Stellar Encounters with Resonant Planetary Systems

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Analysis of *Kepler* data (Fabrycky *et al.* 2014) revealed an absence of 2:1 and 3:2 resonances in planetary systems of $N \leq 6$ planets. Stellar encounters as a mechanism for resonance disruption of $N = 2$ systems is hereby investigated. Parabolic encounters of a solar mass with Earth or Jupiter mass planets orbiting a solar mass do not reproduce the distinct 'gaps' in the period ratio distribution. Period ratios spread around initial resonance values for close encounters of impact parameter $r_0 = 10AU$. Large $r_0 = 100AU$ results are dominated by Hermite integrator error; the apparent conclusion that final resonances reside entirely above initial values is unreliable. Further tests on $N \geq 2$ systems integrated with semi-major axis error of order 10^{-8} are required.

1. INTRODUCTION

Analysis of the first six quarters of the *Kepler* data (Fabrycky *et al.* 2014 [1], hereafter Fabrycky), concluded that 2:1 and 3:2 resonant planetary pairs in the Cygnus-Lyra region [2] show a statistical preference to reside wide of resonances. As mentioned in Fabrycky, such mechanisms that may account for the 'gaps' observed in the period ratio distribution include eccentricity excitation followed by planetary scattering, chaos at low eccentricites or perhaps tidal dissipation in the inner planet causing planetary migration [1]. Here, we propose stellar encounters as a mechanism for resonance disruption, and test whether the observed 'gaps' of Fabrycky can be reproduced in this way.

A mean motion resonant pair of planets has a period ratio, P_{out}/P_{in}, which can be expressed approximately by small integers p and q [3]:

$$\frac{P_{out}}{P_{in}} \approx \frac{p+q}{p}. \tag{1}$$

The resonance order is q, such that 2:1 and 3:2 are 'first-order' resonances. Resonant systems are of significance as they provide a 'locking' mechanism over Myr time-scales [3]: if two planets orbiting in resonance do not experience a close encounter with one another over the course of P_{out}, necessarily, planet-planet scattering will not occur over the lifetime of the planetary system, and the system is long-term stable.

A stellar encounter with a planetary system of N planets leads to changes in orbital elements such as semi-major axis, a, and eccentricity, e, amongst others [4]. Thus, by Kepler's 3rd Law ($P^2 \propto a^3$), a stellar encounter can displace a planetary pair from resonance. The minimum separation of the parent star of the planetary system and the encountering intruder star is the impact parameter, r_0. For small enough r_0, captures and ejections can occur. A captured planet orbits around the intruder star following the encounter, while an ejected planet is bound neither to the intruder nor the parent. In general, a long-term stable system can be perturbed following a stellar encounter, such that on Myr time-scales post-encounter, planet-planet scattering can occur causing collisions and/or ejections (i.e. long-term stability is lost).

The resonant angle, φ, as a function of time, must be considered when formally defining resonance. This parameter incorporates the fine details of an eccentric orbit; the mean anomaly, M, and mean longitude, λ are calculated from the orbital parameters E, the eccentric anomaly, and ϖ, the longitude of pericentre [3].

$$M = E - e\sin E \tag{2}$$

$$\lambda = M + \varpi \tag{3}$$

Resonant angles of the outer planet, φ', and the inner planet, φ are defined as below (outer planet properties are dashed).

$$\varphi' = (p+q)\lambda' - p\lambda - \varpi' \tag{4}$$

$$\varphi = (p+q)\lambda' - p\lambda - \varpi \tag{5}$$

An unchanging resonant angle with respect to time represents exact resonance, whereas a librating resonant angle undergoes small/large oscillations about a constant value and represents small/large amplitude resonance. In contrast, a circulating resonant angle varies over the entire range 0 to 2π [3]. In all, a resonant pair of planets has at least one constant or librating resonant angle [5].

A detailed description of the computer simulation process is laid out in §2. In §3, the final period ratios are presented, and the reliability of these results is addressed. §4 discusses how we might yield more reliable results, and details areas for improvement and further investigation. Finally, we draw overarching conclusions in §5.

2. METHODS

Computer simulations of stellar encounters with resonant planetary systems were run, employing $AMUSE$ software: an astrophysics Python framework and database of community codes [6]. Final period ratios were extracted, and the distribution was compared with that of Fabrycky.

A mathematical tool called an integrator must be employed for numerical N-body simulations evolving according to the force of gravity; for an initial configuration there is no analytic solution to the N-body problem for $N > 2$. A computer uses the integrator to continually update, at each time-step, the positions and velocities of each particle, by calculating the gravitational forces exerted [7]. Time-step must be large enough so as to minimise computation time for practical use, but small enough such that errors are kept to a minimum.

Test runs with time-step as an independent variable were run to determine optimum choice of time-step, as displayed

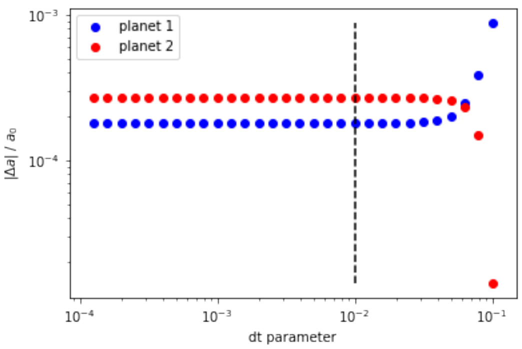

FIG. 1: A 2:1 $Earth$ planetary system is simulated for $t = 1500yrs$ (approx. encounter simulation time) with varied choice of dt parameter, and the fractional changes in semi-major axes of planets 1 and 2 are plotted, with the expectation that changes should be negligible (a_0 denotes initial a). Hermite integrator errors increase significantly for $dt > 0.03$; $dt = 0.01$ chosen for simulations, marked by dashed line

in Fig. 1. A Hermite integrator was implemented for these simulations, which was subject to errors in semi-major axis of order 10^{-4}, over time-scales comparable to the simulation time. This shows planets in the simulation migrate 'naturally' due to integrator error, when they should not.

Two planet configurations were explored. The $Earth$ configuration involved a planetary system of 2 Earth mass planets, where the inner planet was set at a distance $a = 1AU$ from the parent star. The $Jupiter$ configuration bore 2 Jupiter mass planets, with the inner Jupiter set at a distance approx. $a = 5.2AU$ from the parent, the same as we find in our Solar System. Both the parent and intruder stars were solar masses. 2:1 and 3:2 resonances were tested, and all systems were initially circular ($e = 0$), coplanar, and with randomised phase. Because the orbits were initially circular, the relative phase of the 2 planets did not play a role in setting up resonance. Thus, the non-trivial intricacies of setting up resonance for elliptical systems were bypassed.

The encountering star was set on an analytic parabolic ($e = 1$) path with the parent star, where initial separation and r_0 could be set freely. Initial separation was fixed at $500AU$ and r_0 was varied between $(10, 10\sqrt{10}, 100)AU$. Minimum $r_0 = 10AU$ was chosen because captures and ejections consistently occurred in the $Earth$ configuration for $r_0 < 10AU$, as was the case for $r_0 < 10\sqrt{10}AU$ in the $Jupiter$ configuration; $r_0 = 10AU$ was not tested in $Jupiter$ configuration simulations for this reason.

The significance of 'flat' encounters was also investigated. Flat encounters see the intruder moving within the plane of the planetary system, whereas 'inclined' encounters were created by rotating the plane of the planetary system by 3 angles randomised between 0 and 2π: α, β and γ. The matrix found by a combination of 3 standard rotation matrices, and the inverse of this matrix, allows for transformations to and from the default Cartesian coordinates utilised by the computer, **r**, and the coordinates in the rotated frame, **r**$'$.

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$\mathbf{r}' = R_x(\gamma)R_y(\beta)R_z(\alpha)\mathbf{r} \tag{7}$$

$$\mathbf{r} = R_z^{-1}(\alpha)R_y^{-1}(\beta)R_x^{-1}(\gamma)\mathbf{r}' \tag{8}$$

These matrices are versatile as **r** and **r'** can be substituted for velocities in the default frame and the rotated frame, i.e. **v** and **v**$'$.

The $AMUSE$ framework stores mass, position and velocity information, and returns kinetic and potential energies. For a given total energy of a planet, E, it's semi-major axis is found. Trivial calculation of specific angular momentum, h, leads to determination of orbital eccentricity [8]. $\mu = GM$ is the standard gravitational parameter, where M and m are parent and planet mass respectively.

$$E = -\frac{\mu m}{2a} \tag{9}$$

$$a = \frac{h^2}{\mu(1-e^2)} \tag{10}$$

The angles E, θ, f and ϖ can now be found, for calculation of the resonant angle, so as to confirm a system's resonance. Here, r is the planet-parent separation, θ is the true longitude, and f is the true anomaly [3].

$$r = a(1 - e\cos E) \tag{11}$$

$$r = \frac{x'}{\cos\theta} \tag{12}$$

$$r = \frac{a(1-e^2)}{1+e\cos f} \tag{13}$$

$$\theta = f + \varpi \tag{14}$$

Eq. 13 describes both the elliptical orbits of the planets, and the parabolic path of the intruder when used in tandem with Eq. 10.

10,000 simulations were run in all. A given planetary system was allowed to evolve for $100yrs$ without an intruder so as to allow the planets to settle naturally into resonance. Next, the intruder was sent in, and the simulation was run for 2 times the 'encounter time': the time taken for the intruder to reach minimum separation with the parent. The 'encounter time' is analytic and, for a fixed initial separation, depends only on r_0. Simulations were run for around $1500yrs$. Changes to semi-major axis and eccentricity, and final period ratios, were recorded, as well as the occurrences of any unexpected captures/ejections.

3. RESULTS

The final results of 10,000 simulations are displayed in Table 1. 2:1 $r_0 = 10\sqrt{10}AU$ flat encounter simulations were not tested due to time constraints. As mentioned, $r_0 = 10AU$ $Jupiter$ simulations were ignored due to consistent occurrence of captures/ejections. 3:2 $Jupiter$ systems were not tested, as upon visual inspection of resonance angles, resonance was not consistently set up, and this could not be trivially remedied. 2:1 $r_0 = 10\sqrt{10}AU$ $Jupiter$ simulations incurred 53 captures of the outer Jupiter, and 4 ejections of the outer Jupiter, netting 943 useful results. The results that follow focus on the significance of flat encounters, planet configuration and resonance order, followed by an analysis of the Hermite integrator error, and how this significantly reduces the reliability of the results.

TABLE I: Each horizontal line displays the results of 1000 simulations, except in the case of $r_0 = 10\sqrt{10}\ Jupiter$ simulations, with only 943 useful results. Feature denoted '-' implies an $Earth$ configuration with a randomly inclined encounter; the Feature column details deviations from this standard setup. α_P is the standard error in the mean P_{out}/P_{in}. RSD is a measure of the data spread (see Appendix A).

Resonance	$r_0(AU)$	Feature	P_{out}/P_{in}	$\pm$	α_P	$RSD(\%)$
2:1	10	-	1.981	$\pm$	0.002	2.6
2:1	$10\sqrt{10}$	-	2.000771	$\pm$	1×10^{-6}	0.002
2:1	100	-	2.0003294	$\pm$	8×10^{-7}	0.001
3:2	10	-	1.4976	$\pm$	0.0004	0.8
3:2	$10\sqrt{10}$	-	1.501603	$\pm$	2×10^{-6}	0.005
3:2	100	-	1.501599	$\pm$	2×10^{-6}	0.005
2:1	10	Flat	1.826	$\pm$	0.007	12.1
2:1	100	Flat	2.0003220	$\pm$	8×10^{-7}	0.001
2:1	$10\sqrt{10}$	$Jupiter$	2.8	$\pm$	0.3	349.9
2:1	100	$Jupiter$	2.0153	$\pm$	0.0002	0.4

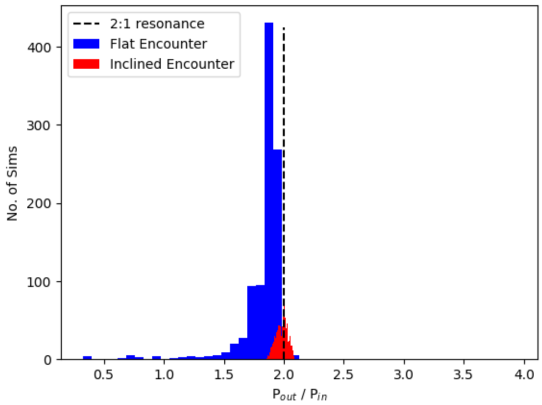

FIG. 2: 2:1 Resonant Systems with $r_0 = 10AU$. Both distributions exhibit a spread of final period ratios *around* the initial resonance value. From Table 1., the flat encounter simulations react more unpredictably than randomly inclined encounters, with an RSD of 12.1% compared to 2.6% (max Flat Encounter value 3.93 too small to be seen).

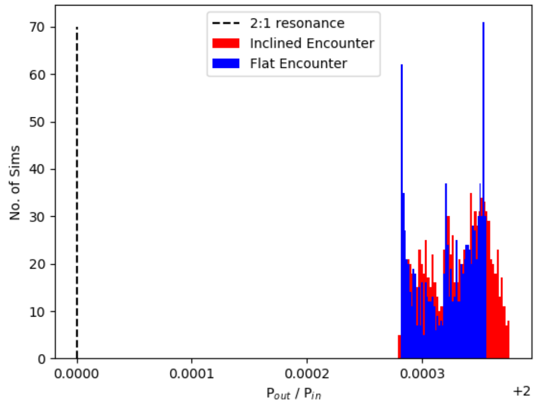

FIG. 3: 2:1 Resonant Systems with $r_0 = 100AU$. Both distributions are spread entirely *above* the initial resonance values. The mean period ratios are not consistent. RSD's to 1 sig. fig. are equal, and several orders of magnitude smaller than the results of associated $r_0 = 10AU$ simulations (average 0.001% compared to 7.4% spread).

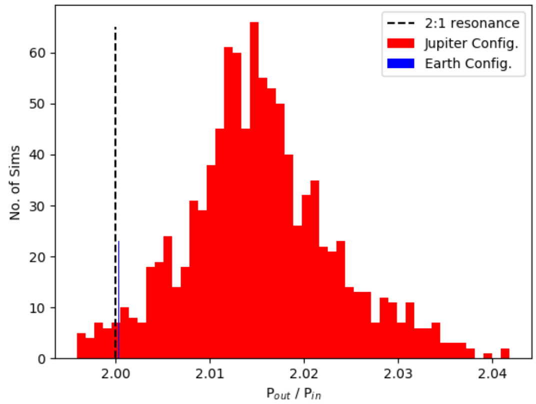

FIG. 4: 2:1 Resonant Systems with $r_0 = 100AU$. $Jupiter$ simulations are more unpredictable than $Earth$ simulations; RSD's of 0.4% and 0.001% respectively. 97.4% of $Jupiter$ results reside *above* initial resonance value.

3.1. Flat and Inclined Encounters

The results of the special case of an encountering star travelling in the plane of the planetary system are displayed in Fig. 2 and Fig. 3. In all, flat encounters are more unpredictable than inclined encounters, as are small r_0 encounters compared to large r_0. A trend that will persist throughout the rest of the results also emerges here: small r_0 simulations produce a spread of period ratios *around* the initial resonance values, whereas large r_0 simulations produce a spread of period ratios *above* the initial resonance values. The latter always has a smaller relative standard deviation (RSD), i.e. a smaller spread in the data, or indeed a more predictable response.

3.2. $Earth$ and $Jupiter$ Configurations

These results examine the significance of the two planetary system configurations ($Earth$, $Jupiter$). Showcased in Fig. 4 are $r_0 = 100AU$ simulations for both configurations. Not shown are the results for simulations of $Jupiter$ $r_0 = 10\sqrt{10}AU$, which reacted with the most unpredictability ($RSD = 349.9\%$), due to a small r_0/a_2 ratio (see Fig. 5). Overall, $Jupiter$ simulations are more unpredictable for a given r_0 due to smaller r_0/a_2 ratio when compared to $Earth$. Trend is broken with $Jupiter$; while 'small' $r_0 = 10\sqrt{10}AU$ results do indeed spread *around* initial resonance value, 2.6% of $r_0 = 100AU$ results reside *below* initial resonance, breaking the trend. This is because a $100AU$ r_0 isn't sufficiently large enough relative to a_2. Indeed, as displayed in Fig. 5, r_0/a_2 for this configuration is less than the next smallest value (that of 2:1 $Earth$ $r_0 = 10\sqrt{10}AU$), the results of which spread entirely *above* initial resonance. Were $Jupiter$ tests run for $r_0 > 100AU$, RSD would decrease, and it is suspected the trend would resurface, with period ratios spreading *above* initial resonance value. Further tests would determine if this hypothesis holds true.

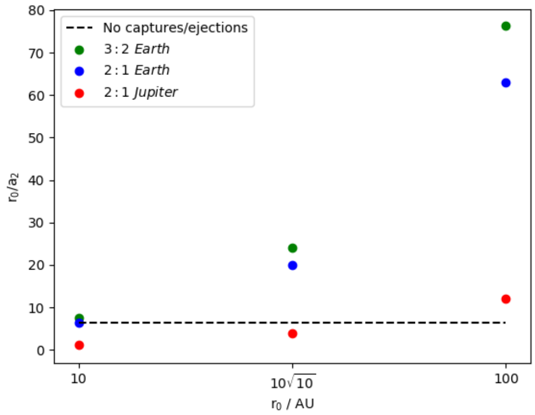

FIG. 5: A rudimentary predictor of whether captures/ejections occur for a given planetary system and a given impact parameter. For 1000 simulations of a 2:1 $Earth$ system with $r_0 = 10AU$, no captures or ejections occurred. Thus, a maximum limit on a hypothetical critical r_0/a_2 value for no captures/ejections is placed here, at 6.3. Simulations with r_0/a_2 below this limit experience captures and ejections, as displayed by 2:1 $Jupiter$ scatter points. Further tests with different r_0 values and planetary system configurations can further refine this limit, and also define the point where *all* simulations experience captures/ejections (if such a critical value exists).

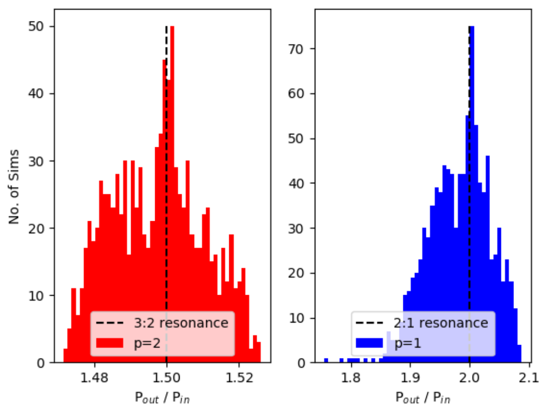

FIG. 6: 3:2 and 2:1 Resonant Systems with $r_0 = 10AU$. Both distributions exhibit a spread of final period ratios *around* the initial resonance values. Spread of 3:2 systems is slightly smaller, with an RSD of 0.8% compared to 2.6% of the 2:1 systems.

3.3. 2:1 and 3:2 Resonances

Results comparing planetary system resonance are displayed in Fig. 6, Fig. 7 and Fig. 8. The trend that small/large r_0's produce results that spread around/above initial resonance values is again apparent. As expected, RSD diminishes with increasing r_0, and interestingly 3:2 system results of $r_0 = (10\sqrt{10}, 100)AU$ are consistent. Otherwise, there are no explicit differences in the results of 2:1 and 3:2 resonant system encounters.

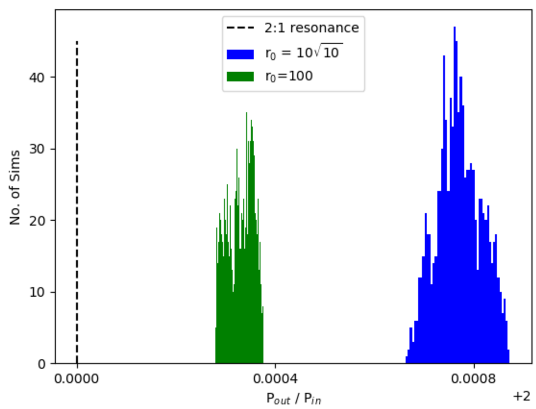

FIG. 7: 2:1 Resonant Systems with $r_0 = (10\sqrt{10}, 100)AU$. Both distributions reside entirely *above* initial resonance value. Spread in data decreases with increasing r_0, with a decline in RSD from 0.002% to 0.001%. As can be seen, mean values are not consistent, and mean is closer to 2 for $r_0 = 100AU$.

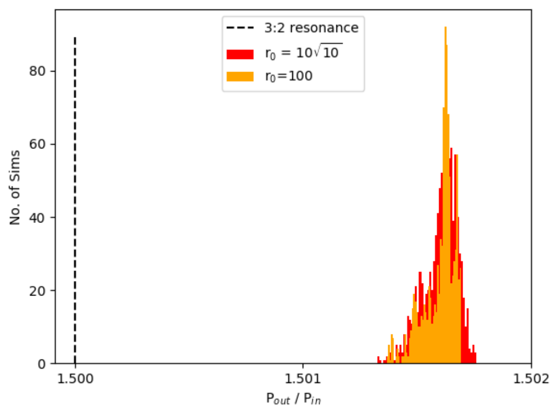

FIG. 8: 3:2 Resonant Systems with $r_0 = (10\sqrt{10}, 100)AU$. Distributions reside *above* initial resonance value, are consistent with one another, and have comparable spread with both $RSD = 0.005\%$ to 1 sig. fig.

3.4. Apparent Conclusions

From these results we might draw the following conclusions: Close encounters of $r_0 = 10AU$ (and $r_0 = 10\sqrt{10}AU$ for $Jupiter$) produce a spread of period ratios *around* the initial resonance values, fail to reproduce 'gaps' below initial resonances, and ultimately do not match Fabrycky. Large encounter simulations produce period ratios spread entirely *above* initial resonance values (bar Fig. 4), thus replicating the observed 'gaps'. However, we must consider how reliable these results truly are.

3.5. Integrator Error and Result Reliability

A control set of simulations is required to ensure results are valid. This is done by simulating a given planetary system over the encounter simulation time *without* a stellar encounter. The final period ratios can be compared with encounter simulation results.

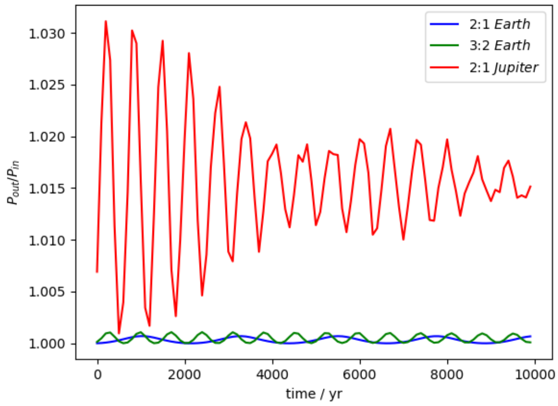

FIG. 9: Variation of normalised period ratios of 3 system configurations over $10kyrs$. Each plot is taken from the average of 100 simulations. Values all start above initial resonance at $t = 0yr$ due to migration occurring in the preceding $100yr$ simulation (the process which allows the resonant angles to stabilise). This graph ultimately shows period ratios are not constant over the simulation time, and this 'natural' migration attributed to Hermite integrator error must be accounted for.

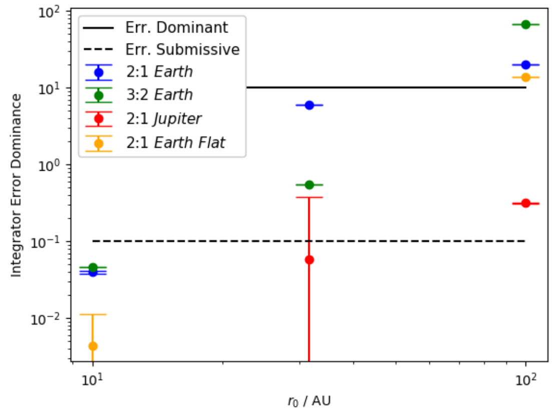

FIG. 10: The integrator error is divided by the stellar perturbation, and the "Integrator Error Dominance" is plotted for all 10 configurations. Here, a dominant value is defined as being greater by at least 1 order of magnitude; while a factor 3 is conventionally used to define a 'dominant' error, this larger overestimated factor is employed to sufficiently account for the migration the planets have *already* experienced by the encounter time, given this migration influences the net effects of the stellar encounter. For close encounters $r_0 = 10AU$, the integrator error is dominated by the effects due to the stellar encounter; thus these results are reliable. For $r_0 = 10\sqrt{10}AU$ encounters, changes due to integrator error and stellar perturbation are comparable, which casts doubt on the results. $Jupiter$ data point suggests reliable results though, because stellar perturbation dominates, and large uncertainty in data point can be entirely attributed to the large spread of period ratios ($RSD = 349.9\%$). As previously discussed, this large spread is expected. $r_0 = 100AU$ $Earth$ results are *all* unreliable, as integrator error dominates the results. $Jupiter$ data point has an integrator error dominance 0.314 ± 0.002, suggesting the associated results are not wholly reliable.

First it is shown in Fig. 9 that even without a stellar encounter, period ratios are not constant on kyr time-scales, and do in fact oscillate with varying amplitudes, depending on the system configuration. This shows we must account for Hermite integrator error in evaluating our results.

Fig. 10 details the dominant source of error in the results. The 'natural' deviation from the initial resonance value at the end of the non-encounter simulation is defined as 'integrator error'. The difference between the period ratios at the end of the encounter simulation and the non-encounter simulation (i.e. the change due to the presence of the intruder) is the 'stellar perturbation'. 100 simulations are run for each configuration to yield integrator error values.

4. DISCUSSION

It is shown in Fig. 10 that results from small r_0 simulations are reliable, and so do not replicate the findings of Fabrycky *et. al* 2014. Results from large r_0 simulations, are unreliable, due to a dominating source of error arising from the Hermite integrator. Here we discuss how the use of a better integrator is essential for producing more reliable results, and specify necessary changes to implement, and interesting areas to explore, that we might expand upon the scope of this investigation.

Planets migrate due to integrator error. It is invalid to remove these 'natural' changes to period ratio from our results, because the planets reside already at a different semi-major axis at the time of encounter compared to when $t = 0$, so will react differently to the intruder; the results cannot be treated, and simulations must be re-run with an integrator that does not allow planetary migration over encounter-simulation time-scales.

Smallest stellar perturbations to period ratio were observed to be of order 10^{-5}, with order 10^{-6} standard error (due to random error). Therefore, for a submissive integrator error (by the same definition as in Fig. 10) we would require 'natural' changes to period ratio to be of order 10^{-6}. However, we can neglect integrator error contribution altogether, provided this error is submissive to the perturbation error (i.e. factor 3 smaller), or by approximation, order 10^{-7}. Given Kepler's 3rd law, this implies changes of order 10^{-8} to semi-major axis (see Appendix B). By contrast, (fractional) changes to semi-major axis of order 10^{-4} were observed for the simulations conducted in this investigation (see Fig. 1).

In addition, we require an integrator that is reliable over Myr time-scales. The behaviour of a resonant system post-encounter is unknown, and must be tested and analysed. Perhaps initial distribution of period ratios differs significantly from a distribution extracted $100Myrs$ after the encounter? Supposing stellar encounters *is* a valid mechanism for resonance disruption, it is naive to assume the planetary systems observed by $Kepler$ have all experienced a stellar encounter 'recently': It is necessary to perform a Myr integration. Fig. 9 demonstrates the integrator is unreliable over kyr time-scales, let alone over Myr time-scales. An inspection of the resonant angles (not shown here) also revealed large-amplitude oscillation over kyr time-scales. An integrator that maintains a constant period ratio and resonant angle over Myr time-scales is required.

To ensure comparisons with Fabrycky's distribution are valid, planetary configuration variables (N, mass, semi-major axis, eccentricity...) should match the system variables as observed by $Kepler$ and analysed by Fabrycky.

It is reasonable to suspect each unique planetary system will react differently to a stellar encounter. This hypothesis is affirmed by our results, particularly Table 1., where the $Jupiter$ configuration reacts with much more uncertainty (RSD) to a given stellar encounter than an $Earth$ system.

If stellar encounters with resonant planetary systems proves to be a viable mechanism for reproducing the 'gaps' observed by Fabrycky, it would then be required to show non-resonant systems cannot settle into these gaps following a stellar encounter either. For tests with non-resonant systems, *many* more simulations would need to be run, given the range of potential period ratios.

It would be interesting to determine the rules behind a 'critical' impact parameter for loss of resonance: the value of r_0 at which a resonant system only just loses its resonance following a stellar encounter. How does this value change in accordance with the configuration of a planetary system (N, mass, semi-major axis, eccentricity...)? It is clear from the comparisons made between the $Earth$ and $Jupiter$ configurations *and* between the 2:1 and 3:2 systems, that the ratio r_0/a_2 is a good indicator of how a system reacts to a stellar encounter of a given r_0 (smaller ratio leads to larger spread, and higher chance of captures/ejections). Following determination of the critical r_0 (if it exists), we can ask: What is the most probable impact parameter of a stellar encounter given the stellar density of the Cygnus-Lyra region? How do these impact parameters compare? The same investigation can be conducted in search of a 'critical' r_0 for capture/ejection, which is explored to a basic level in Fig. 5.

Surplus to these investigations, tests of greater N than that observed by $Kepler$ could be run, with the hypothesis that the planetary systems once held *more* planets than we see today, which have since been captured or ejected. For example, how does an $N = 3$ system evolve if the outermost planet is captured by an intruder star? Are Fabrycky's results reproduced in this case? With a multitude of free parameters and variables, including those not already mentioned such as initial intruder separation, hyperbolic intruder trajectories, non-coplanar systems etc., it seems there is an endless stream of questions yet to be answered.

5. CONCLUSIONS

Stellar encounters with 2:1 and 3:2 resonant planetary systems of 2 unique planetary configurations have been tested for impact parameters $r_0 = (10, 10\sqrt{10}, 100)AU$. It is shown that the reliable results obtained from $r_0 = 10AU$ simulations do not replicate the distinct gaps in the period ratio distribution found in Fabrycky *et. al* 2014. Results from larger r_0 up to $100AU$ are dominated by Hermite integrator error, and the apparent conclusion that these match Fabrycky is unreliable. Further tests must be run for system configurations observed by $Kepler$ with an integrator that is reliable over Myr time-scales, which can maintain a constant period ratio and resonant angle, and is subject to an error in semi-major axis of order 10^{-8}.

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APPENDIX A RSD

The relative standard deviation (RSD) is a statistical parameter that describes the percentage spread of some data about a mean value. It compares the size of the standard deviation, σ, to the mean, μ, where the lower the RSD, the more tightly compact the data is about the mean.

$$RSD = 100 \times \frac{\sigma}{\mu} \tag{15}$$

APPENDIX B IMPROVED INTEGRATOR ERROR

An estimation of the error in semi-major axis of a planet, Δa_i, is made given an estimated integrator error order 10^{-7} in period ratio, here denoted, ΔP. Kepler's 3rd implies a combination of errors like so:

$$\frac{\Delta P}{P} = \sqrt{\left(1.5\frac{\Delta a_1}{a_1}\right)^2 + \left(1.5\frac{\Delta a_2}{a_2}\right)^2}. \tag{16}$$

Taking the approximation that uncertainty in a is the same for both planets, the following expression is obtained:

$$\Delta a = \frac{2a_2}{3} \cdot \frac{\Delta P}{P\sqrt{P^{\frac{4}{3}} + 1}}. \tag{17}$$

This expression evaluated for the 3 unique planetary system configurations takes a minimum with the 2:1 $Earth$ configuration of 2.8×10^{-8}. Thus, for an arbitrary planetary system configuration we require an integrator with error in semi-major axis approximately of order 10^{-8}.

References

[1] Fabrycky, D.C., et al., "Architecture of Kepler's Multi-Transiting Systems: II. New investigations with twice as many candidates", The Astrophysical Journal, Aug. 1, 2014, Vol. 790, No. 2, 146

[2] Kepler mission-Nasa, Jan. 4, 2018, $https : //www.nasa.gov/mission_pages/kepler/overview/index.html$ As of Aug., 2018.

[3] Murray, C.D., Dermott, S.F., *Solar System Dynamics*, Cambridge University Press, New York, 2009

[4] Malmberg, D., Davies, M.B., Heggie, D.C., "The effects of fly-bys on planetary systems", MNRAS, Feb, 2011, Vol. 411, No. 2, pp. 859-877

[5] Malhotra, R., "Orbital Resonances in Planetary Systems", EOLSS, Oct., 2017, Vol 6.119.55

[6] Amuse, Dec., 2017, $amusecode.org/$ As of Aug., 2018.

[7] Greengard, L., "The Numerical Solution of the N-Body Problem", American Institute of Physics, 1990, Computers in Physics Vol. 12, No. 4, p142

[8] J. Binney, S. Tremaine, *Galactic Dynamics*, 3rd Ed, Princeton University Press, New Jersey, 1994, 108