Weingarten maps

Péter Salvi

November 8, 2021

Let S denote an operator on a surface such that $S(u)$ is the (negated) derivative of the unit normal by u, i.e., for a fixed point $\mathbf{p}$ of the surface $\mathbf{x}(u,v)$,

$$S(w) = -\nabla_w \mathcal{G},$$

where $\mathcal{G}$ is the Gauss map of the surface, and w is a vector in the tangent plane. This is the *shape operator*, or Weingarten map—a symmetric, linear operator, expressible by a 2×2 matrix (which is also symmetric when the basis vectors are perpendicular). Note that S is independent of the parameterization of $\mathbf{x}$ as long as the basis it is expressed in is parameterization-independent.

This matrix has very nice properties:

$$\begin{aligned} k(u) &= \langle S(u), u\rangle && \text{(normal curvature)} \\ K &= |S|, && \text{(Gaussian curvature)} \\ H &= \mathrm{tr}(S)/2, && \text{(mean curvature)} \end{aligned}$$

and also the principal curvatures κ_1, κ_2 are the eigenvalues of S, and the principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are the corresponding eigenvectors (expressed in the basis of the matrix).

If we write the matrix of S in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, then the above properties become obvious:

$$S = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \tag{1}$$

But this offers little help when we do not *know* these values.

Luckily S is easily expressible in the basis of the derivatives of $\mathbf{x}$:

$$S = I^{-1} I\!I,$$

where I and $I\!I$ are the first and second fundamental forms:

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad I\!I = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle, & F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle, & G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle, \\ L &= \langle \mathbf{n}, \mathbf{x}_{uu}\rangle, & M &= \langle \mathbf{n}, \mathbf{x}_{uv}\rangle, & N &= \langle \mathbf{n}, \mathbf{x}_{vv}\rangle, \end{aligned}$$
$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

The drawback is that now the matrix of S is not independent of the parameterization.

Take Eq. (1) and add the unit normal $\mathbf{n}$ as a third basis vector, i.e., using the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$, bringing it into 3D space:

$$W = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the *embedded* Weingarten map, sometimes also called the curvature tensor (but this term is abused). All the properties still stand, except for the Gaussian curvature, which can be expressed as

$$K = \frac{\operatorname{tr}(W)^2 - \operatorname{tr}(W^2)}{2},$$

and that there is an extra 0 eigenvalue; normal curvatures are computed based on the projection of the given vector into the tangent plane.

Now we can also use the axes of 3D space as a basis—which is also independent of parameterization—and write the matrix as

$$W = \nabla\mathbf{x}^+ I\!I \, \nabla\mathbf{x}^+,$$

where $\nabla\mathbf{x}^+$ is the pseudoinverse of the gradient (a 3×2 matrix):

$$\nabla\mathbf{x}^+ = \nabla\mathbf{x}^\mathsf{T}(\nabla\mathbf{x}\nabla\mathbf{x}^\mathsf{T})^{-1} = \nabla\mathbf{x}^\mathsf{T} I^{-1} = (I^{-1}\nabla\mathbf{x})^\mathsf{T}.$$

This formulation also has the advantage that its eigenvectors give the principal directions directly expressed in 3D coordinates. Note that while this matrix is exactly the same, independently of the parameterization, in exchange it *does* depend on the coordinates, so e.g. a rotation of 3D changes the elements of the matrix.