G^1 direction blend with approximate twists

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We are given 3 cubic Bézier curves defined by control points $\mathbf{P}_{i,j}$ with $i = 0 \dots 3$ and $j = -1, 0, 1$. These, in turn, define two cubic half-ribbons (one by $j = 0, 1$ and another by $j = 0, -1$). Note that the latter (the 'opposite' side) has different orientation, so $\mathbf{P}_{i,1}$ is opposite $\mathbf{P}_{3-i,-1}$. We can define the following quantities (we omit the v parameter throughout, as it is always 0):

$$\mathbf{R}_u(0) = 3(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}), \qquad \mathbf{R}_u(1) = 3(\mathbf{P}_{3,0} - \mathbf{P}_{2,0}), \tag{1}$$
$$\mathbf{R}_v(0) = 3(\mathbf{P}_{0,1} - \mathbf{P}_{0,0}), \qquad \mathbf{R}_v(1) = 3(\mathbf{P}_{3,1} - \mathbf{P}_{3,0}), \tag{2}$$
$$\mathbf{R}^*{}_v(0) = 3(\mathbf{P}_{3,-1} - \mathbf{P}_{0,0}), \qquad \mathbf{R}^*_v(1) = 3(\mathbf{P}_{0,-1} - \mathbf{P}_{3,0}), \tag{3}$$

and also the slightly longer expressions

$$\begin{aligned}
\mathbf{R}_v(0.5) &= (\mathbf{R}_v(0) + \mathbf{R}_v(1) + 3(\mathbf{P}_{1,1} - \mathbf{P}_{1,0} + \mathbf{P}_{2,1} - \mathbf{P}_{2,0}))/8, \\
\mathbf{R}^*_v(0.5) &= (\mathbf{R}^*_v(0) + \mathbf{R}^*_v(1) + 3(\mathbf{P}_{2,-1} - \mathbf{P}_{1,0} + \mathbf{P}_{1,-1} - \mathbf{P}_{2,0}))/8, \\
\mathbf{T}_0 &= 9(\mathbf{P}_{1,1} - \mathbf{P}_{1,0} - \mathbf{P}_{0,1} + \mathbf{P}_{0,0}), \\
\mathbf{T}_1 &= 9(\mathbf{P}_{3,1} - \mathbf{P}_{3,0} - \mathbf{P}_{2,1} + \mathbf{P}_{2,0}).
\end{aligned}$$

We define a common *direction blend* as

$$\mathbf{D}(u) = (2u^2 - 3u + 1)\bar{\mathbf{R}}_v(0) + 4u(1-u)\bar{\mathbf{R}}_v(0.5) + (2u^2 - u)\bar{\mathbf{R}}_v(1), \tag{4}$$

where $\bar{\mathbf{R}}_v(u) = \frac{1}{2}(\mathbf{R}_v(u) - \mathbf{R}^*_v(u))$ is the averaged cross-derivative function. We want to create a modified cross-derivative of the form

$$\hat{\mathbf{R}}_v(u) = \mathbf{R}_u(u) \cdot \alpha(u) + \mathbf{D}(u) \cdot \beta(u) \tag{5}$$

in such a way that it interpolates the original end tangents, and also approximates the original twists. We assume that the scalar functions α and β are of the form

$$\alpha(u) = (1-u)^3\alpha_0 + 3(1-u)^2u\alpha^*_0 + 3(1-u)u^2\alpha^*_1 + u^3\alpha_1, \tag{6}$$
$$\beta(u) = (1-u)^3\beta_0 + 3(1-u)^2u\beta^*_0 + 3(1-u)u^2\beta^*_1 + u^3\beta_1. \tag{7}$$

Let us define the notation $\Xi(\mathbf{w}; \mathbf{u}, \mathbf{v})$ to be the coordinates of $\mathbf{w}$ in the $(\mathbf{u}, \mathbf{v})$ system. The system is not necessarily perpendicular, and $\mathbf{w}$ may not lie in the plane (in which case it is approximated). This is computed by

$$\Xi(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \left(\frac{\|\mathbf{v}\|^2\langle\mathbf{w}, \mathbf{u}\rangle - \langle\mathbf{u}, \mathbf{v}\rangle\langle\mathbf{w}, \mathbf{v}\rangle}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \langle\mathbf{u}, \mathbf{v}\rangle}, \frac{\|\mathbf{u}\|^2\langle\mathbf{w}, \mathbf{v}\rangle - \langle\mathbf{u}, \mathbf{v}\rangle\langle\mathbf{w}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \langle\mathbf{u}, \mathbf{v}\rangle} \right). \tag{8}$$

With this we can write the constraints on the scalar functions as

$$(\alpha_0,\beta_0)=\Xi(\mathbf{R}_v(0);\mathbf{R}_u(0),\bar{\mathbf{R}}_v(0)),\quad (\alpha_1,\beta_1)=\Xi(\mathbf{R}_v(1);\mathbf{R}_u(1),\bar{\mathbf{R}}_v(1)). \tag{9}$$

As for the twist constraints, we need

$$\frac{\partial}{\partial u}\hat{\mathbf{R}}_v(0)\approx\mathbf{T}_0,\qquad\qquad \frac{\partial}{\partial u}\hat{\mathbf{R}}_v(1)\approx\mathbf{T}_1. \tag{10}$$

Derivating Eq. (5) we get

$$\frac{\partial}{\partial u}\hat{\mathbf{R}}_v(u)=\mathbf{R}_{uu}(u)\cdot\alpha(u)+\mathbf{R}_u(u)\cdot\alpha'(u)+\mathbf{D}'(u)\cdot\beta(u)+\mathbf{D}(u)\cdot\beta'(u).$$

For $u=0$, this becomes

$$\begin{aligned}
&36(\mathbf{P}_{1,0}-\mathbf{P}_{0,0})\alpha_0^*+18(\mathbf{P}_{0,1}-\mathbf{P}_{3,-1})\beta_0^*\approx\\
&36(\mathbf{P}_{1,1}-\mathbf{P}_{1,0}-\mathbf{P}_{0,1}+\mathbf{P}_{0,0})+\\
&12(-2\mathbf{P}_{2,0}+7\mathbf{P}_{1,0}-5\mathbf{P}_{0,0})\alpha_0+\\
&3(\mathbf{P}_{3,1}-11\mathbf{P}_{3,-1}-3\mathbf{P}_{2,1}+3\mathbf{P}_{2,-1}-3\mathbf{P}_{1,1}+3\mathbf{P}_{1,-1}+11\mathbf{P}_{0,1}-\mathbf{P}_{0,-1})\beta_0
\end{aligned} \tag{11}$$

Similarly, for $u=1$ we get

$$\begin{aligned}
&36(\mathbf{P}_{2,0}-\mathbf{P}_{3,0})\alpha_1^*+18(\mathbf{P}_{0,-1}-\mathbf{P}_{3,1})\beta_1^*\approx\\
&36(\mathbf{P}_{3,1}-\mathbf{P}_{3,0}-\mathbf{P}_{2,1}+\mathbf{P}_{2,0})+\\
&12(-5\mathbf{P}_{3,0}+7\mathbf{P}_{2,0}-2\mathbf{P}_{1,0})\alpha_1+\\
&3(-11\mathbf{P}_{3,1}+\mathbf{P}_{3,-1}+3\mathbf{P}_{2,1}-3\mathbf{P}_{2,-1}+3\mathbf{P}_{1,1}-3\mathbf{P}_{1,-1}-\mathbf{P}_{0,1}+11\mathbf{P}_{0,-1})\beta_1
\end{aligned} \tag{12}$$

These can be solved with the Ξ coordinate transformation above. Our only remaining task is to convert the resulting function into Bézier form:

$$\hat{\mathbf{R}}_v(u)=\sum_{i=0}^{5}\mathbf{Q}_iB_i^5(u), \tag{13}$$

where

$$\mathbf{Q}_0=\bar{\mathbf{R}}_v(0)\cdot\beta_0+(\mathbf{P}_{1,0}-\mathbf{P}_{0,0})\cdot 3\alpha_0, \tag{14}$$

$$\begin{aligned}\mathbf{Q}_1=\frac{1}{5}\big(&\bar{\mathbf{R}}_v(0)\cdot(3\beta_0^*-\beta_0)+\bar{\mathbf{R}}_v(0.5)\cdot 4\beta_0-\bar{\mathbf{R}}_v(1)\cdot\beta_0+\\ &(\mathbf{P}_{1,0}-\mathbf{P}_{0,0})\cdot 9\alpha_0^*+(\mathbf{P}_{2,0}-\mathbf{P}_{1,0})\cdot 6\alpha_0\big),\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{Q}_2=\frac{1}{10}\big(&\bar{\mathbf{R}}_v(0)\cdot 3(\beta_1^*-\beta_0^*)+\bar{\mathbf{R}}_v(0.5)\cdot 12\beta_0^*+\bar{\mathbf{R}}_v(1)\cdot(\beta_0-3\beta_0^*)+\\ &(\mathbf{P}_{1,0}-\mathbf{P}_{0,0})\cdot 9\alpha_1^*+(\mathbf{P}_{2,0}-\mathbf{P}_{1,0})\cdot 18\alpha_0^*+(\mathbf{P}_{3,0}-\mathbf{P}_{2,0})\cdot 3\alpha_0\big),\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{Q}_3=\frac{1}{10}\big(&\bar{\mathbf{R}}_v(0)\cdot(\beta_1-3\beta_1^*)+\bar{\mathbf{R}}_v(0.5)\cdot 12\beta_1^*+\bar{\mathbf{R}}_v(1)\cdot 3(\beta_0^*-\beta_1^*)+\\ &(\mathbf{P}_{1,0}-\mathbf{P}_{0,0})\cdot 3\alpha_1+(\mathbf{P}_{2,0}-\mathbf{P}_{1,0})\cdot 18\alpha_1^*+(\mathbf{P}_{3,0}-\mathbf{P}_{2,0})\cdot 9\alpha_0^*\big),\end{aligned} \tag{17}$$

$$\begin{aligned}\mathbf{Q}_4=\frac{1}{5}\big(&-\bar{\mathbf{R}}_v(0)\cdot\beta_1+\bar{\mathbf{R}}_v(0.5)\cdot 4\beta_1+\bar{\mathbf{R}}_v(1)\cdot(3\beta_1^*-\beta_1)+\\ &(\mathbf{P}_{2,0}-\mathbf{P}_{1,0})\cdot 6\alpha_1+(\mathbf{P}_{3,0}-\mathbf{P}_{2,0})\cdot 9\alpha_1^*\big),\end{aligned} \tag{18}$$

$$\mathbf{Q}_5=\bar{\mathbf{R}}_v(1)\cdot\beta_1+(\mathbf{P}_{3,0}-\mathbf{P}_{2,0})\cdot 3\alpha_1. \tag{19}$$