Rendering a black hole

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1 Introduction

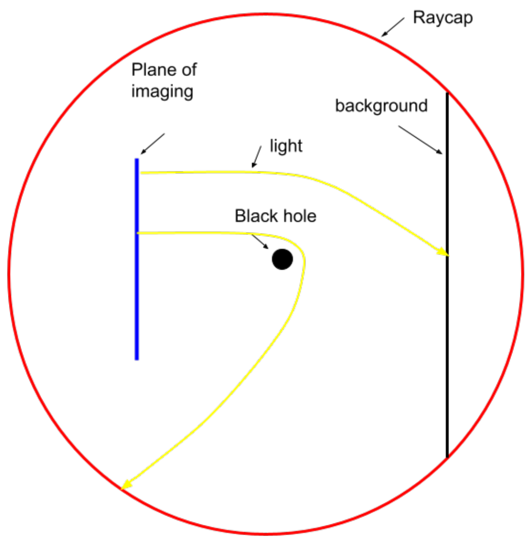
The inspiration for this project came from the rendering of the black hole in the film interstellar and the extreme attention to detail and mathematical accuracy with which it was simulated.

The goal of this project was to code, entirely from the ground up, a rendering engine (in python) which would simulate the warping effect that a black hole has on light as well as the gravitational red-shift experienced by light exiting its gravity well using the particle model for photons.

2 Method

2.1 Ray-Tracing algorithm

To render a black hole using orthographic projection we need a method of ray-tracing with which we are able to warp light. To do this we will re-trace each pixel's path towards the camera moving backwards 1 second (represented by a frame) at a time from the plane where which the image has hypothetically been taken. Since the speed of light is constant and the warping of light is symmetrical when calculated in reverse, the only difference that we must consider is the gravitational red-shift experienced by the light for which we will perform the reverse calculation. this process allows us to trace the source of the light incident on each pixel of the screen and therefore it's colour.our setup also includes a background (to view gravitational lensing on images and to better see its effects) and a ray-cap, to avoid infinite recursion if a photon is bent back towards the camera. A diagram of the setup used can be seen below:



2.2 Gravitational Warping

We can determine how a photon travelling at light speed acts around an object with mass by assuming that the light has dynamic mass, proportional to its energy. (An equation to calculate the dynamic mass of a photon can be found below:

$$m = \frac{\hbar f}{c^2} \tag{1}$$

Where $\hbar$ is Planck's constant and f is the frequency of the photon in Hertz. this can then be substituted into Newtons equation for gravity to find the force acting on the photon:

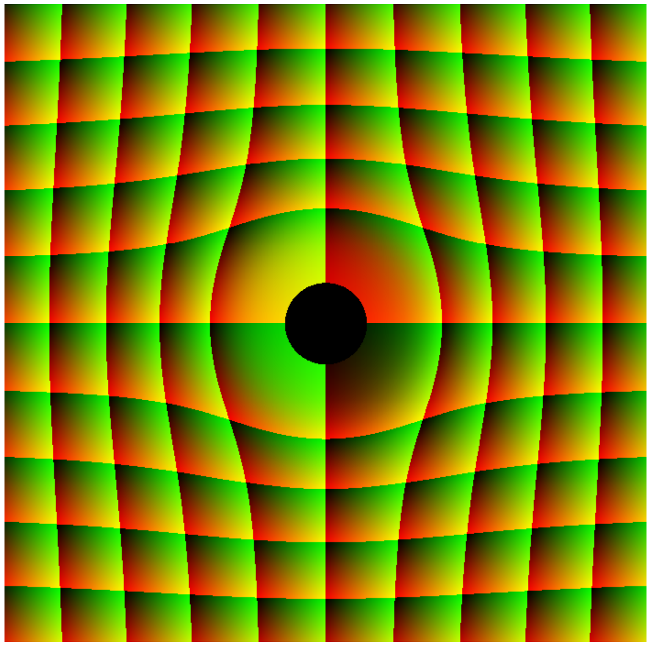
$$F = G\frac{m\hbar f}{r^2c^2} \tag{2}$$

Where m is the mass of the black hole, r is the distance between the black hole and the photon and G is the gravitational constant. This can, again, be substituted into the equation for Newton's third law of motion: $F = ma$. To calculate the acceleration of the light along the vector between itself and the black hole. And helpfully this cancels out the, rather complicated, expression for the dynamic mass of a photon to give:

$$a = G\frac{m}{r^2} \tag{3}$$

This acceleration variable can be added to the velocity vector of the light at each step (as one step is assumed to be 1 second)

The image below shows the magnification effect the black hole has on the textured background behind it.



2.3 Red-shift Calculations

2.3.1 Gravitational Red-shift

Light can not travel faster than the speed of light. However in the section above we explored how to determine lights acceleration around a massive body. This implies that the photons velocity should increase along with its change in direction which preserves the law of conservation of energy. This obviously contradicts the inability of anything to exceed the absolute value of c. And so to allow for both the conservation of energy and the absolute limit of c, a photon moving into a gravitational well experiences a decrease in wavelength (blue-shift) and a photon escaping a gravitational well increases in wavelength (red-shift). The equation to determine the change in frequency is:

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta U}{c^2} \tag{4}$$

Where ΔU is the change in gravitational potential and $\Delta\lambda$ is the change in wavelength. The formula for calculating ΔU is giving as $\Delta U = \frac{Gm}{r_2} - \frac{Gm}{r_1}$ (where r_1 is the initial distance from the singularity and r_2 is the final distance). These formulae can be combined to get an equation for the total change in wavelength:

$$\Delta\lambda = \frac{Gm\lambda}{r_2 c^2} - \frac{Gm\lambda}{r_1 c^2} \tag{5}$$

2.3.2 Velocity Induced Red-shift

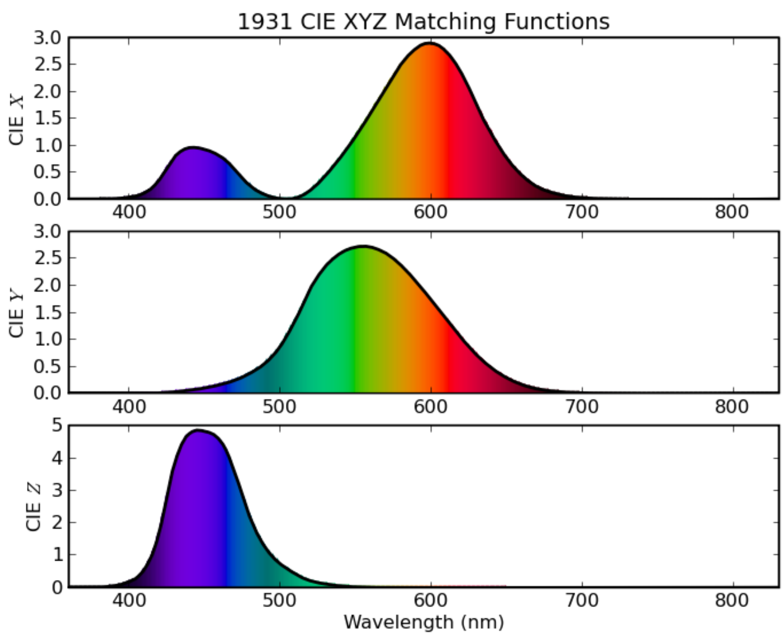
The title of this subsection, although sounding complex, is the common form of red-shift observed when an object emitting radiation is travelling towards or away from the observer (away=red-shift, towards=blue-shift). To calculate the change in wavelength of the emitted light we must first calculate the speed of the emitter (In this case the rotating accretion disk). Since the accretion disk is of negligible mass in comparison to the singularity we can determine that the velocity of each (assumed to be circular) orbit of miscellaneous matter within the belt is approximately:

$$\|\vec{v_o}\| \approx \sqrt{\frac{GM}{r}} \tag{6}$$

This gives us the velocity of the disk tangential to the orbit. However we need to calculate velocity in the direction of emission (or in this case impact as our ray-tracing algorithm works in reverse). To do this, we can find the velocity of the disk in the direction of the emission vector by taking the dot product: $\vec{v_o} \cdot \hat{e} = V$. Now that we have an approximation for the velocity of the the emitter in the desired direction we can use the equation for red-shift, as a function of speed: $\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$.

2.4 wavelength to RGB

This final section is to address a problem that arises from the method of representing colour on a monitor. The human eye detects colour using three types of cone sensors (it is more complicated than this however this is enough to demonstrate the problem). Roughly speaking each type of cone represents red , green and blue light respectively - as shown in the diagram beneath.



Monitors make use of this property of the eye to show combinations of these colors to give the illusion of different wavelengths of light. This poses a problem in our simulation as in the previous section we discussed how to calculate the change in wavelength caused by gravitational red-shift we now need to convert those values into RGB to display them on our screen.

To do this we need to take the spectrograph of our accretion disk, for which I took the 4 visible values in the Hydrogen spectral series[1] (this seemed appropriate as accretion disks are usually composed of Hydrogen predominantly) as the displacement for four superposed normal distributions with arbitrarily small standard deviations. We then multiply our spectrograph with each of the Matching functions and integrate over the visual field (400-700nm) to get a scalar value for Red, Green And Blue.

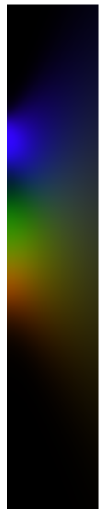
$$C_{red} = \int_{400}^{700} M_{red}(\lambda) \cdot S(\lambda)\, d\lambda \tag{7}$$

Although, due to the lack of a continuous definition for the matching functions, the process is instead done using a discrete sum.

$$C_{red} = \sum_{i=400}^{700} \left[M_{red}(\lambda) \cdot S(\lambda) \right] \tag{8}$$

[1] 656.3nm, 486.1nm, 434.0nm and 410.2nm (3,4,5 and 6 in the Balmer series)

here is a graph of the wavelength spectrum where the y axis shows an decrease in wavelength from 850nm to 300nm, and the x axis shows an increase in the standard deviation of the peak.



2.5 Modelling

The event horizon of the black hole itself is defined using the equation for the schwarzschild radius $R_s = \frac{2GM}{c^2}$ where M is the black hole's mass (we are assuming that the black hole has no spin as well as no charge.). below this threshold nothing can escape the gravity of the black hole including light

In order to model the black holes accretion disk, we must first calculate the ISCO (Innermost Stable Circular Orbit). This is the closest that any object with mass can sustain a stable circular orbit around a black hole. It is therefore logical that this is the inner radius of the accretion disk. The ISCO is defined to be $R_{ISCO} = 3R_s$. Theoretically there is no specific upper bound to the width of the accretion disk of a black hole in the middle of empty space. So in this simulation an arbitrary distance is chosen.