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# PML Theory hw 4

$$p(r_{oil}, r_{air}) = \mathcal{N}\left(\begin{pmatrix} r_{oil} \\ r_{air} \end{pmatrix}; \underbrace{\begin{pmatrix} 0.05 \\ 0.02 \end{pmatrix}}_{\mu}, \underbrace{\begin{pmatrix} 0.08 & -0.045 \\ -0.045 & 0.040 \end{pmatrix}}_{\Sigma}\right)$$

$B$ = budget, $x$ = amount spent on oil
$\Rightarrow B - x$ = amount spent on airlines

$$R = x r_{oil} + (B-x) r_{air} \equiv R(x)$$

(a) we see that

$$\mathbb{E}(R) = \mathbb{E}(x r_{oil} + (B-x) r_{air}) = x\,\mathbb{E}[r_{oil}] + (B-x)\,\mathbb{E}(r_{air})$$
$$= 0.05x + 0.02(B-x) = 0.02B + 0.03x$$
$$= \frac{2B + 3x}{100}$$

$$\mathbb{V}ar(R) = \mathbb{V}ar(x r_{oil} + (B-x) r_{air})$$
$$= x^2\,\mathbb{V}ar[r_{oil}] + (B-x)^2\,\mathbb{V}ar[r_{air}] + 2x(B-x)\,Cov(r_{air}, r_{oil})$$
$$= 0.08x^2 + 0.04(B-x)^2 - 0.09x(B-x)$$

Since the sum of Gaussian random variables is Gaussian, we have

$R(x) \sim \mathcal{N}(\mathbb{E}(R), \mathbb{V}ar(R))$ (as derived above)

(b) Clearly $\mathbb{E}(R(x))$ is maximized if $x = B$
This makes intuitive sense since oil is expected to have a higher return

c) $\partial_x \mathbb{V}ar(R(x)) = 0.16x + 0.04(B-x) - 0.09(B-2x)$

$= 0.42x - 0.17B$

minimizing $\Rightarrow$ $x = \frac{17}{42} B$

d) If $B = 1$, we have

$\mathbb{V}ar(R) = 0.08x^2 + 0.04(1-x)^2 - 0.09x(1-x)$

$= 0.21x^2 - 0.17x + 0.04$

then $\mathbb{V}ar(R) \leq 0.03 \iff 0.21x^2 - 0.17x + 0.01 \leq 0$

$\Rightarrow 21x^2 - 17x + 1 \leq 0$

This quadratic has roots @ $\frac{17 \pm \sqrt{17^2 - 4(21)(1)}}{42} = \frac{17 \pm \sqrt{205}}{42}$

both roots lie in $[0,1]$ so $\cancel{x = \frac{17+\sqrt{205}}{42}}$

$x = \frac{17+\sqrt{205}}{42}$ is the largest ~~x~~ of $x \in (0,1)$ for which $\mathbb{V}ar(R(x)) \leq 0.03$ ~~and~~ (since $\mathbb{E}\left(R\left(\frac{17+\sqrt{205}}{42}\right)\right) > \mathbb{E}\left(R\left(\frac{17-\sqrt{205}}{42}\right)\right)$) whence $x = \frac{17+\sqrt{205}}{42}$ is ~~the~~ such an $x$

e) let $m(x) := \mathbb{E}(R) = 0.02 + 0.03x$

$v(x) := \mathbb{V}ar(R) = 0.21x^2 - 0.17x + 0.04$

then $R \sim N(m(x), v(x))$

$$\Rightarrow p(R > 0) = \int_0^\infty \frac{1}{\sqrt{2\pi}\sqrt{v(x)}} e^{-\frac{1}{2}\left(\frac{\theta - m(x)}{\sqrt{v(x)}}\right)^2} d\theta$$

let $Q = \frac{\theta - m(x)}{\sqrt{v(x)}} \Rightarrow dQ = \frac{d\theta}{\sqrt{v(x)}}$

$$\Rightarrow p(R > 0) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{m(x)}{\sqrt{v(x)}}}^{\infty} e^{-\frac{1}{2}Q^2} dQ$$

now $\partial_x p(R>0) = \partial_x \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\frac{m(x)}{\sqrt{v(x)}}}^{+\infty} e^{-\frac{1}{2}Q^2} \, dQ \right\}$

$$= e^{-\frac{1}{2}\left(\frac{m(x)}{\sqrt{v(x)}}\right)^2} \cdot \partial_x \left[ -\frac{m(x)}{\sqrt{v(x)}} \right]$$

Leibniz Rule

$$\partial_x p(R>0) = 0 \implies 0 = \partial_x \frac{m(x)}{\sqrt{v(x)}} = \frac{m'(x)\sqrt{v(x)} - \frac{m(x)v'(x)}{2\sqrt{v(x)}}}{v(x)}$$

solving gives $x = \frac{58}{135}$

f) Since the average return over the past years indicate a negative corrolation between $r_{oil}$ & $r_{air}$,
This new model predicting a higher return on air ~~could~~ ~~standard deviation~~ should suggest a decrease in oil stocks.
Our uncertainty should also reduce since the variance is lower.