Multivariate Linear Regression

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1 Multivariate Linear Regression

The dataset we will use in this exercise is made up of three columns with the following informations:

1. dimension of the house (in square meters);
2. number of bathrooms;
3. the price of the houses.

Before we can apply the multivariate logistic regression, we have to deal with the problem of the features having a different order of magnitude. We can solve this problem by using a technique called **feature scaling**. If we don't solve the problem, we will end up having a problem with the speed of converging of the gradient descent algorithm.

1.1 Load data

In [1]:
```
import numpy as np
from matplotlib import pyplot as plt

data = np.loadtxt('data/ex1data2.txt', delimiter=',')

print(data.shape)
```

```
(47, 3)
```

In [2]:
```
x = data[:, :2]
y = data[:, -1]

# from n-dimensional vector to m x 1 matrix
y = np.reshape(y, (y.shape[0], 1))

m = x.shape[0]
print("Training examples: {}".format(m))

n = x.shape[1]
print("Features: {}".format(n))

print("10 examples: \n", x[:10, :])
```

```
Training examples: 47
Features: 2
10 examples:
 [[2.104e+03 3.000e+00]
 [1.600e+03 3.000e+00]
 [2.400e+03 3.000e+00]
 [1.416e+03 2.000e+00]
 [3.000e+03 4.000e+00]
 [1.985e+03 4.000e+00]
 [1.534e+03 3.000e+00]
 [1.427e+03 3.000e+00]
 [1.380e+03 3.000e+00]
 [1.494e+03 3.000e+00]]
```

1.2 Features normalization

To prevent our classifier from giving more influence to the larger features, we need to apply a **feature normalization** step before the training phase. Applying this technique will allow the classifier to treat all the features equally while keeping the same knowledge as before. Another highlight of this method is that it speed-up the gradient descent, as it will converge early to the minimum.

We can apply the normalization step as follows:

1. Subtract the mean value of the feature to each feature example;
2. Divide each feature example by the standard deviation of the feature.

```
In [3]: def feature_normalize(x):
            x_norm = x

            mu = np.zeros((1, x.shape[1]))
            sigma = np.zeros((1, x.shape[1]))

            mu = np.mean(x, axis = 0) # mean value
            sigma = np.std(x, axis = 0) # std deviation value

            for i in range(x.shape[1]):
                x_norm[:,i] = (x[:,i] - mu[i])/sigma[i]

            return x_norm, mu, sigma
```

```
In [4]: x_norm, mu, sigma = feature_normalize(x)
        print("10 normalized examples: \n", x_norm[:10, :])
        print("Mean value: {}".format(mu))
        print("Standard deviation value: {}".format(sigma))
```

```
10 normalized examples:
 [[ 0.13141542 -0.22609337]
 [-0.5096407  -0.22609337]
```

```
 [ 0.5079087  -0.22609337]
 [-0.74367706 -1.5543919 ]
 [ 1.27107075  1.10220517]
 [-0.01994505  1.10220517]
 [-0.59358852 -0.22609337]
 [-0.72968575 -0.22609337]
 [-0.78946678 -0.22609337]
 [-0.64446599 -0.22609337]]
Mean value: [2000.68085106    3.17021277]
Standard deviation value: [7.86202619e+02 7.52842809e-01]
```

The chosen hypothesis function is as follow:

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

We need to add a new column of 1 to the dataset to deal with the θ_0 parameter.

```
In [5]: x_norm = np.concatenate([np.ones((m,1)), x_norm], axis = 1)

        print("The shape of the dataset is {}".format(x_norm.shape))
```

```
The shape of the dataset is (47, 3)
```

Let's now introduce two function:

- the former is a function to perform the gradient descent of the cost function;
- the latter is a function used to calculate the value of the cost function.

```
In [6]: def gradient_descent(x, y, theta, alpha = 0.1, iterations = 1500):
            costs = []
            for i in range(0, iterations):
                theta = theta - (alpha / x.shape[0]) * (x.T).dot(x.dot(theta) - y)
                costs.append(calculate_cost(x, y, theta))
            return theta, costs

        def calculate_cost(x, y, theta = [[0],[0],[0]]):
            h = x.dot(theta)
            return 1/(2 * x.shape[0]) * np.sum(np.square(h - y))
```

Let's run the gradient descent using a value of 0.1 for the learning rate (α) while running 1500 iterations:

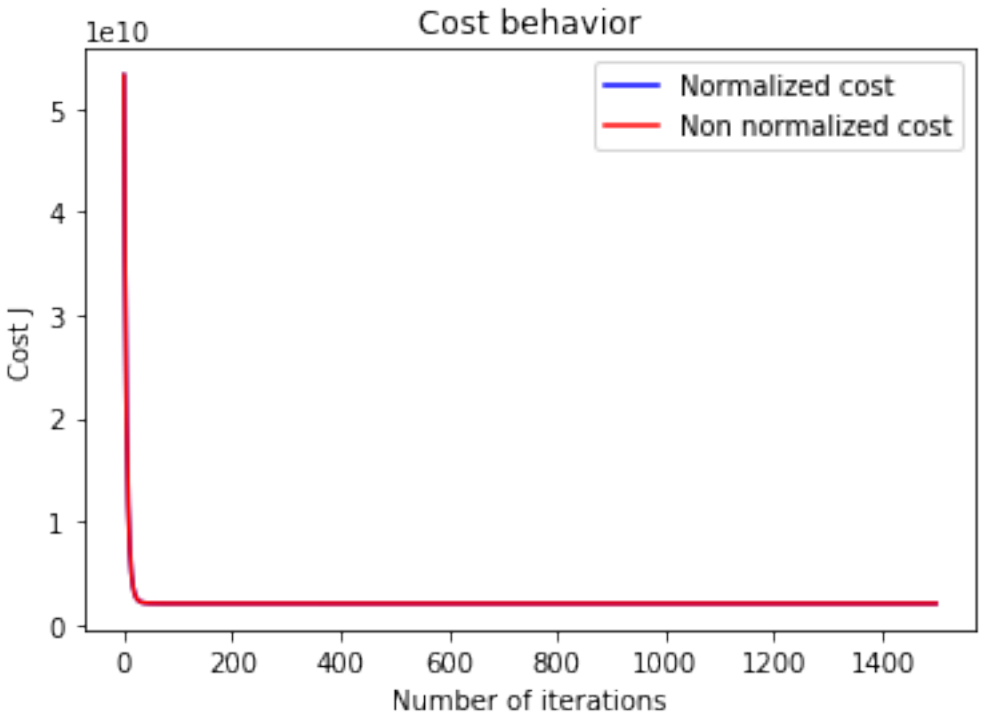
```
In [7]: alpha = 0.1
        num_iters = 1500

        theta = np.zeros((3, 1))
        theta, cost_history = gradient_descent(x_norm, y, theta, alpha, num_iters)
```

```
        theta_no_norm = np.zeros((3, 1))
        x = np.concatenate([np.ones((m,1)), x], axis = 1)
        theta_no_norm, cost_history_no_norm = gradient_descent(x, y,
                                              theta_no_norm, alpha, num_iters)

        print("Non normalized theta:\n {}\n\n".format(theta_no_norm))
        print("Normalized theta:\n {}\n\n".format(theta))
```

```
Non normalized theta:
 [[340412.65957447]
 [109447.79646964]
 [ -6578.35485416]]

Normalized theta:
 [[340412.65957447]
 [109447.79646964]
 [ -6578.35485416]]
```

Let's plot the cost function of the costs calculated on the normalized and non-normalized datasets:

```
In [8]: # Plot the convergence graph
        plt.plot([i for i in range(num_iters)],
                 cost_history, '-b', label = 'Normalized cost')
        plt.plot([i for i in range(num_iters)],
                 cost_history_no_norm, '-r', label = 'Non normalized cost')
        plt.xlabel('Number of iterations') # Set the x-axis label
        plt.ylabel('Cost J') # Set the y-axis label
        plt.title('Cost behavior')
        plt.legend()
        plt.show()
```



In the graph above the functions overlaps. It's for this reason that we can see only the red curve.

1.3 Normal equations

We can use a method called the **Normal Equation** instead of the Gradient Descent iterative method. We can get the same estimate of the parameters using a closed form solution and without applying any feature scaling. The normal equation is as follows:

$$\theta = (X^T X)^{-1} X^T y$$

We can only use this method for smaller dataset without a large number of features, in fact, we need to calculate the inverse matrix of $X^T X$, which can be a resource-consuming task and that can bring numerical instability when the matrix is a singular one (to solve this last problem, we can use the *pseudo-inverse* of the matrix). For big dataset, it is mandatory to use the Batch Gradient Descent or the Stochastic Gradient Descent.
While the **Batch Gradient Descent** computes the next step like:

$$\theta^{(j+1)} = \theta^{(j)} - \frac{\alpha}{m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)} \right)$$

The **Stochastic Gradient Descent** computes at each iteration the θ value using only a small amount of data:

$$\theta^{(j+1)} = \theta^{(j)} - \alpha \left(h_\theta(x^{(j)}) - y^{(j)} \right)$$

Let's implement the equation:

```
In [9]: def normal_equations(x, y):
            return np.linalg.pinv(x.T.dot(x)).dot(x.T).dot(y)
```

Let's use it on our dataset:

```
In [10]: normal_theta = normal_equations(x, y);
         print(normal_theta)
```

```
[[340412.65957447]
 [109447.79646964]
 [ -6578.35485416]]
```

The value of the θ parameter calculated using the normal equation is the same as the one calculated with the gradient descent, even if here we haven't chosen a learning parameter α e we did not do any feature scaling. These two are the strong point for the technique of the normal equations.

1.4 Let's make some predictions

We want to estimate the price for a 1650 square meters house having 3 bedrooms:

```
In [11]: # Let's create the array x (the features used for prediction)
         for_prediction = np.array([1650, 3])
         # Let's use the values of the average value and
         # standard deviation to do feature scaling
         for_prediction = np.array([1,
                                    (for_prediction[0] - mu[0])/sigma[0],
                                    (for_prediction[1] - mu[1])/sigma[1]])
         # Let's add a the second dimensions
         for_prediction = for_prediction.reshape((for_prediction.shape[0], 1))

         # Let's predict the cost using the hypothesis values
         predicted_cost = theta.T.dot(for_prediction)
         print('Predicted cost for a 1650 squared meters and 3 bedrooms house: {}'
               .format(predicted_cost))
```

```
Predicted cost for a 1650 squared meters and 3 bedrooms house: [[293081.4643349]]
```

1.5 Gradient descent behaviors

During the training phase, we use the gradient descent algorithm, which depends on a parameter, the **learning rate**.

The possible scenarios are:

1. If we use a very *little* value for the learning rate, the time needed to the gradient descent to converge will be longer;
2. If we use a *big* value for the learning rate, we can end up overshooting the global (or local) minimum. Eventually, we can end up diverging.

There is no known way to set the learning rate to an optimal value. It's for this reason that it is necessary to try different values for α and keep the one which works better.

Let's see the differences when using different types of learning rates.

1.5.1 Small learning rate

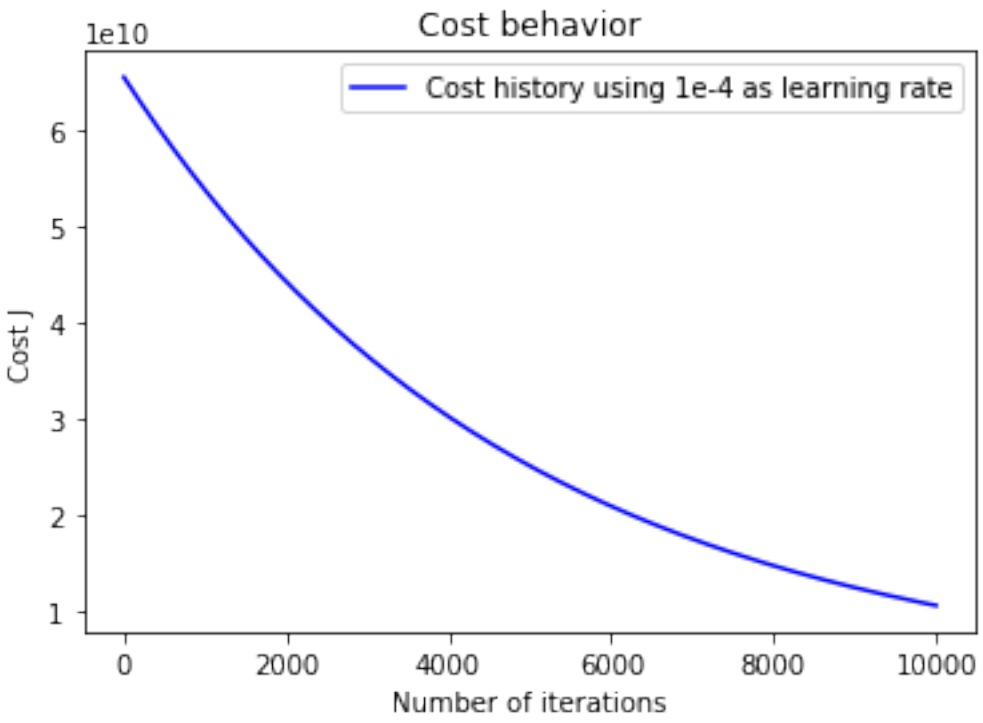
We will use a tiny learning rate, whose value will be 10^{-4} over 10^4 iterations:

```
In [12]: alpha = 0.0001
         num_iters = 10000

         theta_little = np.zeros((3, 1))
         theta_little, cost_history_little = gradient_descent(x_norm, y,
                                                 theta_little, alpha, num_iters)

         print("Theta using a learning rate of 1e-4: \n{}".format(theta_little))
```

```
Theta using a learning rate of 1e-4:
[[215188.10240439]
 [ 61279.52044616]
 [ 19975.26231463]]
```

Let's plot the cost function with respect to the number of iterations:

```
In [13]: # Plot the convergence graph
         plt.plot([i for i in range(num_iters)],
                  cost_history_little, '-b',
                  label = 'Cost history using 1e-4 as learning rate')
         plt.xlabel('Number of iterations')
         plt.ylabel('Cost J')
         plt.title('Cost behavior')
         plt.legend()
         plt.show()
```



As can be seen in the graph, using a small learning rate ($\alpha = 10^{-4}$) slows down the learning phase.

Using such a small learning rate prevents our algorithm from converging to the optimal minimum in 10000 iterations, even if it can be reached in less than 50 iterations as before (using $\alpha = 0.1$).

1.5.2 Big learning rate

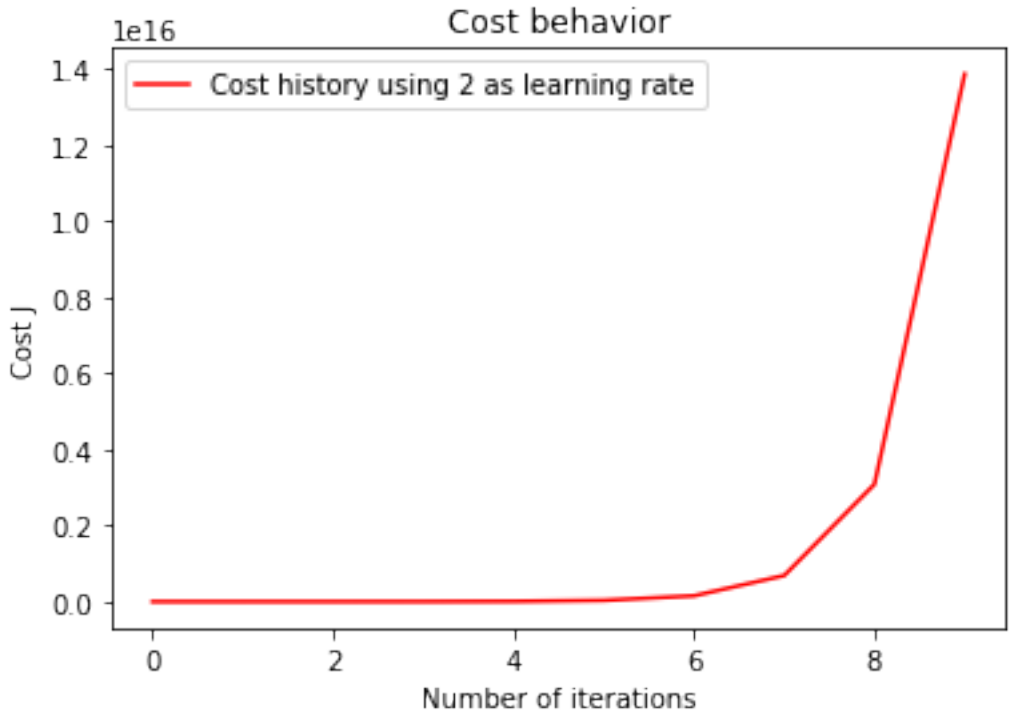
We'll now see the effects of using a high learning rate. The chosen value for the learning rate is 2, and we will use it over 10 iterations.

```
In [14]: alpha = 2
         num_iters = 10
         theta_big = np.zeros((3, 1))
         theta_big, cost_history_big = gradient_descent(x_norm, y,
                                                        theta_big, alpha, num_iters)

         print("Theta using a learning rate of 1: \n{}".format(theta_big))
```

```
Theta using a learning rate of 1:
[[-4.38885763e-08]
 [-9.41838069e+07]
 [-9.42998330e+07]]
```

Let's now plot the graph as we did before:

```
In [15]: # Plot the convergence graph
         plt.plot([i for i in range(num_iters)],
                  cost_history_big, '-r',
                  label = 'Cost history using 2 as learning rate')
         plt.xlabel('Number of iterations')
         plt.ylabel('Cost J')
         plt.title('Cost behavior')
         plt.legend()
         plt.show()
```



The opposing event happens when we choose the learning rate to be high. In this case, the algorithm will overshoot the minimum and will diverge from it. This behavior is represented in the graph, as **the cost increases along with the number of iterations increases**.

1.5.3 Using a different initial condition

Another thing that can modify the algorithm's convergence is the choice of the starting point given to the *gradient descent* algorithm as it follows the descending direction of the gradient calculated in that chosen point.

Let's use the values

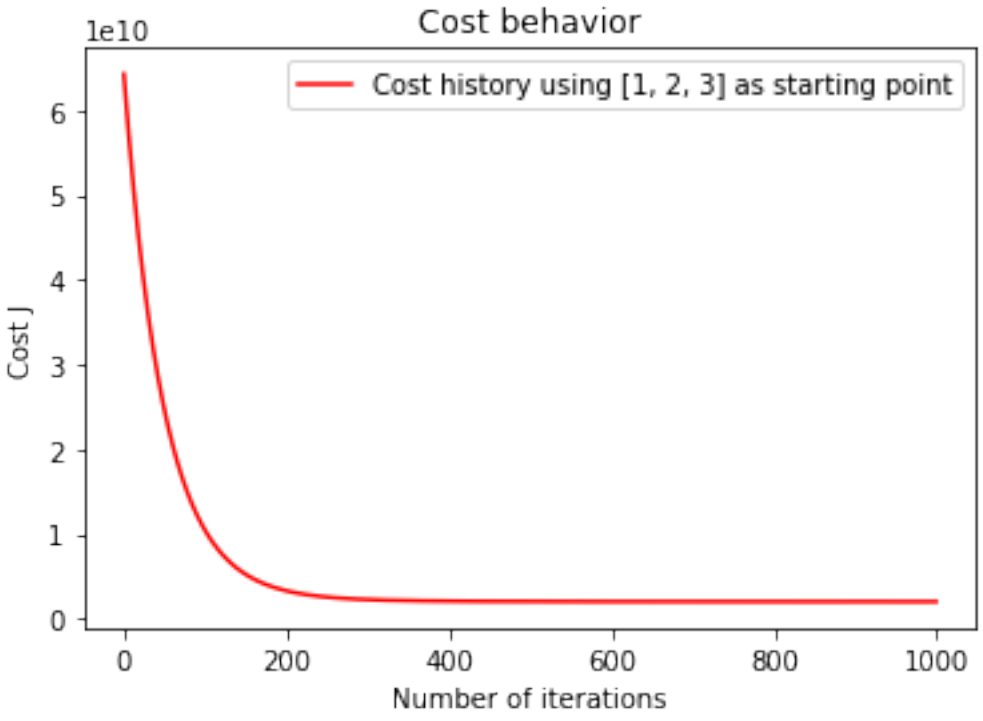
$$\theta_0 = 1 \quad \theta_1 = 2 \quad \theta_3 = 3$$

as the initial conditions:

In [16]:
```
alpha = .01
num_iters = 1000
theta = np.array([1, 2, 3]).reshape((3,1))
theta, cost_history = gradient_descent(x_norm, y,
                                       theta, alpha, num_iters)

print("Theta using a learning rate of 1: \n{}"
      .format(theta_big))
```

```
Theta using a learning rate of 1:
[[-4.38885763e-08]
 [-9.41838069e+07]
 [-9.42998330e+07]]
```

In [17]:
```
# Plot the convergence graph
plt.plot([i for i in range(num_iters)],
         cost_history, '-r',
         label = 'Cost history using [1, 2, 3] as starting point')
plt.xlabel('Number of iterations')
plt.ylabel('Cost J')
plt.title('Cost behavior')
plt.legend()
plt.show()
```



The choice of the starting point doesn't affect the algorithm convergence because the function to be optimized is a **bowl-shaped** function, which has a local minimum that is equal to the global one.

When the function to be optimized has different local and global minima, it can happen that if we choose a starting point near to a local minimum, the algorithm will converge to it, not the global one.

It is usually useful to run the gradient descent algorithm using different choices of the starting point and to save the different values of the cost function calculated. After all of these runs, **we will choose the parameter with the smallest error attached as the optimal value of the parameters**.