A Comparison of Numerical Schemes on Triangular and Quadrilateral Meshes

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Introduction

Finite volume solution procedures for Euler equations have been extensively studied when applied to quadrilateral meshes. Many methods such as mesh adaptation and multi-grid have been developed which make these procedures efficient and highly accurate. Recently, great interest has been focussed on the development and use of unstructured triangular meshes. One of the advantages of these meshes is that they offer great flexibility in the generation of meshes around extremely complex geometries. The other advantage is that in adaptive mesh refinement the mesh always remains an unstructured triangular mesh, and so no modifications of the basic flow algorithm are required. However, a certain amount of doubt exists in the computational fluid dynamics community concerning the performance of triangular mesh solution schemes; in particular a recent paper by Roe [1] proved that the local truncation error is only first order on very irregular meshes. Giles [2] argues, however, that these triangular schemes can still be globally second order accurate.

The goal of this study is to address this question of how well a given computational method can perform on a triangular mesh as compared to the more commonly used quadrilateral meshes. In particular, two node-based finite volume schemes will be examined: the node-based quadrilateral cell Jameson scheme [3] which has been modified for triangular meshes by Mavriplis and Jameson [4], and the quadrilateral cell Lax-Wendroff method developed by Ni [5] which has been modified by Lindquist for use on triangular meshes [6]. Care has been taken to keep the triangular and quadrilateral versions of a scheme similar to provide a fair basis for comparison.

Development of Schemes

Both the quadrilateral and triangular meshes used here are described by an unstructured pointer system. In this system the most important element in the calculation, which in this case is the cell, points to surrounding elements which are needed in the calculation, such as the nodes which make up the cell. By using an unstructured pointer system, a slightly longer computation time and more storage space is required, but greater ease in programming and more flexibility in mesh type is gained.

Both the Jameson and Ni schemes which are examined here are node-based four equation models of the Euler equations. It was found that the numerical smoothing used had a larger effect on the solutions than the base solvers, therefore the reader is referred to the references for more details about the base solvers.

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Numerical Smoothing

Numerical smoothing is a dissipative operator which is added to numerical schemes to damp out oscillations in the solution and provide stability. A fourth difference operator is used for numerical smoothing which is applied throughout the flow field. Another operator is required for shock capturing and will not be discussed here.

To compute the fourth difference smoothing operator, a second difference of a second difference is computed. For both quadrilaterals and triangles two second difference operators are examined. The first is a relatively simple operator which gives a non-zero second difference for a linear function on an irregular mesh. The second operator is more complex, but results in a zero second difference for a linear function. By examining the effect of the second difference operator on a linear function the accuracy of the operator is tested, since for second order or higher accuracy the contribution must be zero.

Typical triangular and quadrilateral cells which will be used to describe the operators are shown in Figure 1 with their corresponding nodes.

The low-accuracy second difference operator is not dependent on the location of the nodes surrounding the node for which the second difference is computed, but merely on the function values at these nodes. For a triangular mesh the contribution from cell A to the second difference at node 1 is

$$(D^2S)_{1A} = (S_3 + S_2 - 2S_1) \tag{1}$$

where S is the variable for which the second difference is computed. For a quadrilateral mesh the contribution from cell A to the second difference at node 1 is

$$(D^2S)_{1A} = (S_4 + S_3 + S_2 - 3S_1) \tag{2}$$

Similar contributions are found for the other nodes of the cell. This second difference is conservative for both triangular and quadrilateral meshes since the total contribution of each cell is zero.

The high-accuracy second difference operator consists of finding the first derivative for each cell and then combining the derivatives on the cells surrounding a node to form a second difference. Unlike the low-accuracy second difference operator, this operator is dependent on the mesh geometry. The operator for a triangular cell mesh is examined first. Referring to Figure 1 the first derivative with respect to x is found for cell A

$$\begin{aligned}
(S_x)_A &= \frac{1}{A_A}\iint_{cell A}\frac{\partial S}{\partial x}dx\,dy \\
&= \frac{1}{A_A}\int_{1-2-3} S dy \\
&= \frac{1}{2\,A_A}[(S_2+S_1)(y_2-y_1) + (S_3+S_2)(y_3-y_2) + (S_1+S_3)(y_1-y_3)]
\end{aligned} \tag{3}$$

and similarly for the derivative with respect to y. A similar process is performed to create a second difference. The integration is taken around all the triangles which surround the node for which the second difference is computed, using the derivative values calculated at the cells. To get a second difference instead of a second derivative, there is no division by the area of the integrated region. The contribution to the second difference at node 1 from cell A is

$$(D^2S)_{1A} = \int_{2-3}[(S_x)_A dy - (S_y)_A dx] = (S_x)_A(y_3 - y_2) - (S_y)_A(x_3 - x_2) \tag{4}$$

This second difference operator is conservative since again the total contribution of each cell is zero.

On all boundaries, solid wall or farfield, boundary conditions must be implemented for the second difference operators. For the low-accuracy operator the contribution to a node on the boundary is simply the contribution from the cells surrounding that node which are inside the domain as given in Equations (1) and (2). The high-accuracy operator involves a line integral in Equation (4) which must be closed when considering a node on the boundary. To do this the integral is continued along the boundary faces on either side of the node in question using the value of (S_x) and (S_y) from the cell directly inside the boundary.

To formulate a second difference operator for a quadrilateral mesh a similar process is employed. It turns out that if the first derivative is found by integrating around the complete quadrilateral, the oscillatory modes are not damped out, and the primary purpose of the operator is not fulfilled. To prevent this problem, the quadrilateral is broken into triangles consisting of three of the four nodes of the quadrilateral, and the triangular operator is applied [6].

The first method of creating a fourth difference is to use the low-accuracy second difference twice by operating first on the state vector and then operating on this second difference. This fourth difference is conservative, but is second order accurate only on a uniform mesh since the second difference operator used is only second order accurate on a uniform mesh. The second method is to compute a second difference of the state vector using the high-accuracy method and operate on this second difference with the low-accuracy second difference. This operator is second order accurate since the first operator is second order accurate and conservative since the second operator is conservative. The second method is more expensive than the first, but the effect per iteration is an increase of only 5-10% which is a small increase for the gain in accuracy. The fourth difference is multiplied by a coefficient, between 0.0001 and 0.01, to control the amount of smoothing which is added to the scheme.

Numerical Accuracy

The accuracy of triangular schemes is of great concern to many people. Triangular schemes by their nature have no sense of directionality, unlike quadrilateral schemes where there are two discrete directions. Different analytical arguments have been made which provide conflicting results concerning the accuracy of triangular schemes. Analytical accuracy studies show that both the Jameson and Ni schemes are second-order accurate for smooth solutions on smooth quadrilateral meshes. The question is whether the accuracy degenerates to first order on triangular meshes and irregular quadrilateral meshes.

A numerical study was performed to determine the accuracy of the schemes and the effect of the numerical smoothing to try to answer this question. For inviscid flow, there should be no total pressure loss for smooth, subsonic flow. Thus, for such flows any total pressure loss is purely numerical in nature. With this in mind, the total pressure loss is defined as the error in subsonic flow with a 0.5 inlet Mach number through a duct with a $\sin^2 x$ bump on the lower surface. This geometry is shown in Figure 2 with Mach number and % total pressure loss contours for the Ni scheme on an irregular triangular mesh. The global error is defined as the root mean square of the total pressure loss. For various meshes on which a typical mesh length h was varied the error was calculated and the order of accuracy for the scheme was found by plotting $\log(h)$ verses $\log(error)$ and finding the slope of the resulting line. An example of this curve fitting is shown in Figure 3 with the mesh used for the computations in Figure 2 indicated. This irregular triangular mesh is shown in Figure 4.

In Table 1 the results of the numerical accuracy study are summarized. It is interesting to note that the absolute value of the error is on the same order of magnitude for the cases shown in Table 1. For the high-accuracy numerical smoothing, both the Ni and Jameson schemes for both quadrilateral and triangular meshes are second order accurate. This applies for both regular and irregular meshes. The low-accuracy numerical smoothing does not produce second order accuracy on either a regular or irregular, quadrilateral or triangular mesh. The decrease in accuracy for a regular mesh is due to dissipative errors caused by enforcing $\frac{\partial u}{\partial n} = 0$ in the second difference operator, which is an improper boundary condition for the base solver. This causes a numerical boundary layer to be formed which is at least one cell wide causing a decrease in accuracy. This effect is further discussed by Lindquist in [6].

Conclusions

The accuracy of node-based Ni and Jameson algorithms on both quadrilateral and triangular meshes are evaluated numerically for a subsonic test case. The results show that all four algorithms can indeed be second order accurate, but that extreme care must be taken when implementing numerical smoothing so the accuracy of the scheme is not reduced by smoothing which is less accurate than the flux calculation. The triangular schemes perform as well as the quadrilateral schemes, and offer flexibility in mesh generation and adaptation.

References

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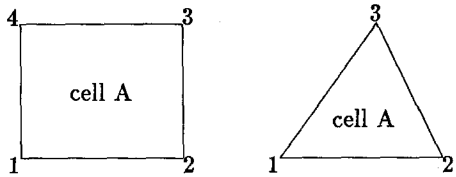

Figure 1: Typical quadrilateral and triangular cells

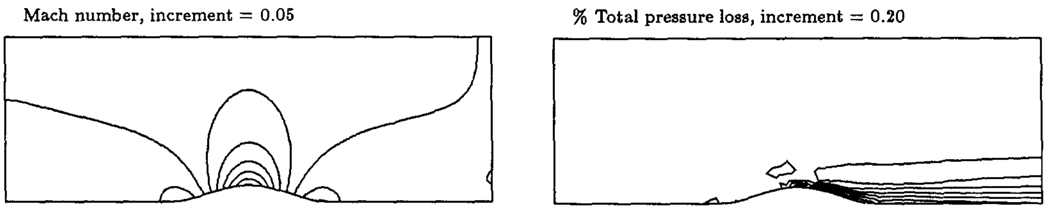

Figure 2: Mach number and % total pressure loss contours for $\sin^2 x$ duct

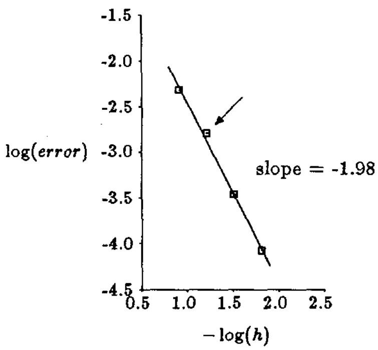

Figure 3: Accuracy data for triangular Ni scheme with high-accuracy smoothing on an irregular mesh

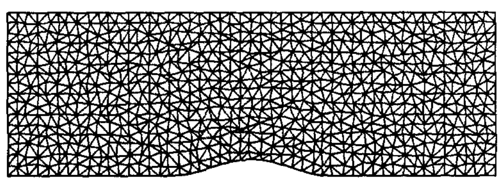

Figure 4: Irregular triangular mesh with 16 faces per unit length for $\sin^2 x$ duxt

Case	Order of Accuracy Regular Mesh	Order of Accuracy Irregular Mesh
Triangular Ni, high-accuracy smoothing	2.33	1.98
Quadrilateral Ni, high-accuracy smoothing	2.04	2.17
Triangular Jameson, high-accuracy smoothing	1.95	2.01
Quadrilateral Jameson, high-accuracy smoothing	2.01	2.05
Triangular Jameson, low-accuracy smoothing	1.55	1.69
Quadrilateral Jameson, low-accuracy smoothing	1.52	1.64

Table 1: Numerical accuracy results