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(a) $$p(f_X \mid y, X) = \frac{p(y \mid f_X, X)\, p(f_X \mid X)}{p(y, X)}$$

$$\propto \prod_{i=1}^{N} \sigma(y_i f(x_i)) \cdot \frac{1}{\sqrt{2\pi}\,|k_{XX}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(f(X) - m(X))^T k_{XX}^{-1} (f(X) - m(X))\right)$$

$$\log p(f_X \mid y, X) = \sum_{i=1}^{N} \log \sigma(y_i f(x_i)) + -\frac{1}{2}(f_X - m_X)^T k_{XX}^{-1}(f_X - m_X) + \text{const.}$$

(b) $$\partial_{(f_X)_i} \log p(f_X \mid y, X) = \frac{1}{\sigma(y_i f(x_i))} \sigma(y_i f(x_i))(1 - \sigma(y_i f(x_i))) y_i - \left(\frac{1}{2} 2 k_{XX}^{-1}(f_X - m_X)\right)_i$$

$$= y_i(1 - \sigma(y_i f(x_i))) - (k_{XX}^{-1}(f_X - m_X))_i$$

$$\Rightarrow \nabla_{f_X} \log p(f_X \mid y, X) = \begin{pmatrix} y_1(1 - \sigma(y_1 f(x_1))) \\ \vdots \\ y_N(1 - \sigma(y_N f(x_N))) \end{pmatrix} - (k_{XX}^{-1} f_X) + (k_{XX}^{-1} m_X)$$

(c) $q(f_X) = N(f_X ; \hat{f}_X , \Sigma)$

We have $\mathbb{E}_q[f_X] = \hat{f}_X$.

we observe that

$$\nabla_{f_X} \log p(y \mid f_X) = \nabla_{f_X} \sum_{i=1}^{N} \log \sigma(y_i f(x_i))$$

$$= \begin{pmatrix} y_1(1-\sigma(y_1 f(x_1))) \\ \vdots \\ y_N(1-\sigma(y_N f(x_N))) \end{pmatrix}$$

so $\overset{(*)}{\nabla_{f_X} \log p(f_X \mid y, X)} = \nabla_{f_X} \log p(y \mid f_X) - k_{XX}^{-1}(f_X - m_X)$

setting (*) equal to zero gives:

$$\nabla_{f_X} \log p(y \mid f_X) = k_{XX}^{-1}(f_X - m_X)$$

$\cancel{\Rightarrow \hat{f}_X = m_X + k_{XX} \nabla_{f_X} \log}$

$$\Rightarrow \hat{f}_X = m_X + k_{XX} \nabla \log p(y \mid \hat{f}_X)$$

$$\Rightarrow \mathbb{E}_q[f(\cdot)] = m_{\cdot} + k_{\cdot X} \nabla \log p(y \mid \hat{f}_X)$$

$\blacksquare$

(d)

$$\nabla \log(y \mid \hat{f}_x) = \begin{pmatrix} y_1(1-\sigma(y_1\hat{f}(x_1))) \\ \vdots \\ y_N(1-\sigma(y_N\hat{f}(x_N))) \end{pmatrix}$$

$$\mathbb{E}_q[f(\cdot)] = m_\cdot + k_{\cdot x} \nabla_{f_x} \log p(y \mid \hat{f}_x)$$

$$= m_\cdot + k_{\cdot x} \begin{pmatrix} y_1(1-\sigma(y_1\hat{f}(x_1))) \\ y_2(1-\sigma(y_2\hat{f}(x_2))) \\ \vdots \\ y_N(1-\sigma(y_N\hat{f}(x_N))) \end{pmatrix}$$

now we observe that if $|\hat{f}(x_i)| >> 1$

then $\sigma(y_i\hat{f}(x_i)) = \dfrac{1}{1+e^{-y_i f(x_i)}} \simeq 1$

(since if $\hat{f}(x_i) << 0$ then $y_i = -1$)

then, in all cases, $e^{-y_i f(x_i)} \simeq 0$

- Therefore the $i^{th}$ component of $\nabla_{f_X} \log p(y \mid \hat{f}_X)$ is close to zero so the dot product $k_{\cdot X}^{T} \nabla_{f_X} \log p(y \mid \hat{f}_X)$ has negligible influence from the $i^{th}$ summand

(e) let

$$\zeta(f) = c\,(r(1;f) + r(-1;f)) \equiv \sum_{y_i \in \{-1,1\}} c\,r(y_i;f)$$

$$= c\,e^{-\max\{0,1+f\}} + c\,e^{-\max\{0,1-f\}}$$

we see that if: (2 cases)

- $f \geq 1 \Rightarrow \zeta(f) = c\,(e^{-(1+f)} + 1)$

$f < -1 \Rightarrow \zeta(f) = c\,(1 + e^{-(1-f)})$

now suppose $\exists\, c \in \mathbb{R} : \zeta(f) = 1 \;\; \forall f \in \mathbb{R}$ (then $c = \frac{1}{\zeta(f)}$)

then we would have

$$1 + e^{-(1+f)} = 1 + e^{-(1-f)}$$

$$\Rightarrow e^{-(1+f)} = e^{-(1-f)}$$

$$\Rightarrow 1+f = 1-f \Rightarrow f = 0 \Rightarrow \Leftarrow$$

(since must hold $\forall f \in \mathbb{R}$) ... thus no such $c$ exists $\blacksquare$