Normal Distribution

Normal Distribution by definition

Standardized Normal Distribution

$Z \sim \mathcal{N}(0,1), \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$\mathbb{E}[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz = \left\| \begin{matrix} u = -\dfrac{z^2}{2} \\ du = -z\, dz \end{matrix} \right\| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^u \, du = \left. \frac{e^u}{\sqrt{2\pi}} \right|_{-\infty}^{\infty} = 0$$

$$\mathbb{E}[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz = \left\| \begin{matrix} u = z, & du = dz \\ dv = z e^{-z^2/2}\, dz, & v = -e^{-z^2/2} \end{matrix} \right\|$$

$$\frac{1}{\sqrt{2\pi}} \left(-z e^{-z^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(-e^{-z^2/2} \right) dz \right) = \frac{1}{\sqrt{2\pi}} \left(0 + \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \right) = 1$$

$$Var[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = 1 - 0 = 1$$

$X \sim \mathcal{N}(\mu, \sigma^2)$

$$Z = \frac{X - \mu}{\sigma} \Rightarrow X = \sigma Z + \mu$$

$$\mathbb{E}[X] = \mathbb{E}[\sigma Z + \mu] = \sigma \times 0 + \mu = \mu$$

$$Var(X) = Var(\sigma Z + \mu) = \sigma^2 \times Var(Z) = \sigma^2 \times 1 = \sigma^2$$

Moment-generating function of Normal Distribution

$Z \sim \mathcal{N}(0,1), \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$M_Z(t) = \mathbb{E}[e^{tz}] = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz - z^2/2}\, dz =$$

$$\left\| tz - \frac{z^2}{2} = \frac{2tz}{2} - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2tz) = -\frac{1}{2}((z-t)^2 - t^2) = -\frac{(z-t)^2}{2} + \frac{t^2}{2} \right\|$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2} + \frac{t^2}{2}}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} \times e^{\frac{t^2}{2}}\, dz =$$

$$e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}}_{=1 \quad \text{pdf of } \mathcal{N}(t,1)} dz = e^{\frac{t^2}{2}}$$

$X \sim \mathcal{N}(\mu, \sigma^2), \quad X = \sigma * Z + \mu, \quad Z \sim \mathcal{N}(0,1)$

$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(\sigma Z + \mu)}] = \mathbb{E}[e^{t\sigma Z + t\mu}] = e^{\mu t} \times \mathbb{E}[e^{t\sigma Z}] = e^{\mu t} \times M_Z(\sigma t) = e^{\mu t} \times e^{\frac{(\sigma t)^2}{2}} = e^{\mu t + \frac{(\sigma t)^2}{2}}$

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