increasing ballot size is then represented by Eq. (3) and $\Delta_k\left(B_{cc}^{(2)}\right) = \frac{1}{2}$. An increase of k affects the PC lower than any of his or her opponents if $\gamma_1 > \gamma_2$. However, to become the electoral winner, not only has the PC to beat one particular candidate but *all* the opponents. Unlike a dice game, one will beat one opponent with probability 1/2, but with probability 1/4, one beats two of them. Therefore, the more candidates are running, the more this harms the PC success probability under Borda's rule.

Although voter blocks as used in our theoretical part are often used in social choice theory, in practice the number of voter groups is large. The term *vote splitting* is used to describe the situation when candidate x would beat y in a one-on-one contest but loses to y when z runs too because z splits off some of the votes that otherwise would go to x (Dasgupta and Maskin, 2020; Sen, 2020). Bluntly speaking, it is unlikely that all voters of the despiser group cast a vote to a single candidate (distinct from A). Even with three candidates on the ballot and a single PC (see profile 6), they split their votes in favour of the non-polarizing candidates. Condition **I** states that a voting rule shall not be affected by vote splitting. However, this precisely happens when the winner is selected by plurality rule.

Rank	γ_1	$\frac{\gamma_2}{2}$	$\frac{\gamma_2}{2}$	$\frac{\gamma_3}{3}$	$\frac{\gamma_3}{3}$	$\frac{\gamma_3}{3}$
1	A	B	C	A	B	C
2	.	C	B	.	.	.
3	.	A	A	.	.	.

Table 1: The effect of vote splitting in a polarized profile

As an example, consider that in a three-candidate/three-voter group profile the PC is not selected due to $\gamma_2 > \gamma_1$. If now not all the despisers rally behind B, but split-off their votes on B and C (with equiprobability), then the PC wins though short of the majority ($\gamma_1 < \gamma_2$).[1] Table 1 illustrates the effect of vote splitting.

We refer to the expected scores in a Borda rule as *Borda scores*. For any $R \in \mathscr{R}$, they are given by

$$B_{pc} = \gamma_1 + k \cdot \gamma_2 + \frac{(1-\gamma_1-\gamma_2)}{k} \cdot \sum_{j=1}^{k} j \tag{6}$$

$$B_{cc} = \frac{\gamma_1}{k-2} \cdot \sum_{j=2}^{k-1} j + \gamma_2 + \frac{(1-\gamma_1-\gamma_2)}{k-1} \cdot \sum_{j=1}^{k-1} j. \tag{7}$$

[1] Since the moderates split their votes uniformly on the three candidates, they constitute a *neutral set* (Saari, 2000).

Now, we use the Gaussian sum, i. e.

$$\sum_{j=1}^{k} j = \frac{k}{2}(k+1) \tag{8}$$

$$\sum_{j=1}^{k-1} j = \frac{k}{2}(k-1) \tag{9}$$

$$\sum_{j=2}^{k-1} j = \frac{(k^2-k-2)}{2} \tag{10}$$

to yield

$$\Delta_k(B_{pc}) = \frac{1}{2}(1-\gamma_1+\gamma_2) \tag{11}$$

$$\Delta_k(B_{cc}) = \frac{1-\gamma_2}{2} \tag{12}$$

An increase in the ballot size harms the consensus candidate more than the PC if $\Delta_k(B_{cc}) > \Delta_k(B_{pc}) \Leftrightarrow \gamma_1 > 2 \cdot \gamma_2$.

Recall the Assumption $\gamma_1, \gamma_2 \in (\gamma_3, 1/2)$. The assumption implies $\gamma_1 < 2 \cdot \gamma_2$. For instance, let $\gamma_1 = \frac{1}{2} - \epsilon$. Then $\gamma_2 = \frac{\gamma_1}{2} = \frac{1}{4} - \epsilon \implies \gamma_3 > \gamma_2$.

2. Simulation results

2.1. Simulation design

We used the possibility of high-performing computing to simulate electoral outcomes in a large electorate with different numbers of alternatives (candidates) and different degrees of polarization. As a measure of polarization, we use the average size of γ_1 and γ_2, viz., $\gamma \equiv \frac{\gamma_1+\gamma_2}{2}$. We are aware that this is neither a well-established, nor a theoretically sound measure. The rough idea behind this measure is that the smaller the group of moderates, the higher the share of individuals with wedged preferences. In the paper, we have demonstrated that γ_3 aligns with Gehrlein' k and r, and we have referenced to Campante and Hojman (2013) who regard the size of the moderates as inverse measure of polarization.

The main argument for using this measure - aside from its simplicity - is that in numerical simulations, the measure well correlates with the well-established an persuasively developed polarization measure by Can et al. (2015). We show this below.

The frequently-used voting schemes and the often-proposed ones are applied to the profiles gained by our numerical simulations. We make use of R by using the `vote` package Sevčíková et al. (2021).

This is not the first attempt to raise electoral results by using extensive simulations (see, e.g. Jones et al., 1995; Dougherty and Heckelman, 2020). However, the growing computing capacities of so-called supercomputers (high-performing computers) allow us to go into deeper detail, in particular, to make cross-comparisons. To our best knowledge, the present paper is the first study to use simulations to analyse and explain the success of polarizing candidates.

In the best case, the PC is equally affected from ballot enlargement like any other single candidate if $\gamma_1 = \gamma_2$. Therefore, we focus on what happens on this margin.

For our simulations, we consider different levels of the polarization factor ranging from 10% to 40% in 5%-steps; Thus: $\gamma \in \{0.10, 0.15, 0.20, \ldots, 0.40\}$.

2.2. Profiles

We simulated a large number of elections, each with an electorate consisting of 20k voters with strict preferences over a set of candidates. The number of candidates on the ballot, k, ranges from $k_{min} = 3$ to $k_{max} = 8$. One of them (we, again, call the candidate A) is a polarizing candidate (PC). A is a PC because a share γ ranks him first, and the same proportion of voters despise him and accordingly rank him last, as outlined in the article. Again, there is a third group of moderate voters in the sense that they rank A randomly. Except for A, all the other candidates are ranked randomly (impartial culture approach) in order to emphasize the effect of polarization.

Since γ can take seven different values and we consider six distinct numbers of candidates on a ballot $(3, \ldots, 8)$, we compute $25k \times 7 \times 6 = 1,050,000$ profiles each consisting of 20k voters. To each profile we apply five different voting rules, which we briefly present below.

2.3. Simulations

In order to focus on the effect of altering the ballot size, we assume the size of supporters and the despiser's size to equal, $\gamma \equiv \gamma_1 = \gamma_2$.

Since γ can take seven different values and we consider six distinct numbers of candidates on a ballot, we computed $25k \times 7 \times 6 = 1,050,000$ profiles each consisting of 20k voters. Aside from the three above-mentioned voting methods, we considered also the absolute-majority rule cum run-off (*plurality runoff-voting*) and the Single-Transferable Voting (Hare method).

Figure 1 illustrates some simulation results. The left-hand side depicts PC's success probability under the Borda count with eight candidates. An increasing γ increases the winning probability of the polarizing candidate. The special exemption is—as expected due to this paper's results—the Borda count. Here, an increasing degree of polarization leads to a significant reduction of PC's success probability. Conversely, the PC becomes a plurality winner regardless of the polarization degree.

As carved out in the previous Section, increasing candidacies reduces PC's (average) probability of being elected. We depicted on the right panel the winning probability of the PC under the Borda rule for different γ-values and different numbers of opponents. The variations indicated by the box plots are due to different γ-values. For an appropriate interpretation of the box plots, consider the box

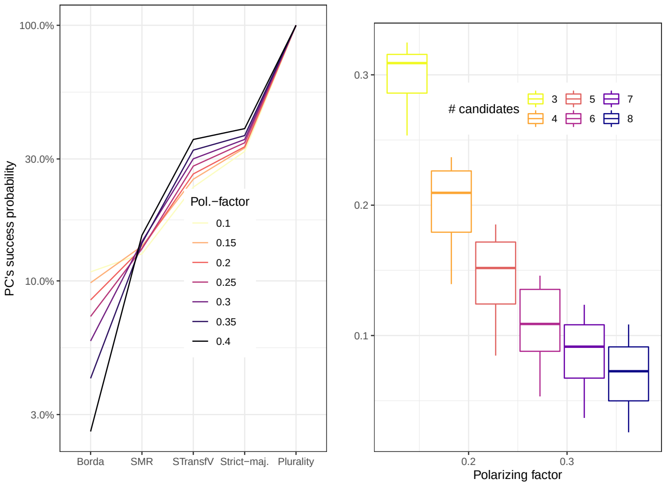

Figure 1: Simulation results: Probability for a PC's electoral success under different voting rules.

plot at $k = 3$ (three candidates). The winning probability lies between 27 and 32 percent, depending γ. With more candidates, the overall winning probability vanishes, but the dispersion within a category increases.

2.4. The polarizing candidate's success under different voting rules

We now provide results on a polarizing candidate's success under different voting rules. We denote by $\phi \in (0, 1) \subset \mathbb{R}$ the probability for an electoral victory of the polarizing candidate (PC).

We introduce a second measure to describe the difference between the probability success of a PC compared to a non-polarizing candidate. For that, we compare the PC's success probability with the success probability of a candidate in a *homogenous situation* without any polarization and ideological predetermination. In such a *homogenous situation*, where the choice between two arbitrary candidates is the choice between Tweedledee and Tweedledum, the probability for a candidate to win an election is given by $^1/_m$. We denote this second measure by

$$\Phi(\phi(\gamma, k)) = k \cdot \phi - 1. \tag{13}$$

For instance, let the probability of a PC' success be $\phi = 60\%$ and consider $k = 5$ candidates on the ballot. The probability of a PC's success is three times higher than in the homogenous situation such that the probability increases by $\Phi = 200\%$.

Let us start by focussing on the ϕ-value. Figure 2 depicts the probability for an electoral victory of the polarizing candidate in four spider-web plots. The plots consider four values of polarization, and the different lines indicate the ϕ-value depending on the number of candidates.

For the sake of clarity, we will enumerate some of our observations in Results.

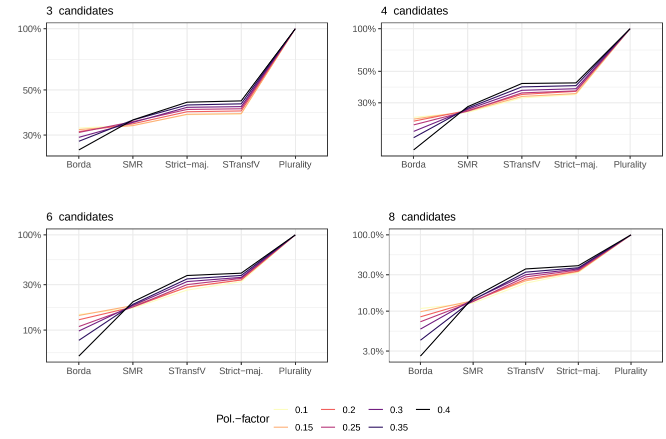

Figure 2: The success of the polarizing candidate under different voting rules - differentiated by the number of candidates.

Our starting point is the plurality rule. It can easily be seen that any non-negative polarization factor increases the success probability of the PC. Since any of the PC's rivals can expect a number of votes equal to $(1 - 2 \cdot \gamma)\frac{1}{k} + \gamma\frac{1}{k-1}$ and the PC itself has an expected number of votes equal to $\gamma + (1 - 2 \cdot \gamma)\frac{1}{k}$, it can readily be seen that the PC's expected amount of votes in his favour exceeds that of any of his rivals whenever $1 > (k-1)^{-1}$. This condition holds for all $k > 2$. In each of the roundabout one million simulated elections, the PC is the plurality winner.

Result 2.1
For any pair (γ, k), the polarizing candidate (PC) is always the plurality winner.

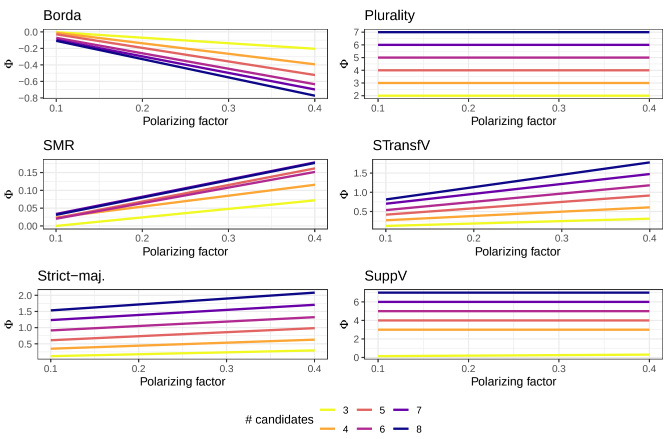

Figure 3: Φ-values for different voting rules and distinct pairs of (γ, k)

Next, we focus on the simple-majority rule (SMR), the strict-majority rule (AV) and the single-transferable vote (STV). They have in common what we note as:

Result 2.2
The PC's probability of being successful under SMR, STV, and AV depends positively on γ and negatively on m.

Thus, a higher polarization increases the PC's success expectation, and more opposing candidates reduce it in absolute terms (ϕ-value). By asking how a larger ballot size affects the relative probability, Φ, we derive a reverse result: the higher the number of candidates, the higher Φ. This is depicted in Figure 3 (where we show the Φ-values also for the other voting rules).

It is important to note that despite Result 2.2, the probability of success (ϕ) differs remarkably among the three voting rules, as depicted in Figure 4. PC's probability for a successful electoral campaign is significantly smaller when pair-wise comparisons determine the electoral outcome.

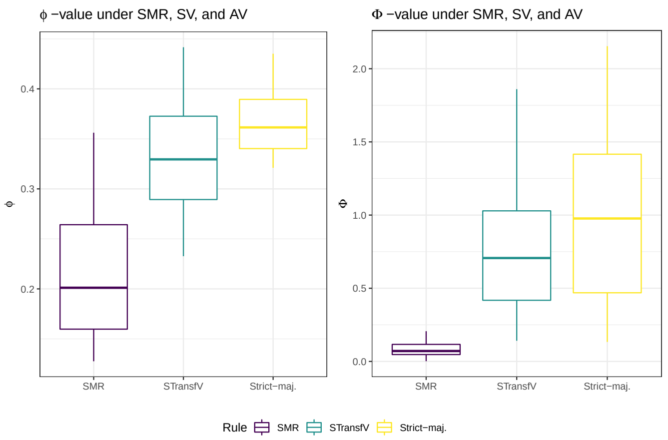

Figure 4: The distribution of ϕ and of Φ for the simple-majority rule, the single-transferable voting and the strict-majority rule.

Thus, it is worth focusing more detailed on the simple-majority rule. Since it brings forth a Condorcet winner, the PC's success under SMR indicates how often the PC is the Condorcet winner. Depending on the polarization factor and the number of candidates on the ballot, the respective value ranges from 12.7% to 35.6%. The highest value occurs with three candidates and $\gamma = 0.4$, the lowest with eight candidates and a polarizing factor equal to 0.1[2].

3. Measures of polarization

To characterize the role of polarizing and consensus candidates we used the k and r parameters from Gehrlein (2005). We have shown that both parameters coincide with γ_3. In a similar vein, Campante and Hojman (2013) argue that the share of 'moderates' can be regarded as an inverse measure of polarization. This section discusses the more detailed measures of polarization.

Facing the spread of polarization and authoritarianism, it is all but surprising that several studies deal with these phenomena. The research comes from different areas of social sciences. Economics-related papers[3] often deal with the modeling and the measurement of polarization. The latter is far from being a new topic. Instead, the question of how to measure the heterogeneity of profiles was addressed several decades ago by Kendall and Smith (1939). They developed what is today known and widely used as the Kendall measure of concordance. The idea is appealingly simple: Let a table represent a profile with the $k > 0$ alternatives (candidates) in the columns and the n voters in the rows. If all the voters have the identical orderings (perfect resemblance), then the columns' sum would be $n, 2 \cdot n, 3 \cdot n, \ldots, k \cdot n$ in some order, and the corresponding squares-of-deviations sum would be as large as possible, i. e., $S = \frac{n^2(k^3-k)}{12}$. The measure is based on the sum of squares of deviations around their mean value and is given by

$$W = \frac{12\,S}{n^2\,(k^3-k)} \subset (0,1) \in \mathbb{R}. \tag{14}$$

Note that W is a measure of resemblance, and therefore a higher W indicates a higher concordance. So, in case of a perfect match (resemblance), $W = 1$.

Some attempts have been made to analyse similarities between W (and Spearman's ρ) and the dispersion measures (e.g. Borroni and Zenga, 2006). However, Esteban and Ray (1994) have persuasively demonstrated that a mass polarization can not and shall not be expressed by using measures of income or wealth dispersion (like the Gini or the Atkinson measure). Instead, what accentuates polarization is the co-occurrence of intra-group homogeneity (cohesion (cf. Alcalde-Unzu and Vorsatz, 2012), "ingroup responsiveness" (cf. Diermeier and Li, 2019)) and inter-group heterogeneity (see also Baldiga and Green, 2011). The latter is represented by an alienation function, the first mentioned by an identification function, whose slope depends on a parameter α, which Esteban and Ray label the *degree of polarizing sensitivity*. A measure for polarization, which has the homotheticity property and satisfies a small set of axioms (in particular: various forms of pooling leads to higher polarization), depends on α only. If $\alpha = 0$, then the polarizing measure coincides with the Gini coefficient. Thus, the more citizens with an identical or similar variable (like income) feel *integrated* to each other, the higher polarization, and the higher the distinction between polarization and dispersion.

Based inter alia on this work, Can et al. (2015) present a polarization measure for strict preference orderings. Let us denote in line with standard notations in social-choice

[2]The PC is even a strong Condorcet winner (a strong Condorcet winner is a candidate who receives more than half of the votes total; see Barberà et al. (2020)) if enough voters from the not-polarized group rank him first. With $\gamma = 0.4$, twenty per cent of the electorate rank the candidates randomly. With three candidates on the ballot only, the PC can expect to get 40% $+20/3$% $\sim$ 46.67% of the votes total. However, the PC is likely to become a Condorcet winner without reaching an absolute majority of the votes. In particular, if a large share of voters in the not-polarized group rank him second.

[3]An overview of the state in the political sciences is provided, e.g. by Hetherington (2009).

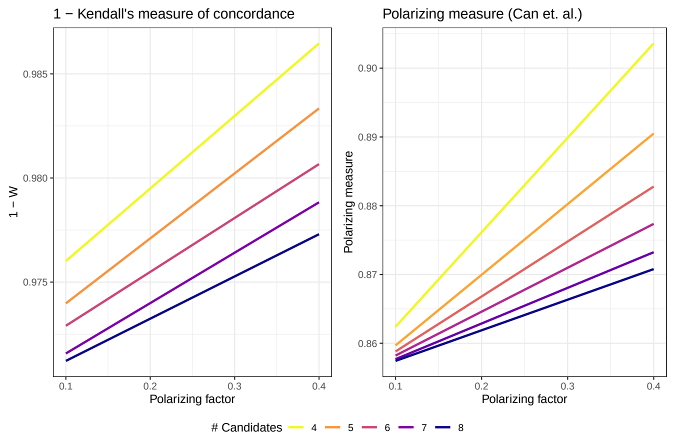

Figure 5: Measures by Kendall and Smith (1939) and Can et al. (2015) and how they correlate with the polarizing factor

theory the number of voters who prefer alternative a over b by $N(a \succ b)$. Then $d_{ab} \equiv |N(a \succ b) - N(b \succ a)|$ can be used to describe the heterogeneity within the (of n voters consisting) electorate with respect to their opinions over any tuple (a, b). Can et al. (2015) show that only the measure

$$\psi(\succ_1, \dots, \succ_n) = \sum_{\{a,b\} \in \bar{\mathscr{B}}}^{n} \frac{n - d_{ab}}{n \cdot \binom{m}{2}} \tag{15}$$

can satisfy a set of reasonable axioms. Here, m denotes the number of alternatives and $\bar{\mathscr{B}}$ the set of all subsets from the set of all alternatives, $\mathscr{B}$, with cardinality 2.

We show that our *polarization factor* is closely related to existing measures presented and discussed in the paper. Figure 5 depicts this. We calculated regression lines over the around one million profiles with respect to $1 - W$ (1 minus Kendall's measure of concordance) and the polarization measure by Can et al. (2015). Though the levels differ, it can be seen that both measures correlate with the polarization factor used in the paper.

4. Condorcet Consistent Voting Rules

Please find on two scripts that ultimately do the same: they run for all possible configurations of γ_i values several Condorcet consistent methods to obtain the winner. Basically, the scripts show that C is never the (unique) winner in Profile 5.

The script relates to the repository by Eric Pacuit (Link). It runs the following Condorcet consistent rules:

1. Split Cycle
2. Copeland
3. Ranked Choice
4. Minimax
5. Beat Path (Schulze)

The script similarly generates all possible configurations of γ_i values and considers

1. Dodgson rule (with 'quick' method)
2. Dodgson rule (with Tideman scores)
3. Copeland
4. Kemmeny-Young
5. Schulze (Beat Path)
6. Ranked Pairs.

The script make use of the `votesys` Package by Jiang Wu, Version 0.0.1.

5. Detailed Rearrangements and Derivations

At certain points, we have omitted detailed derivations. This section provides a step-by-step presentation of some transformations and derivations.

The first regards Section 4.1 (Condorcet-Consistent Rules). We have to show that $\phi_2 > \phi_1$, and $\phi_2 > \phi_3$.

Proof ($\phi_2 > \phi_1$). $\phi_2 > \phi_1 \iff 2(\gamma_1 + \gamma_2) - 1 > 1 - 2\gamma_2$. This inequality can be simplified to $\gamma_1 + 2\gamma_2 > 1$. Because $1 = \sum_i \gamma_i$, we can simplify the expression to $2\gamma_2 > \gamma_2 + \gamma_3 \iff \gamma_2 > \gamma_3$. The last inequality complies with the domain $\gamma_1, \gamma_2 \in (\gamma_3, 1/2)$. □

Proof ($\phi_2 > \phi_3$). $\phi_2 > \phi_3 \iff 2\gamma_1 + 2\gamma_2 - 1 > 1 - 2\gamma_1 \iff 2\gamma_1 + \gamma_2 > 1$. The remainder is identical to the preceding proof. □

Proof ($\phi_1 > \phi_3 \iff \gamma_1 > \gamma_2$). $1 - 2\gamma_2 > 1 - 2\gamma_1 \iff \gamma_1 > \gamma_2$. □

The second part relates to Section 4.2 (Scoring rules).

Proof ($BP^{\lambda}_{2}C$). $BP^{\lambda}_{2}C \iff \lambda\gamma_1 + \gamma_2 > \lambda\gamma_2 + \lambda(1 - \gamma_1 - \gamma_2)$. After some rearrangements we get $\lambda < \frac{\gamma_2}{1 - 2\gamma_1}$. In words: for B to defeat C a scoring rules requires that $\lambda < 1$. □

Proof ($BP^{\lambda}_{5}C$). $\gamma_2 + \lambda\gamma_1 > \lambda\gamma_2 + 1 - \gamma_1 - \gamma_2 \iff \gamma_2 + \lambda\gamma_1 - \lambda\gamma_2 > 1 - \gamma_1 - \gamma_2$. Isolating λ yields $\lambda(\gamma_1 - \gamma_2) > 1 - \gamma_1 - 2\gamma_2 \iff \lambda > \frac{1 - \gamma_1 - 2\gamma_2}{|\gamma_1 - \gamma_2|}$. □

Admittedly, we did not explicitly exclude the case $\gamma_1 = \gamma_2$ in the paper, and the restriction to Borel sets does not rule out this case either. Thus, we are dealing here with a case that occurs with probability $\to 0$, which would need to be addressed separately.

6. Numerical Examples for Scoring-Rule Outcomes

We provide detailed illustrations of Saari triangles corresponding to profiles ① through ⑧, utilizing the parameters $\gamma_1 = 0.42$ and $\gamma_2 = 0.35$ consistently across all instances. These parameters are chosen to reflect the distribution of support that positions the PC A as the plurality winner in profiles ② and ④ through ⑧. The gray shaded areas specifically denote the areas where polarizing candidate(s) secure a victory.

Property 1 (Property PPC). Property **PPC** holds if, for a given profile, a voting rule brings not forth a polarizing candidate as the electoral winner.

The λ-values essential to satisfy the PPC condition exhibit significant variation across different electoral profiles. Notably, in certain cases, such as profiles ③ and ④, the required λ-values are so elevated that, in practical terms, only the application of the anti-plurality rule is effective in precluding the electoral triumph of a PC. This underscores the unique capability of this rule to counteract the polarization dynamics inherent in specific voting scenarios.

Conversely, for other profiles, such as ⑦, the Borda method demonstrates adequacy in meeting the **PPC**. However, it's noteworthy that the Dowdall method does not achieve the same level of effectiveness in this context, indicating a nuanced distinction between scoring rules and their applicability in curbing the success of polarizing figures.

Moreover, the PR predominantly facilitates the electoral victory of a PC, with the exception of profile ①. On the other side of the spectrum, the application of the anti-plurality rule consistently lead to a defeat of the PC(s), indicated by the procedure lines achieving the unshaded area in each profile at $\lambda = 1$.

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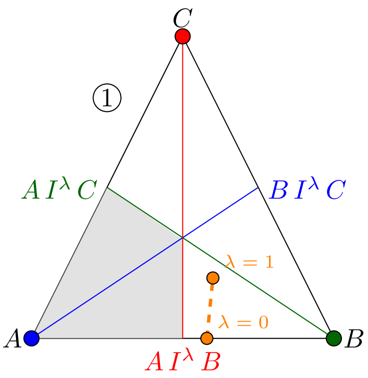
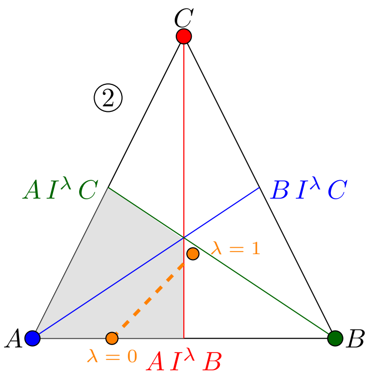
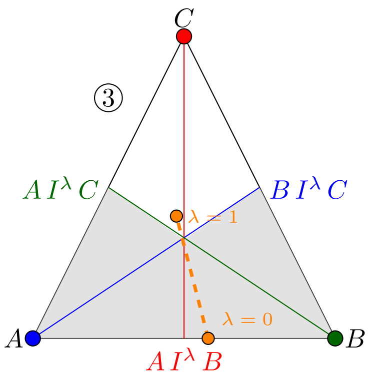
Figure 6: Saari-triangles and respective procedure lines for $\gamma_1 = 0.42, \gamma_2 = 0.35, \gamma_3 = 0.23$



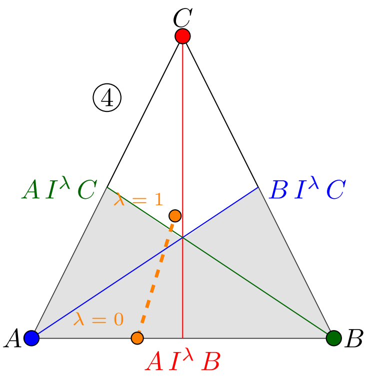
①: **PPC** if $\lambda > 0$.



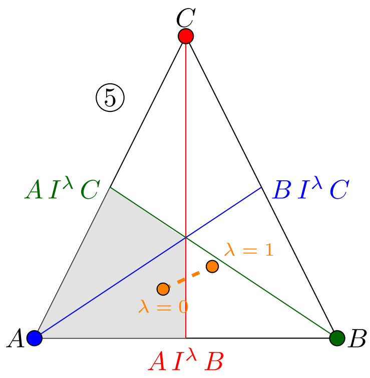
②: **PPC** if $\lambda > 0.741$.



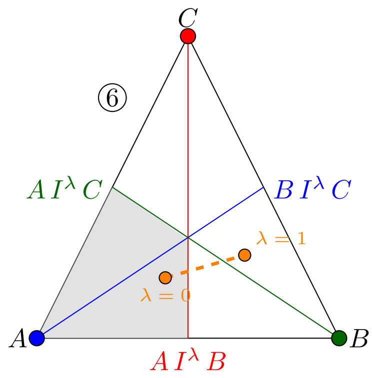
③: **PPC** if $\lambda > 0.\overline{77}$.



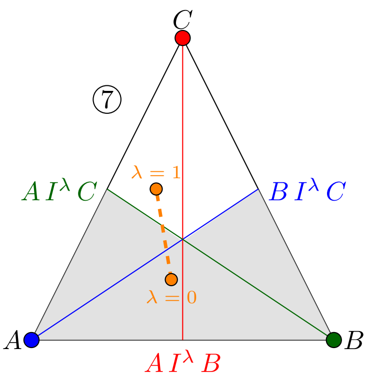
④: **PPC** if $\lambda > 0.844$.



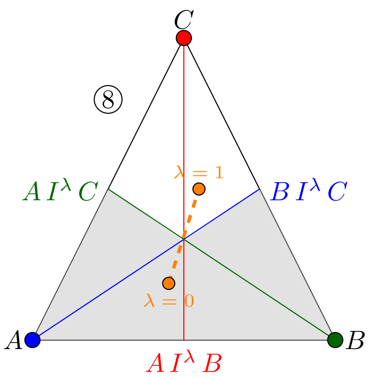
⑤: **PPC** if $\lambda > 0.368$.



⑥: **PPC** if $\lambda > 0.108$.



⑦: **PPC** if $\lambda > 0.352$.



⑧: **PPC** if $\lambda > 0.247$.