Normalization of M-rep evaluations

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Following the paper's notations, the distance function is given as

$$d(P) = \prod_i \sigma_i(M), \tag{1}$$

where $P \in \mathbb{R}^3$, σ_i is the i-th singular value, and I write M for $\mathbb{M}_\nu(P)$ for simplicity. This can also be written as

$$d(P) = \sqrt{\det(MM^T)}, \tag{2}$$

cf. Section 4.3.

Since we want to normalize by the norm of the gradient (treating M as a function of P, the point of evaluation), we need to compute

$$n = \left\| \frac{\partial d}{\partial x}, \frac{\partial d}{\partial y}, \frac{\partial d}{\partial z} \right\|, \tag{3}$$

so the new distance will be

$$\hat{d}(P) = d(P)/n(P). \tag{4}$$

Using Jacobi's formula, the derivative in direction $w \in \{x, y, z\}$ is

$$\frac{\partial d}{\partial w} = \frac{1}{2\sqrt{\det(MM^T)}} \cdot \operatorname{tr}\left(\operatorname{adj}\left(MM^T\right) \cdot \frac{\partial MM^T}{\partial w}\right), \tag{5}$$

where $\operatorname{tr}(A)$ is the trace of A, and

$$\operatorname{adj}(A) = A^{-1} \cdot \det(A) \tag{6}$$

is its adjugate.

Since M is constructed as the pencil

$$M = M_0 + P_x \cdot M_x + P_y \cdot M_y + P_z \cdot M_z, \tag{7}$$

the derivative $\partial MM^T/\partial w$ can be computed as

$$\frac{\partial MM^T}{\partial w} = M_w M^T + MM_w^T. \tag{8}$$