PML Theory:

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a) $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ Let: $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{q \times q}$

Consider the system:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \begin{array}{l} x, \alpha \in \mathbb{R}^p \\ y, \beta \in \mathbb{R}^q \end{array}$$

$$\begin{cases} Ax + By = \alpha & (\mathrm{I}) \\ Cx + Dy = \beta & (\mathrm{II}) \end{cases}$$

we see that then $\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ so solving will give M^{-1}!

We have:

$$Ax = \alpha - By$$

$$\Rightarrow x = A^{-1}(\alpha - By)$$

Subsituting into (II) gives

$$\Rightarrow C(A^{-1}(\alpha - By)) + Dy = \beta$$

$$CA^{-1}\alpha - CA^{-1}By + Dy = \beta$$

$$(\underset{!!}{D - CA^{-1}B})y = \beta - CA^{-1}\alpha$$

$$Qy = \beta - CA^{-1}\alpha$$

$$y = Q^{-1}\beta - Q^{-1}CA^{-1}\alpha$$

$$x = A^{-1}(\alpha - BQ^{-1}\beta + BQ^{-1}CA^{-1}\alpha)$$

we see that:

$$x = \left(A^{-1} + A^{-1}BQ^{-1}CA^{-1}\right)\alpha - \left(A^{-1}BQ^{-1}\right)\beta$$

$$y = -Q^{-1}CA^{-1}\alpha + Q^{-1}\beta$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} A^{-1} + A^{-1}BQ^{-1}CA^{-1} & -A^{-1}BQ^{-1} \\ -Q^{-1}CA^{-1} & Q^{-1} \end{pmatrix}}_{= M^{-1}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

b) ~~we have~~

We make use of our result from part (a)

if $M = LU$ then $M^{-1} = U^{-1}L^{-1}$

we compute $U^{-1}L^{-1}$ and show it equals M^{-1}

$U^{-1}L^{-1}$

$$= \begin{pmatrix} (A^{\frac{1}{2}})^T & (A^{-\frac{1}{2}})B \\ 0 & (Q^{\frac{1}{2}})^T \end{pmatrix}^{-1} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ C(A^{\frac{1}{2}})^T & Q^{\frac{1}{2}} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} ((A^{\frac{1}{2}})^T)^{-1} - ((A^{\frac{1}{2}})^T)^{-1} A^{-\frac{1}{2}} B (Q^{\frac{1}{2}})^T ((A^{\frac{1}{2}})^T)^{-1} & -((A^{\frac{1}{2}})^T)^{-1} A^{-\frac{1}{2}} B (Q^{\frac{1}{2}})^{-1} \\ 0 & (Q^{\frac{1}{2}})^{-1} \end{pmatrix}$$

using schur formula from (a)

$$\times \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ -Q^{-\frac{1}{2}} C (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} & Q^{-\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} (A^{-\frac{1}{2}})^T - (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} B (Q^{\frac{1}{2}})^T (A^{-\frac{1}{2}})^T & -(A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} B Q^{-\frac{1}{2}} \\ 0 & (Q^{\frac{1}{2}})^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ -Q^{-\frac{1}{2}} C (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} & Q^{-\frac{1}{2}} \end{pmatrix}$$

(*) we observe that $A^{\frac{1}{2}}(A^{\frac{1}{2}})^T = A$

$\Rightarrow (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} = A^{-1}$

and similar for Q $\Rightarrow$

$$\Rightarrow \mathcal{U}^{-1}\mathcal{L}^{-1} = \begin{pmatrix} (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} - (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} B (Q^{\frac{1}{2}})^T (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} & -(A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} B Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} \\ -Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} C (A^{-\frac{1}{2}})^T A^{-\frac{1}{2}} & Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} \end{pmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1} B Q C A^{-1} & -A^{-1} B Q^{-1} \\ -Q^{-1} C A^{-1} & Q^{-1} \end{bmatrix}$$

using (*)

$$= M^{-1}$$

Thus $M^{-1} = \mathcal{U}^{-1}\mathcal{L}^{-1} \Rightarrow M = \mathcal{L}\mathcal{U}$

is such an $\mathcal{L}\mathcal{U}$ decomposition of M $\square$

(c) We have from (b) that

$$\cancel{\det} \; M = \mathcal{L} \, \mathcal{U} \qquad (\text{using } \det(PQ) = \det(P)\det(Q))$$

$$\Rightarrow \det(M) = \det(\mathcal{L}) \det(\mathcal{U})$$

$$= \det\left(\begin{bmatrix} A^{1/2} & 0 \\ C(A^{-1/2})^T & Q^{1/2} \end{bmatrix}\right) \det\left(\begin{bmatrix} (A^{1/2})^T & A^{-\frac{1}{2}}B \\ 0 & (Q^{1/2})^T \end{bmatrix}\right)$$

$$= \det(A^{1/2}) \det(Q^{1/2}) \det((A^{1/2})^T) \det((Q^{1/2})^T)$$

$$= \det(A^{1/2}(A^{1/2})^T) \det(Q^{1/2}(Q^{1/2})^T)$$

using identity from hint

$$= \det(A) \det(Q) \qquad \blacksquare$$