UOL Algebra TILL 8 CHAPTER @salyamq2

Logic & Proofs

$$\neg P : \text{not } P, \quad P \wedge Q : \text{and}, \quad P \vee Q : \text{or}, \quad P \Rightarrow Q : \text{if } P \text{ then } Q$$

$$Q \Rightarrow P : \text{converse}, \quad \neg Q \Rightarrow \neg P : \text{contrapositive}$$

Proof by contradiction: assume $\neg P$ and derive falsehood.

Matrices & Operations

$$A + B = [a_{ij} + b_{ij}], \quad kA = [ka_{ij}], \quad (AB)_{ij} = \sum_k a_{ik}b_{kj}$$
$$(AB)^T = B^T A^T, \quad (A^T)^{-1} = (A^{-1})^T, \quad (AB)^{-1} = B^{-1}A^{-1}$$
$$A + 0 = A, \quad AI = IA = A, \quad 0A = A0 = 0$$

Types: diagonal ($a_{ij} = 0,\ i \neq j$), symmetric ($A = A^T$), identity I, zero 0. Addition is commutative; multiplication is associative, not commutative.

Elementary Row Operations

$$\textbf{RO1: } R_i \leftrightarrow R_j \text{ (swap rows)} \Rightarrow \det \text{ changes sign}$$

$$\textbf{RO2: } kR_i \ (k \neq 0) \Rightarrow \det \text{ multiplied by } k$$

$$\textbf{RO3: } R_i + kR_j \Rightarrow \det \text{ unchanged}$$

Elementary matrices E perform one such operation. If $EA = B$, then $A = E^{-1}B$. **Inverse by ERO:** $[A|I] \xrightarrow{\text{row ops}} [I|A^{-1}]$

Vectors, Lines & Planes

$$|v| = \sqrt{v \cdot v}, \quad \hat{v} = \frac{v}{|v|}, \quad u \cdot v = |u||v|\cos\theta$$
$$L : r = a + tv, \quad \Pi : n \cdot (r - r_0) = 0 \Rightarrow Ax + By + Cz + D = 0$$
$$d_{\text{point-plane}} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Linear Systems & Gaussian Elimination

$$Ax = b \Rightarrow \begin{cases} \text{Unique sol.} & \text{consistent} \\ \text{No sol.} & \text{inconsistent} \\ \text{Inf. sol.} & \text{free variables} \end{cases}$$

Row operations (RO1–RO3) preserve solution set.

$$[A|b] \xrightarrow{\text{RREF}} [I|x]$$

Rank Theorem: $\text{rank}(A) = \#$ pivots. System consistent $\Leftrightarrow \text{rank}(A) = \text{rank}([A|b])$.

Homogeneous Systems & Leontief Model

$$Ax = 0 \Rightarrow \text{always solvable by } x = 0$$

$$\text{Null}(A) = \{x : Ax = 0\}, \quad \dim(\text{Null}(A)) + \text{rank}(A) = n$$

Leontief model: $x = Ax + d \Rightarrow (I - A)x = d$, solution: $x = (I - A)^{-1}d$, if $\det(I - A) \neq 0$.

Matrix Inverse

$$A^{-1} : AA^{-1} = A^{-1}A = I$$

Exists $\Leftrightarrow \det(A) \neq 0$. If $\det(A) = 0$, then A is **singular (non-invertible)**.

$$(A^{-1})^T = (A^T)^{-1}, \quad (AB)^{-1} = B^{-1}A^{-1}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A), \quad \operatorname{adj}(A) = (C_{ij})^T$$

Properties:

$$\det(A^{-1}) = \frac{1}{\det(A)}, \quad (A^{-1})^{-1} = A$$

Determinants & Properties

$$\det(A^T) = \det(A), \quad \det(AB) = \det(A)\det(B), \quad \det(kA) = k^n \det(A)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}, \quad \text{triangular: } \det(A) = \prod a_{ii}$$

Equal rows $\Rightarrow \det(A) = 0$, row swap $\Rightarrow$ sign change

Interpretation: $\det(A) = 0 \Rightarrow$ rows (or cols) are linearly dependent.

Cofactors & Cramer's Rule

$$C_{ij} = (-1)^{i+j} \det(M_{ij}), \quad \operatorname{adj}(A) = (C_{ij})^T$$
$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A), \quad x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is A with its i-th column replaced by b. If $\det(A) = 0 \Rightarrow$ no unique solution.

Key Theorems (Ch.1–8)

Invertibility: A invertible $\Leftrightarrow \det(A) \neq 0$
Homogeneous: $Ax = 0$ has nontrivial sol. $\Leftrightarrow \det(A) = 0$
Rank–Nullity: $\dim(\text{Null}(A)) + \text{rank}(A) = n$
Product Inverse: $(AB)^{-1} = B^{-1}A^{-1}$
Leontief Model: $x = (I - A)^{-1}d$, if $\det(I - A) \neq 0$