Accuracy of Node-Based Solutions on Irregular Meshes

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Introduction

This paper presents an analysis of the order-of-accuracy of discrete steady-state solutions obtained using Jameson's node-based time-marching algorithm [1]. Although this algorithm is usually used in the aeronautical community to solve the Euler or Navier-Stokes equations, in this paper it will be assumed for simplicity that the hyperbolic equation being solved is linear and has constant coefficient matrices.

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \tag{1}$$

Attention will be limited to irregular computational grids with triangular cells, since these are the grids for which the order of accuracy is most uncertain. Roe has proved [2] that the truncation error of the discrete steady state operator is only first order in these situations, but numerical results presented in a companion paper by Lindquist [3] show that second order accuracy can be achieved if the numerical smoothing is constructed sufficiently carefully. The aim of this paper is to explain this result, but the analysis is not sufficiently rigorous to be considered a proof.

Analysis of local truncation error

If $U(x,y)$ is an analytic solution of the steady equation

$$L\,U(x,y) \equiv A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{2}$$

then integrating over an arbitrary control volume Ω gives

$$\iint_\Omega A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}\,dx\,dy = \oint_{\partial\Omega}(AU\,dy - BU\,dx) = 0 \tag{3}$$

The control volume Ω for the discrete operator is the union of the triangular cells around one node, and the discrete steady state equation is

$$L^h U^h \equiv \frac{1}{\mathrm{Vol}(\Omega)}\sum_{\partial\Omega}(A\bar{U}\,\Delta y - B\bar{U}\,\Delta x) = 0 \tag{4}$$

with $(\Delta x, \Delta y)^T$ being the vector face length in the counter-clockwise direction, and $\bar{U}$ being the average value of U on the face obtained by a simple arithmetic average of the values at the two end nodes.

The truncation error T^h is obtained by applying the discrete operator L^h to the analytic solution U.

$$T^h \equiv L^h U = \frac{1}{\mathrm{Vol}(\Omega)}\left(\sum_{\partial\Omega}(A\bar{U}\,\Delta y - B\bar{U}\,\Delta x) - \oint_{\partial\Omega}(AU\,dy - BU\,dx)\right) = \frac{1}{\mathrm{Vol}(\Omega)}\sum_{\partial\Omega}T_f \tag{5}$$

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where T_f is the trapezoidal integration error on each face.

$$\begin{aligned} T_f &= \tfrac{1}{2}A(U_1+U_2)(y_2-y_1) - \tfrac{1}{2}B(U_1+U_2)(x_2-x_1) - \int_1^2 (AU\,dy - BU\,dx) \\ &= (An_x + Bn_y)\left(\tfrac{1}{2}(U_1+U_2)\Delta s - \int_{s_1}^{s_2} U\,ds\right) \\ &= \tfrac{1}{12}(\Delta s)^3(An_x + Bn_y)\frac{\partial^2 U}{\partial s^2} + O\left((\Delta s)^5\right) \end{aligned} \tag{6}$$

In this equation, $(n_x, n_y)^T$ is the outward pointing unit normal vector on the face, and Δs is the length of the face.

There are two important points which arise immediately from this truncation error analysis. The first is that if h is a typical cell dimension then $T_f = O(h^3)$ and $\mathrm{Vol}(\Omega) = O(h^2)$, and hence for a general irregular mesh $T^h = O(h)$, so the local truncation error is first order. (If the mesh is sufficently regular then, as noted by Roe, cancellation of truncation errors T_f on opposing faces leads to the truncation error being second order.) The second point is that if one considers a large control volume Ω whose area is $O(1)$, and sums over all of the cells inside Ω then each of the internal faces contributes equal and opposite amounts to the truncation errors in the two cells on either side. Hence the global integrated truncation error is

$$\sum_{\partial\Omega} T_f = O(h^{-1}) \times O(h^3) = O(h^2), \tag{7}$$

since there are $O(h^{-1})$ faces along the perimeter $\partial\Omega$. These two points suggest that the truncation error has a low-frequency component, with wavelength $O(1)$, whose amplitude is second order, and a high-frequency component with wavelength $O(h)$, whose amplitude is first order. This idea is confirmed in the next section in a different manner.

Spectral content of local truncation error

The truncation error, $T(x,y)$, can be defined to be piecewise linear, with its value on each triangle determined by the values at the three corners. Thus,

$$T(x,y) = \sum_j T_j s_j(x,y) \tag{8}$$

where T_j is the truncation error at node j, and $s_j(x,y)$ is the triangular shape function which varies, in a continuous piecewise-linear manner, from a value of 1 at node j to 0 at its immediate neighbors, and is zero on all other triangles.

The next step is to find the spectral content of $T(x,y)$. For simplicity we now assume that the computational domain is a square of size $\pi\times\pi$, and the boundary conditions are treated perfectly so that the truncation error is zero there. Hence, $T(x,y)$ can be expressed as a sine series expansion.

$$T(x,y) = \sum_{m,n} a_{mn} \sin(mx)\sin(ny) \tag{9}$$

The sum is over positive integer values of m and n. The amplitudes a_{mn} are given by

$$a_{mn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi T(x,y)\sin(mx)\sin(ny)\,dx\,dy. \tag{10}$$

The amplitude magnitudes can be bounded by performing an order-of-magnitude analysis. For small, $O(1)$, values of m, n, the integral expression for a_{mn} is first split into a summation over all of the nodes.

$$\int_0^\pi \int_0^\pi T(x,y)\sin(mx)\sin(ny)\,dx\,dy = \sum_j T_j \int_0^\pi \int_0^\pi s_j(x,y)\sin(mx)\sin(ny)\,dx\,dy \tag{11}$$

The contribution from an individual node is then split into two parts, by writing the product of the sine functions as the sum of a constant term and a perturbation.

$$\begin{aligned}\int_0^\pi \int_0^\pi & s_j(x,y)\sin(mx)\sin(ny)\,dx\,dy \\ &= \int_0^\pi \int_0^\pi s_j(x,y)\sin(mx_j)\sin(ny_j)\,dx\,dy \\ &+ \int_0^\pi \int_0^\pi s_j(x,y)\left(\sin(mx)\sin(ny) - \sin(mx_j)\sin(ny_j)\right)\,dx\,dy\end{aligned} \tag{12}$$

If $m, n = O(1)$, the perturbation term is $O(h)$ and its integral is $O(h^3)$. Since T_j is $O(h)$ and there are $O(h^{-2})$ nodes, the overall contribution to a_{mn} is $O(h^2)$. The other term can be integrated since

$$\int_0^\pi \int_0^\pi s_j(x,y)\,dx\,dy = \tfrac{1}{3}\mathrm{Vol}(\Omega_j) \tag{13}$$

where $\mathrm{Vol}(\Omega_j)$ is the sum of the areas of the triangles which meet at node j, and is the same area that was used earlier in defining the discrete difference operator. Hence, the expression for a_{mn} now becomes

$$a_{mn} = \tfrac{1}{3}\sum_j \mathrm{Vol}(\Omega_j) T_j \sin(mx_j)\sin(ny_j) + O(h^2) \tag{14}$$

Substituting the definition of the truncation error, the area terms cancel and the summation over the nodes can be changed to a summation over faces.

$$a_{mn} = \tfrac{1}{3}\sum_f T_f \left(\sin(mx_{j-})\sin(ny_{j-}) - \sin(mx_{j+})\sin(ny_{j+})\right) + O(h^2) \tag{15}$$

where T_f is the trapezoidal integration error on an individual face, whose magnitude was estimated earlier, and the nodes j_+ and j_- are the neighboring nodes above and below the face, relative to the direction of the face's unit normal vector $(n_x, n_y)^T$.

If $m, n = O(1)$, then $\sin(mx_{j-})\sin(ny_{j-}) - \sin(mx_{j+})\sin(ny_{j+}) = O(h)$. $T_f = O(h^3)$ and the number of faces is $O(h^{-2})$ so the sum over all of the faces is $O(h^2)$. Hence, $a_{mn} = O(h^2)$ for $m, n = O(1)$.

The same analysis for $m, n = O(1/h)$ says that $a_{mn} = O(h)$ or smaller. Because of Parseval's theorem,

$$\sum_{m,n} a_{mn}^2 = \frac{4}{\pi^2}\int_0^\pi \int_0^\pi T^2(x,y)\,dx\,dy = O(h^2), \tag{16}$$

there are two possible extremes. If the grid is extremely random then a_{mn} is fairly continuous and so to satisfy Parseval's theorem $a_{mn} = O(h^2)$ for $m, n = O(1/h)$ and decays rapidly for larger m, n. At the other extreme, if the grid is non-uniform in an extremely regular form, then $a_{mn} = O(h)$ at a few values of m, n and is practically zero elsewhere. In either case, the truncation error can be split conceptually into two parts; a low frequency part defined by the amplitudes a_{mn} for

$m, n = O(1)$ and the remaining high frequency part. The low frequency part has a root-mean-square amplitude which is $O(h^2)$ whereas the amplitude of the high frequency part is $O(h)$. The next section determines the corresponding split in the solution error which is produced by this truncation error.

Spectral content of solution error

If U is a solution to the differential equation

$$LU = 0, \tag{17}$$

and U^h is the correponding discrete solution to the difference equation

$$L^h U^h = 0. \tag{18}$$

then the solution error ϵ^h, defined by

$$\epsilon^h = U - U^h, \tag{19}$$

satisfies the discrete equation

$$L^h \epsilon^h = L^h(U - U^h) = L^h U = T^h \tag{20}$$

As shown in the last section, T^h can be split into two parts, a low-frequency part T_1^h and a high-frequency remainder, T_2^h, and the error can be split into two corresponding parts ϵ_1^h and ϵ_2^h. Examing the low-frequency part first, the discrete equation

$$L^h \epsilon_1^h = T_1^h \tag{21}$$

can be accurately approximated by the differential equation

$$L\epsilon_1 = T_1 \tag{22}$$

since L^h is a consistent approximation to L and ϵ_1^h and T_1^h are both smooth. Since this equation is independent of h, it shows that ϵ_1 is of the same order as T_1, and so is second order. (Note: this is the basis of the standard assumption that the solution error is of the same order as the truncation error, because on smooth grids the truncation error is also smooth.)

In calculating ϵ_2^h , the solution error due to the high-frequency truncation error, it is assumed that a fourth difference smoothing is used in the numerical calculation, and that this smoothing has three important properties. The first two are that it is conservative and does not produce a truncation error which is larger than that produced by the physical part of the discrete operator. The third assumption is that it is effective in smoothing all high-frequency oscillations. Given these assumptions, the equation

$$L^h \epsilon_2^h = T_2^h \tag{23}$$

will be approximated in an order-of-magnitude sense by the equation

$$L' \epsilon_2 = T_2 \tag{24}$$

where L' is equal to L plus a term due to the smoothing.

$$L' \equiv A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + h^3\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right) \tag{25}$$

Because T_2 is high-frequency, with a typical length scale $O(h)$, boundary effects can be neglected, and the response ϵ_2 can be estimated by considering a solution error which is a constant vector multiplied by a high-frequency sinusoidal function.

$$L' \sin(mx)\sin(ny)\, u_{const} = \Big(m\cos(mx)\sin(ny)A + n\sin(mx)\cos(ny)B + h^3(m^4+n^4)\sin(mx)\sin(ny) \Big)\, u_{const} \qquad (26)$$

This equation shows that if $m, n = O(h^{-1})$, then an $O(1)$ solution error is produced by an $O(h^{-1})$ truncation error, and hence T_2 which is first order produces a solution error ϵ_2 which is second order.

To summarize, the low-frequency component of the truncation error is second order and produces a solution error which is also second order. The high-frequency component of the truncation error is first order, but it produces a solution error which is also second order.

Conclusions

The overall conclusion is that the solution error is second order, even though the local truncation error is first order. The key assumptions in the argument are that boundary condition errors are negligible, and that a fourth difference smoothing is being used which does not increase the order of the truncation error but does smooth high-frequency oscillations.

In discussing the accuracy the root-mean-square error has been used as the indicator of accuracy. It seems likely that the maximum error will behave in a similar manner. In applications one is often most interested in obtaining integral quantities such as lift and drag, and the integration implicit in these will tend to further filter, or suppress, the high frequency component in the error. On the other hand, if the important physical quantity is a derivative variable, such as the surface pressure gradient, then this will amplify the high frequency component.

References

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