Univariate Linear Regression

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1 Univariate Linear Regression

In this exercise, we will try to accomplish a linear regression using only one variable of the dataset used.

Let's start by importing some libraries that we will use:

```
In [1]: # %load ../../standard_import.txt
        import pandas as pd
        import numpy as np
        import matplotlib.pyplot as plt

        from sklearn.linear_model import LinearRegression
        from mpl_toolkits.mplot3d import axes3d

        pd.set_option('display.notebook_repr_html', False)
        pd.set_option('display.max_columns', None)
        pd.set_option('display.max_rows', 150)
        pd.set_option('display.max_seq_items', None)

        #%config InlineBackend.figure_formats = {'pdf',}
        %matplotlib inline

        import seaborn as sns
        sns.set_context('notebook')
        sns.set_style('white')
```

1.1 Linear Regression with one variable

In a linear regression with only one variable, we can write the hypothesis function as follows:

$$h_\theta(x) = \theta_0 x_0 + \theta_1 x_1 = \theta_0 + \theta_1 x_1$$

where the last equality is due to the convention of x_0 being chosen to be 1.

Let's parse our dataset and let's show its dimensions:

```
In [2]: data = np.loadtxt('data/ex1data1.txt', delimiter=',')
        print("The dimension of the dataset is {}".format(data.shape))
```

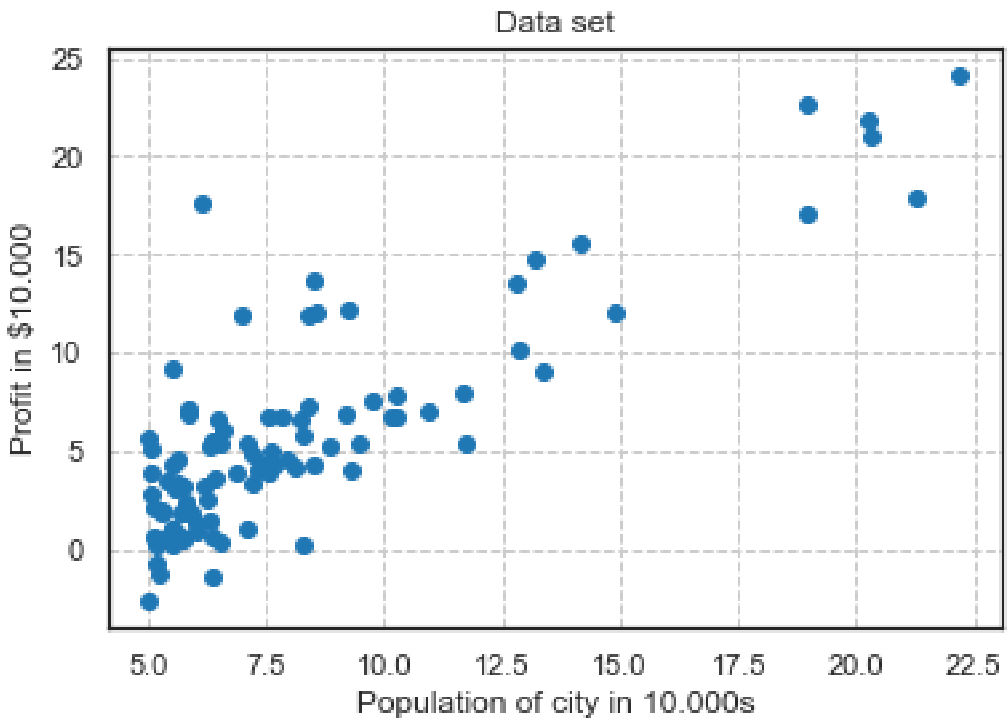
```
The dimension of the dataset is (97, 2)
```

Let's now add a column of 1 to our training features and let's treat the target feature as a separate variable:

```
In [3]: x = np.c_[np.ones(data.shape[0]), data[:, 0]]
        y = np.c_[data[:, 1]]
```

Let's plot the dataset:

```
In [4]: plt.scatter(x[:,1],y)
        plt.title('Data set')
        plt.xlabel('Population of city in 10.000s')
        plt.ylabel('Profit in $10.000')
        plt.grid(linestyle="--")
```



1.2 Cost function

We will use a cost function as a metric of goodness to track how the training phase goes. The cost function has the following form:

$$J(\theta) = \frac{1}{2m}\sum_{j=1}^{m}(h_\theta(x^{(j)}) - y^{(j)})^2$$

where:

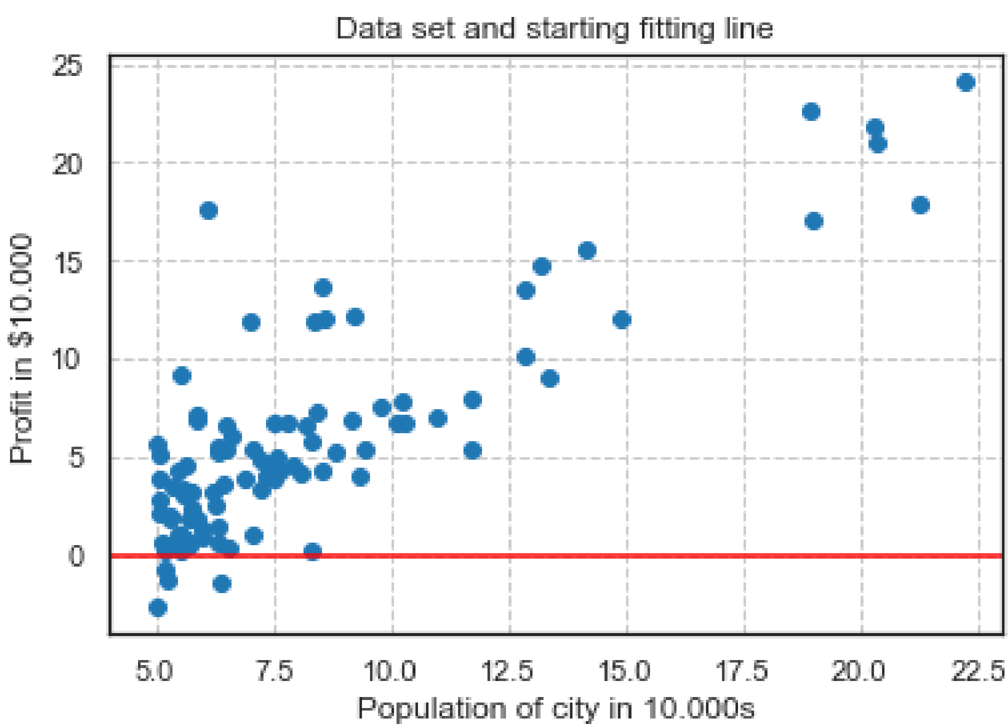
- $h_\theta(x^{(j)})$ represents the prediction that our model does;

- $y^{(j)}$ is the true value for the j-th example. In the best case, we'd like our model to predict this value;
- m is the number of training samples at our disposal.

```
In [5]: def calculate_cost(x, y, theta = [[0],[0]]):
            h = x.dot(theta)
            return 1/(2 * data.shape[0]) * np.sum(np.square(h - y))
```

```
In [6]: theta = [[0],[0]]
        plt.scatter(x[:,1],y)
        plt.plot(np.arange(0, 24) * theta[1] + theta[0], 'r')
        plt.title('Data set and starting fitting line')
        plt.xlabel('Population of city in 10.000s')
        plt.ylabel('Profit in $10.000')
        plt.grid(linestyle="--")
        plt.xlim(4, 23)
```

```
Out[6]: (4, 23)
```



In the graph above, the red line represents the line representing the parameters vector (having both the intercept and the angular coefficient set to 0). **The model doesn't fit the data as we will train it later**.

1.3 Batch Gradient Descent

Let's use the *batch gradient descent*, which uses the whole training set to predict the parameters that approximate the data in the best possible way. The problem with this approach is that it becomes slower as the dataset becomes bigger. For this reason, it is best to use this approach only with small training sets.

Since we're trying to reduce the cost function, we will consider the gradient of this function, which can be written in two version:

- Scalar version;
- Vectorized version.

1.3.1 Gradient function

Scalar version

$$\frac{\partial J}{\partial \theta_i} = \frac{1}{m}(h_\theta x^{(i)} - y^{(i)})x^{(i)}$$

Vectorized version

$$\frac{\partial J}{\partial \theta}(\theta) = \frac{1}{m}x^T(x\theta - y)$$

Gradient descent The gradient just introduced has to be used in a method (an iterative one). We will implement an algorithm that uses the following formula:

$$\theta^{(k)} = \theta^{(k-1)} - \frac{\alpha}{m}\frac{\partial J}{\partial \theta_i}(\theta^{(k)})$$

At each iteration, the current estimate of the parameters gets updated using their last estimate and a term that depends on a learning rate α and the gradient value.

```
In [7]: iterations_to_do = 1500
        initial_cost = calculate_cost(x, y)
        costs = [initial_cost]
```

```
In [8]: def gradient_descent(x, y, alpha = .1, theta=[[0],[0]], iterations = 1500):
            for i in range(0, iterations):
                theta = theta - (alpha / data.shape[0]) * (x.T).dot(x.dot(theta) - y)
                costs.append(calculate_cost(x, y, theta))
            return theta
```

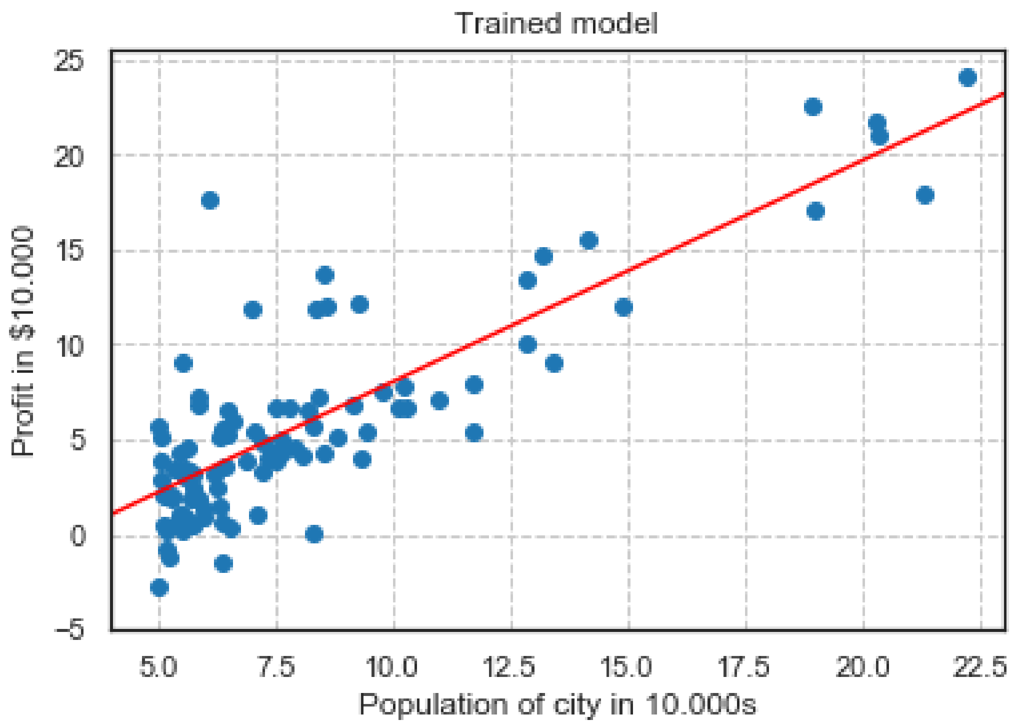
```
In [9]: final_theta = gradient_descent(x, y, alpha = .01,
                    theta=[[0],[0]], iterations = iterations_to_do)

        print("The estimated parameters that fit our date are: \n{}"
              .format(final_theta))
```

```
The estimated parameters that fit our date are:
[[-3.63029144]
 [ 1.16636235]]
```

1.4 Final results

Let's now plot our dataset and let's see how our trained model fits the data:

```
In [10]: plt.scatter(x[:,1],y)
         plt.plot(np.arange(0, 24) * final_theta[1] + final_theta[0], 'r')
         plt.title('Trained model')
         plt.xlabel('Population of city in 10.000s')
         plt.ylabel('Profit in $10.000')
         plt.xlim(4,23)
         plt.grid(linestyle='--')
```



By reducing the cost function (see the next section), we end up having a couple of parameters. The red line is a representation of the current estimate of these parameters that we can use to predict new values of the target feature.

1.5 Errors

We can see how the error is affected when the parameters change. Let's see the differences between the first and the last iteration.

```
In [11]: print("The error at the first iteration is: {}"
                 .format(initial_cost))
```

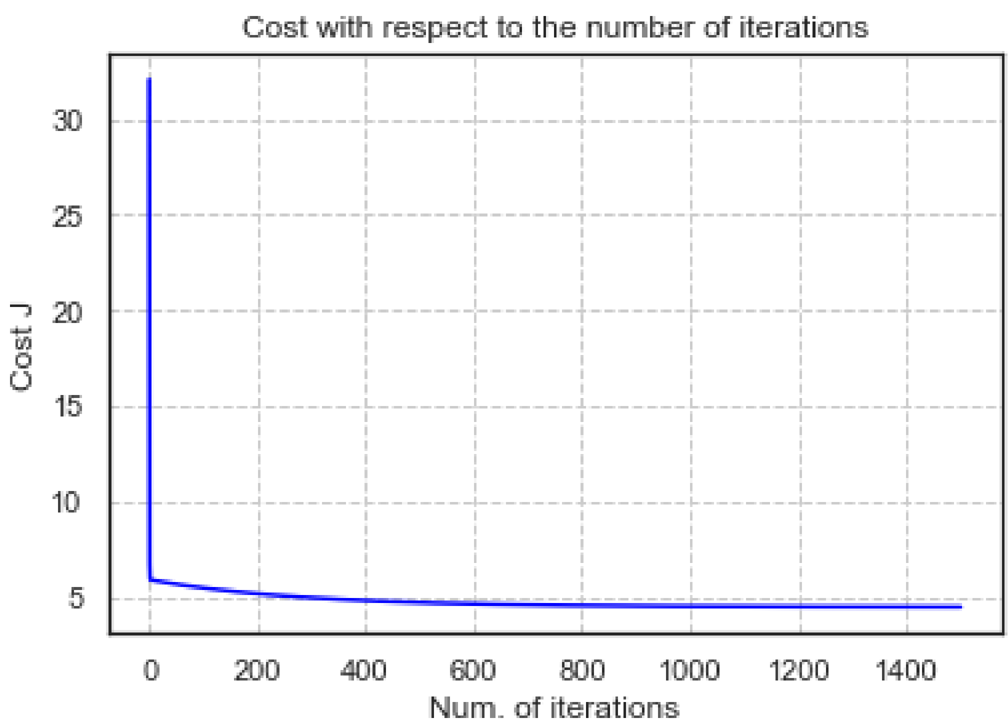
```
The error at the first iteration is: 32.072733877455676
```

```
In [12]: print("The error at the last iteration is: {}"
               .format(calculate_cost(x, y, final_theta)))
```

```
The error at the last iteration is: 4.483388256587725
```

Overall cost on different iteration Even if the differences between the costs is explanatory, we usually want to see how the gradient descent affects the parameters vector and how the cost behaves at each iteration. Let's plot this behaviour:

```
In [13]: plt.plot(np.arange(0, iterations_to_do + 1, 1), costs, 'b')
         plt.title('Cost with respect to the number of iterations')
         plt.xlabel('Num. of iterations')
         plt.ylabel('Cost J')
         plt.grid(linestyle='--')
```



1.6 Prediction from learned parameters

Let's use the parameters found to predict:

1. The profit for a population of 3.5 thousand people;
2. The profit for a population of 7 thousand people.

We will use the vectorized version of the hypothesis function to perform the prediction, using the following formula:

$$h_\theta(x) = \theta^T x$$

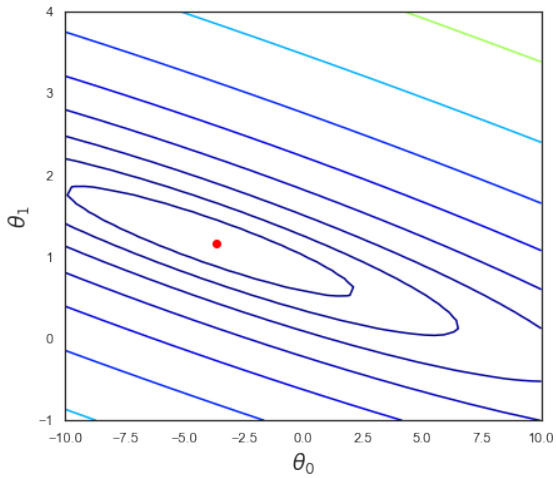
```
In [14]: print("The profit for a population of 3.5k people is: {} $"
               .format(final_theta.T.dot([1, 3.5])*10000))
         print("The profit for a population of 7k people is: {} $"
               .format(final_theta.T.dot([1, 7])*10000))
```

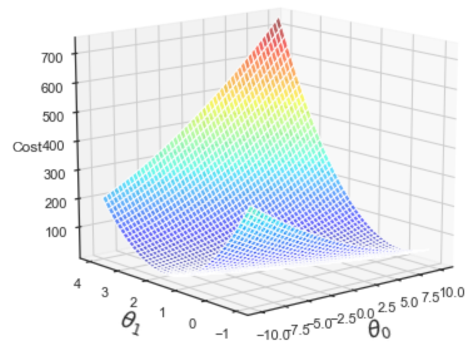
```
The profit for a population of 3.5k people is: [4519.7678677] $
The profit for a population of 7k people is: [45342.45012945] $
```

1.7 Show if the optimization has gone right

To see if the optimization of the cost function worked, we can use a contour plot of the cost function with respect to the parameters in the θ vector.

```
In [15]: # Create grid coordinates for plotting
         B0 = np.linspace(-10, 10, 50)
         B1 = np.linspace(-1, 4, 50)
         xx, yy = np.meshgrid(B0, B1, indexing='xy')
         Z = np.zeros((B0.size,B1.size))

         # Calculate Z-values (Cost) based on grid of coefficients
         for (i,j),v in np.ndenumerate(Z):
             Z[i,j] = calculate_cost(x,y, theta=[[xx[i,j]], [yy[i,j]]])

         fig = plt.figure(figsize=(15,6))
         ax1 = fig.add_subplot(121)
         ax2 = fig.add_subplot(122, projection='3d')

         # Left plot
         CS = ax1.contour(xx, yy, Z, np.logspace(-2, 3, 20), cmap=plt.cm.jet)
         ax1.scatter(final_theta[0],final_theta[1], c='r')

         # Right plot
         ax2.plot_surface(xx, yy, Z, rstride=1, cstride=1, alpha=0.6, cmap=plt.cm.jet)
         ax2.set_zlabel('Cost')
         ax2.set_zlim(Z.min(),Z.max())
         ax2.view_init(elev=15, azim=230)

         # settings common to both plots
         for ax in fig.axes:
             ax.set_xlabel(r'$\theta_0$', fontsize=17)
             ax.set_ylabel(r'$\theta_1$', fontsize=17)
```





In the left image, the red dot represents the parameters found using the gradient descent algorithm. It is possible to see that it is near the center of the smaller ellipsis, which represents the region where we have the minimum value of the cost function.

In the right image, it is possible to see a 3D plot of the cost function.