G^1 direction blend with approximate twists

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We are given 3 cubic Bézier curves defined by control points $\mathbf{P}_{i,j}$ with $i = 0\dots3$ and $j = -1, 0, 1$. These, in turn, define two cubic half-ribbons (one by $j = 0, 1$ and another by $j = 0, -1$). Note that the latter (the 'opposite' side) has different orientation, so $\mathbf{P}_{i,1}$ is opposite $\mathbf{P}_{3-i,-1}$. We can define the following quantities (we omit the v parameter throughout, as it is always 0):

$$\begin{aligned}
\mathbf{R}_u(0) &= 3(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}), & \mathbf{R}_u(1) &= 3(\mathbf{P}_{3,0} - \mathbf{P}_{2,0}), && (1)\\
\mathbf{R}_v(0) &= 3(\mathbf{P}_{0,1} - \mathbf{P}_{0,0}), & \mathbf{R}_v(1) &= 3(\mathbf{P}_{3,1} - \mathbf{P}_{3,0}), && (2)\\
\mathbf{R}^*{}_v(0) &= 3(\mathbf{P}_{3,-1} - \mathbf{P}_{0,0}), & \mathbf{R}^*_v(1) &= 3(\mathbf{P}_{0,-1} - \mathbf{P}_{3,0}), && (3)
\end{aligned}$$

and also the slightly longer expressions

$$\begin{aligned}
\mathbf{R}_v(0.5) &= (\mathbf{R}_v(0) + \mathbf{R}_v(1) + 9(\mathbf{P}_{1,1} - \mathbf{P}_{1,0} + \mathbf{P}_{2,1} - \mathbf{P}_{2,0}))/8,\\
\mathbf{R}^*_v(0.5) &= (\mathbf{R}^*_v(0) + \mathbf{R}^*_v(1) + 9(\mathbf{P}_{2,-1} - \mathbf{P}_{1,0} + \mathbf{P}_{1,-1} - \mathbf{P}_{2,0}))/8,\\
\mathbf{T}_0 &= 9(\mathbf{P}_{1,1} - \mathbf{P}_{1,0} - \mathbf{P}_{0,1} + \mathbf{P}_{0,0}),\\
\mathbf{T}_1 &= 9(\mathbf{P}_{3,1} - \mathbf{P}_{3,0} - \mathbf{P}_{2,1} + \mathbf{P}_{2,0}).
\end{aligned}$$

We define a common *direction blend* as

$$\mathbf{D}(u) = (2u^2 - 3u + 1)\bar{\mathbf{R}}_v(0) + 4u(1-u)\bar{\mathbf{R}}_v(0.5) + (2u^2 - u)\bar{\mathbf{R}}_v(1), \tag{4}$$

where $\bar{\mathbf{R}}_v(u) = \frac{1}{2}(\mathbf{R}_v(u) - \mathbf{R}^*_v(u))$ is the averaged cross-derivative function. We want to create a modified cross-derivative of the form

$$\hat{\mathbf{R}}_v(u) = \mathbf{R}_u(u) \cdot \alpha(u) + \mathbf{D}(u) \cdot \beta(u) \tag{5}$$

in such a way that it interpolates the original end tangents, and also approximates the original twists. We assume that the scalar functions α and β are of the form

$$\begin{aligned}
\alpha(u) &= (1-u)^3\alpha_0 + 3(1-u)^2u\alpha^*_0 + 3(1-u)u^2\alpha^*_1 + u^3\alpha_1, && (6)\\
\beta(u) &= (1-u)^3\beta_0 + 3(1-u)^2u\beta^*_0 + 3(1-u)u^2\beta^*_1 + u^3\beta_1. && (7)
\end{aligned}$$

Let us define the notation $\Xi(\mathbf{w}; \mathbf{u}, \mathbf{v})$ to be the coordinates of $\mathbf{w}$ in the $(\mathbf{u}, \mathbf{v})$ system. The system is not necessarily perpendicular, and $\mathbf{w}$ may not lie in the plane (in which case it is approximated). This is computed by

$$\Xi(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \left(\frac{\|\mathbf{v}\|^2\langle\mathbf{w}, \mathbf{u}\rangle - \langle\mathbf{u}, \mathbf{v}\rangle\langle\mathbf{w}, \mathbf{v}\rangle}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \langle\mathbf{u}, \mathbf{v}\rangle}, \frac{\|\mathbf{u}\|^2\langle\mathbf{w}, \mathbf{v}\rangle - \langle\mathbf{u}, \mathbf{v}\rangle\langle\mathbf{w}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \langle\mathbf{u}, \mathbf{v}\rangle}\right). \tag{8}$$

With this we can write the constraints on the scalar functions as

$$(\alpha_0, \beta_0) = \Xi(\mathbf{R}_v(0); \mathbf{R}_u(0), \bar{\mathbf{R}}_v(0)), \quad (\alpha_1, \beta_1) = \Xi(\mathbf{R}_v(1); \mathbf{R}_u(1), \bar{\mathbf{R}}_v(1)). \tag{9}$$

As for the twist constraints, we need

$$\frac{\partial}{\partial u}\hat{\mathbf{R}}_v(0) \approx \mathbf{T}_0, \qquad \frac{\partial}{\partial u}\hat{\mathbf{R}}_v(1) \approx \mathbf{T}_1. \tag{10}$$

Derivating Eq. (5) we get

$$\frac{\partial}{\partial u}\hat{\mathbf{R}}_v(u) = \mathbf{R}_{uu}(u) \cdot \alpha(u) + \mathbf{R}_u(u) \cdot \alpha'(u) + \mathbf{D}'(u) \cdot \beta(u) + \mathbf{D}(u) \cdot \beta'(u).$$

For $u = 0$, this becomes

$$\begin{aligned}
&36(\mathbf{P}_{1,0} - \mathbf{P}_{0,0})\alpha_0^* + 18(\mathbf{P}_{0,1} - \mathbf{P}_{3,-1})\beta_0^* \approx \\
&36(\mathbf{P}_{1,1} - \mathbf{P}_{1,0} - \mathbf{P}_{0,1} + \mathbf{P}_{0,0}) + \\
&12(-2\mathbf{P}_{2,0} + 7\mathbf{P}_{1,0} - 5\mathbf{P}_{0,0})\alpha_0 + \\
&3(\mathbf{P}_{3,1} - 11\mathbf{P}_{3,-1} - 3\mathbf{P}_{2,1} + 3\mathbf{P}_{2,-1} - 3\mathbf{P}_{1,1} + 3\mathbf{P}_{1,-1} + 11\mathbf{P}_{0,1} - \mathbf{P}_{0,-1})\beta_0
\end{aligned} \tag{11}$$

Similarly, for $u = 1$ we get

$$\begin{aligned}
&36(\mathbf{P}_{2,0} - \mathbf{P}_{3,0})\alpha_1^* + 18(\mathbf{P}_{0,-1} - \mathbf{P}_{3,1})\beta_1^* \approx \\
&36(\mathbf{P}_{3,1} - \mathbf{P}_{3,0} - \mathbf{P}_{2,1} + \mathbf{P}_{2,0}) + \\
&12(-5\mathbf{P}_{3,0} + 7\mathbf{P}_{2,0} - 2\mathbf{P}_{1,0})\alpha_1 + \\
&3(-11\mathbf{P}_{3,1} + \mathbf{P}_{3,-1} + 3\mathbf{P}_{2,1} - 3\mathbf{P}_{2,-1} + 3\mathbf{P}_{1,1} - 3\mathbf{P}_{1,-1} - \mathbf{P}_{0,1} + 11\mathbf{P}_{0,-1})\beta_1
\end{aligned} \tag{12}$$

These can be solved with the Ξ coordinate transformation above. Our only remaining task is to convert the resulting function into Bézier form:

$$\hat{\mathbf{R}}_v(u) = \sum_{i=0}^{5} \mathbf{Q}_i B_i^5(u), \tag{13}$$

where

$$\mathbf{Q}_0 = \bar{\mathbf{R}}_v(0) \cdot \beta_0 + (\mathbf{P}_{1,0} - \mathbf{P}_{0,0}) \cdot 3\alpha_0, \tag{14}$$

$$\begin{aligned}\mathbf{Q}_1 = \frac{1}{5}\big(&\bar{\mathbf{R}}_v(0) \cdot (3\beta_0^* - \beta_0) + \bar{\mathbf{R}}_v(0.5) \cdot 4\beta_0 - \bar{\mathbf{R}}_v(1) \cdot \beta_0 + \\ &(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}) \cdot 9\alpha_0^* + (\mathbf{P}_{2,0} - \mathbf{P}_{1,0}) \cdot 6\alpha_0\big),\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{Q}_2 = \frac{1}{10}\big(&\bar{\mathbf{R}}_v(0) \cdot 3(\beta_1^* - \beta_0^*) + \bar{\mathbf{R}}_v(0.5) \cdot 12\beta_0^* + \bar{\mathbf{R}}_v(1) \cdot (\beta_0 - 3\beta_0^*) + \\ &(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}) \cdot 9\alpha_1^* + (\mathbf{P}_{2,0} - \mathbf{P}_{1,0}) \cdot 18\alpha_0^* + (\mathbf{P}_{3,0} - \mathbf{P}_{2,0}) \cdot 3\alpha_0\big),\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{Q}_3 = \frac{1}{10}\big(&\bar{\mathbf{R}}_v(0) \cdot (\beta_1 - 3\beta_1^*) + \bar{\mathbf{R}}_v(0.5) \cdot 12\beta_1^* + \bar{\mathbf{R}}_v(1) \cdot 3(\beta_0^* - \beta_1^*) + \\ &(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}) \cdot 3\alpha_1 + (\mathbf{P}_{2,0} - \mathbf{P}_{1,0}) \cdot 18\alpha_1^* + (\mathbf{P}_{3,0} - \mathbf{P}_{2,0}) \cdot 9\alpha_0^*\big),\end{aligned} \tag{17}$$

$$\begin{aligned}\mathbf{Q}_4 = \frac{1}{5}\big(&-\bar{\mathbf{R}}_v(0) \cdot \beta_1 + \bar{\mathbf{R}}_v(0.5) \cdot 4\beta_1 + \bar{\mathbf{R}}_v(1) \cdot (3\beta_1^* - \beta_1) + \\ &(\mathbf{P}_{2,0} - \mathbf{P}_{1,0}) \cdot 6\alpha_1 + (\mathbf{P}_{3,0} - \mathbf{P}_{2,0}) \cdot 9\alpha_1^*\big),\end{aligned} \tag{18}$$

$$\mathbf{Q}_5 = \bar{\mathbf{R}}_v(1) \cdot \beta_1 + (\mathbf{P}_{3,0} - \mathbf{P}_{2,0}) \cdot 3\alpha_1. \tag{19}$$