Well-Balanced Methods for Computation of the Standing Accretion Shock Instability (SASI)

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Standing Accretion Shock Instability (SASI)

- Accretion shocks in several astrophysical phenomena
  - star formation, white dwarf/neutron stars, core-collapse supernovae
- Unlike planar shocks, geometry could trap perturbations
- Potential instability could be of interest in driving the evolution of core-collapse supernovae [Blondin2003]
- An advective-acoustic feedback cycle has been hypothesized as a candidate [Foglizzo2009, Sato2009]
  - Used an external potential to model fluid deceleration
  - Should be amenable to a well-balanced method [Käppeli2014]

Euler Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \rho \boldsymbol{v}) + \nabla p = -\rho \nabla \phi,$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[(E + p) \boldsymbol{v} \right] = -\rho \boldsymbol{v} \cdot \nabla \phi.$$

With the Energy and Equation of State (EOS)

$$E = \rho e + \frac{\rho}{2} v^2 \qquad\qquad p = p(\rho, e)$$

A More Convenient Form (1D)

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v_x \\ E \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ (E+p) v_x \end{bmatrix} \quad \text{and} \quad \boldsymbol{S} = \begin{bmatrix} 0 \\ -\rho \\ -\rho v_x \end{bmatrix} \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \xRightarrow{\text{steady state}} \quad \boxed{\frac{\partial \boldsymbol{F}}{\partial x} \stackrel{?}{=} \boldsymbol{S}}$$

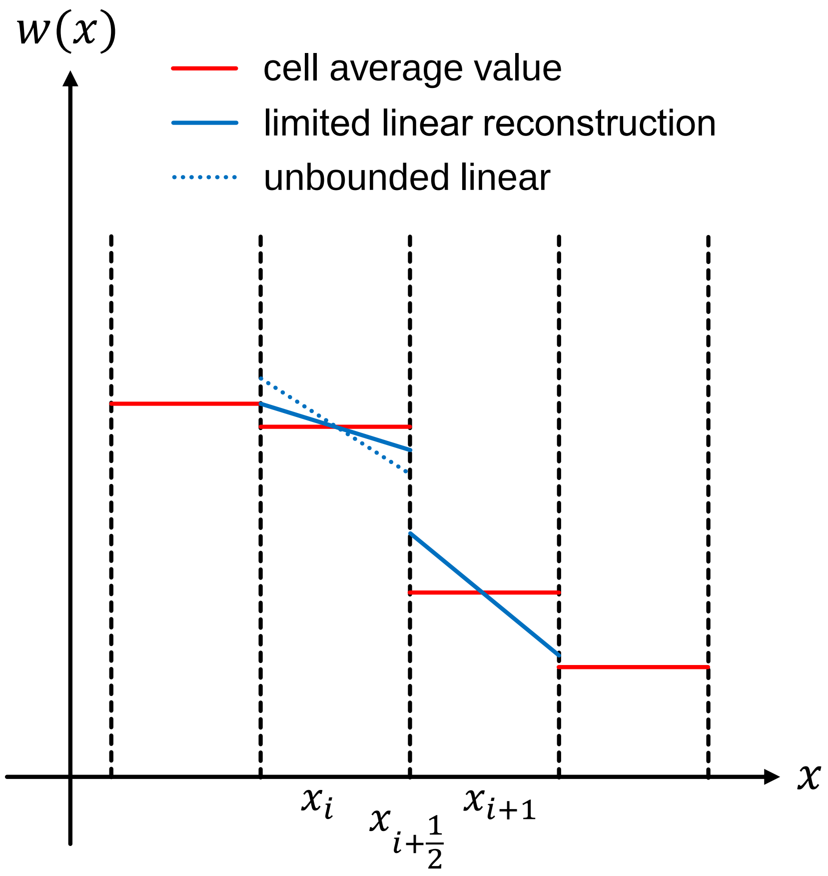
With the primitive vector and ideal gas EOS

$$\boldsymbol{w} = [\rho, v_x, p]^T \qquad p = \rho e(\gamma - 1)$$

Numerical Method

- Finite Volume Method (FVM)
- Solve for cell averages
  - Reconstruct to faces
  - Solve Riemann problems

- However, we desire a better reconstruction for steady states



Steady States

Weak solutions; require *entropy condition* for uniqueness.

For our case, *isentropic flow* → *Bernoulli constant*

$$\frac{\mathrm{v}^2}{2} + \phi + h = \text{constant} \equiv b$$

With specific enthalpy

$$h = e + \frac{p}{\rho}$$

Use this constant to compute the reconstruction

Well-Balanced Scheme

A few additional constants

$$\rho \boldsymbol{v} \equiv m = \text{const} \qquad p = p(K, \rho) = K\rho^{\gamma}$$

Recast Bernoulli constant as function of just one unknown

$$\frac{1}{2}\frac{m^2}{\rho(x)^2} + \frac{\gamma}{\gamma - 1} K\rho(x)^{\gamma-1} + \phi(x) = b$$

Iteratively solve for $\rho_{0,i}^{n}(x)$ at the cell interfaces, then compute other primitives

Discretization of the Source Terms

$$S^n_{\rho v,i} = \frac{\left(\rho^n_{0,i}\left(v^n_{0,i}\right)^2 + p^n_{0,i}\right)\Big|^{x_{i+1/2}}_{x_{i-1/2}}}{\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + O(\Delta x^2)$$

$$S^n_{E,i} = -\rho v^n_{x,i} \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho v_x \frac{\partial \phi}{\partial x} \mathrm{d}x + O\left(\Delta x^2\right)$$

$$\boldsymbol{S}^n_i = \begin{bmatrix} 0 \\ S^n_{\rho v,i} \\ S^n_{E,i} \end{bmatrix}$$

Implementation

- foam-extend-4.0 fork of OpenFOAM
  - dbnsFoam solver
  - Several limiters and Riemann solvers already implemented
- Written in C++
- Provides a lot of general field manipulation capability
- Uses a very general FVM mesh description
  - Few restrictions on number and orientation of cell faces
- Runtime selection of many parameters

- But it’s large and complicated (>1 000 000 lines of code!)

Time-Marching Scheme

$$\mathbf{U}_t = \mathbf{R}(\mathbf{U}) \qquad i = 1, \ldots, 4$$

$$\mathbf{U}^{(0)} \equiv \mathbf{U}^n,$$

$$\mathbf{U}^{(i)} = \mathbf{U}^{(0)} + \beta_i \Delta t \mathbf{R}\left(\mathbf{U}^{(i-1)}\right),$$

$$\mathbf{U}^{n+1} \equiv \mathbf{U}^{(4)},$$

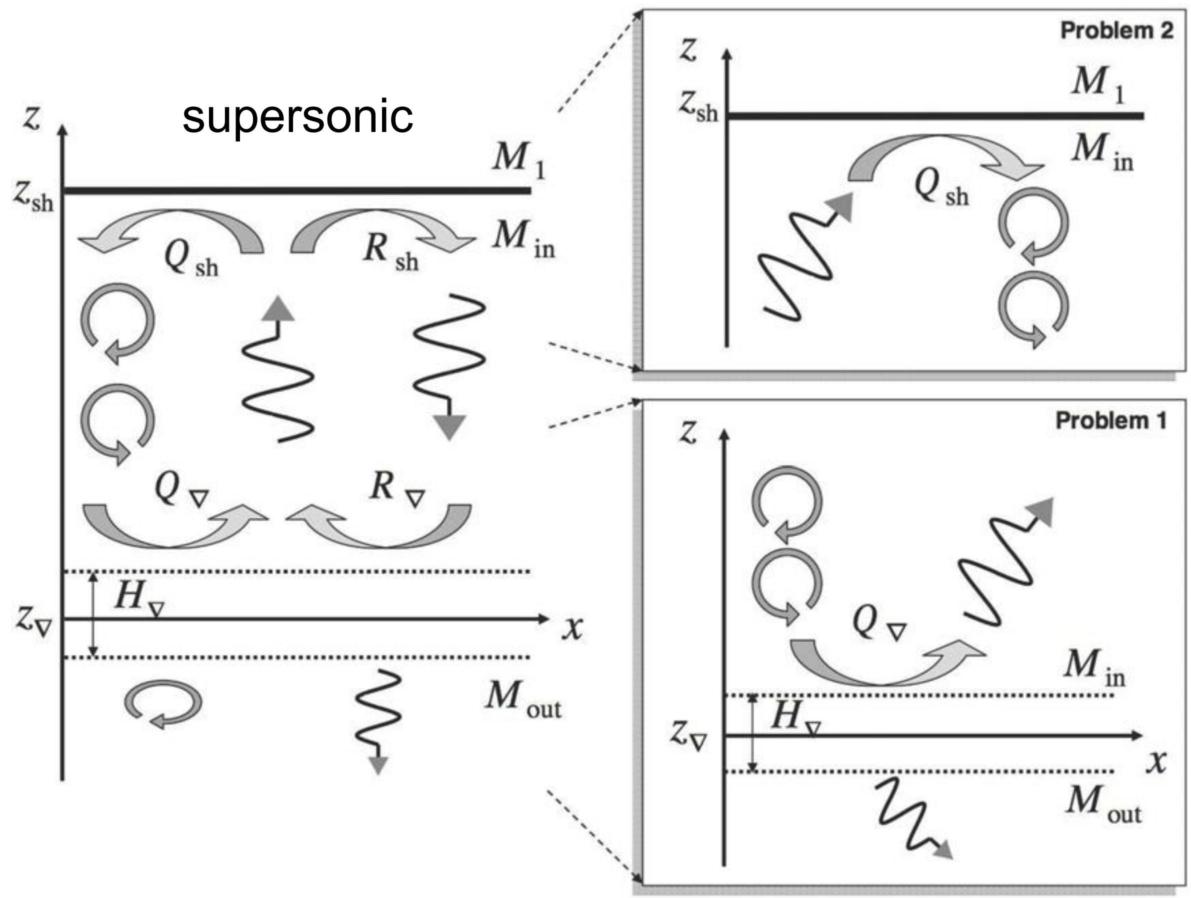
$$\beta_1 = 0.11, \quad \beta_2 = 0.2766, \quad \beta_3 = 0.5, \quad \beta_4 = 1$$

Low storage Runge-Kutta: 4 stages, 2nd order accurate, maximizes CFL number [Lallemand1990] and strong stability preserving (SSP) [Shu1988]

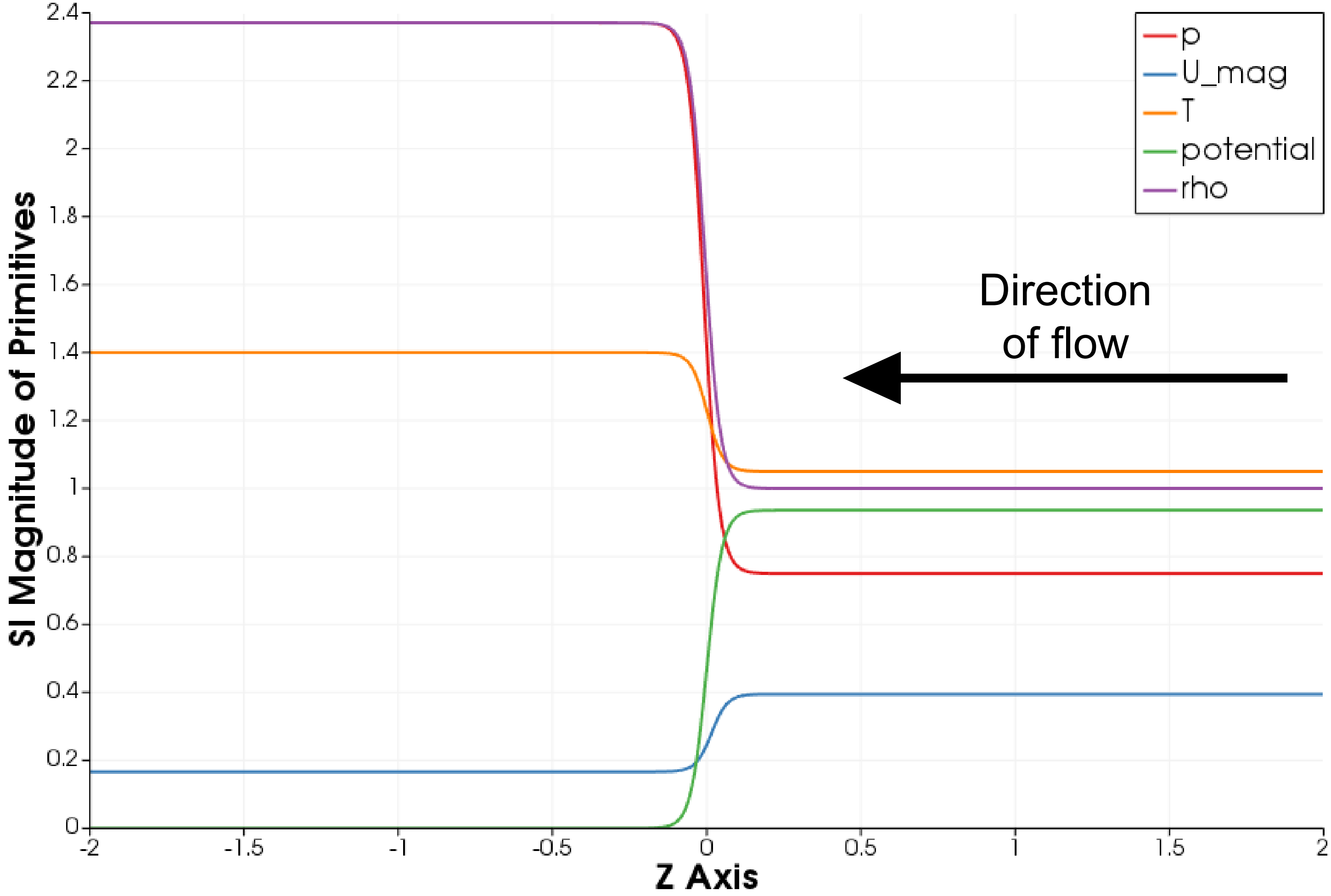
Test Problem

[image:Sato2009]

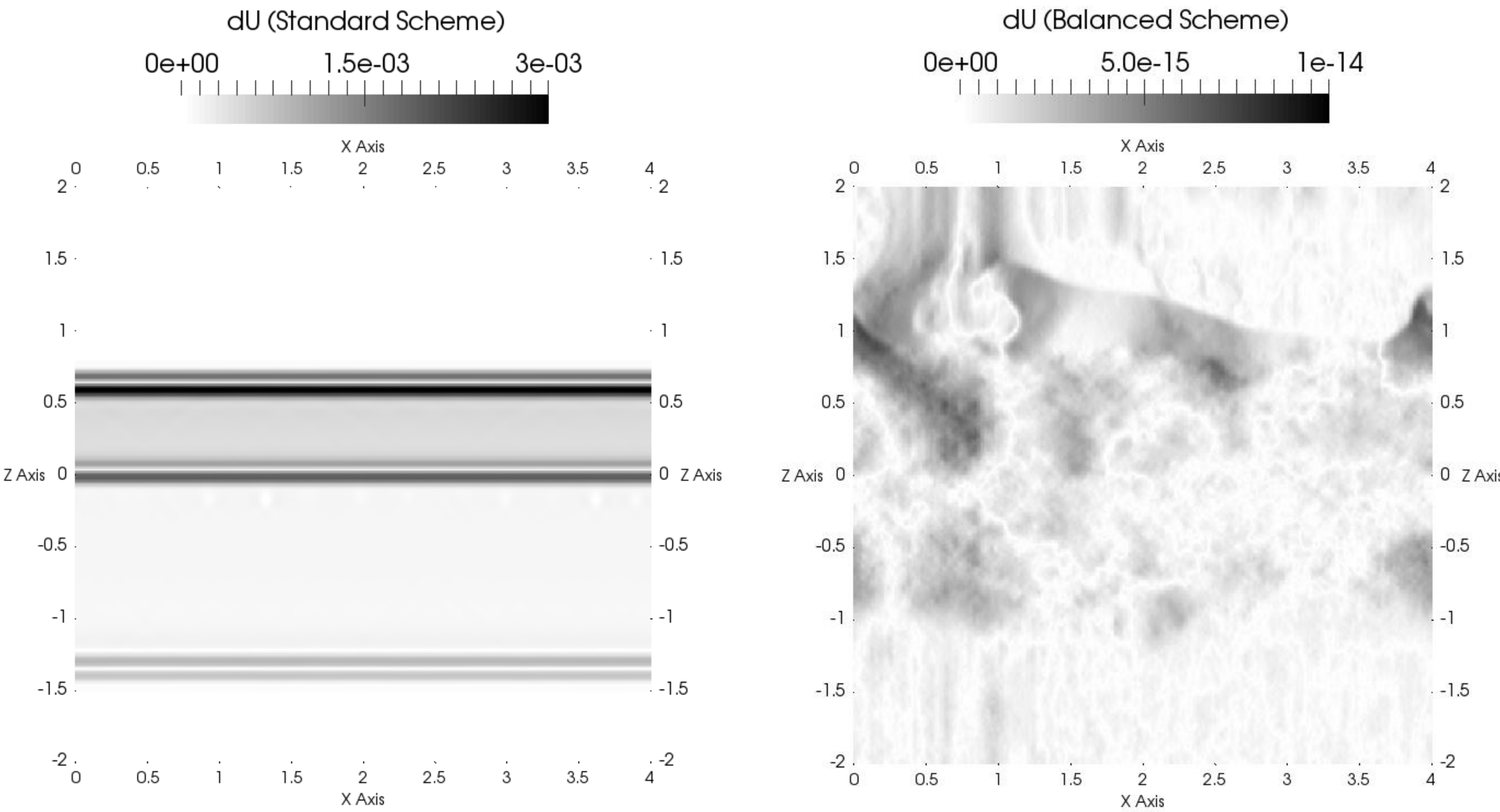
Direction
of flow



Sub Problem 1 – Initial Conditions

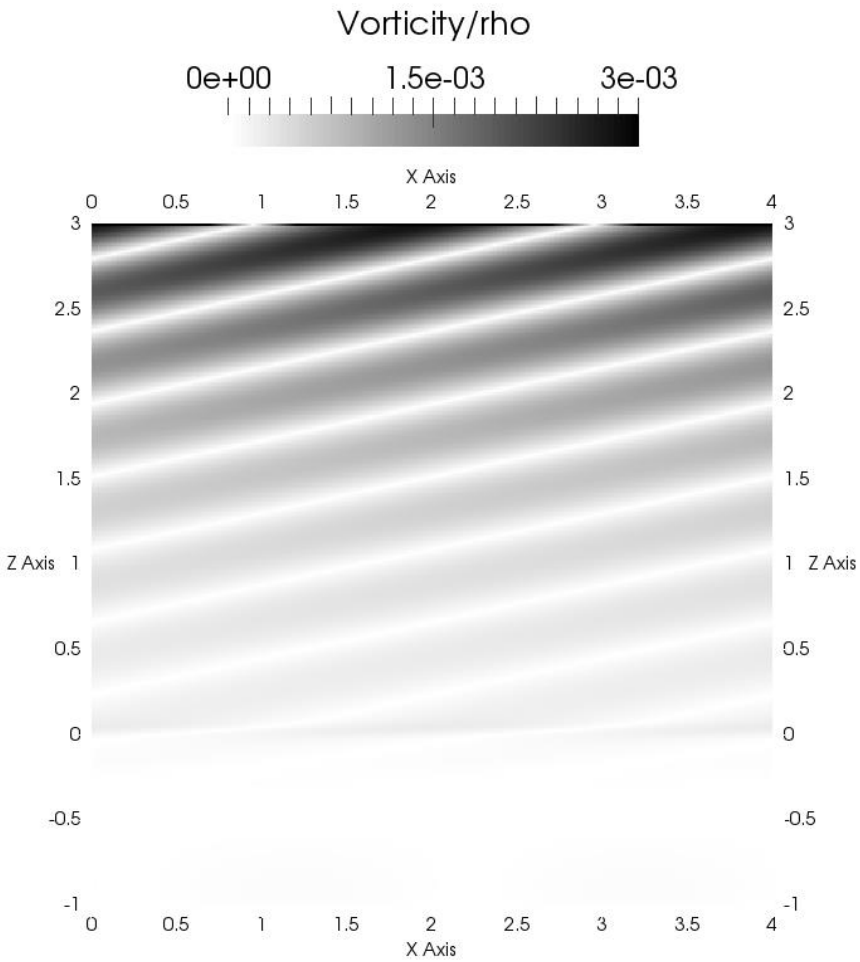


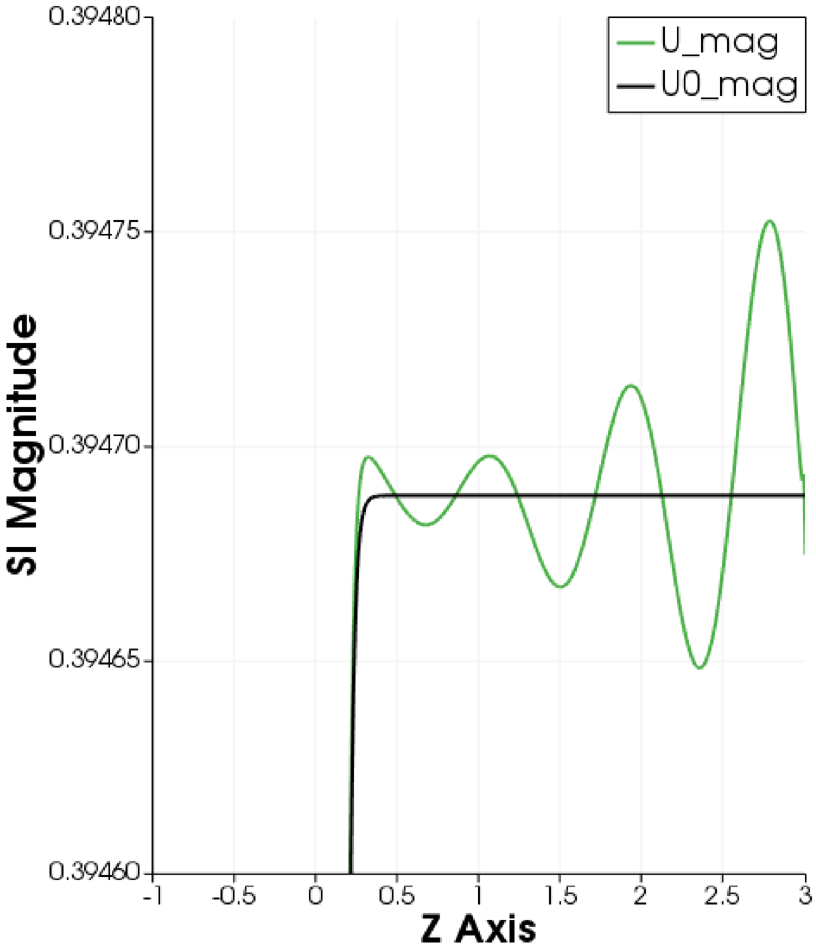
Sub Problem 1 – Equilibrium $t = 1, \quad dt = 0.001$



Sub Problem 1 – Perturbed

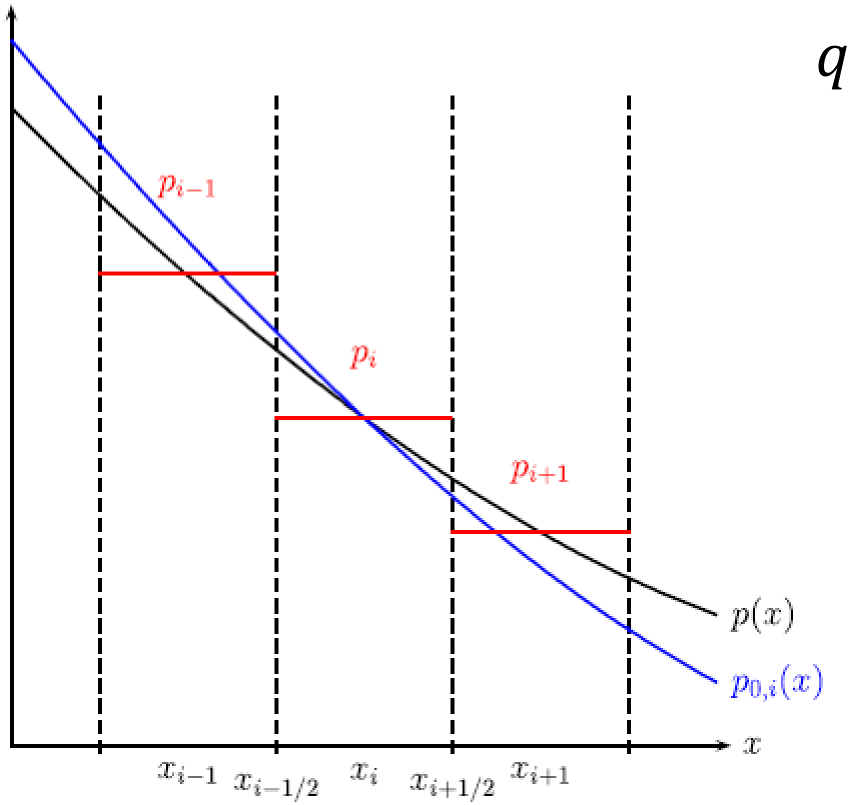
$t = 13.14, \quad dt = 0.001$





Second Order Extension

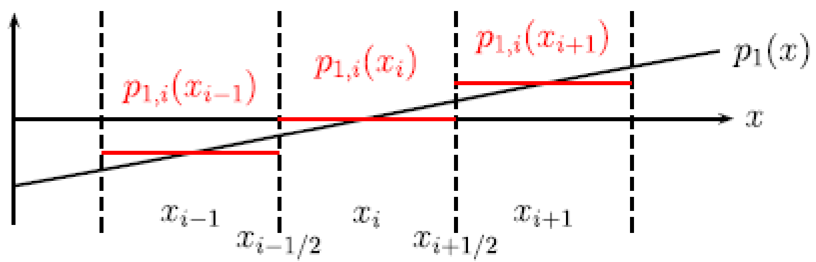
[images:Käppeli2014]



$$q_i(x) = q_{0,i}(x) + Dq_{1,i}(x - x_i)$$

$$q_{1,i}(x) = q(x) - q_{0,i}(x)$$

$$q_{1,i}(x_i) = q_i - q_{0,i}(x_i) = 0$$

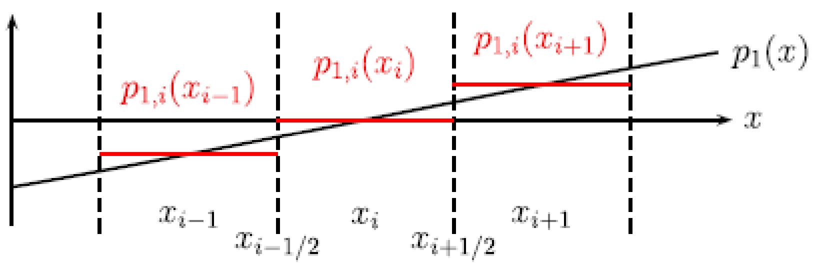


Second Order Extension

$$Dq_{1,i} = \text{limiter}\left(\frac{q_{1,i}(x_i) - q_{1,i}(x_{i-1})}{\Delta x}, \frac{q_{1,i}(x_{i+1}) - q_{1,i}(x_i)}{\Delta x}\right)$$

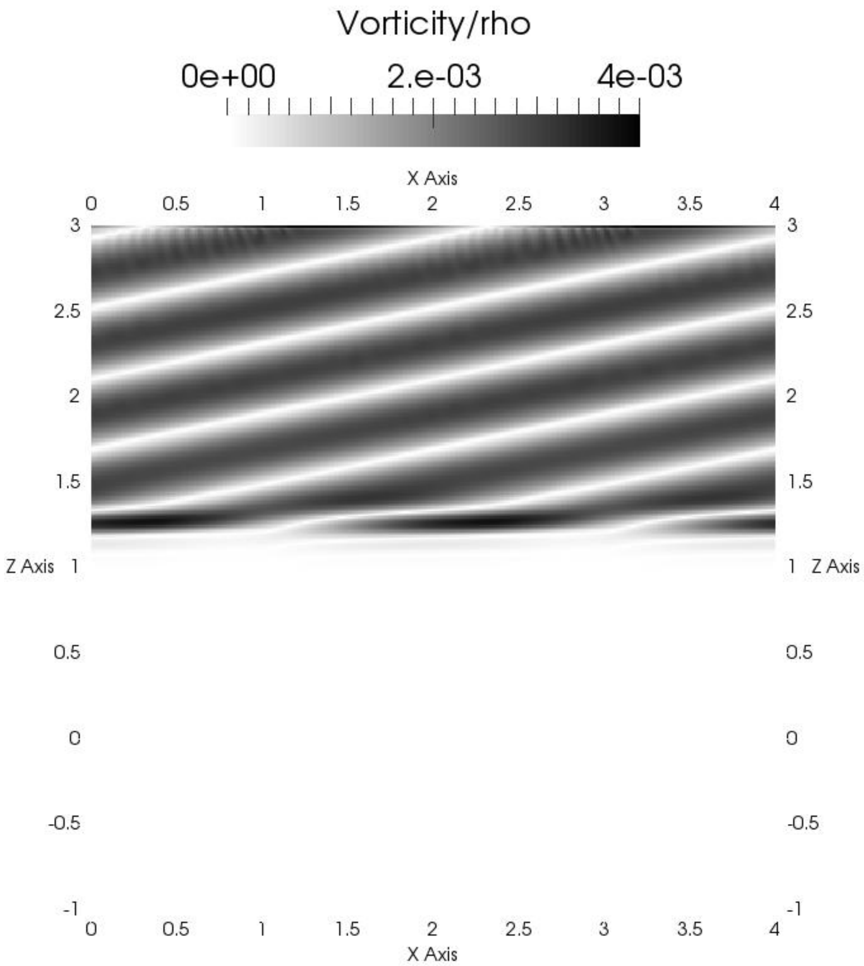
$$q_{1,i}(x_{i-1}) = q_{i-1} - q_{0,i}(x_{i-1}), \qquad q_{1,i}(x_{i+1}) = q_{i+1} - q_{0,i}(x_{i+1})$$

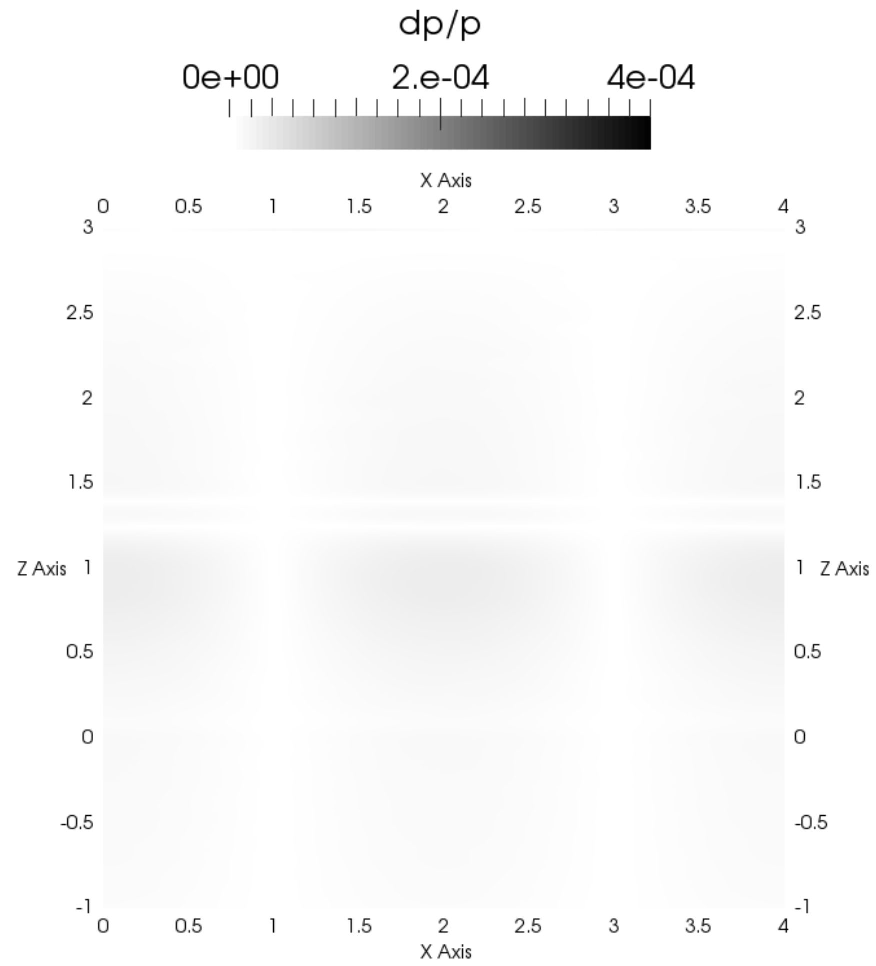
$$q_i(x) = q_{0,i}(x) + \text{limiter}\left(-\frac{q_{1,i}(x_{i-1})}{\Delta x}, \frac{q_{1,i}(x_{i+1})}{\Delta x}\right)(x - x_i)$$



Sub Problem 1 – 2nd Order

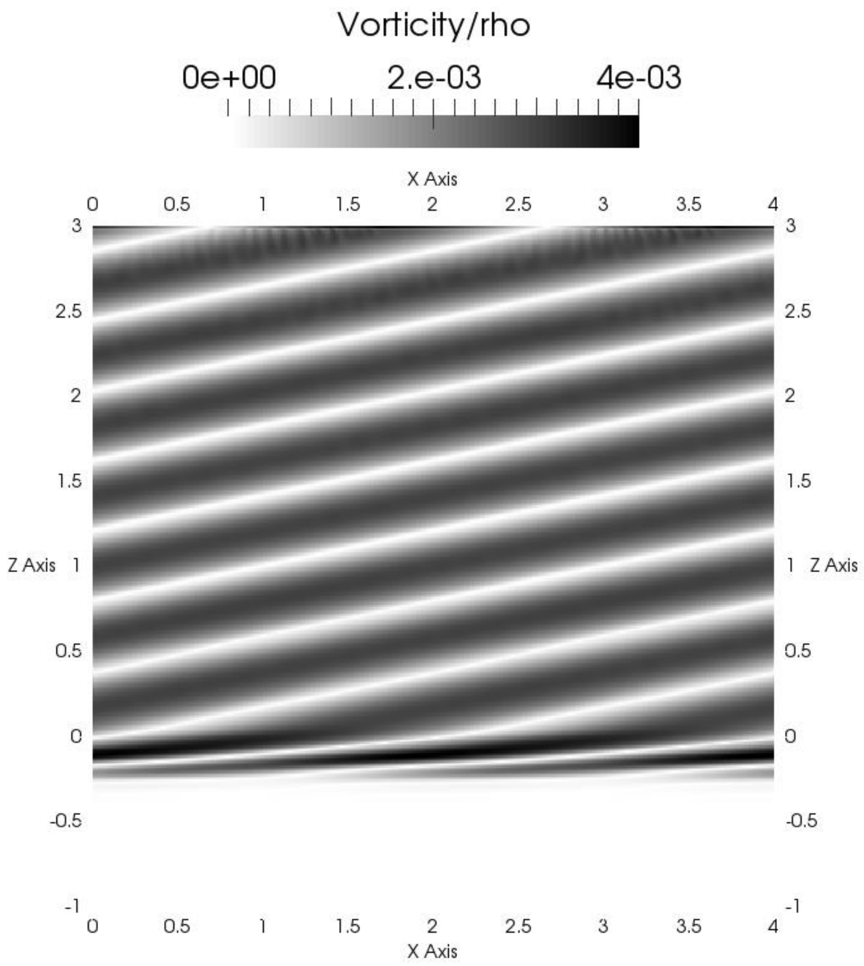
$t = 4.368, \quad dt = 0.001$

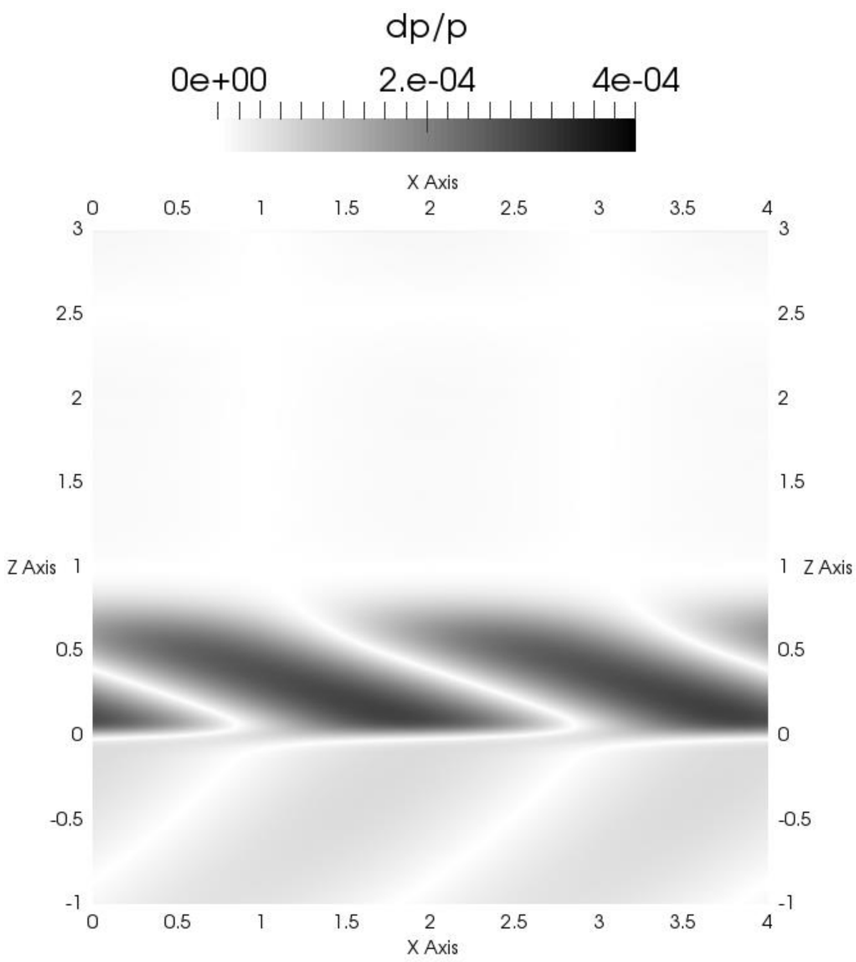




Sub Problem 1 – 2nd Order

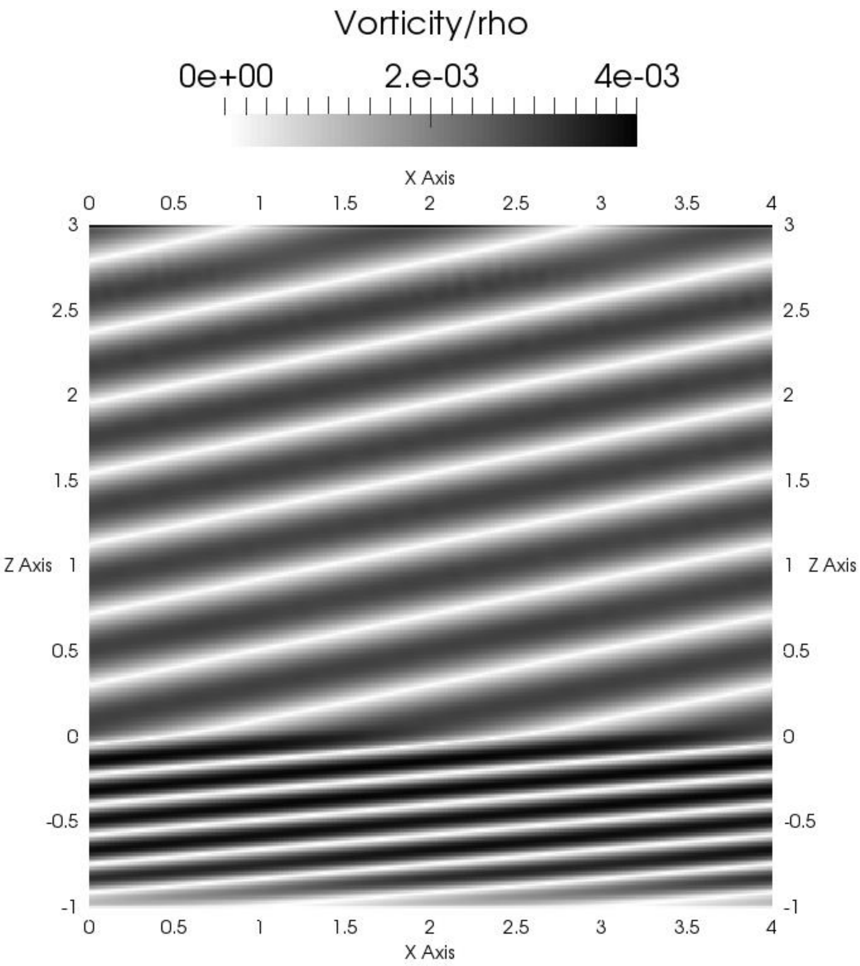
$t = 8.736, \quad dt = 0.001$

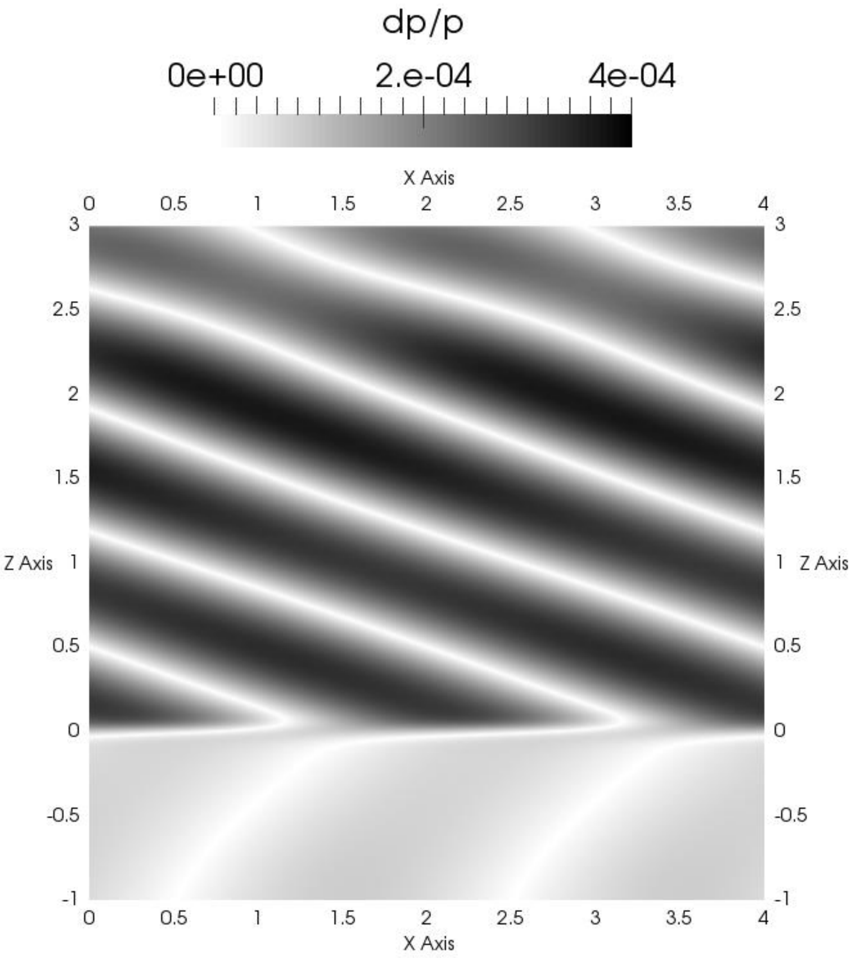




Sub Problem 1 – 2nd Order

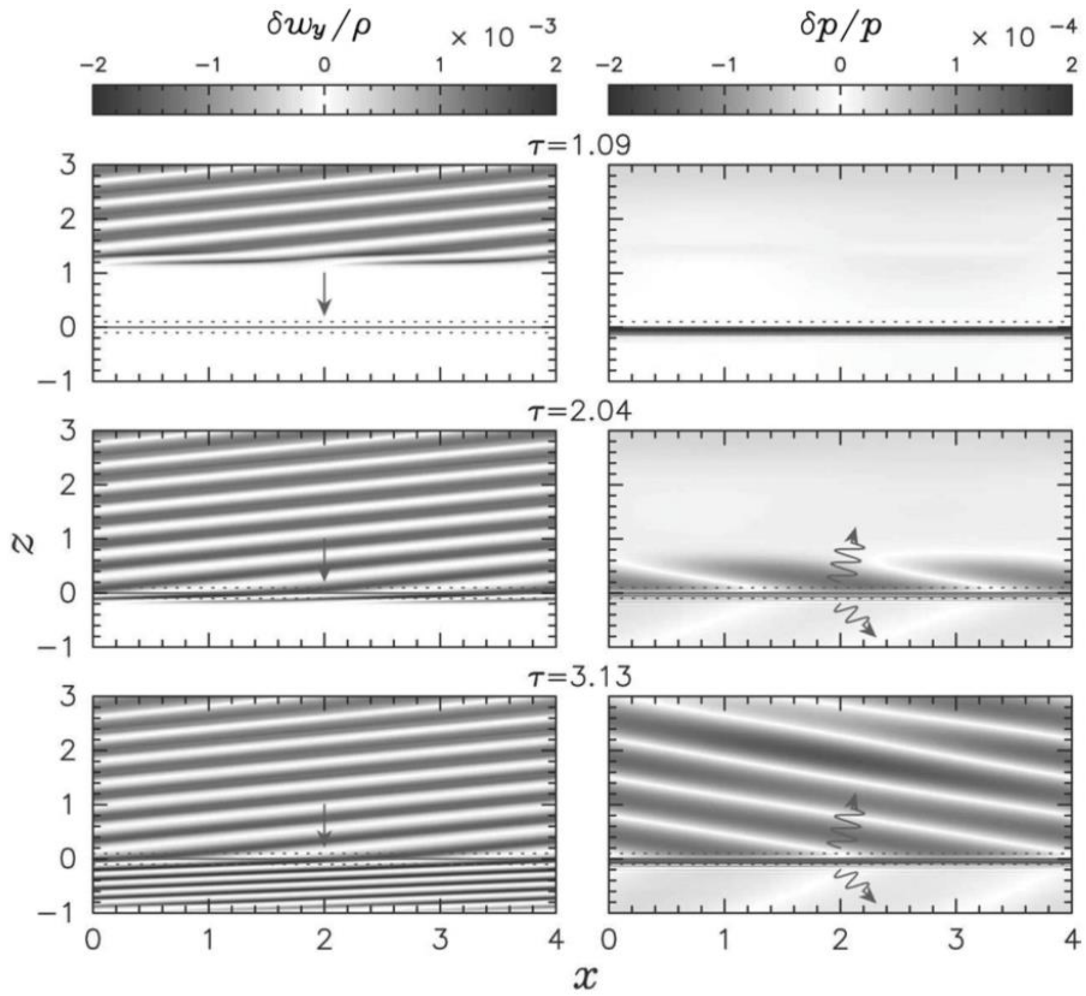
$t = 13.14, \quad dt = 0.001$





Comparison to Paper Results

[image:Sato2009]



Next Steps

- Convergence study
- Clean up code
  - Improve use of OpenFOAM's generalizability
  - Separate as stand alone library plug-in
  - Possible optimizations
- Simulate the full test problem
  - Attempt on an annular geometry if time permits

- And of course, write it all down! ☺

References

1. J. M. Blondin, A. Mezzacappa, C. DeMarino. Stability of standing accretion shocks, with an eye toward core-collapse supernovae. *The Astrophysical Journal*, 584(2):971–980, February 2003.
2. T. Foglizzo. A simple toy model of the advective-acoustic instability. I. Perturbative approach. *The Astrophysical Journal*, 694(2):820–832, March 2009.
3. J. Sato, T. Foglizzo, S. Fromang. A simple toy model of the advective-acoustic instability. II. Numerical simulations. *The Astrophysical Journal*, 694(2):833–841, March 2009.
4. R. Käppeli, S. Mishra. Well-balanced schemes for the Euler equations with gravitation. *Journal of Computational Physics*, 259:199–219, February 2014.
5. M. Lallemand. *Dissipative Properties of Runge-Kutta Schemes with Upwind Spatial Approximation for the Euler Equations*. PhD thesis, Rapport de Recherche 1179, INRIA Sophia-Antipolis, Valbonne, January 1990.
6. Chi-Wang Shu and Stanley Osher. Efficient implementation of essentially non-oscillatory shock-capturing schemes. *Journal of Computational Physics*, 77(2):439–471, August 1988.

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Thank you for listening!

Questions?