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Simplicial Homotopy Theory

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Abstract

We provide an introductory text to the field of simplicial homotopy theory suitable for undergraduate students with minimal experience in category theory.
We show that the category of simplicial sets and the category of topological spaces can both be equipped with a model structure and that the geometric realization functor is a Quillen equivalence.

A Classical Introduction

Simplicial homotopy theory originated in the field of algebraic topology.
In algebraic topology, one attaches to a topological space algebraic invariants which depend on the space. In studying these algebraic invariants, one hopes to obtain a deeper understanding of the original space in question. A natural question is then to ask is exactly how deep an understanding can we hope to obtain.

Naively, one could hope to understand topological spaces in the conventional sense; up to homeomorphism using this approach. Unfortunately, it turns out that this is not possible. In some sense, the notion of a homeomorphism is too rigid and these invariants are always too coarse; no matter how many are attached. Homotopy theory is then a concession on this point, we concede that homeomorphism is too strong a condition to aim for and we look for a weaker one from which we can still make some strong statements.

We present some of the background material along with some important definitions in order to motivate the theory from a classical perpective.
The material here follows Hatcher [2].

2.1 Topological Homotopy

The notion of homotopy was first defined in a topological setting and is of central importance in the field of algebraic topology. Understanding homotopy from a topological perspective provides necessary intuition for the more general definition of homotopy we discuss later.

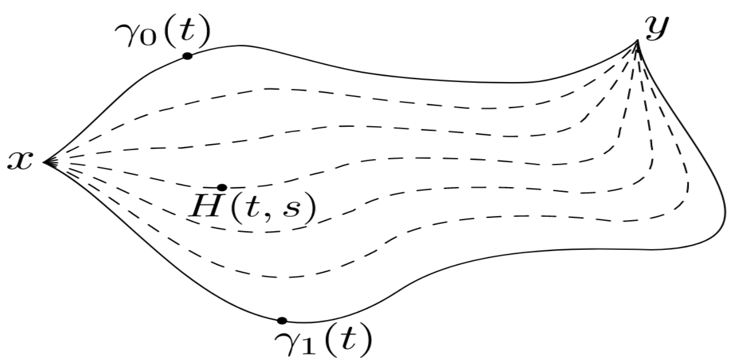
Definition Let $f, g : X \to Y$ be continuous mappings between topological spaces.
We say that f is *homotopic to* g if there exists a continuous mapping $H : X \times [0,1] \to Y$ for which $H_{|X\times\{0\}} \equiv f$ and $H_{\ |X\times\{1\}} \equiv g$.
We often denote $f \sim g$ as shorthand for f is homotopic to g.
An alternative characterization is to say there exists a commutative diagram of the form:

$$\begin{tikzcd}
X \times \{0\} \arrow[d] \arrow[drr, "f"] & & \\
X \times [0,1] \arrow[rr, "H"] & & Y \\
X \times \{1\} \arrow[u] \arrow[urr, "g"'] & &
\end{tikzcd}$$

We will see that this equivalent definition can be used to define the notion of homotopy in a more general sense.

Remark One often thinks of such a homotopy as a continuous family $(\gamma_t : X \to Y)_{t\in[0,1]}$ of continuous maps indexed by $[0,1]$ where $\gamma_0 = f$ and $\gamma_1 = g$.
The explicit homotopy here is then the mapping $(x,t) \mapsto \gamma_t(x)$. This alternative characterization gives a better intuition for what a homotopy actually is in the topological setting.

Example To give an familiar example, consider the case where $X = [0,1]$ and $Y = \mathbb{C}$. Then continuous mappings $[0,1] \to \mathbb{C}$ are simply paths in $\mathbb{C}$. Two such paths f,g are homotopic if there exists a continuous family $(\gamma_s : [0,1] \to \mathbb{C})_{s\in[0,1]}$ with $\gamma_0 = f, \gamma_1 = g$. We suppose for simplicity here that $f(0) = g(0) = x$ and $f(1) = g(1) = y$ but this is not required in general. The following illustration is then helpful to clarify things:



Remark It is well known (Hatcher [2]) that "homotopic to" defines an equivalence relations on
$\mathrm{Hom}_{\mathbf{Top}}(X,Y) := \{X \to Y \mid \text{continuous}\}$ for arbitrary topological spaces X,Y.

Definition Let X,Y be topological spaces. A continuous mapping $f : X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $f \circ g \sim \mathbb{1}_Y$ and $g \circ f \sim \mathbb{1}_X$

We observe that this definition looks remarkably similar to the definition of a homeomorphism (simply replace the homotopic to symbols with equality signs). So homotopy equivalence is indeed a weaker condition than that of a homeomorphism. This concession allows

for a more relaxed notion of topological "sameness" which is often sufficient to study certain topological properties and generally elicits some more tractable approaches.
Our definition of homotopy allows us to define an important class of algebraic invariants of a particular topological space: the homotopy groups!

Definition Given a topological space X, We define $\pi_0(X)$ as the the set of connected components of X.
Let $x_0 \in X$. For each $n \geq 1$, we define the *n-th homotopy group* $\pi_n(X, x_0)$ of X to be the set of homotopy classes of pointed maps $S^n \to X$.
To be more explicit, fix a point $p \in S^n$ and $x_0 \in X$. We define $\pi_n(X, x_0)$ to be the set of homotopy classes of continuous maps $f : S^n \to X$ for which $f(p) = x_0$. We equip this set with a binary operation as follows:
We observe that the n-sphere S^n is homeomorphic to the n-cube $[0,1]^n$ for each $n \geq 0$. It is then easy to see that we may equivalently define the n^{th} homotopy group $\pi_n(X, x_0)$ as the set of homotopy classes of continuous maps $[0,1]^n \to X$ which send the boundary of $[0,1]$ to $\{x_0\}$. The upshot here is that it is much easier to define the group operation explicitely using this definition. On the level of maps we define the product $f \bullet g$ of two such $f, g : [0,1]^n \to X$ satisfying $f(\partial[0,1]^n) = \{x_0\} = g(\partial[0,1]^n)$ by

$$(f \bullet g)(t_1, \ldots, t_n) := \begin{cases} f(2t_1, t_2, \ldots, t_n), & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{if } \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

It can be shown (Hatcher [2]) that the operation $[f][g] := [f \bullet g]$ is well defined on the equivalence classes so we really have a group structure. The identity is given by the homotopy class of the constant map $[0,1]^n \to \{x_0\} \subset X$ at the point x_0. The inverse of an element $[f]$ is denoted $[\overline{f}]$ and is defined by $\overline{f}(t_1, \ldots, t_n) := f(1 - t_1, \ldots, t_n)$.
The homotopy groups are *functorial* in nature. Meaning that given a continouous mapping $f : X \to Y$, we have an induced group homomorphism $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ for any $x \in X$. Explicitely this group homomorphism is given by postcomposition by f on homotopy classes: $\pi_n(f, x)([\gamma]) := [f \circ \gamma]$.
Functors will be discussed in more detail in the category theory chapter (3).

Remark It should be noted that it is possible to define an equivalent group operation in terms of the original definition involving spheres:
Let $f, g : S^n \to X$ be continuous maps such that $f(p) = x_0 = g(p)$. In this case the operation is defined in by collapsing the equator of S^n to the basepoint p (sometimes called the "pinch map") and then taking the wedge product of two spheres along p. On the northern hemisphere, we define the mapping as f and on the southern hemisphere g. It can be shown that this operation is well defined on homotopy classes and moreover that it defines a group isomorphic to the one defined in terms of products of intervals.
This definition can often lead to computational difficulties and is generally less tractable than the definition involving products of intervals. However it can sometimes be better for conceptual understanding.

It turns out that for $n \geq 2$, $\pi_n(X,x)$ is abelian for any topological space $X \ni x$. There is a wonderful illustration in Hatcher [2] (page 340) which explains visually why this is the case.

2.2 CW Complexes and Whitehead's Theorem

While the homotopy groups are straighforward to define, they are generally difficult to compute. There is a particular class of topological spaces whose homotopy groups are easy to define. This class of spaces are known as CW complexes and are constructed combinarially.

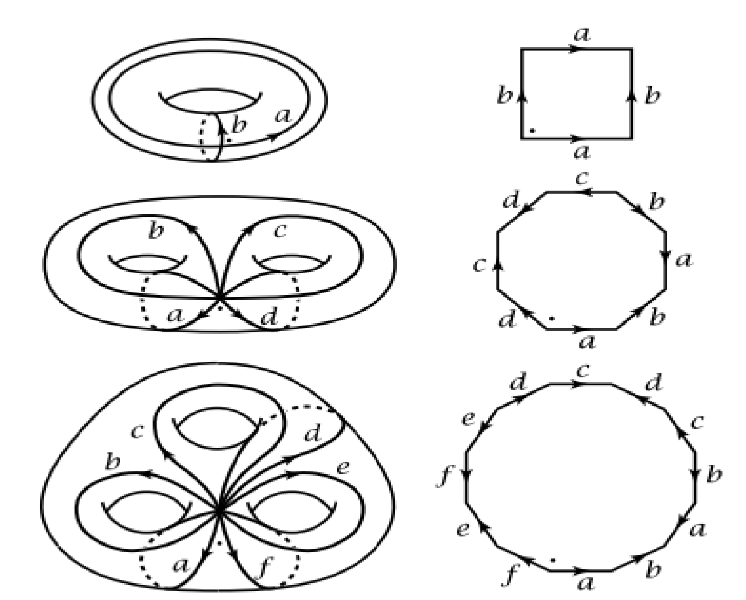
Definition A *CW complex* is defined in terms of its inductive construction

- The 0-*skeleton* X^0 is a discrete set of points. A point $x_0 \in X^0$ is called a *zero cell*
- The $n-$skeleton X^n is constructed from the $n-1$ skeleton by attaching so called *n-cells* via *attaching maps* $\varphi_\alpha : S^{n-1} \to X^{n-1}$. Explicitely, given our collection $(\varphi_\alpha : S^{n-1} \to X^{n-1})_\alpha$, we define the n-skeleton as the disjoint union of X^{n-1} with a collection of copies $(D_\alpha)_\alpha$ of the n-disk where we identify the boundary of each D_α along the attaching map φ_α :

$$X^n := (X^{n-1} \bigsqcup_\alpha D^n_\alpha)/_{x \sim \varphi_\alpha(x) \ \forall x \in \partial D^n_\alpha} \quad \forall n \geq 1$$

If this inductive process consists of finitely many steps then we may define a *finite dimensional CW complex* $X := X^n$ (where n is the largest $n \geq 0$ for which $X^{n-1} \neq X^n$)
We also allow countably many steps. In this case the CW complex $X = \bigcup_{n\geq 0} X^n$ is equipped with the weak topology.

The following (Hatcher [2]) is an illustration of three examples of finite dimensional CW complexes:



The first is of a torus; its zero skeleton consists of a single point p. We have attaching cells a and b in which we glue an interval's endpoints to p. Finally we glue the boundary of D^2 to the square represented in the figure. We therefore obtain a two dimensional CW complex.
Similarly, for the other two figures we begin with a single point. We then attach the boundary of four (resp. six) intervals to that point. Finally we attach the boundary of D^2 to the corresponding polygon again obtaining a two dimensional CW complex.
Generally we can construct an orientable manifold of genus $k \geq 1$ by a similar construction using a $4k$ sided polygon.

Remark This definition can be restated more elegantly using category theoretic language (see (3) for preliminaries)
An $n-$skeleton is a pushout in the category **Top** of topological spaces of the following form:

$$\begin{tikzcd} \bigsqcup_\alpha S_\alpha^{n-1} \arrow[r, hook, "\sqcup i_\alpha"] \arrow[d, "(\varphi_\alpha)_\alpha"] & \bigsqcup_\alpha D_\alpha^{n-1} \arrow[d] \\ X^{n-1} \arrow[r] & X^n \end{tikzcd}$$

We obtain the CW complex through a countable iteration of pushouts of this form. This will be discussed in greater generality later.

It turns out that even the conventional notion of homotopy equivalence is too strong a condition to strive for in general. The Freyd Uncertainty Principle (as outlined in Peterson [9]) tells us that given any set-valued, homotopy invariant functor $T : \mathbf{Top}_* \to \mathbf{Set}$ (from pointed topological spaces), one can always find a continuous mapping $f : X \to Y$ which is not null homotopic but its image is equal to the image of a null homotopic map.
This effectively tells us that we cannot hope to even achieve the more modest goal of complete classification up to homotopy equivalence. For this reason, we make yet a further concession. The following is a weaker notion of topological sameness which is of great importance.

Definition A continuous mapping $f : X \to Y$ between topological spaces X and Y is said to be a *weak homoptopy equivalence* if

1. $\pi_0 f : \pi_0(X) \to \pi_0(Y)$ is a bijection

2. For each $n \geq 1$ and $x \in X$, $\pi_n f : \pi_n(X,x) \to \pi_n(Y, f(x))$ is a group isomorphism

Theorem 2.2.1 *Every homotopy equivalence is, in particular, a weak homotopy equivalence. This justifies its name.*

Proof This is fairly easy to see:
Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. We use the

functorial property of the homotopy group along with the diagramatic definition of a homotopy.
We have the following commutative diagrams by definition of homotopy equivalence

$$\begin{tikzcd}
X\times\{0\} \arrow[d] \arrow[drr, "gf"] & & \\
X\times[0,1] \arrow[rr] & & X \\
X\times\{1\} \arrow[u] \arrow[urr, "\mathbb{1}_X"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y\times\{0\} \arrow[d] \arrow[drr, "fg"] & & \\
Y\times[0,1] \arrow[rr] & & Y \\
Y\times\{1\} \arrow[u] \arrow[urr, "\mathbb{1}_Y"'] & &
\end{tikzcd}$$

Now let $x \in X, y \in Y$ and $n \geq 1$ be arbitrarily chosen. We see, by applying the functorial property of the homotopy groups that we have :

$$\begin{tikzcd}
\pi_n(X\times\{0\}) \arrow[d] \arrow[drr, "\pi_n(g)\pi_n(f)"] & & \\
\pi_n(X\times[0,1]) \arrow[rr] & & \pi_n(X) \\
\pi_n(X\times\{1\}) \arrow[u] \arrow[urr, "\mathbb{1}_{\pi_n(X)}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_n(Y\times\{0\}) \arrow[d] \arrow[drr, "\pi_n(f)\pi_n(g)"] & & \\
\pi_n(Y\times[0,1]) \arrow[rr] & & \pi_n(Y) \\
\pi_n(Y\times\{1\}) \arrow[u] \arrow[urr, "\mathbb{1}_{\pi_n(Y)}"'] & &
\end{tikzcd}$$

(we choose not to include the basepoints in the diagram to avoid notational clutter and since they can be arbitrarily chosen anyway.)
We observe that the vertical arrows in these diagrams are always isomorphisms. This is since the homotopy group of a product of spaces is isomorphic to the product of their homotopy groups (with arbitrarily chosen basepoints).
Then for arbitrary $(x,t) \in X \times [0,1]$, we have

$$\pi_n(X \times [0,1], (x,t)) \simeq \pi_n(X,x) \times \pi_n([0,1]) \simeq \pi_n(X,x) \times \{0\}$$

Then it is clear that the group homomorphism induced from the inclusion must actually be an isomorphism. Similarly the horizontal arrow is an isomorphism.
In each diagram, the vertical arrows are equal and isomorphisms and the horizontal arrow is also an isomorphism. Therefore it follows that for each $n \geq 1$, we have

$$\pi_n(g)\pi_n(f) = \mathbb{1}_{\pi_n(X)} \quad \text{and} \quad \pi_n(f)\pi_n(g) = \mathbb{1}_{\pi_n(Y)}$$

and thus $\pi_n(f)$ is a group isomorphism for each $n \geq 1$. Also, the connected components of X and Y are obviously in bijection. Therefore f is a weak homotopy equivalence. ∎

CW complexes were developed by the British mathematician, John Henry Whitehead. He was a very important figure in the development of homotopy theory. Whitehead discovered an important property of his CW complexes in relation to weak homotopy equivalence. Namely that the converse to the above theorem also holds whenever the spaces in question are *CW* complexes.

Theorem 2.2.2 (Whitehead's theorem) *Let X, Y be CW complexes and $f : X \to Y$ a weak homotopy equivalence. Then f is a homotopy equivalence.*

In other words, Whitehead showed that for CW complexes, a weaker condition suffices to define homotopy equivalence.

2.3 A Combinatorial Approach

We have seen that CW complexes are a combinatorial construction, however they still carry an explicit topology. This can sometimes lead to challenges since one has to be careful to study the topology in question too. There is another means by which one can combinatorially build a topological spaces.
Namely, in 6, we will define a category called the *simplicial sets* (**sSet**) whose objects and morphisms are purely combinatorial in nature. There is then a means by which one can "realize" any simplicial set as a topological space. The advantage here is that one can construct simplicial sets with no reference to a topology and by doing so we can consider a purely combinatorial problem. We can also construct from an arbitrary topological space a canonical (or singular) simplicial set.
It turns out that, on the level of weak homotopy equivalence, one can view these two processes as, in some sense, inverse to one another and it is this viewpoint allows us to understand the problem of classification up to weak homotopy equivalence.

We first need to build up some abstract machinery to better equip us to attack this problem. We first provide the necessary backround material on category theory, followed by a discussion on model categories. We will show that we can build a theory of homotopy on the simplicial sets and that this theory allows us to make some pleasant topological statements.

Some Category Theory Preliminaries

3.1 Introduction

We begin by considering the following quote from Alexander Grothendieck. He is widely regarded as one of the most influential mathematicians of the 20^{th} century. His most important work was in the area of modern algebraic geometry into which he introduced a plethora of new ideas involving category theory. However, his ideas and approoach reshaped a great deal of modern mathematics.
Grothendieck describes the following analogy in which he likens the process of constructing a concrete mathematical proof to the cracking of a nut

> "I can illustrate the second approach with the same image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!"

Grothendieck's believed that instead of approaching a problem directly, one could alternatively build up abstract theory surrounding that problem and in doing so let the problem "soak" and "soften". Appraoching the problem in this way can often lead to unforseen and deeper insights and, within the context of the built up theory, the original problem can become almost tautological.

Remark Vakil [11] is an algebraic geometry textbook which extends this analogy in a tribute to Grothendieck.

Category theory provides us with an extremely convenient language to describe many of the important phenomena in algebraic topology. We abstract many of the constructions in simplicial homotopy theory using category theory. In doing so we can circumvent the challenge of constructing ad hoc proofs in a classical setting. Often these results become almost obvious when viewed from the right perspective.

Remark The theory of categories was developed by American mathematicians Saunders Mac Lane and Samuel Eilenberg in 1945. These mathematicians also did important work in the field of algebraic topology in which their inspiration for inventing category theory originated.

We provide some important basic definitions and results from category theory along with some illustrative examples. The material in this section follows Leinster [6] along with Mac Lane [7]

Definition A *category* $\mathscr{C}$ consists of the following data

1. A class of *objects* denoted by $\mathrm{Ob}\mathscr{C}$
2. For any two objects X,Y in $\mathrm{Ob}\mathscr{C}$, a class $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ of arrows or *morphisms*. We generally denote any such morphism $f \in \mathrm{Hom}$ by $f : X \to Y$ or sometimes by $X \xrightarrow{f} Y$ in diagrams.
3. Given any morphisms f in $\mathrm{Hom}(X,Y)$ and g in $\mathrm{Hom}(Y,Z)$ there is a so called *composition* gf in $\mathrm{Hom}(X,Z)$:

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "gf"] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

We require that for each X in $\mathrm{Ob}\mathscr{C}$ there is an identity morphism $\mathbb{1}_X \in \mathrm{Hom}(X,X)$ and that composition of morphisms is associative.

Remark When convenient, we will sometimes denote $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ simply by $\mathscr{C}(X,Y)$ to avoid clutter.

In principle, category theory is a abstract formalism of mathematical structure. Roughly speaking, the idea of category theory is that after abstracting out the particulars of a mathematical theory, what one is left with is a collection of objects with relations between them.

Within the class of morphisms of a particular category $\mathscr{C}$ we distinguish an important subclass:

Definition A morphism $X \xrightarrow{f} Y$ is said to be an *isomorphism* if there exists a morphism $Y \xrightarrow{g} X$ for which $fg = \mathbb{1}_Y$ and $gf = \mathbb{1}_X$. We say objects X and Y are *isomorphic* (denoted $X \simeq Y$) if there exists an isomorphism $f : X \to Y$

Within the context of the category $\mathscr{C}$, objects X and Y are in some sense indistinguishable if they are isomorphic. To use another analogy, through the lenses of whatever category we are working with, one cannot "see" any distinguishable difference.
Loosely speaking, in Greek "isomorphism" translates to "same shape".

Examples With this in mind, we consider the following examples

1. The category of sets **Set**. It's objects are sets and its morphisms are functions. The isomorphisms in this category are bijections. This is because we have imposed no further structure on the objects in this category.

2. The category of topological spaces **Top**. It's objects are topological spaces and its morphisms are continuous mappings. Isomorphisms within this cateogory are homeomorphisms. We remark that if two topological spaces are homeomorphic, then there is nothing topologically interesting that can be said about one space that cannot also be said about the other.
   In an algebraic topology context, the category of *pointed topological spaces*, denoted by $\mathbf{Top}_*$, is often considered. It's objects are topological spaces equipped with a basepoint (denoted (X,x) for $x \in X$). A morphism $f : (X,x) \to (Y,y)$ is a continuous mapping $f : X \to Y$ such that $f(x) = y$.

3. The category of groups **Grp**. It's objects are groups and its morphisms are group homomorphisms. Bijective group morphisms are the isomorphisms of this category. One could also consider the category of Abelian groups **AbGrp** whose objects are Abelian groups and whose morphisms are still group homomorphisms. The latter is a so called *full sub-category* of the former.

4. The category **Ring** of rings with ring homomorphisms.

5. The category of k-vector spaces $\mathbf{Vect}_k$ (for a chosen field k) with k-linear maps as morphisms

6. The category of A modules ${}_A\mathbf{Mod}$ (for a chosen ring A) with A-linear maps as morphisms.

7. The category of finite simple graphs with graph morphisms between them (where a graph morphism is defined as a mapping between the vertex sets which preserves edges). One could also consider directed graphs.

8. Let G be a group. Consider the category BG consisting of a single object: the group G itself. It's morphisms are elements $g \in G$ and composition of morphisms is given by left multiplication.
   We observe that every morphism in this category is an isomorphism (since by definition, every $g \in G$ is invertible).
   Whenever a category has the property that every morphism is an isomorphism, we call the category in question a *groupoid*. BG is then an example of a groupoid with a single object.

9. The category of smooth manifolds with smooth maps as morphisms. It's isomorphisms are diffeomorphisms.

10. Let $(P, \leq)$ be a poset (for instance $(\mathbb{N}, \leq)$ or $\mathcal{P}(X)$ for a set X with set inclusion as a partial order). It's objects are elements $x \in P$ and there is a morphism from $x \in P$ to $y \in P$ iff $x \leq y$. Note that objects x and y are isomorphic in this category if and only if $x = y$

The purpose here was to demonstrate the ubiquitous nature of category theory in mathematics. In almost any mathematical theory one can abstract the structure using category theoretic language. The advantage to this approach is that if one proves statements pertaining to arbitrary categories, they can be applied to a wide class of mathematical areas.

3.2 Functors and Natural Transformations

In a category theory context, whenever presented with a candidate class of objects, it is generally productive to consider whether one has arrows between such objects.

Definition (Functors)

1. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$ associates to each object $X \in \mathrm{Ob}\mathscr{C}$, an object $F(X) \in \mathrm{Ob}\mathscr{D}$ and associates to each morphism $f : X \to Y$ in $\mathscr{C}$, a morphism $F(f) : F(X) \to F(Y)$.

   We require that for each $X \in \mathrm{Ob}\mathscr{C}$, $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ and given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $F(gf) = F(g)F(f)$

2. A *contravariant functor* F between categories $\mathscr{C}$ and $\mathscr{D}$ associates to each $X \in \mathrm{Ob}\mathscr{C}$, an object $F(X) \in \mathrm{Ob}\mathscr{D}$. To each morphism $f : X \to Y$, F associates a morphism $F(f) : F(Y) \to F(X)$.
   We again require that $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ and we require that $F(gf) = F(f)F(g)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{C}$.

Remark We define the *opposite* or *dual category* $\mathscr{C}^{\mathrm{op}}$ of a category $\mathscr{C}$ as the category in which $\mathrm{Ob}\mathscr{C} = \mathrm{Ob}\mathscr{C}^{\mathrm{op}}$ and for any objects X, Y in $\mathscr{C}$, $\mathscr{C}^{\mathrm{op}}(X,Y) := \mathscr{C}(Y,X)$
This definition is made mostly for convenience so we can denote a contravariant functor between $\mathscr{C}$ and $\mathscr{D}$ as a covariant functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ thus allowing us to unambiguously refer to a functor using standardized notation.
As a general rule, most theorems in category theory have a dual version which can be obtained by reversing the directions of the morphisms in quesion.

In other words, a functor is then a portal between categories. It was in the context of algebraic topology where the notion of a functorial construction first arose. As mentioned in the introduction, the algebraic invariants attached to a particular topological space are almost always functorial in nature.

Examples Functors appear everywhere in modern mathematics. We mention the following examples:

- Each homotopy group π_n for $n \geq 1$ defines a covariant functor π_n :$\mathbf{Top}_* \to \mathbf{Grp}$. To each pointed topological space (X,x), the functor assigns the group $\pi_n(X,x)$ and to a pointed continuous map $f : (X,x) \to (Y, f(x))$, a group morphism $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y, f(x))$ defined by $\pi_n(f)([\varphi]) := [f \circ \varphi]$

- Similarly each singular homology group H_n defines a covariant functor H_n :**Top**$\to$ **AbGrp**. This functor however, does not require an explicit choice of basepoint.
- Simplicial Cohomology (the dualized notion of simplicial homology) unsurprisingly defines a functor H^n : $\mathbf{Top}^{\mathrm{op}} \to$ **AbGrp** for each $n \geq 0$. The reason for constructing cohomology is that it has nicer algebraic properties than its non-dualized counterpart. One can define a so called cup product $H^j(X) \times H^k(X) \to H^{j+k}(X)$ which allows us to define the graded cohomology ring $H^\bullet(X) := \bigoplus_{j\geq 0} H^j(X)$ of a space X.
This graded cohomology ring induces a contravariant functor **Top** $\to$ **Ring**. The extra structure usually yields a stronger invariant than homology and allows for a number of more straighforward computations.
- Given a group G, a group action is simply a functor from the category BG to **Set**. Similarly, a linear representation of a finite group G is simply a functor $G[1] \to \mathbf{Vect}_k$
- A boring but ocasionally useful example is the so called *identity functor* and is defined exactly as one might expect. Given an aribrary category $\mathscr{C}$. The identity functor $\mathbb{1}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ simply sends every object to itself and every morphism to itself.

The following are two particularly important functors studied in category theory

Definition Let $\mathscr{C}$ be a *locally small category* (meaning that for any two objects X and Y, $\mathrm{Hom}(X,Y)$ is a set). Fix some object X in $\mathscr{C}$. We define the so called *hom functors*:

1. The functor $\mathrm{Hom}(X,-) : \mathscr{C} \to$**Set** which associates to $Y \in \mathrm{Ob}\mathscr{C}$, the set $\mathrm{Hom}(X,Y)$ and associates to each morphism $f : Y \to Z$ in $\mathscr{C}$ the set valued function $f_* : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ by postcomposition with f ($g \mapsto fg$)
2. The functor $\mathrm{Hom}(-,X) : \mathscr{C}^{\mathrm{op}} \to$ **Set** associates the set $\mathrm{Hom}(Y,X)$ to each object Y in $\mathscr{C}$ and the function $f^* : \mathrm{Hom}(Z,X) \to \mathrm{Hom}(Y,X)$ (defined by precomposition with f) to each morphism $f : Y \to Z$ in $\mathscr{C}$

We sometimes denote these Hom functors by h_X and h^X respectively

These functors equip us to discuss many important theorems and ideas in category theory.

We can also conceptualize the notion of a morphism of two functors $F, G : \mathscr{C} \to \mathscr{D}$ between two categories.

Definition Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* $\eta : F \to G$ is a collection $(\eta_X : F(X) \to G(X))_{X\in\mathrm{Ob}\mathscr{C}}$ of morphisms in $\mathscr{D}$ such that, for every morphism $f : X \to Y$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

If $\eta_X : F(X) \to F(Y)$ is an isomorphism for each $X \in \mathrm{Ob}\mathscr{C}$, then η is said to be a *natural isomorphism*

We can then build the category of functors between two categories

Definition We denote by $\mathrm{Fun}(\mathscr{C},\mathscr{D})$ the functor category. Its objects are functors $\mathscr{C} \to \mathscr{D}$ and its morphisms are natural transformations. Sometimes this category is also denoted by $[\mathscr{C},\mathscr{D}]$ or even $\mathscr{D}^{\mathscr{C}}$

The definition of a natural isomorphism allows us to define another notion of great importance. We have discussed that an isomorphism of objects within a particular category defines some sort of indistinguishability. An important question to ask is whether an analogous notion can be considered for entire categories. One could easily define an isomorphism of categories $\mathscr{C}$ and $\mathscr{D}$ as a functor $F : \mathscr{C} \to \mathscr{D}$ with an *inverse functor* $G : \mathscr{D} \to \mathscr{C}$. Namely such that $FG = \mathbb{1}_{\mathscr{D}}$ and $GF = \mathbb{1}_{\mathscr{C}}$.
This turns out to be far too rigid a criterion to be meaningful or useful. Instead of requiring equality on the nose for these compositions, we can merely require that they are naturally isomorphic to the respective identites. Somehow this is good enough for preserving most of the properties of interest of a category. This is the notion of an equivalence and is spelled out more precisely below

Definition An *equivalence* of categories $\mathscr{C}$ and $\mathscr{D}$ consists of functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : \mathbb{1}_{\mathscr{C}} \to GF$, $\varepsilon : FG \to \mathbb{1}_{\mathscr{D}}$
In particular, for arbitrary morphisms $f : X \to X'$ in $\mathscr{C}$ and $g : Y \to Y'$ in $\mathscr{D}$, the following diagrams commute:

$$\begin{array}{ccc} FGY & \xrightarrow{FGg} & FGY' \\ \downarrow{\scriptstyle \varepsilon_Y} & & \downarrow{\scriptstyle \varepsilon_{Y'}} \\ Y & \xrightarrow{g} & Y' \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X'}} \\ GFX & \xrightarrow{GFf} & GFX' \end{array}$$

If such functors exist, we say that the categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*.

In order to prove that two categories are equivalent, it can be a little bit cumbersome to explicitely construct functors in both directions as in the above definition. For this reason, one usually makes use of the following theorem (Mac Lane [7]) which provides some friendlier equivalent conditions to check.

Theorem 3.2.1 *A functor $F : \mathscr{C} \to \mathscr{D}$ between locally small categories is an equivalence if and only if*

1. *F is* faithful*, meaning that for any objects $X,Y \in Ob\mathscr{C}$ the map*

$$Hom_{\mathscr{C}}(X,Y) \to Hom_{\mathscr{D}}(F(X),F(Y)), \quad f \mapsto F(f) \quad (*)$$

is an injection

2. *F is* full*, meaning that* $(*)$ *defines a surjection for all* $X,Y \in Ob\mathscr{C}$

3. *F is* essentially surjective*, meaning that for any object* $Y \in \mathscr{D}$*, there is some* $X \in \mathscr{C}$ *for which* $F(X) = Y$

The following example illustrates why the notion of equivalence is a better idea than an isomorphism for qualifying categorical sameness:
Let k be a field. Consider the category $\mathscr{K}$ whose objects are all finite direct sums of the form $\oplus_j k$ and whose morphisms are k-linear maps.
It is fairly obvious that the category $\mathbf{Vect}_k^{\mathrm{f,d}}$ of finite dimensional vector spaces over k is in some sense the same as $\mathscr{K}$. Indeed given such a finite dimensional space V, we can consider a basis $\{e_1,\ldots,e_n\}$ of V. Then we have an explicit isomorphism $V = \bigoplus_{j=1}^n ke_j \simeq \bigoplus_{j=1}^n k$.
These categories are not isomorphic! Somehow the requirement of an explicit inverse is too strong since once can consider a number of different bases of a particular space so we don't get a bijective correspondence.
However, these categories are equivalent to one another. Explicitely the inclusion functor $i : \mathscr{K} \to \mathbf{Vect}_k^{\mathrm{f,d}}$ is an equivalence. It would be difficult to define a an explicit functor in the other direction as in the definition but it is easy to see that i is faithful, full and essentially surjective.
We will establish an important equivalence between the homotopy category of simplicial sets and the homotopy category of topological spaces. This equivalence allows us to make some insightful statements about topological spaces up to weak homotopy equivalence.

3.3 Adjoint Functors

Often in modern mathematics, one has the situation in which there are categories $\mathscr{C}$ and $\mathscr{D}$ and a pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. Often whenever such a pair of functors of interest appear in mathematics, they satisfy a certain abstract criterion.

Remark Interestingly enough, adjoints are one of the few cornerstone concepts of category theory not invented by Eilenberg and MacLane. It was Dutch mathematician Daniel Kan who originally discovered them. Kan actually worked in simplicial homotopy theory and proved many of the results we discuss in this paper.

Definition Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. We say that F is *left adjoint* to G (resp. G *right adjoint* to F) if

$$\mathscr{D}(F(X),Y) \simeq \mathscr{C}(X,G(Y))$$

naturally in $X \in \mathrm{Ob}\mathscr{C}$, $Y \in \mathrm{Ob}\mathscr{D}$. In other words, the functors

$$\mathscr{D}(F(-),-), \mathscr{C}(-,G(-)) : \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$$

are naturally isomorphic.
An *adjunction* is an explicit choice of natural isomorphism φ. Often we refer to the triple (F,G,φ) as an adjunction.

Remark The term adjoint was inspired by a similar phenomenon observed in operator theory where the Hermitian adjoint A^* of a linear operator A is characterized by the property that $\langle Ax, y\rangle = \langle x, A^*y\rangle$

The idea here is that we want to somehow say that morphisms $X \to G(Y)$ are the "same as" morphisms $F(X) \to Y$.

Example There are a large number of examples of adjoints which emerge throughout mathematics, one of which we will study later in this project. We provide several illustrative examples:

1. The following is a very simple yet illustrative example:
   Consider the poset categories $\mathscr{C} = (\mathbb{R}, \leq)$ and $\mathscr{D} = (\mathbb{Z}, \leq)$. Consider the inclusion functor $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ and the floor functor $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$. Claim that the functor i is left adjoint to $\lfloor - \rfloor$. Indeed given $a \in \mathbb{Z}$ and $b \in \mathbb{R}$, we have $a \leq b \iff a \leq \lfloor b \rfloor$ and because of this we therefore have $\mathbb{R}(i(a), b) \simeq \mathbb{Z}(a, \lfloor b \rfloor)$ naturally in $a \in \mathbb{Z}, b \in \mathbb{R}$

2. From commutative algebra, we have the famous so called "tensor-hom" adjunction.
   Let A be a commutative ring and ${}_A\mathbf{Mod}$ be the category of A-modules. Fix some module M.
   We have endofunctors $- \otimes M$ which associates to a module L, the tensor product $L \otimes M$.
   We also have the hom functor $\mathrm{Hom}(M, -)$ where we equip each hom set with the obvious module structure (defined pointwise).
   The Tensor-Hom adjunction tells us that $-\otimes$ is left adjoint to $\mathrm{Hom}(M, -)$.
   Namely that for modules L, M, N over A, we have $\mathrm{Hom}(L \otimes M, N) \simeq \mathrm{Hom}(L, \mathrm{Hom}(M, N))$

3. Given a category whose objects are sets with some underlying structure and whose morphisms are maps which preserve that structure (a so called *concrete* category), there is always a so called *forgetful functor*.
   For instance, consider the category **Top**. The functor U : **Top** $\to$ **Set** "forgets" the structure by sending a topological space $(X, \mathcal{T})$ to its underlying set X.
   This forgetful functor often has a right adjoint; namely the functor **Set** $\to$ **Top** which attaches to each set X the chaotic topology ($\mathcal{T}_X = \mathcal{P}(X)$)

3.4 Limits and Colimits

The definition of a limit is extremely general so to provide some motivation for it via several important examples.
Let us first consider the following example from topology.
Suppose that we have two topological spaces X and Y. In point set topology one traditionally defines the product space $X \times Y$ equipped with the so called product topology (the topology generated by unions of products of open sets $U \times V \subseteq X \times Y$. To a category theorist, this definition is entirely unsatisfactory. This is because it depends, a priori, on the properties of the spaces X and Y. This is problematic since we would like something more general.

In order to obtain a category theory friendly definition, one ought to define the product in a way that depends only on morphisms within the category of topological spaces. We make the following observation:
We have canonical continuous projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ for which whenever we have continuous maps $Z \to X$ and $Z \to Y$, they factor uniquely through $X \times Y$. It can be shown that the product topology, as defined above, it the finest topology which makes the projections continuous. With this in mind we make the following definition for the product of two objects in a general category

Definition Let $\mathscr{C}$ be a category and let $X, Y \in \mathrm{Ob}\mathscr{C}$. A product of X and Y (if it exists) is an object P in $\mathscr{C}$ along with morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ for which given any morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique morphism $f_Z : Z \to P$ making the following diagram commute:

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f_X"'] \arrow[d, dashed, "\exists f_Z"] \arrow[dr, bend left, "f_Y"] & \\ & P \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$

Note that the product is not necessarily unique. However it is unique up to isomorphism (Mac Lane [7]). For this reason we will often refer to "the product" of objects and denote this by $X \times Y$.

This notion can be generalized to the product over an arbitrarily indexed collection of objects.

Definition Let $(X_\alpha)_{\alpha \in A}$ be a collection of objects in $\mathscr{C}$ indexed by a set A. A product of (X_α) consists of an object P and a collection of morphisms $(\pi_\alpha : P \to X_\alpha)$ for which any collection of morphisms $(f_\alpha : Z \to X_\alpha)$ factors uniquely through P.
That is, there exists a morphism $g : Z \to P$ for which $f_\alpha = \pi_\alpha g$ for all $\alpha \in A$
"The product" of such a collection is usually denoted by $\prod_{\alpha \in A} X_\alpha$.

For most notions in category theory, there exists a dual notion obtained from reversing the direction of the arrows in the corresponding diagram. In the case of products there is a dual notion of a so called coproduct:

Definition Let X, Y be objects in a category $\mathscr{C}$. A coproduct of X and Y consists of an object C in $\mathscr{C}$ along with morphisms $i_X : X \to C$, $i_Y : Y \to C$ such that given anymorphisms $f : X \to Z$ and $Y \to Z$, there exists a unique morphism $f_Z : C \to Z$ for which the folloing diagram commutes

$$\begin{tikzcd}
X \arrow[dr, "i_X"] \arrow[ddr, bend right, "f_X"'] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "f_Y"] \\
& C \arrow[d, dashed, "\exists! f_Z"] & \\
& Z &
\end{tikzcd}$$

This definition also generalizes in the the obvious way to a collection or morphisms. The coproduct of a collection $(X_\alpha)_{\alpha\in A}$ is denoted by $\bigsqcup_{\alpha\in A} X_\alpha$ or just $X \sqcup Y$ for two objects X, Y

Remark To give some familiar examples: the coproduct **Top** is the disjoint union. The coproduct in a category like $\mathbf{Vect}_k$ or ${}_A\mathbf{Mod}$ is the direct sum.

Let us consider another familiar example of a limit:
Let $f : V \to W$ be a linear map between vector spaces. The kernal of this map is the set of all vectors which are sent to zero. Obviously the kernal is a vector subspace and thus there is in inclusion morphism $\ker(f) \hookrightarrow V$. We always have the zero map $0 : V \to W$ defined by $v \mapsto 0\ \forall v \in V$. We make the following observation which may seem unusual a priori:
$\ker(f) \hookrightarrow V$ is such that the compositions $\ker(f) \overset{i}{\hookrightarrow} V \xrightarrow{f} W$ and $\ker(f) \overset{i}{\hookrightarrow} V \xrightarrow{0} W$ are equal. Moreover, it can be shown that the pair $(\ker(f), i)$ satisfy a universal property. Often we denote this by the following commutative diagram (called a *fork*):

$$\begin{tikzcd}
\ker(f) \arrow[r, hook, "i"] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & W
\end{tikzcd}$$

The generalization of this observation to arbitrary categories is as follows:

Definition Let $\mathscr{C}$ be a category and let $f, g : X \to Y$ be morphisms between objects X, Y in $\mathscr{C}$.

1. An *equalizer* of f and g consists of an object $E \in \mathrm{Ob}\mathscr{C}$ and a morphism $i : E \to X$, for which $fi = gi$ and given any other object H and morphism $j : H \to X$ for which $fj = gj$, there is a unique morphism $k : H \to E$ for which the following diagram commutes

$$\begin{tikzcd}
E \arrow[r, hook, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
H \arrow[u, dashed, "k"] \arrow[ur, "j"'] & &
\end{tikzcd}$$

2. Dually, an object C and a morphism $\ell : Y \to C$ is said to be a *coequalizer* of f and g if $\ell f = \ell g$ and given any other object D and morphism $m : Y \to D$ for which $mf = mg$, there is a unique morphism $C \to D$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\ell"] \arrow[dr, "m"'] & C \arrow[d, dashed, "\exists"] \\
& & D
\end{tikzcd}$$

Remark Provided that our category is *pointed*, meaning that there exists a object $*$ unique up to isomorphism, for which given any object X, there are unique morphisms $X \to *$ and $* \to X$. This can be restated more elegantly by saying that the initial and terminal object are isomorphic (these terms will be explained shortly).
The kernel of a morphism $f : X \to Y$ is therefore the equalizer of the zero map $0 : X \to Y$ and f
Dually the cokernel of a morphism $f : X \to Y$ is the coequalizer of the zero map and f

With these observations, we are now ready to give the general definition of a model category:

Definition Let $\mathcal{J}$ be a small category (meaning that the objects of $\mathcal{J}$ form a set and $\mathcal{J}$ is locally small). Given a category $\mathscr{C}$ and a functor $D : \mathcal{J} \to \mathscr{C}$ which we call a *diagram*

1. A *limit* of the diagram consists of an object $L \in \mathrm{Ob}\mathscr{C}$ and a collection of morphisms $(\varphi_J : L \to D(J))_{J\in\mathcal{J}}$ indexed by $\mathcal{J}$ for which given any morphism $f : J_1 \to J_2$ in $\mathcal{J}$ following diagram commutes :

$$\begin{tikzcd} & L \arrow[dl, "\varphi_{J_1}"'] \arrow[dr, "\varphi_{J_2}"] & \\ D(J_1) \arrow[rr, "D(f)"] & & D(J_2) \end{tikzcd}$$

   We also require that a so called *universal property* is satisfied meaning that given any other object M and collection of morphisms $(\lambda_J : M \to D(J))_{J\in\mathcal{J}}$ which satisfy this condition, there is a unique morphism $M \to N$ which makes the following diagram commute

$$\begin{tikzcd} & M \arrow[d, dotted, "\exists"] \arrow[ddl, bend right, "\lambda_{J_1}"'] \arrow[ddr, bend left, "\lambda_{J_2}"] & \\ & L \arrow[dl, "\varphi_{J_1}"'] \arrow[dr, "\varphi_{J_2}"] & \\ D(J_1) \arrow[rr, "D(f)"] & & D(J_2) \end{tikzcd}$$

2. Dually, a *colimit* of the diagram (obtained from reversing the directions of the arrows) consists of an object C and a collection of morphisms $(\alpha_J : D(J) \to C)_{J\in\mathcal{J}}$ such that given any morphism $f : J_1 \to J_2$ in $\mathcal{J}$, the following diagram commutes

$$\begin{tikzcd} & C & \\ D(J_1) \arrow[ur, "\alpha_{J_1}"] \arrow[rr, "D(f)"] & & D(J_2) \arrow[ul, "\alpha_{J_2}"'] \end{tikzcd}$$

and given any other object E and collection of morphisms $(\beta_J : D(J) \to E$ there is a unique morphism $C \to E$ making the following diagram commute:

$$\begin{tikzcd} & E & \\ & C \arrow[u, dashed, "\exists"] & \\ D(J_1) \arrow[uur, bend left, "\beta_{J_1}"] \arrow[ur, "\alpha_{J_1}"] \arrow[rr, "D(f)"] & & D(J_2) \arrow[uul, bend right, "\beta_{J_2}"'] \arrow[ul, "\alpha_{J_2}"'] \end{tikzcd}$$

We denote the limit of a diagram D by $\lim(D)$ and the colimit by $\mathrm{colim}(D)$

Limits and colimits are not unique but it can be easily seen that they are unique up to isomormophism. For this reason, we will often refer to "the" limit or colimit of a particular diagram. Almost every category theory textbook (for instance Mac Lane [7]) proves this straightforward result.

Remark The limit/colimit of a particular diagram need not exist for a general category $\mathscr{C}$. We say that a category is *complete* if it has all limits and *cocomplete* if it has all colimits.

We see then for instance that the product of two objects X and Y in is the limit of the diagram $D : \mathfrak{J} \to \mathscr{C}$ where $\mathfrak{J}$ consists of two objects J_1 and J_2 with only identity morphisms and where $D(J_1) = X, D(J_2) = Y$. The coproduct of X and Y is simply the colimit of this same diagram.

Similarly, the equalizer and coequalizer are the respective limit and colimit of a diagram correpsonding to:

$$\bullet \rightrightarrows \bullet$$

(where we omit the identity arrows for simplicity)
This kind of notation is extremely useful and it allows us to easily define several other prevelant examples of (co)limits.

Definition Given a complete and cocomplete category $\mathscr{C}$ and a diagram $D : \mathfrak{J} \to \mathscr{C}$ we note the following important examples:

- The colimit of the diagram corresponding to

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

is called a *pushout*

- The limit of the diagram corresponding to

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

  is called a *pullback*

- The limit (resp. colimit) of the empty diagram is called the *terminal* (resp. *initial*) object of $\mathscr{C}$.
  We usually denote the terminal object of a category $\mathscr{C}$ by 1 and we remark explicitely that for each $X \in \mathrm{Ob}\mathscr{C}$, there is a unique morphism $X \to 1$. In the category **Grp**, the terminal object is the trivial group (containing just the identity). In the category **Set** or **Top** , the terminal object is a singleton set.
  Similarly, we usually denote the initial object by 0. Then for each $X \in \mathrm{Ob}\mathscr{C}$, there is a unique morphism $0 \to X$. In the category **Set**, the initial object is the empty set. In the category **Ring**, the initial object is the integers.

3.5 Representables and The Yoneda Lemma

One of the most important places where the Hom functors appear is when discussing representables.
In short, by fixing an object X in a locally small category $\mathscr{C}$, the functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ give us information about how X "interacts with" every other object within $\mathscr{C}$.
Certain functors (whose target is **Set**) have the pleasant property that, for a certain choice of object, the Hom functor gives us all the information we could need. These functors are known as representables and their definition is as follows:

Definition We call a functor $F : \mathscr{C} \to$ **Set** (where $\mathscr{C}$ locally small) *representable* if there exists an object $X \in \mathrm{Ob}\mathscr{C}$ for which F is naturally isomorphic to $\mathrm{Hom}(X,-)$

So if a functor is representable, we can understand it entirely in terms of a Hom functor which is already well understood. This kind of thinking leads us to an extremely important result known as Yoneda's lemma.
The idea here is somehow analagous to an idea in functional analysis. Rather than studying a complex Banach or Hilbert space X by itself, it can often be productive to consider it's dual $X^* := \mathrm{Hom}(X,\mathbb{C})$ (the Hom set into the simplest non-trivial space). In doing so we often obtain important information about X and gain insights which could not have been arrived at by studying the space X by itself.
The categorical analogue here is that given a locally small category $\mathscr{C}$, it can often be fruitful to study the category of functors from $\mathscr{C}$ to **Set**. The Yoneda Lemma (formulated by Japanese mathematician and computer scientist Nobou Yoneda) gives an important bijective correspondence of interest.

Lemma 3.5.1 *(Yoneda) Let $\mathscr{C}$ be a locally small category.*
Then for any object $X \in Ob\mathscr{C}$, there is a natural bijection between the objects of $Fun(h_X, F)$ and the elements of the set $F(X)$.

3.6 Additional Important Definitions

There are several other important definitions and concepts which we will avail of later on.

We discussed the notion of an isomorphism, but category theory also has analogues of injective and surjective morphisms. We note that the definitions below depend only on morphisms between objects and do not depend on the structure of the objects in question.

Definition Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$

1. f is said to be a *monomorphism* if for all $A \in \mathrm{Ob}\mathscr{C}$ and morphisms $g_1, g_2 : A \to X$ such that $fg_1 = fg_2$, we have $g_1 = g_2$

2. Dually f is said to be an *epimorphism* if for all $B \in \mathrm{Ob}\mathscr{C}$ and morphisms $h_1, h_2 : Y \to B$ such that $h_1 f = h_2 f$, we have $h_1 = h_2$.

The material from here follows Hovey [4]

Definition Firstly we consider the following important notions relating to ordinals:

1. An *ordinal* λ is a well-ordered set.

   - Every ordinal λ has a so called *successor ordinal* which is usually denoted $\lambda + 1$. It is the smallest ordinal larger than λ
   - An ordinal λ is said to be a *limit ordinal* if there exists an ordinal $\alpha < \lambda$ and for every ordinal $\gamma < \lambda$ there exists an ordinal β such that $\gamma < \beta < \lambda$. That is λ is an ordinal which is not the sucessor of any other ordinal.
   - We can then consider an ordinal as a category in the usual sense that we would any poset.

2. Given some ordinal λ, A λ*-sequence* in some category $\mathscr{C}$ is a colimit preserving functor $F : \lambda \to \mathscr{C}$. We have the following illustration:

$$X_1 \to X_2 \to X_3 \to \ldots \to X_\alpha \to \ldots$$

   Provided that $\mathscr{C}$ is cocomplete, we can take the colimit of the diagram.

3. In particular, for any ordinal $\alpha < \lambda$, we have an induced map $\mathrm{colim}_{\beta<\alpha} X_\beta \to X_\alpha$. The morphism $X_0 \to \mathrm{colim}_{\beta<\alpha} X_\beta$ is said to be the *composition* of a λ-sequence.

4. Suppose I is a class of morphisms in $\mathscr{C}$. Then we say that $X_0 \to \mathrm{colim}_{\beta<\alpha} X_\beta$ is a *transfinite composition* of morphisms in I if $X_\beta \to .X_{\beta+1}$ is in I for every ordinal β such that $\beta + 1 < \lambda$

Definition Let $\mathscr{C}$ be a locally small category which is cocomplete.
An object $X \in \mathrm{Ob}\mathscr{C}$ is said to be *small* (or sometimes *compact*) if for any diagram $F : \mathfrak{J} \to \mathscr{C}$ indexed over an arbitrary ordinal λ, we have an isomorphism

$$\mathrm{colim}_j(\mathrm{Hom}(X, F(j))) \to \mathrm{Hom}(X, \mathrm{colim}_j(F_j)) \quad (*)$$

Let $I \subset \mathrm{Map}\mathscr{C}$ be a distinguished class of morphisms. If given any λ-sequence $Y_0 \to Y_1 \to \ldots$ such that $Y_\alpha \to Y_{\alpha+1}$ is in I for all ordinals α: $\alpha + 1 < \lambda$ and $(*)$ is satisfied, we say that X is small relative to I.

Remark We note that this is not the most rigourous and general possible definition of a small object. However, in the context of dealing with simplicial sets (**sSet**) and topological spaces (**Top**), which will be the focus of this document, this definition is equivalent and easier to work with.

Model Categories

4.1 Localization

In order to classify topological spaces up to homotopy equivalence, one natural approach is to turn to category theory and try to relate the notion of homotopy equivalence to an isomorphism of objects. However there is a snag in this approach:
We observe that two spaces being homotopy equivalent does not guarantee that they are isomorphic in the category **Top** of topological spaces (homeomorphic). Indeed the complex plane and the origin are homotopy equivalent (via a deformation retract) as topological spaces but are certainly not isomorphic in this category. We therefore need to construct a canonical category in which the objects are topological spaces but where we can view homotopy equivalences as isomorphisms
In fact, better yet, given a category $\mathscr{C}$ and some distinguished class of morphisms W, one can construct a canonical category in which the objects are the same but we can view these distinguished morphisms as isomorphisms by formally adding inverses for each of our distinguished morphisms. This construction is known as a *localization* and its formal definition is outlined below in the form of a theorem.

Theorem 4.1.1 *Let $\mathscr{C}$ be a category and W a class of morphisms in $\mathscr{C}$.*
There exists a category $\mathscr{C}[W^{-1}]$, called the localization of $\mathscr{C}$ with respect to W*, and a functor $\lambda : \mathscr{C} \to \mathscr{C}[W^{-1}]$ for which:*

1. *The image under λ of any morphism $w \in W$ is an isomorphism in $\mathscr{C}[W^{-1}]$*

2. *Given a category $\mathscr{D}$ and a functor $F : \mathscr{C} \to \mathscr{D}$ which sends every morphism in W to an isomorphism, there is a unique (up to natural isomorphism) functor $\tilde{F} : \mathscr{C}[W^{-1}] \to \mathscr{D}$ such that $F = \tilde{F} \circ \lambda$.*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathscr{D} \\ \mathscr{C}[W^{-1}] \arrow[ur, dotted, "\tilde{F}"] & \end{tikzcd}$$

Moreover this category is unique up to equivalence.

Proof This proof is based on the proof discussed in Milicic [8] with additional illustration. We provide an explicit construction of this category in order to prove its existence.
The objects in this category are, of course, the same as the objects in $\mathscr{C}$. The construction of morphisms in $\mathscr{C}[W^{-1}]$ is more involved:
Let us fix two objects $X, Y \in \mathrm{Ob}\mathscr{C}[W^{-1}] = \mathrm{Ob}\mathscr{C}$. We define a *path* between X and Y to consist of a finite collection of objects $\{X_j\}_{j=0}^n$ with $X_0 = X$ and $X_n = Y$ along with a concatenation of morphisms $(\varphi_j)_{j=1}^n \equiv (\varphi_1, \dots, \varphi_n)$ where each φ_j is either an arbitrary morphism $f_j : X_{j-1} \to X_j$ or a morphism $w_j : X_j \to X_{j-1}$ from our distinguished class W. (i.e. we consider the morphisms as letters and consider arbitrary finite words consisting of these letters).
An example of such a path is as follows:

$$
\begin{tikzcd}
X_0 \bullet \arrow[dr, "f_1"] & & \bullet X_2 \arrow[dl, "w_2"] & & \cdots & & X_n \bullet \arrow[dl, "w_n"] \\
& \bullet X_1 & & \bullet X_3 \arrow[ul, "w_3"] \arrow[ur, "f_4"] & & \cdots \bullet X_{n-1} &
\end{tikzcd}
$$

(here we have $\varphi_1 = f_1$, $\varphi_2 = w_2, \varphi_3 = w_3, \varphi_4 = f_4, \dots \varphi_n = f_n$)
Denote the class of all such paths by $\mathbb{P}(X,Y)$
For any $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, we have an obvious associative binary operation $\mathbb{P}(X,Y) \times \mathbb{P}(Y,Z) \to \mathbb{P}(X,Z)$ defined by concatenation of paths We now define an equivalence relation on $\mathbb{P}(X,Y)$. This equivalence relation is defined in terms of the following preliminary definition:
An *elementary transformation* of paths in $\mathbb{P}(X,Y)$ is one of the following

1. A switch of $(\dots, f_i, f_{i+1}, \dots)$ and $(\dots, f_{i+1} \circ f_i, \dots)$
2. A switch of $(\dots, w_i, w_i, \dots)$ and $(\dots, \mathbb{1}, \dots)$ for $w_i \in W$
3. A switch of $(\dots, \mathbb{1}, f_i, \dots)$ or $(\dots, f_i, \mathbb{1}, \dots)$ and $(\dots, f_i, \dots)$

We say that two paths in $\mathbb{P}(X,Y)$ are *equivalent* if there exists a finite sequence of elementary transformations between them. It is easy to see that this is an equivalence relation.
A morphism in $\mathscr{C}[W^{-1}]$ is then defined as an equivalence class of paths. It is clear that concatenation is well defined on these classses and thus we have an associative composition operation. Clearly the identity on each object X is given by $[(\mathbb{1}_X)]_\sim$.
We then define the functor $\lambda : \mathscr{C} \to \mathscr{C}[W^{-1}]$ in the obvious way:
It acts as the identity on objects and maps a morphism $f : X \to Y$ in $\mathscr{C}$ to the equivalence class $[(f)]_\sim$. Then, by construction, we have a formal inverse for any equivalence class $[(w)]_\sim$; namely $[(w)]_\sim$ but in the other direction. That is:

$$[X\bullet \xrightarrow{w} \bullet Y]_\sim \quad \text{has inverse} \quad [X\bullet \xleftarrow{w} \bullet Y]_\sim$$

Moreover, if some other category $\mathscr{C}$ and functor $F : \mathscr{C} \to \mathscr{D}$ satisfies this condition. It is then clear that the functor $T : \mathscr{C}[W^{-1}] \to \mathscr{D}$ defined by $P = (\varphi_1, \dots, \varphi_n) \mapsto T(\varphi_n) \cdots T(\varphi_1)$ where

$$T(\varphi_j) = \begin{cases} F(w_j)^{-1}, & \text{if } \varphi_j = w_j :\in W \\ F(f_j), & \text{otherwise} \end{cases}$$

Then T clearly defines an isomorphism so (2) is satisfied.
Therefore we have established the existence of such a category. It remains to prove uniqueness but this is not difficult.
Indeed suppose that we have $(\mathscr{D}, \lambda)$ and $(\mathscr{E}, \varphi)$ that both satisfy (1) and (2). Then it follows that there exist functors $G : \mathscr{D} \to \mathscr{E}$ and $H : \mathscr{E} \to \mathscr{D}$ for which the following diagram commutes

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & \mathscr{D} \arrow[dl, bend left, "H"] \\
\mathscr{E} \arrow[ur, bend left, "G"] &
\end{tikzcd}$$

This then implies that we have $\varphi = (HG)\lambda$ and $\lambda = (GH)\varphi$. Uniqueness (up to natural isomorphism) of such F and G then implies that $HG \simeq \mathbb{1}_{\mathscr{E}}$ and $GH \simeq \mathbb{1}_{\mathscr{D}}$ and thus these categories are isomorphic $\blacksquare$

Therefore localization of a category with respect to some class of morphisms affords us an arena in which we can view objects as isomorphic if there exists a distinguished morphism from W between them in our original category $\mathscr{C}$.
While it is possible to work with such a localization (with no further assumptions), a number of challenges can arise. For one, even if $\mathscr{C}$ has small limits and colimits, there is no garuntee that $\mathscr{C}[W^{-1}]$ will have this property. Additionally, even if $\mathscr{C}$ is locally small, there is no guarantee that $\mathscr{C}[W^{-1}]$ has this property for an arbitrary choice W.
Without such guarantees it can be rather unpleasant to work with this category a priori.

On the other hand, by making a few assumptions about the structure of $\mathscr{C}$ and being slightly selective in our choice of W, one can ameliorate some of the biggest challenges. In particular, an elegant way to circumvent this issue is through the use of model categories. This abstraction allows one to factor morphisms in our category $\mathscr{C}$ in such a way that simplifies matters in the localization.

We develop some of the basic theory of model categories. The proofs of any of the admitted theorems can be found in Hovey [4]

4.2 Statement of the Axioms

Remark Given a category $\mathscr{C}$, the class of all morphisms Map$\mathscr{C}$, of $\mathscr{C}$, admits a category structure. The objects in this category are morphisms between objects in $\mathscr{C}$ and the morphisms are commutative squares

Definition A *model structure* on a category $\mathscr{C}$ consists of three distinguished sub-categories of $\mathrm{Map}(\mathscr{C})$, known as *weak equivalences*, *cofibrations* and *fibrations*, satisfying the following conditions

1. Given arrows $f, g \in \mathrm{Map}(\mathscr{C})$ such that the composition fg is defined. If any two of these arrows are weak equivalences then so is the third. This is often referred to as the *"two out of three property"*

2. Given morphisms $f, g \in \mathrm{Map}(\mathscr{C})$, if g is a retract of f. That is if there exists a commutative diagram of the following form:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathbb{1}_A"] & B \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\
C \arrow[r] \arrow[rr, bend right, "\mathbb{1}_C"'] & D \arrow[r] & C
\end{tikzcd}$$

   And if f is a weak equivalence (resp. cofibration or fibration), then g is a weak equivalence (resp. cofibration, fibration)

3. Acyclic cofibrations have the left lifting property with respect to fibrations and cofibrations have the left lifting property with respect to acyclic fibrations In other words, given any commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "\alpha"] & B \arrow[d, two heads, "\beta"] \\
C \arrow[r] & D
\end{tikzcd}$$

   If α or β is a weak equivalence then there exists a morphism $C \to B$ which makes the diagram commute.

   Remark We define an *acyclic cofibration* (resp. *acyclic fibration*) to be a cofibration (resp. fibration) which is also a weak equivalence.
   We annotate weak equivelences using $\xrightarrow{\sim}$, cofibrations using $\rightarrowtail$ and fibrations using $\twoheadrightarrow$

4. Any morphism $f \in \mathrm{Map}\mathscr{C}$ can be factorized into a cofibration followed by an acyclic fibration and into an acyclic cofibration followed by an fibration. Explicitely, for any $f : A \to B$ we always have commutative diagrams of the form:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[rr, bend right, "f"'] & B' \arrow[r, two heads, "\sim"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail, "\sim"] \arrow[rr, bend right, "f"'] & A' \arrow[r, two heads] & B
\end{tikzcd}$$

   Moreover. this factorization is functiorial in $\mathrm{Map}\mathscr{C}$ meaning that given a commutative square of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "g"] & D
\end{tikzcd}$$

There exist factorizations of the form

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] \arrow[rr, bend left, "f"] & B' \arrow[r, two heads, "\sim"] \arrow[d] & B \arrow[d] \\
C \arrow[r, tail] \arrow[rr, bend right, "g"'] & D' \arrow[r, two heads, "\sim"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail, "\sim"] \arrow[d] \arrow[rr, bend left, "f"] & A' \arrow[r, two heads] \arrow[d] & B \arrow[d] \\
C \arrow[r, tail, "\sim"] \arrow[rr, bend right, "g"'] & C' \arrow[r, two heads] & D
\end{tikzcd}$$

A category $\mathscr{C}$ equipped with a model structure is said to be a *model category* if it has all small limits and colimits.

Remark In reference to our goal, the weak equivalences are the important class of morphisms since these are the ones we wish to formally invert. The cofibrations and fibrations are simply a convenient tool for computations and proofs.

4.3 Cofibrant and Fibrant Replacement

Since a model category has all small limits and colimits, in particular it has an intial and terminal object (being the respective colimit and limit of the empty diagram).

Definition With this in mind, we state the following definition:

1. An object $X \in \mathrm{Ob}\mathscr{C}$ is said to be *cofibrant* if the unique morphism from the initial object to X is a cofibration

2. An object $Y \in \mathrm{Ob}\mathscr{C}$ is said to be *fibrant* if the unique morphism from Y into the terminal object is a fibration

Definition We denote by $\mathscr{C}_c$ the subcategory of cofibrant objects of $\mathscr{C}$, $\mathscr{C}_f$ the subcategory of fibrant objects and $\mathscr{C}_{c,f}$ the subcategory of objects which are both cofibrant and fibrant.

This allows us to define the notion of a *fibrant* (resp. *cofibrant*) replacement functor. Given an object $X \in \mathrm{Ob}\mathscr{C}$, we can factor the unique morphism $X \to 1$ to the terminal object as an acyclic cofibration followed by a fibration:

$$\begin{tikzcd}
X \arrow[r, tail, "\sim"] \arrow[rr, bend right] & \mathrm{Fib}(X) \arrow[r, two heads] & 1
\end{tikzcd}$$

In this way we can always find a fibrant object which is weakly equivalent to X
Similarly one can factor the unique morphism $0 \to X$ from the cofibrant object as

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[rr, bend right] & \mathrm{Cof}(X) \arrow[r, two heads, "\sim"] & X \end{tikzcd}$$

thereby "replacing" X by a cofibrant object which is weakly equivalent to it.
Moreover fibrant and cofibrant replacement are covariant functors:
Indeed, given a morphism $f : X \to Y$ in $\mathscr{C}$, by axiom #4 of the model category definition, there exists a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "\sim"] & Y \arrow[d, tail, "\sim"] \\ Fib(X) \arrow[r, dashed, "Fib(f)"] \arrow[d, two heads] & Fib(Y) \arrow[d, two heads] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

and thus an induced morphism $\mathrm{Fib}(f) : \mathrm{Fib}(X) \to \mathrm{Fib}(Y)$.
Dually, we have an induced morphism $\mathrm{Cof}(X) \to \mathrm{Cof}(Y)$
In practice, we usually denote the fibrant replacement functor as $R : \mathscr{C} \to \mathscr{C}_f$ and the cofibrant replacement functor by $Q : \mathscr{C} \to \mathscr{C}_c$.
We now discuss the central construction which motivates the use of model structures.

4.4 The Homotopy Category

Given a model category $\mathscr{C}$, we consider the localization of $\mathscr{C}$ with respect to the subcategory of weak equivalences W of $\mathrm{Map}(\mathscr{C})$. For reasons that will become clear shortly, this unique category is denoted by $\mathrm{Ho}(\mathscr{C})$. The functor $\mathscr{C} \to \mathrm{Ho}\mathscr{C}$ will be denoted by γ when conventient and sometimes just by $\mathrm{Ho}(-)$. For any functor $F : \mathscr{C} \to \mathscr{D}$ which sends weak equivalences to isomorphisms, we denote the unique functor $\mathrm{Ho}F$ for which $F = \mathrm{Ho}F \circ \gamma$ (previously denoted in $\tilde{F}$ in section 2)
We will see that the use of model categories affords us a straighforward way of constructing homotopies of morphisms.
We begin by proving an obvious but very useful result which will minimize our efforts later

Proposition 4.4.1 *Given a model category $\mathscr{C}$, the opposite category $\mathscr{C}^{op}$ ha the structure of a model category*

Proof Consider the dual category $\mathscr{C}^{\mathrm{op}}$. We have $\mathscr{C}^{\mathrm{op}}(X,Y) = \mathscr{C}(Y,X)$ for all $X, Y \in \mathrm{Ob}\mathscr{C}$. Then let our subcategory of weak equivalences remain the same. That is, every weak equivalence $f \in \mathscr{C}^{\mathrm{op}}(Y,X)$ corresponds to a unique weak equivalence in $\mathscr{C}(X,Y)$
Given a cofibration in $\mathscr{C}$, as the name suggests, it is the dual of a fibration in $\mathscr{C}^{\mathrm{op}}$. Similarly

a fibration in $\mathscr{C}$ is the dual of a cofibration in $\mathscr{C}^{\mathrm{op}}$.
Indeed, one easily sees that using this construction one can simply reverse the directions of the arrows in the corresponding diagrams for each of the four axioms of a model category to obtain a model structure on $\mathscr{C}^{\mathrm{op}}$.
Finally this category also has small limits and colimits since $\mathscr{C}$ does. ∎

It is precisely the functorial factorizations (axiom 4 of the definition) availible to us which affords a generalized notion of a homotopy between morphisms of an arbitrary model category. We require the following preliminary definitions:

Definition Let $\mathscr{C}$ be a model category.

1. A *path object* of $X \in \mathrm{Ob}\mathscr{C}$ is a factorization of the diagonal map $\Delta : Y \to Y \times Y, y \mapsto (y,y)$

$$\begin{tikzcd} Y \arrow[r, tail, "\sim"] \arrow[rr, bend left, "\Delta"] & Y' \arrow[r, two heads, "(p_0\ p_1)"] & Y \times Y \end{tikzcd}$$

   into an acyclic cofibration followed by a fibration

2. Dually, a *cyclinder object* of $X \in \mathrm{Ob}\mathscr{C}$ is a factorization of the codiagonal map $\nabla : X \sqcup X \to X$ (which is simply the identity on each summand)

$$\begin{tikzcd} X \sqcup X \arrow[r, "i_0+i_1"] \arrow[rr, bend left, "\nabla"] & X' \arrow[r, two heads, "\sim"] & X \end{tikzcd}$$

   into a cofibration followed by an acyclic fibration.

Remark The term "cyclinder/path object" is a slight abuse of language since the definition also includes the specific factorization used.

Definition Let $f, g : X \to Y$ be morphisms in a model category $\mathscr{C}$.

1. A *right homotopy* between f and g is a morphism $H_r : X \to Y'$ for some path object Y' of Y such that $p_0 H_r \equiv f$ and $p_1 H_r \equiv g$.
   If such a right homotopy exists, we say that f is *right homotopic to* g (denote $f \sim_r g$)

2. A *left homotopy* between f and g is a morphism $H_\ell : X' \to Y$ for some cyclinder object X' of X such that $i_0 H_\ell \equiv g$ and $i_1 H_\ell \equiv g$.
   If such a left homotopy exists, we say that f is *left homotopic to* g (denote $f \sim_\ell g$)

f and g are said to be *homotopic* if they are both left and right homotopic. We denote f is homotopic to g by the shorthand $f \sim g$.

Since the dual of a model category is itself a model category, we see that a right homotopy between morphisms in $\mathscr{C}$ is precisely a left homotopy between those same objects in its dual $\mathscr{C}^{op}$. For this reason, it suffices to establish left homotopies (or indeed right homotopies) between morphisms in order to prove that two morphisms are homotopic.
It is, by a large extent, more natural to work with cyclinder objects so we will usually try to deal exclusively with left homotopies.
We can now define the notion of a homotopy equivalence for an arbitrary model category which is, unsurprisingly, a generalization of the definition for topological spaces.

Definition A morphism $f : X \to Y$ in a model category $\mathscr{C}$ is said to be a *homotopy equivalence* if there exists a morphism $g : Y \to X$ for which $fg \sim \mathbb{1}_Y$ and $gf \sim \mathbb{1}_X$

As we noted in the Classical Introduction chapter, this definition looks remarkably similar to the definition of an isomorphism of objects and indeed we will see that homotopy equivalences in a model category $\mathscr{C}$ correspond precisely to isomorphims in Ho$\mathscr{C}$.
Just as in the classical topological setting, these homotopies have certain equivalence relation properties. We admit the following lemma from Hovey [4]:

Lemma 4.4.2 *We have the following:*

1. *If $B \in Ob\mathscr{C}_c$ is a cofibrant object, then "left homotopic to" is an equivalence relation on $Hom_{\mathscr{C}}(B,X)$ for any $X \in Ob\mathscr{C}$*

2. *If $X \in Ob\mathscr{C}_f$ is a cofibrant object, then right homotopy is an equivalence relation on $Hom_{\mathscr{C}}(B,X)$ for any $B \in Ob\mathscr{C}$*

3. *Given a cofibrant object B and an acyclic fibration $h : X \to Y$. Then the functor $Hom(B,-)$ induces a bijection on the equivalence classes with respect to left homotopy*

4. *Dually, given a fibrant object X and an acyclic cofibration $h : A \to B$, the functor $Hom(-,X)$ induces a bijection on the equivalence classes with respect to right homotopy*

5. *Let $B \in Ob\mathscr{C}_c$ and suppose $h : X \to Y$ is either an acyclic fibration or a weak equivalence of fibrant objects, then h induces a bijection*

$$\mathscr{C}(B,X)/\sim_\ell \to \mathscr{C}(B,Y)/\sim_\ell$$

The following result (Hovey [4]) allows us to simplify matters:

Theorem 4.4.3 *Given a model category $\mathscr{C}$, the inclusion functors*

$$\mathscr{C}_{c,f} \hookrightarrow \mathscr{C}_c \hookrightarrow \mathscr{C} \qquad \mathscr{C}_{c,f} \hookrightarrow \mathscr{C}_f \hookrightarrow \mathscr{C}$$

Induce equivalences of categories

$$Ho\mathscr{C}_{c,f} \longrightarrow Ho\mathscr{C}_c \longrightarrow Ho\mathscr{C} \qquad Ho\mathscr{C}_{c,f} \longrightarrow Ho\mathscr{C}_f \longrightarrow Ho\mathscr{C}$$

where the "inverses" are induced by the fibrant and cofibrant replacement functors. In other words, up to homotopy we need only consider fibrant and cofibrant objects. This significantly simplifies matters when proving that a particular category satisfy the model axioms since we can restrict our attention to a much smaller class of spaces.

We discussed Whitehead's theorem in a topological setting in the introduction chapter. There is a generalized version of this theorem for an arbitrary model category:

Theorem 4.4.4 *(Whitehead)*
Let $\mathscr{C}$ be a model category and let $X,Y \in Ob\mathscr{C}_{c,f}$ be objects which are both fibrant and cofibrant.
Then $f : X \to Y$ is a weak equivalence if and only if f is a homotopy equivalence.

Proof We argue as in Goerss and Jardine [1].
Given a weak equivalence $f : X \to Y$ between $X,Y \in \mathrm{Ob}\mathscr{C}_{c,f}$, by axiom 4 of the model category definition, we can find a factorization

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, two heads, "p"] & Y \end{tikzcd}$$

with either i is an acyclic cofibration or p is a acyclic fibration. But the two out of three axiom then implies that both i and p must be acyclic. It then follows that $Z \in \mathrm{Ob}\mathscr{C}_{c,f}$ since cofibrations (resp. fibrations) are closed under compositions (since the class of (co)fibrations are subcategories).

$$0 \rightarrowtail X \rightarrowtail Z \qquad\qquad Z \twoheadrightarrow Y \twoheadrightarrow 1$$

It therefore suffices to show that every acyclic cofibration $f : X \to Y$ between $X,Y \in \mathrm{Ob}\mathscr{C}_{c,f}$ is a homotopy equivalence (since we can dualize the argument to prove the same for acyclic cofibrations).
Let $f : X \to Y$ be such an acyclic fibration and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "f"] \\ Y \arrow[r] & Y \end{tikzcd}$$

By the left-lifting axiom, there exists a morphism $g : Y \to X$ making the diagram commute. Then if we consider a cylinder object of X of the form

$$\begin{tikzcd} X \bigsqcup X \arrow[r, tail, "(i_0\ i_1)"] \arrow[rr, bend right, "\nabla"'] & X' \arrow[r, two heads, "h"] & X \end{tikzcd}$$

It follows that we can form a commutative diagram of the form:

$$\begin{tikzcd} X \coprod X \arrow[r, "(gf, \mathbb{1}_X)"] \arrow[d, tail] & X \arrow[d, two heads, "f"] \\ X' \arrow[r, "fh"] & Y \end{tikzcd}$$

and again by the lifting property, there exists a morphism $X' \to X$ making the diagram commute. Therefore f is a homotopy equivalence with homotopy inverse g. ∎

4.5 Quillen Functors and Quillen Equivalences

So far we have studied ideas pertaining to a single category equipped with a model structure. However, as discussed in the cateogory theory chapter, it is often fruitful when presented with any kind of mathematical object to consider arrows between such objects.
In this case, we consider model categories themselves as objects and consider functors between them as "morphisms" of sorts. To make this kind of statement precise one considers a so called 2-category in which the objects are model categories, the morphisms are so called Quillen adjunctions and the morphisms between morphisms are natural transformations. This is simply a formalism which avoids some rather nasty set theoretical issues.

Definition Let $\mathscr{C}$ and $\mathscr{D}$ be model categories.

1. A *left Quillen functor* $F : \mathscr{C} \to \mathscr{D}$ is a functor which preserves cofibrations and acyclic cofibrations and is a left adjoint

2. A *right Quillen functor* $G : \mathscr{D} \to \mathscr{C}$ is a functor which preserves fibrations and acyclic fibrations and is a right adjoint

3. A *Quillen adjunction* is a triple (F, G, φ) (where φ is an explicit choice of natural isomorphism $\mathscr{D}(F(-), -) \simeq \mathscr{C}(-, G(-))$) for which F is a left Quillen functor

It can easily be shown (Hovey [4]) that (F, G, φ) is a Quillen adjunction if and only if G is a right Quillen functor. So it need not be included in the definition. In other words, we can choose to work exclusively with either left of right Quillen functors depending on circumstances.
Now let us restrict our attention to functors on homotopy categories induced by such functors. We will need the following definitions to proceed:

Definition Let $\mathscr{C}, \mathscr{D}$ be model categories.

1. Given a left Quillen functor $F : \mathscr{C} \to \mathscr{D}$, we define its *total left derived functor* LF : $\mathrm{Ho}\mathscr{C} \to \mathrm{Ho}\mathscr{D}$ by

$$\begin{tikzcd} \mathrm{Ho}\mathscr{C} \arrow[r, "\mathrm{Ho}Q"] \arrow[rr, bend right, "LF"'] & \mathrm{Ho}\mathscr{C}_c \arrow[r, "\mathrm{Ho}F"] & \mathrm{Ho}\mathscr{D} \end{tikzcd}$$

2. Given a right Quillen functor $G : \mathscr{D} \to \mathscr{C}$, we define its *total right derived functor* $RG : \mathscr{D} \to \mathscr{C}$ by

$$\begin{tikzcd} \mathrm{Ho}\mathscr{D} \arrow[r, "\mathrm{Ho}R"] \arrow[rr, bend right, "RG"'] & \mathrm{Ho}\mathscr{D}_f \arrow[r, "\mathrm{Ho}G"] & \mathrm{Ho}\mathscr{D} \end{tikzcd}$$

As discussed in Hovey [4], most of the nice properties one might expect or hope for are satisfied:

Given natural transformations between Quillen functors $F \xrightarrow{\eta} F' \xrightarrow{\eta'}\to F''$ we have
$L(\eta'\eta) = L(\eta')L(\eta)$ and $R(\eta'\eta) = R(\eta')R(\eta)$
Also we have a natural isomorphism $L(1_{\mathscr{C}}) \simeq 1_{\mathrm{Ho}\mathscr{C}}$ We can then define a derived adjuntion as follows:

Definition Given a Quillen adjunction (F,G,φ) between model categories $\mathscr{C}$ and $\mathscr{D}$, We define the adjunction $L(F,G,\varphi) := (LF,RG,\Phi)$ between $\mathrm{Ho}\mathscr{C}$ and $\mathrm{Ho}\mathscr{D}$ is called the *total left derived adjunction*

We are interested in the case where the total left derived adjunction of a Quillen adjunction is an equivalence.

Definition A Quillen adjunction (F,G,φ) is said to be a *Quillen equivalence* if the total left derived functor $L(F,G,\varphi)$ is an equivalence of categories.

We admit from Hovey [4] the following:

Lemma 4.5.1 *A Quillen adjunction (F,G,φ) is a Quillen equivalence if and only if for each cofibrant X in $\mathscr{C}$ and fibrant Y in $\mathscr{D}$,*

$$f : FX \to Y \text{ is a weak equivalence in } \mathscr{D} \iff \varphi(f) : X \to GY \text{ is a weak equivalence in } \mathscr{C}$$

Remark The second condition here is actually how Hovey [4] defines a Quillen equivalence.

This allows us to prove the following:

Lemma 4.5.2 *Let $\mathscr{C}$, $\mathscr{D}$ be model categories and $F : \mathscr{C} \to \mathscr{D}$ a left Quillen functor.*

1. *If (F,G,φ) and (F,G',φ') are Quillen both Quillen adjunctions then (F,G,φ) is a Quillen equivalence if and only if (F,G',φ') is a Quillen equivalence*

2. *Dually, given a right Quillen functor $G : \mathscr{D} \to \mathscr{C}$, if both (F, G, φ) and (F', G, φ') are Quillen adjunctions then (F, G, φ) is a Quillen equivalence if and only if (F', G, φ') is a Quillen equivalence.*

Proof We observe that since (F, G, φ) is a Quillen equivalence, we have $L(F, G, \varphi)$ is such that $LF : \mathrm{Ho}\mathscr{C} \to \mathrm{Ho}\mathscr{D}$ is an equivalence of categories. It then immediately follows that both RG and RG' equivalences of categories (since adjoints are essentially unique). ∎

This allows us to unambiguously refer to the left (resp. right) Quillen functor.

4.6 The Small Object Argument

The following section provides us with an essential tool which will be used to prove the existence of factorizations in both the category of topological spaces and the category of simplicial sets. As per most this chapter, we follow Hovey [4] here.

Remark Unless otherwise stated, we assume that all categories are locally small and are cocomplete.

Let us start with the following essential definitions:

Definition Let $I \subset \mathrm{Map}\mathscr{C}$ be a distinguished class of morphisms.
A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be

1. *I-injective* if f has the right lifting property with respect to every $i \in I$. That is, for every commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"'] & Y \end{tikzcd}$$

with $i \in I$, there exists a morphism $B \to X$ making the diagram commute.

2. *I-projective* if f has the left lifting property with respect to every $i \in I$. That is given any commutative square of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "i"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

with $i \in I$, there exists a morphism $Y \to A$ making the diagram commute.

3. An *I-fibration* if f has the right lifting property with respect to every I-projective morphism

4. An *I-cofibration* if f has the left lifting property with respect to all I-injective morphisms.

The motivation for the names I-fibations and I-cofibrations will become more clear later.

Remark Hovey [4] sometimes denotes by the class of I-injective morphisms (resp. I-projective, I-fibrations, I-cofibrations) by I-inj (resp. I-proj, I-fib, I-cof). This notation is convenient so we elect to adopt it too.

The next definition is an essential construction for defining cofibrations in **Top**. The construction looks markedly similar to that of a CW complex in **Top** but allows for transfinite "attachings" rather than just countable ones.

Definition A morphism $f : A \to B$ in $\mathscr{C}$ is said to be *relative I-cell complex* if there is an ordinal λ and a λ-sequence $X : \lambda \to \mathscr{C}$ so that for every ordinal α: $\alpha + 1 < \lambda$, we have a pushout square of the form (where $i_\alpha \in I$)

$$\begin{tikzcd} S_\alpha \arrow[r] \arrow[d, "i_\alpha"] & X_\alpha \arrow[d] \\ D_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$

and f is obtained from the composition of X.
More succintly, we say a relative I-cell complex is a transfinite composition of pushouts of morphisms i_α in I.
If an object Y is initial and $0 \to Y$ is a relative cell complex, then we say that the object Y is a *cell complex*.

We admit the following lemma from Hovey [4]

Lemma 4.6.1 *Given some $I \subset Map\mathscr{C}$, every I-cell is closed under transfinite compositions.*

We can, in some sense, "speed up the process" by considering pushouts of coproducts. The following lemma (Hovey [4] justifies doing so:

Lemma 4.6.2 *Given $I \subset Map\mathscr{C}$, and a family $(i_\alpha : S_\alpha \to D_\alpha)_{\alpha \in A}$ (for some set A) of morphisms in Ii. Then a pushout of the coproduct of (i_α) is an I-cell*

Proof Suppose that a morphism $f : X \to Y$ forms the following pushout square

$$\begin{tikzcd} \coprod_\alpha S_\alpha \arrow[r] \arrow[d, "\sqcup_\alpha i_\alpha"'] & X \arrow[d, "f"] \\ \coprod_\alpha D_\alpha \arrow[r] & Y \end{tikzcd}$$

It must be shown that f is a relative I-cell complex.
Firstly, every set is isomorphic to an ordinal (Jech et al. [5]) so without loss of generality we

can assume that A is some ordinal λ.
Then, by transfinite induction, we construct a λ-sequence as follows:
Let $X_0 := X$. For each limit ordinal α, we set $X_\alpha := \mathrm{colim}_{\beta<\alpha} X_\alpha$ and given some X_β, we define $X_{\beta+1}$ as the pushout $X_\beta \bigsqcup_{S_\beta} D_\beta$.
Then the composition $X \to X_\lambda$ is by construction, isomorphic to f ∎

We discussed the notion of a functorial factorization when defining a model category. Let us make this explicit:

Definition Given a category $\mathscr{C}$. A *functorial factorization* is a pair of endofunctors (φ, ψ) on Map$\mathscr{C}$ for which given any morphism $f : X \to Y$ in $\mathscr{C}$, there exists a factorization of the form

$$\begin{tikzcd} X \arrow[r, "\varphi(f)"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "\psi(f)"] & Y \end{tikzcd}$$

and given a commutative square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

there exist objects C, Z and a morphism $C \to Z$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi(f)"'] \arrow[rr, bend left, "f"] \arrow[d] & C \arrow[r, "\psi(f)"'] & B \arrow[d] \\ X \arrow[r, "\varphi(g)"] \arrow[rr, bend right, "g"'] & Z \arrow[r, "\psi(g)"] & Y \end{tikzcd}$$

We now have the terminology necessary to understand the meaning of the small object argument

Theorem 4.6.3 *(The Small Object Argument)*
Given a cocomplete category $\mathscr{C}$ and some $I = \{S_\alpha \to D_\alpha\}_{\alpha \in A} \subset Map\mathscr{C}$ is such that S_α is small relative to the class of I-cells. Then there exists a functorial factorization (φ, ψ) such that given any morphism $f \in Map\mathscr{C}$, $\varphi(f)$ is an I-cell and $\psi(f)$ is I-injective.

Remark The functorial factorizations we will consider are of course in the context of model categories.

4.7 Cofibrantly Generated Model Categories

We now define the important notion of a cofibrantly generated model category. Essentially, we begin with a smaller class of cofibrations and trivial cofibrations which garuntee that any morphism can be factored and then we allow for addition to these classes so that all of the axioms are satisfied.

Definition A model category $\mathscr{C}$ is said to be *cofibrantly generated* if there exist $I = \{i_\alpha : S_\alpha \to D_\alpha\}_{\alpha \in A}, J = \{j_\beta : K_\beta \to L_\beta\}_{\beta \in B} \subset \mathrm{Map}\mathscr{C}$ (called the *generating cofibrations* and *generating acyclic cofibrations* respectively) for which the following conditions are satisfied:

1. S_α is small relative to the class of I-cells for each $\alpha \in A$
2. K_β is small relative to the class of J-cells for each $\beta \in B$
3. The class of fibrations are J-injective
4. The class of acyclic fibrations are I-injective

We admit the following results from Hovey [4]:

Lemma 4.7.1 *Let $\mathscr{C}$ be a cofibrantly generated model category with I,J generating . Then we have the following:*

1. *The class I-cof is precisely the class of cofibrations.*
2. *The class J-cof is precisely the class of acyclic cofibrations.*
3. *If $f : X \to Y$ is a cofibration then f is a retract of some relative I-cell complex.*
4. *If $f : X \to Y$ is an acyclic cofibration then f is a retract of some J-cell complex.*

Justifying the names I-fibration and J-cofibration

We now provide the result (Hovey [4]) which equips us to engineer a desired model structure on a category having appropriately chosen a class of weak equivalences and I, J. It is the cornerstone for proving that both simplicial sets and topological spaces admit model category structures.

Theorem 4.7.2 *Let $\mathscr{C}$ be a category which is complete and cocomplete. Fix some subcategory $W \subset Map\mathscr{C}$ and subsets $I = \{i_\alpha : S_\alpha \to D_\alpha \mid \alpha \in A\}, J = \{j_\beta : K_\beta \to L_\beta \mid \beta \in B\} \subset Map\mathscr{C}$*
Then the following are both necessary and sufficient conditions for the existence of a model structure on $\mathscr{C}$ where W is the subcategory of weak equivalences, I is the set of generating cofibrations, J is the set of generating acyclic cofibrations

1. *W is closed under retracts and satisfy the two out of three property.*

2. *For each $\alpha \in A$, S_α is small relative to the class of I-cells*

3. *For each $\beta \in B$, K_β is small relative to the class of J-cells.*

4. *The class of I-cells is contained in the interesection of W and J-inj and the class of J-cells is contained in the intersection of W and I-cof.*

5. *We have either J-cof $\supset W \cap I$-cof or I-inj$\supset W \cap J$-inj.*

To establish a model structure on a suitably chosen category, it therefore suffices to elect a subcategory of weak equivalences and suitable choices for I and J and then check these conditions. Finally we have a useful condition for checking whether a functor with whose domain is a cofibrantly generated model category is a Quillen functor.

Lemma 4.7.3 *(Hovey [4])*
Let $\mathscr{C}$ be a cofibrantly generated model category (with I and J as before) and $\mathscr{D}$ be a model category.
Then an adjunction (F,G,φ) is Quillen if and only if for each $f \in I$, Ff is a cofibration and for each $g \in J$, Fg is an acyclic cofibration.

Remark This lemma is proven using the fact left adjoints commute with colimits (Mac Lane [7])

An Example In the upcoming chapters, we will discuss the model structures of the category of simplicial sets and the category of topological spaces in detail. Both are examples of cofibrantly generated model categories, however there are a number of other important examples.
One important example in algebraic topology is the category of chain complexes $\mathrm{Ch}_{\geq 0}(R)$ over an arbitrary ring R. Its objects are chains of R-modules

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots$$

where each X_n is an R-module and $(\partial_n)_{n\geq 0}$ are a collection of module homomorphisms for which $\partial_n\partial_{n+1} \equiv 0$ for every $n \geq 0$. We have the usual notion of the homology of this chain complex

$$H_n(X) := \ker(\partial_n)/\mathrm{Im}(\partial_{n+1}) \quad \forall n \geq 0$$

A morphism $f : X \to Y$ between two chain complexes X and Y is a collection of R-module morphisms $(f_n : X_n \to Y_n)_{n\geq 0}$ for which the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_n"] & \cdots \\
\arrow[r] & Y_{n+1} \arrow[r, "\partial_{n+1}"'] & Y_n \arrow[r, "\partial_n"'] & Y_{n-1} \arrow[r] &
\end{tikzcd}$$

We say that $f : X \to Y$ is a weak equivalence if f induces an isomorphism on every homology group. That is $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism for each $n \geq 0$.
For each $n \geq 0$, we define S^n as the chain complex with $S^n_k = \delta_{n,k} R$ and ∂_k is the zero map for each k.

$$\cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \cdots$$

Also we define the chain complex D^n by $D^n_k = R$ if $k \in \{n-1, n\}$ and 0 otherwise with $\partial_k = \mathbb{1}_R$ if $k = n$ and 0 otherwise.

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathbb{1}_R} R \longrightarrow 0 \longrightarrow \cdots$$

By defining $I = \{S^n \to D^n\}_{n \geq 0}$ and $J = \{0 \to D^n\}_{n \geq 0}$ and checking the equivalent conditions of 4.7.2, it can be shown (Hovey [4]) that $\mathrm{Ch}_{\geq 0}(R)$ is a cofibrantly generated category.

The Classical Model Struture on Topological Spaces

The category of topological spaces admits a model structure. In particular **Top** can be equipped with a cofibrantly generated model structure. We provide a brief overview of the proof admitting results from both Hirschhorn [3] and Hovey [4].

5.1 Preliminary Definitions

We will require several preliminary definitions. We begin with discussing how we define fibrations in this category.

(Serre) Fibrations

Definition We denote by

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subset \mathbb{R}^{n+1} \quad , \quad D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

as the n-dimensional unit sphere and the n-dimensional unit disk respectively.
We have the obvious inclusion mapping $S^{n-1} \hookrightarrow D^n$ for each $n \geq 1$.

Definition A continuous mapping $f : X \to Y$ of topological spaces is said to be a *Serre fibration* if given any commutative diagram of the form

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

there is a lift $D^n \to X$ making the diagram commute.

Remark There are several alternative definitions we could make use of which make reference to simplicial sets as discussed in 6:
We note that for any $n \geq 1$, $S^{n-1} \simeq |\Lambda^n_k|$ for all $k \in [n]$ and $|\Delta^n| \simeq D^n$. For this reason, it is easy to see that we could equivalently require right lifting with respect to all inclusions of the form $|\Lambda^n_k| \hookrightarrow |\Delta^n|$. Finally, by adjointness of the geometric realisation and singular functor, we have $f : X \to Y$ is a Serre fibration if and only if $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration.

Cofibrations

Next, we discuss how we define cofibrations. We define some notions which are really topologically specific examples of ideas we defined in the model category chapter.

Definition Given spaces $X \subset Y$, we say that Y is *obtained from X by attaching a cell* if there exists a pushout square of the form

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ D^n \arrow[r] & Y \end{tikzcd}$$

We call such a commutative diagram an *attaching cell*.

Remark In the language of the geometric realization, we could equivalently replace S^{n-1} with $|\Lambda^n_k|$ and D^n with $|\Delta^n|$ in the above diagram

Definition We say that an inclusion $f : X \hookrightarrow Y$ is a *relative cell complex* if there exist a process of iterated attaching cells (possibly an uncountably infinite one)
It is possible to "speed up" the process in which cells are attached by considering multiple attachings via a pushout of the following diagram

$$\begin{tikzcd} \bigsqcup_{t\in T} S^{n_t-1} \arrow[r] \arrow[d] & X \\ \bigsqcup_{t\in T} D^{n_t} & \end{tikzcd}$$

(over some set T)
Through transfinite iterations of such pushouts, one can obtain a relative cell complex
If $\emptyset \to Z$ is a relative cell complex then we call the space Z a *cell complex*.

Remark We note that our definition of CW complexes in the Introduction chapter is extremely similar to a relative cell complex. The only difference is that for a CW complex, we allow only a countable number of pushouts whereas here there is no such restriction. Thus all CW complexes are cell complexes but the reverse inclusion need not hold.

Remark In terms of the model category chapter, if we define $I := \{S^{n-1} \hookrightarrow D^n \mid n \geq 0\}$ then we are really considering relative I-cell complexes.

Definition The relative cell complexes are really what we are interested in, but since the model category axioms require closure under retracts, we elect to use the following definition:
A continuous map $f : X \to Y$ is a *cofibration* if it is a retract of a relative cell complex.

5.2 The Model Structure

Theorem 5.2.1 *The category **Top** of topological spaces admits a model structure. Its weak equivalences are homotopy equivalences, its fibrations are Serre fibrations and its cofibrations are retracts of relative cell complexes. Moreover, this model model category is cofibrantly generated with* $I := \{S^{n-1} \hookrightarrow D^n \mid n \geq 1\}$ *and* $J := \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] \mid n \geq 0\}$ *as the respective generating cofibrations and generating acyclic cofibrations.*

We note that the category of topological spaces is both complete and cocomplete. Therefore if we can show that this defines a model structure on **Top** then it immediately follows that **Top** is actually a model category. We prove each of the axioms but in order to do so we will need several lemmas.

We shall first show that weak homotopy equivalences satisfy the two out of three property and are closed under retracts since this is a necessary condition.

Two out of three axiom (Hovey [4])

There is only one non trivial case. Namely let $f : X \to Y$ be a weak equivalence and suppose that $g : Y \to Z$ is a continuous map for which $gf : X \to Z$ is a weak equivalence.

$$\begin{tikzcd} X \arrow[r, "gf"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

It is clear that $\pi_0(g)$ must be a bijection.
Let $y \in Y$, since f is a weak homotopy equivalence, the connected components of X and Y are in bijection and so there exists $x \in X$ for which there is a path joining $f(x)$ and y. That is a continuous mapping $\gamma : [0,1] \to Y$ with $\gamma(0) = f(x)$ and $\gamma(1) = y$.
Therefore $\pi_n(Y,y) \simeq \pi_n(Y, f(x))$ and $\pi_n(Z, g(y)) \simeq \pi_n(Z, g(f(x)))$ (where the explicit isomorphisms given by the respective conjugations $\Phi_\gamma : [\beta] \mapsto [\gamma^{-1}\beta\gamma]$ and $\Phi_\gamma : [\psi] \mapsto [(g\gamma)^{-1}\psi(g\gamma)]$).
We then obtain the following commutative diagram for every $n \geq 1$

$$\begin{tikzcd} \pi_n(Y,y) \arrow[r, "\pi_n(g,y)"] \arrow[d, "\Phi_\gamma"'] & \pi_n(Z, g(y)) \arrow[d, "\Phi_{g\gamma}"] \\ \pi(Y, f(x)) \arrow[r, "\pi_n(g, f(x))"'] & \pi_n(Z, g(f(x))) \end{tikzcd}$$

The lower horizontal arrow is an isomorphism since gf is a weak equivalence by hypothesis. Since the vertical arrows are also isomorphisms, it then follows that $\pi_n(g, f(x))$ is an isomorphism. Thus g is a weak equivalence ∎

Retract

It is obvious that weak equivalences are closed under retracts.
Indeed suppose that $f : B \to Y$ is a weak homotopy equivalence and we have the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\mathbb{1}_A"] \arrow[d, "g"] & B \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ X \arrow[r] \arrow[rr, bend right, "\mathbb{1}_X"'] & Y \arrow[r] & X \end{tikzcd}$$

Then we have the following commutative diagram (where we omit the choice of basepoints in the diagram to avoid clutter) induced by π_n for each $n \geq 1$:

$$\begin{tikzcd} \pi_n A \arrow[r] \arrow[rr, bend left, "\mathbb{1}_{\pi_n A}"] \arrow[d, "\pi_n g"] & \pi_n B \arrow[r] \arrow[d, "\pi_n f"] & \pi_n A \arrow[d, "\pi_n g"] \\ \pi_n X \arrow[r] \arrow[rr, bend right, "\mathbb{1}_{\pi_n X}"'] & \pi_n Y \arrow[r] & \pi_n X \end{tikzcd}$$

It is then clear that the horizontal arrows are group isomorphisms (since $\pi_n(f)$ is a group isomorphism) and therefore that $\pi_n(g)$ is a group isomorphism $\blacksquare$

The lifting and factorization axioms are proved in extensive detail in several sources such as Hovey [4] and Hirschhorn [3]. The argument followed in doing so is almost identical to the one we use in proving the same axioms for simplicial sets (6.7) (namely involving the small object argument and generating (acyclic) cofibrations) so we elect to omit the proof here.

We now move our attention to the discussion of simplicial sets and the model structure with which they can be equipped with. We will approach and view simplicial sets through a category theoretic lense.

The Kan model structure on Simplicial Sets

6.1 Simplicial Sets

Simplicial sets provide us with a combinatorial means of constructing and understanding topological spaces. It turns out that we may "realize" a simplicial set as a topological space using some categorical machinery.
The eventual goal of this chapter is to demonstrate that the category of simplicial sets is a model category though this will take some work. The material here follows Goerss and Jardine [1]

Definition We define the *simplex category* Δ as follows:
It's objects are finite totally ordered sets of the form $[n] := \{0 \leq 1 \leq \ldots \leq n\}$ and its morphisms are order preserving functions. In other words, $f : [n] \to [m]$ is a morphism in Δ if and only if for all $i, j \in [n]$ such that $i \leq j$, we have $f(i) \leq f(j)$

From this category we can build our category of interest.

Definition We define the category of simplicial sets **sSet** as follows:
It's objects are contravariant functors from the simplex category Δ to the category of sets and its morphisms are natural transformations. (i.e $\mathbf{sSet} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$)
It is common practice to denote a simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ as a collection of sets $(X_n)_{n \geq 0}$ where we understand implicitely that for each $\phi : [m] \to [n]$ increasing there is a mapping of sets $X\phi : X_n \to X_m$ and use the notational shorthand $X_n := X([n])$ for each $n \geq 0$

Definition A natural transformation $(f_n : X_n \to Y_n)_{n \geq 0}$ between simplicial sets $X = (X_n)_{n \geq 0}$ and $Y = (Y_n)_{n \geq 0}$ is called a *simplicial morphism*.
In particular, given any increasing map $\phi : [m] \to [n]$, the following square commutes

$$\begin{array}{ccc} X_n & \xrightarrow{X\phi} & X_m \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_m} \\ Y_n & \xrightarrow{Y\phi} & Y_m \end{array}$$

One particularly important class of simplicial sets are the so called *standard $n-$simplicies*

Definition For each $n \geq 0$, we denote by Δ^n the functor $\mathrm{Hom}_\Delta(-,[n]) : \Delta^{\mathrm{op}} \to \mathbf{Set}$ called the *standard $n-$simplex* in **sSet**
Explicitely, each $[m] \in \mathrm{Ob}(\Delta)$ is mapped to the Hom set $\mathrm{Hom}_\Delta([m],[n])$ and given a strictly increasing mapping $\varphi : [j] \to [k]$, Δ^n induces a morphism $\Delta^n\varphi :\ \mathrm{Hom}([k],[n]) \to \mathrm{Hom}([j],[n])$ defined by precomposition with φ

Remark The 0^{th} simplicial set Δ^0 is a terminal object in **sSet**. Indeed, for each $n \geq 0$, we have $\mathrm{Hom}_\Delta([n],[0]) = \{j \mapsto 0\ \forall j \in [n]\}$: a singleton set. Thus for each $\phi : [m] \to [n]$ increasing, $\Delta^0(\phi)$ is simply the unique map between two singleton sets. It then follows that any simplicial morphism $f : X \to \Delta^0$ must, for every $\phi : [m] \to [n]$ increasing, have the following corresponding commutative square

$$\begin{array}{ccc} X_n & \xrightarrow{X\varphi} & X_m \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_m} \\ \Delta^0_n & \xrightarrow{\Delta^0\varphi} & \Delta^0_m \end{array}$$

Obviously there is a unique such simplicial morphism f regardless of the choice of X and thus Δ^0 is the terminal object in **sSet**. We therefore will sometimes denote Δ^0 by 1.

There are two important sub-complexes of Δ^n which we define below:

Definition (Boundaries and Horns)

1. The boundary $\partial\Delta^n$ of Δ^n is the smallest simplicial set containing each face $d_i([n] \xrightarrow{\mathbb{1}} [n])$. Equivalently, it is the simplicial set obtained by removing the unique n-cell
2. The *k-horn* Λ^n_k of Δ^n is defined as the simplicial set generated by all faces $d_i([n] \xrightarrow{\mathbb{1}_n} [n])$ such that $i \neq k$. Alternatively, Λ^n_k can be obtained by removing the k^{th} face of the boundary $\partial\Delta^n$ of the standard n-simplex

6.2 Generating Simplicial Sets

We can understand the category **sSet** based solely on the so called *coface* and *codengeneracy* maps. They are so named since for any simplicial set X, the respective functorial images are the face and codegeneracy maps.

Definition (Coface and Codegeneracy)

1. For each $n \geq 0$ and $i \in [n]$, we define the i^{th} *coface map* $d^i : [n-1] \to [n]$ by

$$d^i(j) := \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases}$$

In other words, the i^{th} coface map of $[n]$ skips the i^{th} element of $[n-1]$ thereby imbedding the string $(0 \leq 1 \leq \ldots \leq j-1 \leq j+1 \leq \ldots \leq n)$ in $[n]$

2. Similarly for each $n \geq 0$, $i \in [n]$, we define the i^{th} *codegeneracy map* $s^i : [n+1] \to n$ defined by
$$s^i(j) := \begin{cases} j, & \text{if } j \leq i \\ j, & \text{if } i = j+1 \\ j-1, & \text{if } j+1 > i \end{cases}$$
Thereby embedding a string $(0 \leq 1 \leq \ldots \leq j \leq j \leq \ldots \leq n)$ of length $n+1$ in $[n]$

From Hatcher [2] we quote the following well established result:

Proposition 6.2.1 *(The Cosimplicial Identities)*
We have the following identities

$$\begin{cases} 1) & d^j d^i = d^i d^{j-1} & \forall i < j \\ 2) & s^j s^i = s^i s^{j+1} & \forall i \leq j \\ 3) & s^j d^i = d^i s^{j-1} & i < j \\ 4) & d^i s^j = \mathbb{1} & \textit{if } i \in \{j, j+1\} \\ 5) & s^j d^i = d^{i-1} s_j & \forall i > j+1 \end{cases}$$

Proof We prove just the first two identities and the rest are similar
1) We note that both $[n-1] \xrightarrow{d^i} [n] \xrightarrow{d^j} [n+1]$ and $[n-1] \xrightarrow{d^{j-1}} [n] \xrightarrow{d^i} [n+1]$ skip i and j. However there is obviously a unique increasing map $[n-1] \to [n]$ with this property
2) Clearly both $s^j s^i([n+1])$ and $s^i s^{j+1}([n+1])$ are chains in $[n-1]$ of length $n+1$ containing two copies of both i and $j+1$. There is a unique non-decreasing map $[n+1] \to [n-1]$ satisfying this property and therefore the mappings must be equal ∎

Given a simplicial set $X \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, for each coface map $d^i : [n-1] \to [n]$ and each codegeneracy map $s^j : [n+1] \to [n]$ we denote the corresponding functorial images $Xd^i : X_n \to X_{n-1}$ and $Xs^j : X_n \to X_{n+1}$ by d_i and s_j respectively (provided the choice of simplicial set is unambiguous)
These face and degeneracy maps satisfy the so called *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i, & \forall i < j \\ s_i s_j = s_{j+1} s_i & \forall i \leq j \\ d_i s_j = s_{j-1} d_i & \forall i < j \\ s_j d_i = \mathbb{1} & \text{if } i \in \{j, j+1\} \\ d_i s_j = s_j d_{i-1} & \forall i > j+1 i > j+1 \end{cases}$$

The respective proofs follows directly from the cosimplicial identities and the contravariance of simplicial sets ($X(fg) = X(g)X(f)$ for all $[k] \xrightarrow{g} [m] \xrightarrow{f} [n]$ increasing)

Remark It can be shown that every increasing map $[m] \to [n]$ in Δ can be expressed as a composition of coface and codegeneracy maps. Therefore, for a simplicial set $X = (X_n)_{n \geq 0}$. It suffices to define the behaviour of the face and degeneracy maps.

6.3 Geometric Realization

Our objective is to define a functor $|-|$: **sSet** $\to$ **Top** which associates to each simplicial set X, a topological space $|X|$.
It turns out that our distinguished class $(\Delta^n)_{n\geq 0}$ of standard simplicial complexes uniquely determine this functor.

Definition We define the topological $n-$simplex

$$|\Delta^n| := \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} \mid \sum_j t_j = 1 \text{ and } t_j \geq 0\ \forall j\} \subset \mathbb{R}^{n+1}$$

as the realization of the standard $n-$simplex in **sSet**

Remark We note that there is an obvious functor from the simplex category to **Top** defined by $[n] \mapsto |\Delta^n|$ on objects and for each increasing map $f:[m] \to [n]$, a continuous map $f_*:|\Delta^m| \to |\Delta^n|$ defined by

$$f_*(t_0,\ldots,t_m) := (s_0,\ldots,s_n) \quad , \quad s_j := \sum_{i\, \in f^{-1}(\{j\})} t_i \quad \forall j \in [n]$$

In order to define the geometric realization of a particular simplicial set, we need to make one more preliminary definition

Definition Given a simplicial set X, we define the category $\Delta \downarrow X$ (frequently referred to as the *simplex category* of X). It's objects are simplicial maps $\Delta^n \to X$ (over all $n \geq 0$) and a morphism of objects $\alpha:\Delta^m \to X$ and $\beta:\Delta^n \to X$ is a commutative diagram of the form

$$\begin{tikzcd} \Delta^m \arrow[rr] \arrow[dr, "\alpha"'] & & \Delta^n \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

The category $\Delta \downarrow X$ is always small. Indeed the simplicial morphisms $\Delta^n \to X$ for each $n \geq 0$ form a set and a finite number of increasing maps $[m] \to [n]$ for each $m,n \in \mathbb{N}$. Thus the objects are a set of countable cardinal and the Hom set of any two simplicial morphisms $\Delta^m \to X$ and $\Delta^n \to X$ is finite.
Since $\Delta \downarrow X$ is a small category and any functor $\Delta \downarrow X \to$**Set** is a colimit of a representable functor (Mac Lane [7]), it follows that the following colimit exists and that we have the following isomorphism:

$$X \simeq \operatorname{colim}(\Delta^n)_{n\geq 0}$$

over $\{\sigma:\Delta^n \to X \ \mid \sigma \in \operatorname{Map}(\Delta \downarrow X)\}$

Definition We may then define the *geometric realization* of a simplicial set X as the following colimit in **Top**:

$$|X| := \operatorname{colim}(|\Delta^n|)_{n\geq 0}$$

over $\{\sigma:\Delta^n \to X \mid \sigma \in \operatorname{Map}(\Delta \downarrow X)\}$

So it suffices to define geometric realization explicitely on just the standard n-simplices and then this clever colimit allows us to easily define realization on all simplicial sets.

Remark As discussed in the opening section of this chapter, it is easy to see that the geometric realization of any simplicial set is in fact a CW complex.

There is also a functor in the opposite direction which is of great importance

Definition We define the *singular functor* Sing : **Top** $\to$ **sSet** (often abbreviated to just S) The image of any topological space X is defined as the contravariant functor

$$\mathrm{Sing}(X) \equiv SX : \Delta^{\mathrm{op}} \to \mathbf{Set} \quad : \quad [n] \mapsto \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, X\right)$$

N.B: We note that **Top** is a locally small category:
The cardinality of $\mathrm{Hom}_{\mathbf{Top}}\left(X,Y\right)$ is at most $(\#Y)^{\#X}$ for any topologcial spaces X,Y

This functor is covariant since given any continuous map $f : X \to Y$ we obtain a simplicial map $Sf : SX \to SY$ defined by postcomposition with f:

$$Sf : \mathrm{Hom}(|\Delta^n|, X) \to \mathrm{Hom}(|\Delta^n|, Y) \quad : \quad (|\Delta^n| \xrightarrow{\rho} X) \mapsto (|\Delta^n| \xrightarrow{\rho} X \xrightarrow{f} Y)$$

Remark (Hatcher [2])
It is precisely this functor that gives us the classical notion of singular homology.
For each $n \geq 0$, one considers the free abelian group $C_n(X)$ generated by $S(X)_n = \mathrm{Hom}(|\Delta^n|, X)$.
We obtain a chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

where each ∂_n is defined in terms of the face maps $\partial_n := \sum_{j=0}^{n} (-1)^j d_j$.
Using the simplicial identity $d_i\ d_j = d_{j-1} d_i :\ \forall i < j$ it can be shown that $\partial^2 \equiv 0$, so we always have $\mathrm{Im}\partial_{n+1} \subseteq \ker\partial_n$.
The homology groups $H_n(X)$ of the space X are then defined in the usual way:

$$H_n(X) := \ker\partial_n / \mathrm{Im}\partial_{n+1}$$

One can alternatively define the notion of simplicial homology. It is an important and highly non-trivial result that simplicial and singular homology are equivalent.

Proposition 6.3.1 *(Goerss and Jardine [1])*
The functor $|-|$ is left adjoint to Sing. That is, we have

$$Hom_{\boldsymbol{Top}}\left(|X|, Y\right) \simeq Hom_{\boldsymbol{sSet}}\left(X, SY\right)$$

naturally in $X \in Ob(\boldsymbol{sSet})$ and $Y \in Ob(\boldsymbol{Top})$

Proof The proof follows from noting that for every $n \geq 0$, we have
$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Hom}_{\mathbf{sSet}}(\Delta^-, Y)) \simeq (SY)_n$ by the Yoneda lemma.
Then $(SY)_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$ by definition. Therefore we have $\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, SY) \simeq \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y)$ for any $n \geq 0$. The result then follows from taking the colimit over $(\Delta^n \to X)$ in $\Delta \downarrow X$. ∎

Therefore we have an adjunction between the category of simplicial sets and the category of topological spaces. By equipping the respective categories with model structures, we will see that this adjunction is actually a Quillen adjunction.

6.4 Kan Fibrations and Kan Complexes

We define the subcategory of fibrations in the classical model structure on **sSet**.

Definition Let X, Y be simplicial sets. A simplicial morphism $p : X \to Y$ is called a *Kan fibration* (or sometimes just a fibration for brevity) if, given any commutative diagram in **sSet** of the form:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

there exists a morphism $\Delta^n \to X$ which makes this diagram commute.
Said differently, a Kan fibration is a morphism which has the right lifting property with respect to all inclusions of the form $\Lambda^n_k \hookrightarrow \Delta^n$

Definition A *Kan complex* is a fibrant object in **sSet**.
Recall that a simplicial set X is fibrant if and only if the unique map $X \to \Delta^0$ is a fibration

Remark The term "Kan complex" is in honor to Dutch mathematician Daniel Kan who made large contributions to the categorification of simplicial homotopy theory.

We have an alternative way of characterizing Kan complexes which is sometimes helpful

Lemma 6.4.1 *Let X be a simplicial set, The following conditions are equivalent*

1. *X is a Kan complex*
2. *For every simplicial morphism $f : \Lambda^n_k \to Y$, there exists a simplicial morphism $\tilde{f} : \Delta^n \to X$ for which the following triangle commutes*

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

3. *Given any n-tuple $(x_0, \dots, \hat{x}_j, \dots, x_n)$ of simplices (where this notation means that we leave out the j^{th} simplex) satisfying $d_i x_k = d_{k-1} x_i \ \forall i < k, i, k \neq j$, there exists an $n-$simplex x for which $d_i x = x_i$*

Proof 1 $\iff$ 2: This is clear since by definition we have a left lift for any pushout square of the form:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] & \Delta^0 \end{tikzcd}$$

and the morphisms $X \to \Delta^0$ and $\Delta^n \to \Delta^0$ are unique.
2 $\iff$ 3 : We admit from Goerss and Jardine [1] that there is a bijective correspondence between $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_j, X)$ and $\{(x_0, \dots, \hat{x}_j, \dots, x_n) \mid d_i x_k = d_{k-1} x_i,\ i < k, k \neq j\}$. The result then follows ∎

Remark Sometimes the second condition outlined here is used to define a Kan fibration but in the context of model categories, the original definition is more appropriate.

6.5 Anodyne Extensions

The discussion of the homotopy theory of simplicial sets is greatly simplified by the use of anodyne extensions. This categorical machinery facilitates us to equip simplicial sets with our desired model category structure
We require the following preliminary definition:

Definition We say that a subclass C of simplicial monomorphisms is *saturated* if we have the following:

1. Every simplicial isomorphism (natural isomorphism of simplicial sets) is a member of the class C

2. Given any pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ Z \arrow[r] & Y \bigsqcup_X Z \end{tikzcd}$$

   if f is in C then g is in C

3. For simplicial morphisms α, β if α is a retract of β and α is in C then β is in C (The corresponding diagram is the same as discussed in condition 2 of the definition of a model category)

4. Given a countable collection simplical sets $(X^{(j)})_{j \in J \subseteq \mathbb{N}}$ and a sequence of simplicial morphisms
$$X^{(1)} \xrightarrow{f_1} X^{(2)} \xrightarrow{f_2} X^{(3)} \xrightarrow{f_3} \dots \quad (*)$$
the canonical morphism from $X^{(1)}$ to the colimit of $(*)$ is in C

5. Given a collection of simplicial morphisms $(X_j \xrightarrow{f_j} X_j \to Y_j)_{j \in I}$ indexed over a (possibly uncountable) set I, the corresponding coproduct map
$$\bigsqcup_{j \in I} X_j \xrightarrow{\sqcup f_j} \bigsqcup_{j \in I} Y_j$$
is in C

Definition Given any collection of simplicial monomorpisms D, we define its *saturation* as the intersection of all saturated classes containing D. In other words, the saturation is the minimal saturated class containing D

The saturation of a particular class of monomorphisms is of interest to us:

Definition The saturation of the class of inclusions $\{\Lambda^n_k \hookrightarrow \Delta^n \mid k \in \{0,\dots,n\}, n \geq 0\}$ of all n,k-horns is called the class of *anodyne extensions*.

Remark Such a construction is garuanteed to exist since, for instance, the collection of all monomorphisms is trivially saturated.

In the context of cofibrantly generated model categories, the construction of a saturated class and in particular, anodyne extensions in conjunction with the small object argument, precisely garuntee the existence of functorial factorizations

6.6 Simplicial Homotopy and Homotopy Groups

In order to show that **sSet** admits a model structure it helps to define the notion of homotopy first. We will see that this notion agrees with the general definition of homotopy defined in the Model Categories chapter.

Definition We say that simplicial morphisms $f, g : B \to X$ are *homotopic* if there exists a morphism $H : B \times \Delta^1 \to X$ (referred to as a *homotopy*) for which the following diagram commutes

$$\begin{tikzcd}
B \times \Delta^0 \arrow[d] \arrow[dr, "f"] & \\
B \times \Delta^1 \arrow[r, "H"] & X \\
B \times \Delta^0 \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Where the vertical arrows are the obvious inclusions.

Remark Notice that this diagram is almost identical to the alternative definition of homotopy provided in the introductory chapter. This is no coincidence; the geometric realization preserves diagrams of this form.
We observe that, in our definition of homotopy of an arbitrary model category, objects of the form $B \times \Delta^1$ are cylinder objects of **sSet**. Similarly (and perhaps more intuitively) products of the form $X \times [0,1]$ are cylinder objects in the category of topological spaces.

There is also the stronger notion of homotopy relative to a simplicial subset:

Definition Given an inclusion $A \overset{i}{\hookrightarrow} B$, we say that $f, g : B \to X$ are *homotopic relative to A* if the following conditions are satisfied:

1. There exists a homotopy $H : B \times \Delta^1 \to X$ between f and g
2. We have $f_{|\ A} = g_{|\ A}$
3. The following diagram commutes

$$\begin{tikzcd} A \times \Delta^1 \arrow[r] \arrow[d, hook, "i\times\mathbb{1}"'] & A \arrow[d, "f_{|\ A}=g_{\ A}"] \\ B \times \Delta^1 \arrow[r, "H"] & X \end{tikzcd}$$

Where the upper horizontal arrow is the canonical epimorphism onto A.

It can be shown that homotopy is an equivalence relation on simplicial morphisms whenever their shared codomain is a fibrant object but not in general. Similarly homotopy relative to a simplicial subset is an equivalence relation provided that this condition is met. One notes that, in contrast, for topological spaces homotopic to is always an equivalence relation regardless of choice of object (space).
We can define the homotopy groups of a fibrant simplicial set at a chosen vertex. The geometric realization functor then "realizes" each simplicial homotopy group as a classical topological homotopy group.

Definition Let X be a fibrant simplicial set and $x \in X_0$ an arbitrary 0-simplex. For any $n \geq 1$ we define the *n^{th} homotopy group $\pi_n(X,x)$* as the homotopy classes (relative to $\partial\Delta^n \hookrightarrow \Delta^n$) of maps $\gamma : \Delta^n \to X$, which make the following diagram commute:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \Delta^0 \arrow[d, "x"] \\ \Delta^n \arrow[r, dashed, "\alpha"] & X \end{tikzcd}$$

We also define $\pi_0(X)$ as the set of path components. Just like in the topological setting, there is no natural group operation with which we can equip this set. Analagously we call a fibrant set X *connected* if $\pi_0(X)$ is a singleton set.

This definition seems a bit opaque upon initial inspection (as is the following definition of the group operation) but it is important to note that it was defined with the original topological homotopy groups and the geometric realization in mind.

Let $[\alpha],[\beta] \in \pi_n(X,x)$ for fibrant X. As discussed, there is a canonical simplicial morphism $\Delta^0 \to X$ uniquely defined by the element $x \in X_0$. As an abuse of notation, we usually just denote this morphism by x also. We may then define the simplicial morphism $(x_0,x_1,\ldots,x_{n-1},\hat{x}_n,x_{n+1}) : \Lambda^{n+1}_n \to \Delta^{n+1}$ where

$$x_j = \begin{cases} x, & \text{if } j \in \{0,\ldots,n-2\} \\ \alpha, & \text{if } j = n-1 \\ \beta, & \text{if } j = n+1 \end{cases}$$

(Note that this uniquely determines a simplicial morphism since $d_i x_j = d_{j-1} x_i$ whenever $i < j$, $i,j \in [n+1] \setminus \{n\}$ by the third equivalent condition (see 6.4.1)
Since X is a Kan complex, there exists a "left lift" $\gamma : \Delta^{n+1} \to X$ induced from the inclusion $\Lambda^{n+1}_n \hookrightarrow \Delta^{n+1}$. Moreover we see, from the simplicial identities, that $d_n\gamma$ is constant on x.
It can be shown (Goerss and Jardine [1]) that such $d_n\gamma$ is independent of choice of representatives and therefore the operation

$$[\alpha][\beta] := [d_n\gamma]$$

is well defined on equivalence classes.

Remark The addition of x_j for $j \in \{0,\ldots,n-2\}$ is simply there so that we can include the n-cells α and β into the boundary of a $(n+1)$-simplex. Then the omitted cell from this construction is precisely $d_n\gamma$.

The following lemma (Goerss and Jardine [1]) will be useful when proving the Quillen equivalence.

Lemma 6.6.1 *Given a pushout square in **sSet** of the form*

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d, two heads] \\ \Delta^0 \arrow[r] & Y \end{tikzcd}$$

There is a long exact sequence of the form

$$\ldots \to \pi_{n+1}(Y,y) \to \pi_n(f^{-1}(y),x) \to \pi_n(X,x) \to \pi_n(Y,y) \to \ldots$$

for any $x \in f^{-1}(y)$

Remark We have (Hovey [4]) that given a fibrant simplicial set X and some $x \in X$, the existence of a natural isomorphism $\pi_n(X,x) \simeq \pi_n(|X|,|x|)$.

6.7 The Model Structure on Simplicial Sets

We now prove the main result of this chapter. We will lean on many of the results and definitions discussed above.
Before proceeding, we need to define what our class of weak equivalences is first. We would like the definition to be analagous to the classical one. That is that it induces an isomorphism on all homotopy groups. However we have the slight issue that homotopy groups are only defined on fibrant objects. For this reason we need to somehow bootstrap whatever definition we use so that it applies for any possible simplicial set. There are several equivalent ways of doing so, we use the following definition:

Definition Let $f : X \to Y$ be a simplicial morphism. We say that f is a *weak equivalence* if the image under the geometric realization functor $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.
For morphisms between fibrant objects one can simply define the a weak homotopy equivalence directly in the obvious way.

Theorem 6.7.1 *The category* ***sSet*** *of simplicial sets admits a model structure. It's cofibrations are inclusions, its fibrations are as defined in 5.4 and its weak equivalences are weak homotopy equivalence.*
This model structure is called the Quillen *or* Kan *model structure or sometimes just the* classical model structure

Two Out of Three

This immediately follows from the result on topological spaces since by definition, a weak equivalence in **sSet** induces a weak equivalence in **Top** via the geometric realization functor.

Retracts

The case for fibrations is straighforward. Indeed suppose we have the following commutative diagram and f is a Kan fibration:

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\mathbb{1}_A"] \arrow[d, "g"] & B \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ X \arrow[r] \arrow[rr, bend right, "\mathbb{1}_X"'] & Y \arrow[r] & X \end{tikzcd}$$

f has the right lifting property with respect to all inclusions $\Lambda^n_k \to \Delta^n$ so therefore we have that for every diagram of the form:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathbb{1}_A"] & B \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\
\Delta^n \arrow[r] & X \arrow[r] \arrow[rr, bend right, "\mathbb{1}_X"'] & Y \arrow[r] & X
\end{tikzcd}$$

there is a lift $\Delta^n \to B$ since f is a Kan fibration by hypothesis.
Then clearly the composition $\Delta^n \to X \to B \to A$ describes a right lift of g. Thus g is a Kan fibration.
The case for cofibrations (inclusions) is clear. Indeed it easily follows since the horizontal arrows in the corresponding diagram must be inclusions since they compose to the identity.
The case for weak equivalences again follows directly from the fact it is true in **Top**.

In order to prove the other two axioms, we need the following lemma (Goerss and Jardine [1]):

Lemma 6.7.2 *If $f : X \to Y$ is an acyclic fibration then f has the right lifting property with respect to all inclusions $\partial\Delta^n \to \Delta^n$*

Let us denote by I the collection of all inclusions $(\partial\Delta^n \to \Delta^n)$ and by J the collection of anodyne extensions.

Factorizations

We apply the small object argument (since by the previous lemma, every cofibration is generated by a transifinite sequence of pushouts of morphisms $\partial\Delta^n \hookrightarrow \Delta^n$) to conclude that there exist functorial factorizations (α,β) and (γ,δ) (applied to I and J respectively) such that given any morphism $f : X \to Y$, there exist simplicial sets A, B and commutative diagrams of the form

$$\begin{tikzcd}
X \arrow[r, "\alpha(f)"] \arrow[rr, bend right, "f"'] & A \arrow[r, "\beta(f)"] & Y
\end{tikzcd} \qquad \begin{tikzcd}
X \arrow[r, "\gamma(f)"] \arrow[rr, bend right, "f"'] & B \arrow[r, "\delta(f)"] & Y
\end{tikzcd}$$

where $\alpha(f)$ is anodyne, $\beta(f)$ is a fibration, $\gamma(f)$ is an inclusion and $\delta(f)$ is an acyclic fibration.
Anodyne extensions are inclusions, it can be shown (Goerss and Jardine [1]) that anodyne extensions are weak equivalences. Therefore the first of the above factorizations is an acyclic cofibration followed by a fibration and the second is a cofibration followed by an acyclic fibration.

Lifting

Suppose we have a commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "g"'] & X \arrow[d, two heads, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

and either f or g is acyclic
The case where f is acyclic is true by our lemma since I generate all cofibrations.
The case where g is acyclic requires slightly more work.
By the factorization lemma, there is a factorization of $g : A \to B$ of the form

$$\begin{tikzcd} A \arrow[r, tail, "\alpha(g)"'] \arrow[rr, bend left, "g"] & C \arrow[r, two heads, "\beta(g)"'] & B \end{tikzcd}$$

With $\alpha(g)$ an anodyne extension and $\beta(g)$ an acyclic fibration.
We can then construct a commutative diagram of the form:

$$\begin{tikzcd} A \arrow[r, tail, "\alpha(g)"] \arrow[d, tail, "g\sim"] & C \arrow[d, two heads, "\beta(g)"] \\ B \arrow[r, "\mathbb{1}_B"] & B \end{tikzcd}$$

By the previous lifting property ($\beta(g)$ is a trivial fibration) we have a lift $r : B \to C$ making the diagram commute. Therefore we have a retract diagram of the form

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A"] \arrow[d, tail, "g"] & A \arrow[r, "\mathbb{1}_A"] \arrow[d, tail, "\alpha(g)"] & A \arrow[d, tail, "g"] \\ B \arrow[r, "r"] \arrow[rr, bend right, "\mathbb{1}_B"'] & C \arrow[r, "\beta(g)"] & B \end{tikzcd}$$

whence g is a retract of an anodyne extension and thus has the left lifting property with respect to fibrations.

Remark One should note that it is possible to equip **sSet** with a number of different model structures. Another for example is the so called Joyal model structure which is also popular in the study of quasi-categories.

Remark The fact that all inclusions are cofibrations tell us that every object in **sSet** is cofibrant.

6.8 The Quillen Equivalence

We now discuss the important relation between the two model categories that we have spent so much time on. We have already seen that the geometric realization functor is a left adjoint whose respective right adjoint is given by the singular functor. It turns out that this adjunction is actually a Quillen equivalence.

Theorem 6.8.1 *(Goerss and Jardine [1])*
*The geometric realization functor $|-|$: **sSet**→**Top** induces an equivalence of categories $Ho|-|$:Ho(**sSet**)→ Ho(**Top**)*
In particular, the adjunction

$$\begin{tikzcd} \textbf{\textit{sSet}} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \textbf{\textit{Top}} \arrow[l, bend left, "Sing"] \end{tikzcd}$$

induces the equivalence

$$\begin{tikzcd} Ho(\textbf{\textit{sSet}}) \arrow[r, bend left, "L|-|"] \arrow[r, phantom, "\perp"] & Ho(\textbf{\textit{Top}}) \arrow[l, bend left, "R(Sing)"] \end{tikzcd}$$

In order to make sense of the proof of this statement we need to introduce the following definitions

Definition Given simplicial sets X,Y, we define the *function complex* $\mathbf{Hom}(X,Y)$ by

$$\mathbf{Hom}\ (X,Y)_n := \mathrm{Hom}(X\times\Delta^n, Y)$$

Definition Given a Kan complex (fibrant simplicial object) X and a vertex x of X. we define the path space PX as the pullback of the following diagram

$$\begin{tikzcd} & \mathbf{Hom}(\Delta^1,X) \arrow[d, "(d_0)^*"] \\ \Delta^0 \arrow[r, "x"] & \mathbf{Hom}(\Delta^0,X) \end{tikzcd}$$

We define the morphism $\pi : PX\to X$ by the composition:

$$PX \to \mathbf{Hom}\ (\Delta^1,X) \xrightarrow{(d^1)^*} \mathbf{Hom}\ (\Delta^0,X) \xrightarrow{\simeq} X$$

PX is a fibrant object in **sSet**

Finally, we define the *loop space* ΩX as the pullback of the diagram

$$\begin{tikzcd} & PX \arrow[d, "\pi"] \\ \Delta^0 \arrow[r, "x"] & X \end{tikzcd}$$

In other words, the fibre over $\pi : PX \to X$

This theorem is proven by showing that the induced functor $|-|$ is full, faithful and essentially surjective on homotopy classes. The following lemma (Goerss and Jardine [1]) shows that the functor is essentially surjective on homotopy classes.

Lemma 6.8.2 *The morphism $\eta_X : X \to S|X|$ is a weak equivalence for every Kan fibration.*

Proof We provide a brief sketch of the proof.
The functors $|-|$ and S both preserve fibrations and pullbacks. Therefore their composition also does (Goerss and Jardine [1]). It thus follows that the induced morphism $S|\pi| : S|PX| \to S|X|$ is a fibration and that applying $S|-|$ to the pullback diagrams defining the path and loop spaces give well defined objects $S|PX|$ and $S|\Omega X|$.
We argue that the induced group morphism from the n^{th} homotopy functor is an isomorphism by strong induction on $n \geq 0$.
It is clear that this is the case for $n = 0$
Suppose now that there exists some $n \geq 1$ for which $\pi_j(\eta_X, x) : \pi_j(X, x) \to \pi_j(S|X|, \eta(x))$ is an isomorphism for all $j \leq n$ and $x \in X$
As shown in Goerss and Jardine [1], we have a diagram of the form

$$\begin{tikzcd} \pi_{n+1}(X,x) \arrow[r, "\eta_X"] \arrow[d, "\simeq"] & \pi_{n+1}(S|X|, \eta x) \arrow[d] \\ \pi_n(\Omega X, x) \arrow[r, "\eta_{\Omega X}"] & \pi_n(S|\Omega X|, \eta x) \end{tikzcd}$$

where by the induction hypothesis, the lower horizontal arrow is an isomorphism.
We show that the right vertical arrow is an isomorphism which then implies that the upper horizontal arrow is an isomorphism.
By Lemma 6.6.1, there is a long exact sequence over the fibre of $S|PX| \to S|X|$ of the form:

$$\cdots \to \pi_{n+1}(S|X|, \eta x) \to \pi_n(S|\Omega X|, \eta x) \to \pi_n(S|PX|, \eta x) \to \pi_n(S|X|, \eta x) \to \cdots \quad (*)$$

Goerss and Jardine [1] shows that PX is contractible and it therefore follows that $\pi_n(S|PX|, \eta x) = \{0\}$. Exactness then forces an isomorphism $\pi_{n+1}(S|X|, \eta x) \to \pi_n(S|\Omega X|, \eta x)$ completing the proof ∎

As proved in Quillen [10], the category Ho(**Top**) is equivalent to the category of CW complexes. Therefore any topological space is weak homotopy equivalent to a CW complex.

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