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PML theory:

(a) By Baye's Thm, we have:

$$p(\mathcal{C}_1|x) = \frac{p(x|\mathcal{C}_1)p(\mathcal{C}_1)}{p(x|\mathcal{C}_1)p(\mathcal{C}_1) + p(x|\mathcal{C}_2)p(\mathcal{C}_2)}$$

$$= \frac{1}{1 + \frac{p(x|\mathcal{C}_2)p(\mathcal{C}_2)}{p(x|\mathcal{C}_1)p(\mathcal{C}_1)}}$$

$$= \frac{1}{1 + \frac{p(\mathcal{C}_2|x)}{p(\mathcal{C}_1|x)}} = \frac{1}{1 + e^{-\log \frac{p(\mathcal{C}_1|x)}{p(\mathcal{C}_2|x)}}}$$

$p(x)$ ~~[illegible]~~ dividing numerator & denominator of fraction

(b) We have

Bayes

$$a(x) = \log \frac{p(\mathcal{C}_1|x)}{p(\mathcal{C}_2|x)} = \log \frac{p(x|\mathcal{C}_1)p(\mathcal{C}_1)}{p(x|\mathcal{C}_2)p(\mathcal{C}_2)}$$

$$= \log\left[\exp\left[\varphi(x)^T(w_1 - w_2) + \log\frac{Z(w_2)}{Z(w_1)}\right]\right] + \log\frac{p(\mathcal{C}_1)}{1-p(\mathcal{C}_1)}$$

$$= \varphi(x)^T(w_1 - w_2) + \log\frac{Z(w_2)}{Z(w_1)} + \log\frac{p(\mathcal{C}_1)}{1-p(\mathcal{C}_1)}$$

which is of the form $\varphi(x)^T\theta + \theta_0$

with $\theta := w_1 - w_2$, $\theta_0 = \log\frac{Z(w_2)\,p(\mathcal{C}_1)}{Z(w_1)(1-p(\mathcal{C}_1))}$

(c) Revisiting Lecture 5, we see that

$$p(x|\mathcal{C}_k) = \exp\left([x, -\tfrac{1}{2}x^2] \begin{bmatrix} \mu_k/\sigma_k^2 \\ 1/\sigma_k^2 \end{bmatrix} - \left(\frac{\mu_k^2}{2\sigma_k^2} + \log\sqrt{2\pi\sigma_k^2} \right) \right)$$

is the exponential-family-friendly expression for $x|\mathcal{C}_k$

$$\Rightarrow a(x) = [x, -\tfrac{1}{2}x^2] \begin{bmatrix} \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \\ \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \end{bmatrix}$$

$$+ \left(\frac{\mu_2^2}{2\sigma_2^2} - \frac{\mu_1^2}{2\sigma_1^2} + \log\frac{\sqrt{2\pi\sigma_2^2}}{\sqrt{2\pi\sigma_1^2}} \right) + \log\frac{p(\mathcal{C}_1)}{1-p(\mathcal{C}_1)}$$

i.e.

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \\ \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \end{pmatrix}$$

$$\theta_0 = \frac{\mu_2^2}{2\sigma_2^2} - \frac{\mu_1^2}{2\sigma_1^2} + \log\frac{\sigma_2}{\sigma_1} + \log\frac{p(\mathcal{C}_1)}{1-p(\mathcal{C}_1)}$$

if $\theta_0, \theta_1, \theta_2$ are given:
We have a system of 3 equations in 5 unknowns:

$$\begin{cases} \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} = \theta_1 \\ \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} = \theta_2 \\ \frac{\mu_2^2}{2\sigma_2^2} - \frac{\mu_1^2}{2\sigma_1^2} + \log\frac{\sigma_2}{\sigma_1} + \log\frac{p(\mathcal{C}_1)}{1-p(\mathcal{C}_1)} = \theta_0 \end{cases}$$

- which clearly has no unique solution so we cannot recover the parameters

if $p(\mathcal{C}_1) = 1/2$ & $\sigma_1, \sigma_2 = 1$ (assuming this is what $\sigma = 1$ means ?)

our system looks like:

$$
\begin{cases}
\mu_1 - \mu_2 = \theta_1 \\
0 = \theta_2 \quad \dots \text{ if } \theta_2 \neq 0 \text{ the variance assumption is wrong!} \\
\mu_2^2 - \mu_1^2 + 0 + 0 = \theta_0
\end{cases}
$$

$$\Rightarrow \mu_2^2 - \mu_1^2 = (\mu_2 - \mu_1)(\mu_2 + \mu_1) = \theta_0$$

$$\Rightarrow -\theta_1 (\mu_2 + \mu_1) = \theta_0$$

from first eqn.

$$\Rightarrow \mu_2 + \mu_1 = -\frac{\theta_0}{\theta_1}$$

$$\& \quad \mu_1 - \mu_2 = \theta_1$$

$$
\Rightarrow
\begin{cases}
\mu_1 = \frac{1}{2}\left(\theta_1 - \frac{\theta_0}{\theta_1}\right) \\
\mu_2 = -\frac{1}{2}\left(\theta_1 + \frac{\theta_0}{\theta_1}\right)
\end{cases}
$$

- So we recover the parameters in this case.