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**ON THE ACCURACY OF LIMITERS AND
CONVERGENCE TO STEADY STATE SOLUTIONS**

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Abstract

This paper addresses the practical problem of obtaining convergence to steady state solutions when limiters are used in conjunction with upwind schemes on unstructured grids. The base scheme forms a gradient and limits it by imposing monotonicity conditions in the reconstruction stage. It is shown by analysis in one dimension that such an approach leads to various schemes meeting TVD requirements in one dimension. It is further shown that these formally second order accurate schemes are less than second order accurate in practice because of the action of the limiter function in smooth regions of the solution. Modifications are proposed to the limiter that restore the second order accuracy. In multiple dimensions these schemes produce steady state solutions that are monotone and devoid of oscillations. However, convergence stalls after a few orders of reduction in the residual. With the modified limiter, on the other hand, it is shown that converged steady state solutions can be obtained.

1 Introduction

Impressive progress has been made in the area of upwind schemes in the last decade. In their most popular form, upwind schemes are implemented in two stages:

1. A reconstruction stage which obtains a representation of the solution surface given either pointwise or cell-averaged data. Monotonicity principles are also invoked at this stage to enable discontinuities to be captured without oscillations.
2. The reconstructed variables on either sides of the interfaces between the control volumes are interpreted as initial data for approximate Riemann solvers.

In recent years, many upwind schemes have been extended to unstructured grids. For upwind schemes to

work well on unstructured grids it stands to reason that true multi-dimensionality be reflected both in the reconstruction and the Riemann solver stages. While truly multi-dimensional Riemann solvers have been investigated by a number of researchers with varying degrees of success, in this work the grid-aligned one-dimensional approximate Riemann solver of Roe [1] will be used since the issues addressed in this paper are independent of this choice. Some of the upwind schemes in use for unstructured grids are rather formal extensions of one-dimensional models [2, 3] even in the reconstruction stage, without much rigorous theory to support them. However, on relatively uniform triangular grids, many of these schemes seem to produce good resolutions of shocks and other features. Barth and Jespersen [4] introduced a novel upwind scheme for unstructured grids. This scheme first performs a linear reconstruction to interpolate data to the control volume faces and then employs an approximate Riemann solver to compute the fluxes. In the reconstruction stage monotonicity principles are enforced to ensure that the reconstructed values be bounded by the values of a cell and its neighbors. To this end, multi-dimensional limiter functions are used. In contrast to the previous efforts, the reconstruction is truly multi-dimensional. Barth and Jespersen were able to compute smooth oscillation-free transonic flow solutions even on highly irregular triangular meshes.

It has long been known that the use of limiters can severely hamper the convergence of upwind codes to steady state. This particularly nagging problem has somewhat limited the use of these codes in production environments. The convergence problems are even more pronounced in the case of limiters that make use of nondifferentiable functions such as the Max or the Min functions. The multi-dimensional limiter of Barth and Jespersen [4] is an example of such a limiter. While these limiters have been carefully designed to avoid oscillations, they also inhibit convergence of nonlinear problems to steady state. Convergence problems with such limiters have not been adequately addressed in the literature. For structured grids, Venkatakrishnan [5] among others, has obtained converged steady state solutions using differentiable limiters. Convergence was achieved by modifying the limiter function whereby the limiter is effectively turned off when

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the oscillations are below a certain threshold. This modification is similar to that used by van Albada et al. [6] in a different context viz., the problem of capturing smooth extrema without clipping.

In this paper, the base scheme of Barth and Jespersen is analyzed in one dimension. It is shown that several upwind schemes that obey the monotonicity conditions given by Spekreijse [7] can be derived from the base scheme. It is shown that in practice these schemes yield less than the formal second order accuracy. This is demonstrated by advecting a smooth Gaussian distribution in one dimension. Modifications are proposed to the schemes which restore the second order accuracy. The modifications proposed are such that they also alleviate the convergence problems with the base scheme in two dimensions. While the base scheme fails to converge to steady state for these problems, the modified scheme is shown to converge quite well.

2 The base scheme

The scheme of Barth and Jespersen [4] will be considered as the base scheme and is outlined briefly in one dimension. Consider a distribution of cell-averaged quantities $u_i, i = 1, \dots, n$ on a uniform grid with spacing Δx . x_i is the coordinate of the centroid for cell i defined by the interval $[x_{i-1/2}, x_{i+1/2}]$. Since the distinction between cell-averaged and pointwise values at the centroid of the cell is of no consequence when dealing with second order accurate schemes, the u_i 's can also be interpreted as the pointwise values at the x_i 's. The reconstructed distribution $U_i(x)$ within cell i is

$$U_i(x) = u_i + \Phi_i \nabla u_i (x - x_i), \quad \Phi_i \in [0, 1] \quad (1)$$

In this formula the vector ∇u_i is the best estimate of the solution gradient in cell i computed from surrounding cell-averaged values and x is the coordinate at the point of interest. In one dimension this gradient is taken by Barth and Jespersen to be a central difference approximation. The key idea in their paper is to find the largest possible Φ_i while invoking the monotonicity principle that the values of the linearly reconstructed function within cell i be bounded by the maximum and minimum of the neighboring cells (including cell i). That is, if $u_i^{max} = \max(u_{i-1}, u_i, u_{i+1})$ and $u_i^{min} = \min(u_{i-1}, u_i, u_{i+1})$ then they require that

$$u_i^{min} \leq u(x) \leq u_i^{max} \quad (2)$$

Barth and Jespersen mention that this definition of monotonicity coincides with that of Spekreijse [7] for structured grids in multi-dimensions. The value of Φ_i that satisfies Eqn. (2) is given by

$$\Phi_i = \min(\Phi_{i+1/2}, \Phi_{i-1/2}) \quad (3)$$

where

$$\Phi_{i+1/2} = \begin{cases} \min(1, \frac{u_i^{max} - u_i}{U(x_{i+1/2}) - u_i}), & \text{if } U(x_{i+1/2}) - u_i > 0 \\ \min(1, \frac{u_i^{min} - u_i}{U(x_{i+1/2}) - u_i}), & \text{if } U(x_{i+1/2}) - u_i < 0 \\ 1 & \text{if } U(x_{i+1/2}) - u_i = 0 \end{cases} \quad (4)$$

$U(x_{i+1/2})$ in the Eqn. (4) is determined by using Eqn. (1) with $\Phi \equiv 1$. $\Phi_{i+1/2}$ is obtained by replacing $i + 1/2$ with $i - 1/2$ in Eqn. (4). The reconstructed value $U_{i+1/2}^L$ to the left of the interface $i + 1/2$, which is determined from the reconstruction in cell i , is then

$$U_{i+1/2}^L = u_i + \left(\frac{u_{i+1} - u_{i-1}}{2\Delta x} \right) \left(\frac{\Delta x}{2} \right) \Phi_i \quad (5)$$

where the term $\frac{(u_{i+1} - u_{i-1})}{2\Delta x}$ is the central difference approximation to $\frac{du}{dx}$ at cell i .

It will be shown that the scheme outlined above meets the conditions required by Spekreijse [7] for obtaining monotone steady state solutions in two dimensions. Spekreijse has also shown that the schemes that satisfy the conditions for monotonicity in two dimensions obey the TVD property in one dimension. The scheme of Barth and Jespersen will be shown to be a well known scheme in one dimension. Furthermore, it will be shown that two more schemes which also obey the TVD property in one dimension can be derived by replacing the central difference gradient approximation by upwinded and downwinded approximations. These schemes are all second order accurate away from extrema.

The analysis of Spekreijse[7] is reproduced below for a two-dimensional structured grid. Consider the following conservation law in two dimensions:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) + \frac{\partial}{\partial y} g(u) = 0 \quad (6)$$

A discretization of Eqn. (6) on an equidistant mesh with mesh size h is given by

$$U_{i,j}^{n+1} = U_{i,j}^n + \lambda (A_{i+1/2,j}^n (U_{i+1,j}^n - U_{i,j}^n) + B_{i,j+1/2}^n (U_{i,j+1}^n - U_{i,j}^n) + C_{i-1/2,j}^n (U_{i,j}^n - U_{i-1,j}^n) + D_{i,j-1/2}^n (U_{i,j}^n - U_{i,j-1}^n)) \quad (7)$$

where $\lambda = \Delta t/h$ and

$$\begin{aligned} A_{i+1/2,j}^n &= A(\dots, U_{i-1,j}^n, U_{i,j}^n, U_{i+1,j}^n, \dots) \\ B_{i,j+1/2}^n &= B(\dots, U_{i,j-1}^n, U_{i,j}^n, U_{i,j+1}^n, \dots) \\ C_{i-1/2,j}^n &= C(\dots, U_{i-1,j}^n, U_{i,j}^n, U_{i+1,j}^n, \dots) \\ D_{i,j-1/2}^n &= D(\dots, U_{i,j-1}^n, U_{i,j}^n, U_{i,j+1}^n, \dots) \end{aligned}$$

A scheme is called monotone if it has positive, uniformly bounded coefficients, i.e. if

$$A_{i+1/2,j}^n, B_{i,j+1/2}^n, C_{i-1/2,j}^n, D_{i,j-1/2}^n \geq 0 \quad (8)$$

and there exists a bound $G > 0$ such that for all (i, j)

$$A_{i+1/2,j}^n, B_{i,j+1/2}^n, C_{i-1/2,j}^n, D_{i,j-1/2}^n \leq G \quad (9)$$

A solution U is called monotone if for all (i, j)

$$\begin{aligned} \min(U_{i-1,j}, U_{i+1,j}, U_{i,j-1}, U_{i,j+1}) &\leq U_{i,j} \leq \\ \max(U_{i-1,j}, U_{i+1,j}, U_{i,j-1}, U_{i,j+1}) \end{aligned} \quad (10)$$

Dropping the subscripts on the coefficients A, B, C and D for convenience, with the scheme given by Eqn. (7), at steady state we have

$$U_{i,j} = \frac{AU_{i+1,j} + BU_{i,j+1} + CU_{i-1,j} + DU_{i,j-1}}{A + B + C + D} \quad (11)$$

which meets the monotonicity requirements of Eqn. (10) if the scheme obeys the conditions given by Eqns. (8) and (9). Spekreijse also shows for λ sufficiently small ($\lambda \leq \frac{1}{2G}$) that the scheme given by Eqn. (7) is TVD in one space dimension. Given the cell-averaged values $U_{i,j}$, Spekreijse considers reconstructions of the form

$$\begin{aligned} U_{i+1/2,j}^L &= U_{i,j} + \frac{1}{2}\psi(R_{i,j})(U_{i,j} - U_{i-1,j}) \\ U_{i-1/2,j}^R &= U_{i,j} + \frac{1}{2}\psi\left(\frac{1}{R_{i,j}}\right)(U_{i,j} - U_{i+1,j}) \\ U_{i,j+1/2}^L &= U_{i,j} + \frac{1}{2}\psi(S_{i,j})(U_{i,j} - U_{i,j-1}) \\ U_{i,j-1/2}^R &= U_{i,j} + \frac{1}{2}\psi\left(\frac{1}{S_{i,j}}\right)(U_{i,j} - U_{i,j+1}) \\ R_{i,j} &= \frac{u_{i+1,j} - u_{i,j}}{u_{i,j} - u_{i-1,j}} \\ S_{i,j} &= \frac{u_{i,j+1} - u_{i,j}}{u_{i,j} - u_{i,j-1}} \end{aligned}$$

The main contribution of Spekreijse lies in the formulation of the conditions on the limiter function ψ to obtain second order accuracy and monotone steady state solutions using a flux-split finite volume formulation. He also has shown that ψ has to be a piecewise linear or a non-linear function in order to meet these conditions. He has derived the following conditions on the limiter function $\psi(r)$ to meet the positivity and boundedness conditions given by Eqns. (8) and (9). These are

$$\alpha \leq \psi(r) \leq M, \forall r \quad (12)$$

$$-M \leq \frac{\psi(r)}{r} \leq 2 + \alpha, \forall r \quad (13)$$

where $\alpha \in [-2, 0]$ and $M < \infty$. This monotonicity region is depicted in Figure 1.

The conditions given by Eqns. (12) and (13) are very general and many schemes can be derived that meet these conditions. Spekreijse also shows that a spatial discretization is second order accurate away from the points where $\frac{du}{dx} = 0$ if $\psi(1) = 1$ and $|\psi'|$ is uniformly bounded.

In one dimension the second order accurate κ schemes of van Leer [8] are given by

$$U_{i+1/2}^L = u_i + \frac{1+\kappa}{4}(u_{i+1} - u_i) + \frac{1-\kappa}{4}(u_i - u_{i-1}) \quad (14)$$

where $\kappa \in [-1, 1]$. Setting $\kappa = -1$ yields a fully one-sided upwind scheme, $\kappa = 0$ yields the Fromm scheme

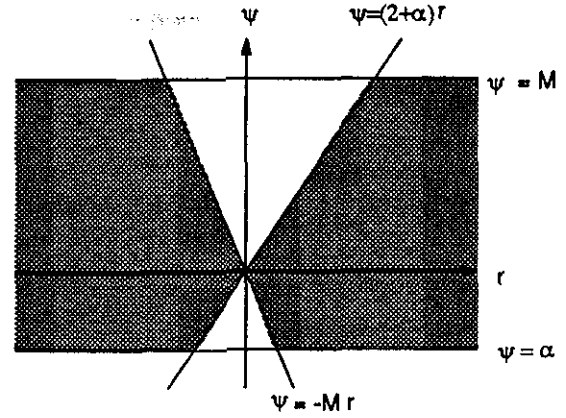


Figure 1: Monotonicity region.

and $\kappa = 1$ yields the central difference scheme. These schemes do not obey the monotonicity conditions since as the function $\psi(r)$ is linear:

$$\psi(r) = \frac{1-\kappa}{2} + \frac{1+\kappa}{2}r \quad (15)$$

We will now carry out an analysis of the base scheme and reformulate it in Spekreijse's framework. We will show that depending on whether we choose central, upwind or downwind difference approximations to the gradient at cell i , three different schemes that meet the monotonicity requirements can be derived. The scheme of Barth and Jespersen can be recast as follows:

$$\begin{aligned} U_{i+1/2}^L &= u_i + \frac{1}{4}(u_{i+1} - u_{i-1}) \min(f1, f2) \quad (16) \\ f1 &= \min\left(1, \frac{u_{i+1} - u_i}{\frac{u_{i+1} - u_{i-1}}{4}}\right) \\ f2 &= \min\left(1, \frac{u_{i-1} - u_i}{\frac{-u_{i+1} - u_{i-1}}{4}}\right) \end{aligned} \quad (17)$$

This is equivalent to

$$U_{i+1/2}^L = u_i + \frac{1}{2}\psi(r_i)(u_i - u_{i-1}), \quad (18)$$

$$\psi(r) = \frac{1}{2}(r+1) \min\left[\min\left(1, \frac{4r}{r+1}\right), \min\left(1, \frac{4}{r+1}\right)\right] \quad (19)$$

$$r_i = \frac{u_{i+1} - u_i}{u_i - u_{i-1}} \quad (20)$$

If u_i is a maximum or a minimum, i.e. $u_i > u_{i-1}, u_{i+1}$ or $u_i < u_{i-1}, u_{i+1}$, either $f1$ or $f2$ equals zero and hence $\psi(r) \equiv 0, \forall r \leq 0$. The scheme thus reduces to first order accuracy at extrema. The function $\psi(r)$ is shown in Figure 2. It clearly satisfies the conditions of Spekreijse with $M = 2$. In one dimension, the scheme of Barth and Jespersen is thus the $\kappa = 0$ or the Fromm scheme with the min-mod limiter applied to enforce monotonicity whenever $r \leq 1/3$ or $r \geq 3$. If the central difference gradient

in the Eqn. (5) were replaced by one-sided gradients, we obtain two more schemes that also obey the conditions of Spekreijse. These are shown in Figure 2 as well. With the gradient approximated by the upwind approximation $\frac{u_i - u_{i-1}}{\Delta x}$, the limiter function can be shown to be

$$\psi(r_i) = \text{Min}(1, 2r_i) \quad (21)$$

With the gradient approximated by the downwind approximation $\frac{u_{i+1} - u_i}{\Delta x}$, the limiter function is

$$\psi(r_i) = \text{Min}(r_i, 2) \quad (22)$$

For the limiters discussed above, $\psi(r) \equiv 0, \forall r \leq 0$. Finally Figure 2 also shows the van Albada limiter which is

$$\psi(r) = \frac{r^2 + r}{r^2 + 1} \quad (23)$$

Note that all the schemes discussed above are second order accurate away from points where $\frac{du}{dx} = 0$.

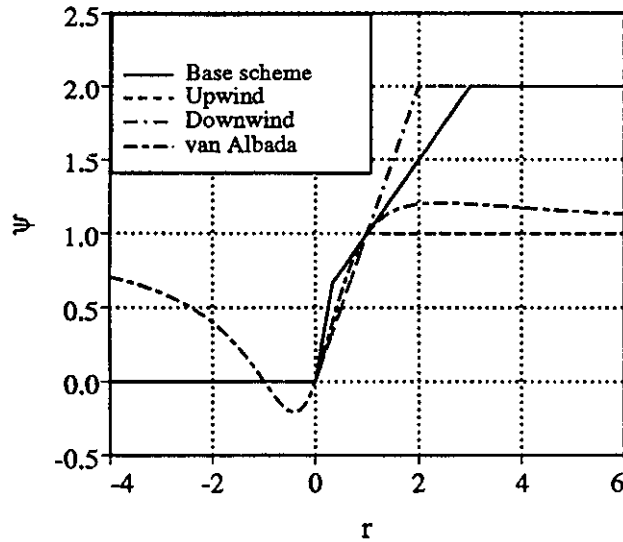


Figure 2: Limiter functions for various schemes.

3 Limiters and accuracy

In this section the issue of accuracy will be addressed for the schemes discussed above by considering a simple one-dimensional problem. The problem is the advection of a Gaussian distribution on a uniform grid. The speed of advection is taken to be 1. The time discretization is done using a four stage Runge-Kutta scheme that is second order accurate in time for nonlinear problems. Note that the schemes with the limiters are nonlinear even for linear problems. The spatial accuracy of all the schemes will be assessed by marching to a fixed time of $T = 0.2$ at a constant CFL number of 0.5 using a sequence of successively refined uniform grids. The initial profile is the Gaussian distribution

$$u(x) = .1e^{-100(x-.4)^2} \quad (24)$$

The L_2 and the L_∞ norm of the error in the numerical solution are measured and are plotted as functions of the inverse of the number of grid points in Figures 3 and 4. Figure 3 shows that all the formally second order accurate monotonicity preserving schemes exhibit only about 1.65 order of accuracy in the L_2 norm i.e., the slopes of the schemes with limiters in Figure 3 are roughly 3.3. The Fromm scheme, on the other hand, clearly exhibits order accuracy. Another observation that can be made from Figure 3 is that the scheme that bases the derivative on an upwind approximation has larger errors than the one that uses a downwind approximation. The base scheme which uses a central difference approximation for the derivative is superior to the schemes which use one-sided approximations, but has larger errors than the Fromm scheme. However, only with the three schemes that employ limiters are the advected distributions (not shown) truly nonoscillatory and nonnegative everywhere, as is the initial distribution. On the other hand, with the Fromm scheme, the advected distribution, while exhibiting no odd-even decoupling, possesses a few small negative values whose magnitudes decrease with grid refinement. The reason for the lack of the formal second order accuracy with the schemes which employ limiters is that the limiters are active in the near-constant regions of the solution. The clipping of extrema at these points leads to first order accuracy locally and less than second order accuracy globally. The effects of the limiter on convergence are well known and what has been shown here is that the limiter affects the accuracy of even smooth solutions. The situation is worse when the L_∞ norm is considered (Figure 4). All the schemes with the limiters exhibit only slightly better than first order accuracy.

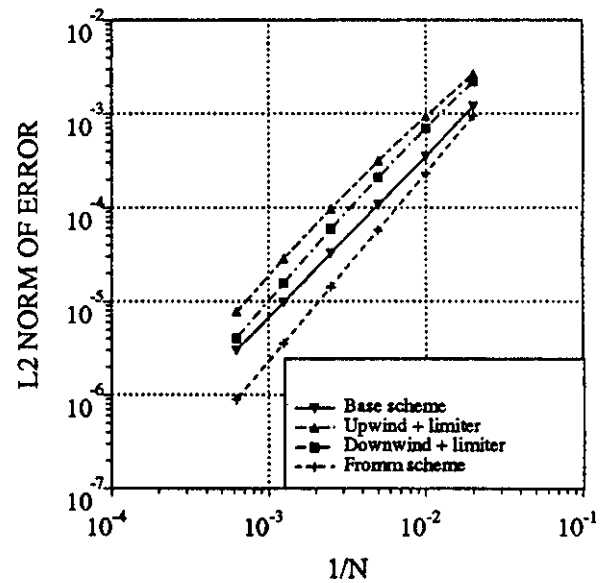


Figure 3: L_2 norm of error in solution with various schemes.

In the following, we will first show that the scheme that uses the van Albada limiter can be modified to elimi-

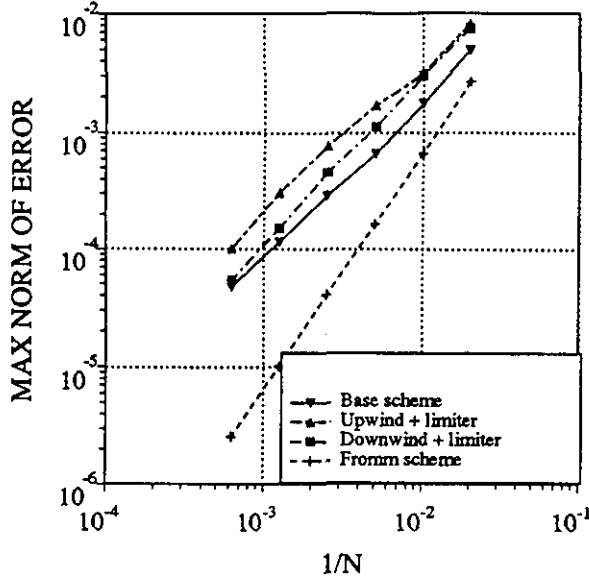


Figure 4: L_∞ norm of error in solution with various schemes.

nate the undesirable effects of limiting in the near-constant regions of the solution. It is then possible to recover true second order accuracy. Modifications to the base scheme, for use in unstructured grids, will then be devised based on the insight gained from modifying the van Albada limiter. The modification to the limiter is the same as the one proposed by van Albada et al. [6] to avoid clipping smooth extrema. The scheme with van Albada limiter is given by

$$u_{i+1/2}^L = u_i + \frac{1}{2} \left(\frac{r^2 + r}{r^2 + 1} \right) (u_i - u_{i-1}) \quad (25)$$

$$r = \frac{\Delta_+}{\Delta_-}$$

$$\Delta_+ = u_{i+1} - u_i$$

$$\Delta_- = u_i - u_{i-1}$$

The modified van Albada scheme is

$$u_{i+1/2}^L = u_i + \frac{1}{2} \frac{(\Delta_+^2 + \epsilon^2)\Delta_- + (\Delta_-^2 + \epsilon^2)\Delta_+}{\Delta_+^2 + \Delta_-^2 + 2\epsilon^2} \quad (26)$$

ϵ^2 is made proportional to $(\Delta x)^3$. In the smooth regions of the solution $(\Delta_+, \Delta_-) \sim \Delta x$, and in the near-constant regions ϵ^2 dominates Δ_+^2, Δ_-^2 and we recover the $\kappa = 0$ scheme without limiting. We set $\epsilon^2 = (K\Delta x)^3$ where K is a constant. If K is set to zero, the original scheme is recovered. A large value of K implies no limiting at all and the $\kappa = 0$ scheme is obtained. Table 1 shows the errors with the van Albada limiter with the constant $K = 0.3$ for the one-dimensional advection of the Gaussian. This value of K will be used later on in the two-dimensional calculations. Second order accuracy is approached as the mesh gets progressively finer. However, the advected distribution exhibits a few negative values. If ϵ^2 were instead set to be a constant value, say 10^{-12} for 64-bit

arithmetic computations, second order accuracy will be realized when Δ_+^2 and Δ_-^2 are below this threshold. For uniform grids it is thus immaterial how ϵ^2 is computed.

Table 1: L_2 and L_∞ norms of errors with van Albada limiter.

No. of cells	L_2 norm	L_∞ norm
50	2.163E-3	8.853E-3
100	7.604E-4	3.714E-4
200	2.345E-4	1.452E-3
400	6.797E-5	5.414E-4
800	1.904E-5	1.954E-4
1600	5.211E-6	6.883E-5
3200	1.401E-6	2.375E-5

The modification to the van Albada limiter immediately suggests a modification for the limiter of Barth and Jespersen. While the modification to the van Albada limiter is smooth in suppressing the action of the limiter in the near-constant regions, the proposed modification is nonsmooth. This modification eliminates the action of the limiter whenever the absolute values of the numerator and the denominator in Eqn. (4) are less than a threshold, which is set equal to $(K\Delta)^{1.5}$. This modification will be referred to as *conditional threshold*. Table 2 shows that the modified scheme with $K = 0.3$ exhibits second order accuracy in both L_2 and L_∞ norms as the grid is refined. The second order accuracy does come at a cost; the solution possesses a few negative values.

Table 2: L_2 and L_∞ norms of errors of the base scheme with and without conditional threshold.

No. of cells	No threshold		Threshold	
	L_2 norm	L_∞ norm	L_2 norm	L_∞ norm
50	1.203E-3	4.916E-3	1.205E-3	4.862E-3
100	3.446E-4	1.752E-4	3.257E-4	1.613E-4
200	1.068E-4	6.578E-3	9.670E-5	6.153E-4
400	3.220E-5	2.894E-4	2.261E-5	1.633E-4
800	9.777E-6	1.153E-4	4.745E-6	4.117E-5
1600	2.984E-6	4.768E-5	9.033E-7	2.699E-6
3200	9.162E-7	1.993E-5	2.231E-7	6.354E-7

The nonsmooth modification given above for the base scheme, while restoring second order accuracy for the advection problem, is unsuitable for obtaining steady state solutions in multi-dimensions. As will be demonstrated in the next section, convergence stalls with the base scheme after a few orders of magnitude reduction in the residual when marching to steady state. When solving nonlinear problems to steady state, the smoothness of the limiter is an important consideration. The conditional threshold modification only makes matters worse in this respect. The modified van Albada limiter, on the other hand, handles the oscillations in the near-constant regions in a smooth manner and has been shown to aid in the convergence to steady state [5]. It is desirable to devise a

modification to the base scheme that handles these oscillations likewise in a smooth manner. Such a modification is also expected to improve convergence when computing steady state solutions on unstructured grids while maintaining second order accuracy.

The base scheme in the form given by Eqns. (1-4) needs to be modified. It is this form, given by Barth and Jespersen, that is suitable for unstructured grids. Therefore, we seek an approximation $\phi(y)$ to the function $\text{Min}(1, y)$. Eqn. (19) then becomes

$$\psi(r) = \frac{1}{2}(r+1) \text{Min} \left[\phi \left(\frac{4r}{r+1} \right), \phi \left(\frac{4}{r+1} \right) \right] \quad (27)$$

It is imperative that this new function $\psi(r)$ be completely contained under the original curve shown in Figure 2 for the base scheme. Otherwise, the monotonicity requirements are violated and oscillations are amplified. Since a modification similar to that for the van Albada limiter is desired, $\phi(y)$ is taken to be a ratio of second degree polynomials. The condition $\psi(1) = 1$ for second order accuracy implies $\phi(2) = 1$. In addition, we stipulate that the curve for $\phi(y)$ be contained under the curve $z(y) = y$. This condition is required in order for $\psi(r)$ to be contained under the curve for the original limiter. It is further required that $\phi(\infty) = 1$ and $\phi(0) = 0$. The function ϕ satisfying these conditions is given by

$$\phi(y) = \frac{y^2 + 2y}{y^2 + y + 2} \quad (28)$$

The functions $\phi(y)$ and $\text{Min}(1, y)$ intersect at $y = 2$ and $\phi(y) > 1, \forall y > 2$. However, this is of no consequence for the function $\psi(r)$ which is given by

$$\psi(r) = \frac{1}{2}(r+1) \text{Min} \left[\frac{4r(3r+1)}{11r^2 + 4r + 1}, \frac{4(r+3)}{r^2 + 4r + 11} \right] \quad (29)$$

The new limiter and the old limiter are displayed in Figure 5. The curve for the new limiter is completely contained under that for the original limiter. The new limiter has a slope discontinuity at $r = 1$.

The new limiter can easily be modified to handle oscillations in the near-constant regions of the solution in a smooth manner. In the original scheme given by Eqns. (1-4), the key step is the determination of the function $\Phi_{i+1/2}$. With the function $\text{Min}(1, y)$ approximated by $\phi(y)$, Eqn. (4) becomes

$$\Phi_{i+1/2} = \begin{cases} \phi \left(\frac{u_i^{\max} - u_i}{U(x_{i+1/2}) - u_i} \right), & \text{if } U(x_{i+1/2}) - u_i > 0 \\ \phi \left(\frac{u_i^{\min} - u_i}{U(x_{i+1/2}) - u_i} \right), & \text{if } U(x_{i+1/2}) - u_i < 0 \\ 1, & \text{if } U(x_{i+1/2}) - u_i = 0 \end{cases} \quad (30)$$

Without loss of generality assume $U(x_{i+1/2}) - u_i > 0$ since the other cases can be handled in a similar manner. Letting $\Delta_- = U(x_{i+1/2}) - u_i$ and $\Delta_+ = u_i^{\max} - u_i$,

$$\begin{aligned} \Phi_{i+1/2} &= \phi \left(\frac{\Delta_+}{\Delta_-} \right) \\ &= \left(\frac{1}{\Delta_-} \right) \Delta_- \phi \left(\frac{\Delta_+}{\Delta_-} \right) \end{aligned}$$

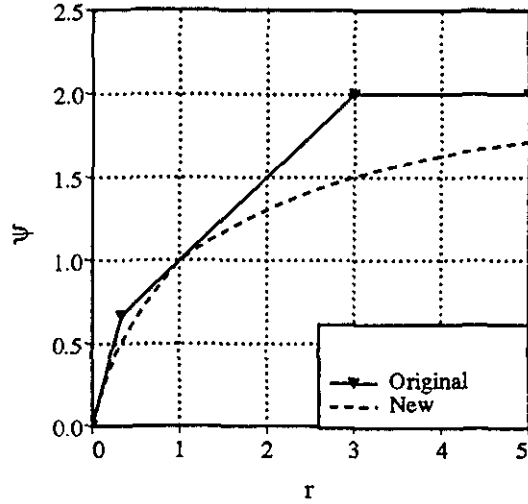


Figure 5: The new limiter and the old limiter.

Akin to the modification for the van Albada limiter, the term $\Delta_- \phi \left(\frac{\Delta_+}{\Delta_-} \right)$ is modified. With $\phi(y)$ defined by Eqn. (28), $\Phi_{i+1/2}$ becomes

$$\Phi_{i+1/2} = \frac{1}{\Delta_-} \left[\frac{(\Delta_+^2 + \epsilon^2)\Delta_- + 2\Delta_-^2\Delta_+}{\Delta_+^2 + 2\Delta_-^2 + \Delta_-\Delta_+ + \epsilon^2} \right] \quad (31)$$

Once again ϵ^2 is taken to be $(K\Delta x)^3$. In the smooth regions of the solution $\Delta_-, \Delta_+ \sim \Delta x$. In the near-constant regions ϵ^2 dominates Δ_-^2, Δ_+^2 and $\Delta_-\Delta_+$, and $\Phi_{i+1/2}$ reduces to 1. Thus the scheme reverts to the scheme without limiting in the smooth regions. As a practical matter, to avoid division by a very small value of Δ_- in Eqn. (31), Δ_- should be replaced by $\text{Sign}(\Delta_-)(|\Delta_-| + w)$ where w is taken to be 10^{-12} for 64-bit arithmetic computations. The results from using the scheme with $K = 0.3$ for the problem of the advection of the Gaussian distribution are shown in Table 3. The table demonstrates that the new scheme still maintains second order accuracy in the L_2 norm. In comparing Tables 2 and 3, it may be observed that the errors are higher with the new limiter in comparison with old limiter with conditional threshold. This is because the new limiter is more diffusive which is reflected in Figure 5.

In summary, there appears to be conflict between achieving second order accuracy and true monotone solutions. With the schemes that employ limiters in their original forms, the advected Gaussian distributions are truly nonoscillatory and nonnegative everywhere (as is the initial distribution), but the methods are less than second order accurate. The modified schemes, on the other hand, achieve second order accuracy, but the advected distributions possess a few negative values, whose magnitudes decrease with grid refinement.

Table 3: L_2 and L_∞ norms of errors with the new limiter.

No. of cells	L_2 norm	L_∞ norm
50	1.512E-3	6.150E-3
100	4.668E-4	2.312E-4
200	1.333E-4	8.213E-3
400	3.597E-5	2.792E-4
800	9.150E-6	9.113E-5
1600	2.254E-6	2.897E-5
3200	5.389E-7	8.908E-6

4 Limiters and convergence

It has long been known that the use of limiters can stall convergence to steady state solutions. In this section the problem of obtaining converged solutions to supersonic flows is addressed. While in the case of flows with extremely weak shocks it may be possible to obtain solutions without using limiters, for flows with strong shocks limiters are absolutely essential. Barth and Jespersen [4] demonstrated the effectiveness of their multi-dimensional limiter by computing oscillation-free transonic flow solutions on highly irregular triangular meshes. They do not address the issue of convergence in their paper, however.

We use the spatial discretization of Barth and Jespersen and use an implicit scheme similar to that used by Venkatakrishnan and Mavriplis [9] for temporal discretization. A sparse linear system arises at each time step from an approximate linearization of the nonlinear problem. This sparse linear system is solved by using the GMRES algorithm [10] with incomplete LU factorization as the preconditioner. Since the implicit side is only discretized to first order accuracy, it does not involve the limiter.

Supersonic flow over an NACA0012 airfoil at a free-stream Mach number of 2 and 0° angle of attack is first considered. The mesh has 4224 vertices and the far field boundary is located 50 chords away from the airfoil. A limiter is essential to compute this flow; in its absence, the solution procedure becomes unstable after a few iterations. The CFL number is allowed to vary between 1 and 100 inversely proportional to the L_2 norm of the residual. Figure 6 shows the convergence histories with the scheme of Barth and Jespersen and with a first order accurate scheme. While the first order accurate scheme converges rapidly, the second order accurate scheme stalls after about two orders of magnitude reduction in the residual. Even though the higher order scheme has not converged, the solution obtained is quite good and is devoid of oscillations. Figure 7 shows the Mach contours using the base scheme.

We believe the limiter of Barth and Jespersen is active in the near-constant regions of the flow and responds to oscillations at the noise level. This effectively reduces the radius of convergence of the quasi-Newton methods and is responsible for the convergence stall. Venkatakrishnan and Barth [2] found that the radius of convergence

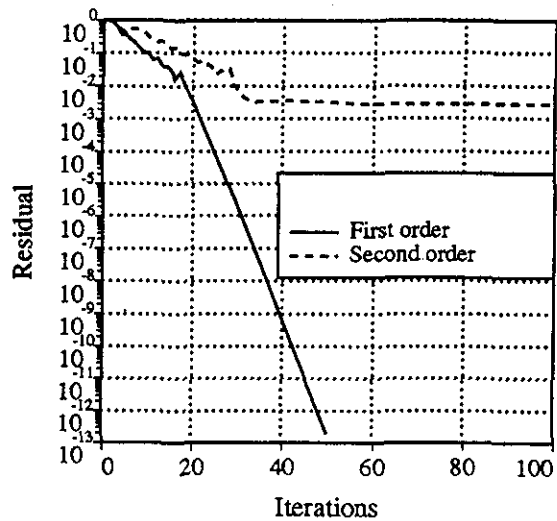


Figure 6: Convergence histories obtained with the first order and the second order accurate schemes for the supersonic flow case.

was indeed very small when limiters are employed in the context of a fully-implicit Newton's method.

The new limiter given by Eqn. (31) is expected to ameliorate convergence. On nonuniform triangular grids we take $\epsilon^2 = (K\bar{\Delta})^3$ where $\bar{\Delta}$ is an average grid size and K is a constant. $\bar{\Delta}$ is taken to be the length of an equilateral triangle in an equivalent uniform triangulation with the same number of triangles covering the same physical area as the original grid. K is a parameter and signifies a threshold. Oscillations below this threshold are allowed to exist in the solution and are not dealt with by the limiter. A value of zero implies that the limiter is active even in the near-constant regions, whereas a very high value for K implies effectively no limiting at all. While setting K to be a large value is acceptable for subcritical flows, for flows with discontinuities this will cause the solution procedure to become unstable. Thus, for solutions with discontinuities, as K is increased from 0, the convergence of the scheme will improve until the solution procedure becomes unstable for very large values of K . Figure 8 shows the convergence histories obtained with the modified scheme for the supersonic flow case with K set to 0.1 and 0.3. Notice the improvement in convergence as the value of K is increased. Figure 9 shows the Mach contours of the steady state solution with the K set to 0.3. Also shown in Figure 8 is the convergence history obtained with the base scheme with the limiter modified using conditional threshold. Here ϵ is set to $(0.3\bar{\Delta})^{1.5}$. Convergence worsens in comparison with the base scheme since the modification to the limiter is not smooth.

To demonstrate that the phenomenon of convergence stall is independent of the time discretization, an explicit scheme is also used solve the supersonic flow problem. A four-stage Runge-Kutta scheme is used for this pur-

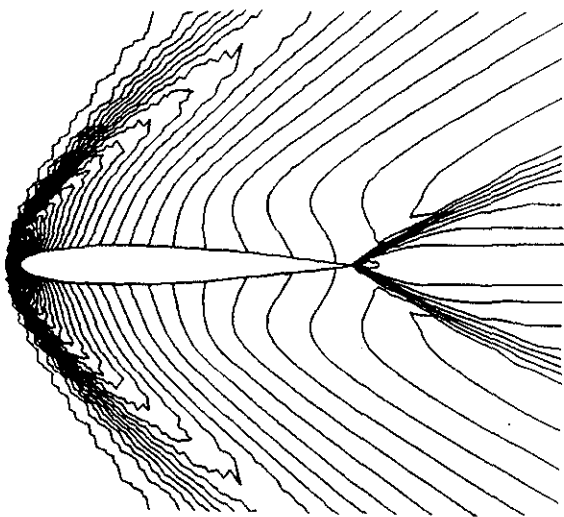


Figure 7: Mach contours obtained with the second order accurate base scheme.

pose with a CFL number of 1.2. Figure 10 shows the convergence histories obtained with the base scheme and with the new limiter. Once again convergence stalls after about four orders of magnitude reduction in the residual with the original limiter, whereas it is markedly improved with the new limiter ($K = 0.3$). Also shown in Figure 10 is the convergence history with the original limiter modified using a conditional threshold, with ϵ set to $(0.3\bar{\Delta})^{1.5}$. Convergence deteriorates again in comparison with the base scheme. It may also come as a surprise that the original scheme achieves nearly four orders of reduction in the residual with the explicit scheme, whereas it stalled after only about two orders with the implicit scheme. A plausible explanation for this is that for a rapidly varying nonlinear function (as is the case with the original limiter), the explicit scheme takes much smaller steps and is able to respond to the function better than the implicit scheme which takes larger steps.

The phenomenon of convergence slowdown also occurs in subcritical flows when limiters are used. Here limiters are not essential to obtain solutions and the effects of the limiters, if used, should be benign. We consider flow over a four-element airfoil in landing configuration at a freestream Mach number of 0.2 and angle of attack of 5° . The triangular mesh has 6019 vertices. Figure 11 shows the convergence histories without the limiter, with the original limiter and with the modified limiter. Convergence stalls with the original limiter, whereas convergence with the new limiter is comparable to the scheme without limiting. Convergence is only shown with the parameter K set to 0.1. Convergence with $K = 0.3$ is identical to that without limiting.

Rather than use a global average grid size to base the threshold for turning off the limiter, it is also possible to use the local grid size. At grid point i , ϵ_i^2 is taken

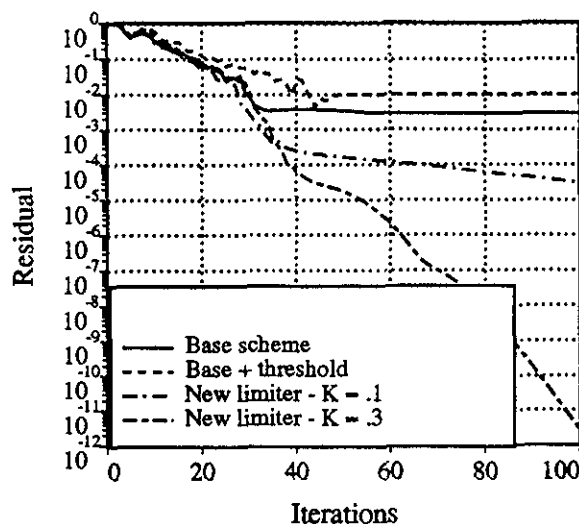


Figure 8: Convergence histories with the new limiter and the old limiter with conditional threshold modification.

to be $(K\bar{\Delta}_i)^3$, where $\bar{\Delta}_i$ is the local grid size. Since a cell-vertex scheme is used, the control volume is a polygon made up of segments of the medians of the triangles. This control volume can be approximated as a circle and $\bar{\Delta}_i$ can be taken to be the diameter of this circle. Results are presented for the case of transonic flow over an NACA0012 airfoil at a freestream Mach number of 0.8 and angle of attack of 1.25° using this approach. Figure 12 shows the convergence histories and Figure 13 displays the surface pressure profiles obtained with the base scheme and with the new limiter for two values of K . These figures clearly show the tradeoff between convergence and obtaining monotone steady state solutions. With $K = 1$, the convergence hardly improves and the pressure profile matches that of the base scheme. With $K = 5$, convergence improves dramatically, but the surface pressure on either side of the shock on the upper surface shows a discrepancy. The weaker shock on the lower surface on the other hand exhibits no such discrepancy. Figure 14 shows the lift histories with the three schemes during the latter iterations. The base scheme exhibits an odd-even mode as does the new limiter with $K = 1$, whereas the new limiter with $K = 5$ converges monotonically. However, even with the base scheme the variation is only in the fourth decimal place. Thus, when using the base scheme, it is possible to declare convergence once the mean lift coefficient no longer varies. For the subcritical flow discussed earlier, the constant K had to be increased to 100 in order to obtain convergence comparable to the scheme without limiting. Otherwise, all the trends observed when using a global threshold hold true when using a local threshold. One way to determine the value of K for use in supercritical flows is to solve a subcritical problem on the same grid and find the smallest value of K that yields convergence

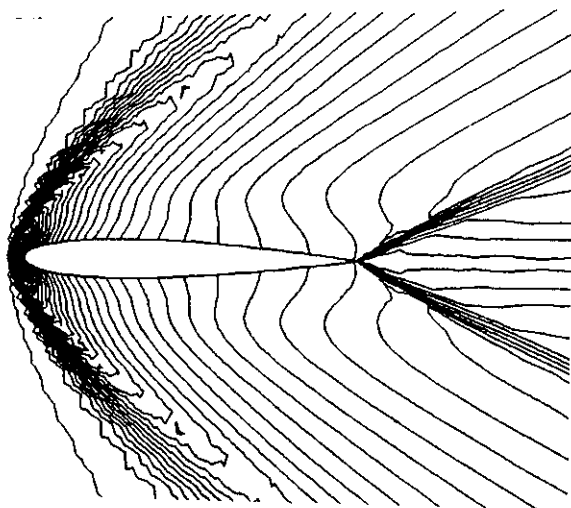


Figure 9: Mach contours obtained with the modified scheme.

comparable to the scheme without limiting.

5 Conclusions

It is shown in this paper that the limiters currently in use with upwind schemes have two drawbacks. One of these is that the formally second order accurate schemes are less than second order accurate in practice. This is demonstrated by advecting a smooth Gaussian distribution in one dimension. The second shortcoming is that the limiters inhibit convergence to steady state solutions. The reason for both these problems is the way the limiters handle oscillations in the smooth regions of the flow. Modifications are proposed to the limiter which solves both these problems. While the original scheme fails to converge for supersonic flow computations on unstructured grids, the modified scheme is shown to converge quite well. The improvement in convergence comes at a cost; the solution exhibits deviations from the monotone solution near strong shocks.

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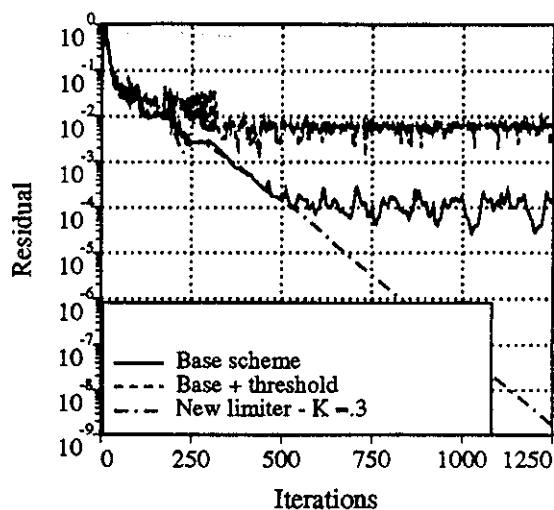


Figure 10: Convergence histories with the explicit scheme.

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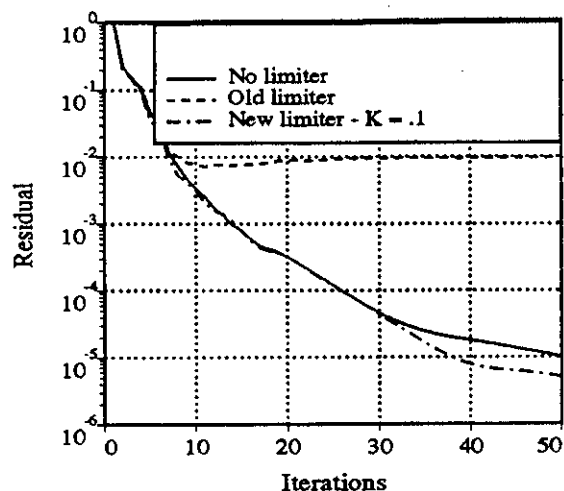


Figure 11: Convergence histories for the subcritical flow case.

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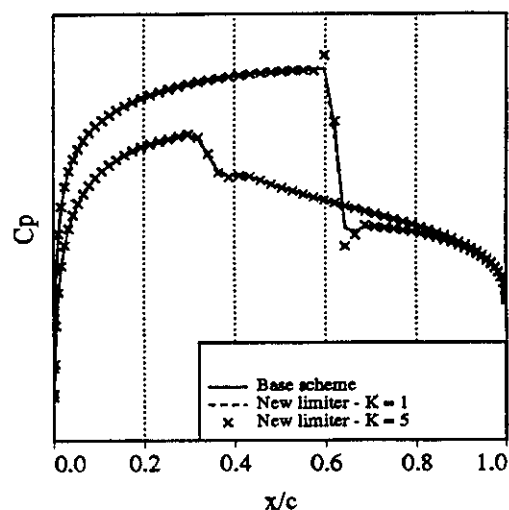


Figure 13: Surface pressure profiles for transonic flow over an NACA0012 airfoil.

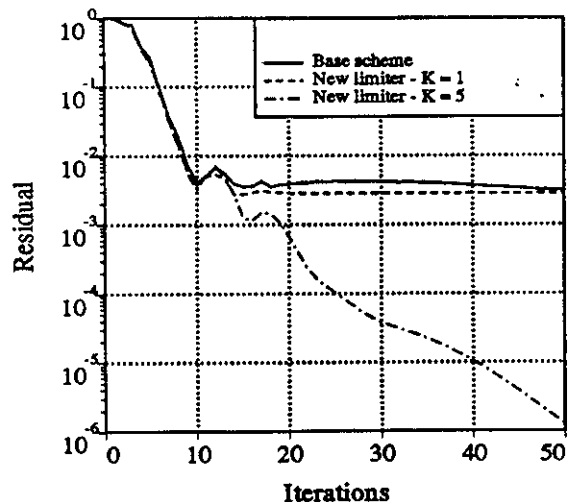


Figure 12: Convergence histories for transonic flow over an NACA0012 airfoil.

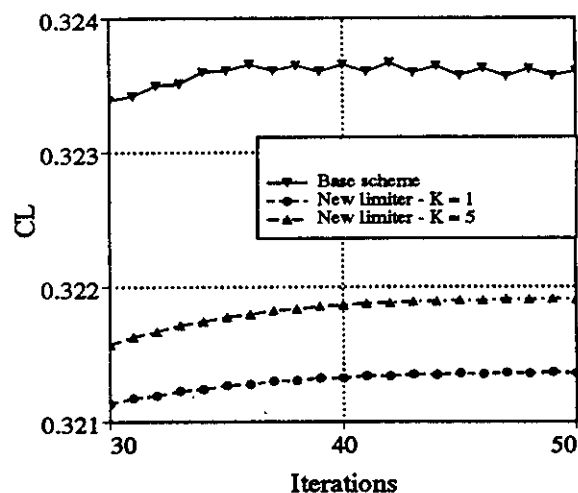


Figure 14: Lift coefficient histories for the transonic flow case.