Generalized Likelihood Ratio Tests

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Jan-April 2024

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Introduction

- Suppose a random variable X has a pdf $f(x_1, \theta_1, \theta_2, ..., \theta_k)$ depending on the unknown parameters $\theta_1, \theta_1, ..., \theta_k$
- The set of all parameters is denoted by Ω known as the parameter space.
- Let Ω_0 be a subset of Ω consider the hypothesis:

$$H_0 : (\theta_1, \theta_2, ..., \theta_k) \in \Omega_0 \ \ vs \ \ H_a : (\theta_1, \theta_2, ..., \theta_k) \in \Omega - \Omega_0$$

- The above case is a composite hypothesis against a composite one. In this case we do not use Neyman Pearson lemma but we use a more generalized form of it generalized likelihood ratio test(LRT)

- We denote the likelihood function of a sample $x_1, x_2, ..., x_n$ by

$$L(\Omega) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2, ..., \theta_k)$$

- Let $L(\hat{\Omega}) = \max L(\Omega)$ be maximized $L(\Omega)$ and $L(\Omega_0) = \max(L(\Omega_0))$ be maximized $L(\Omega_0)$ with Ω_0 only
- We take the likelihood ratio as a quotient

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}$$

- Since λ is the quotient of non-negative functions it is greater than or equal to zero. Further since $\Omega_0 \in \Omega$ then

$$L(\hat{\Omega}) \leq L(\hat{\Omega})$$

hence $0 \leq \lambda \leq 1$

Likelihood Ratio Test

- To test $H_0 : \theta \in \Omega_0$ against $H_a : \theta \in \Omega - \Omega_0$ the critical region for the likelihood ratio test is the set of points in the sample for which

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} \leq k$$

Where k is a constant determined so that the level of significance is α

$$P\left[\lambda \leq k | H_0 \ \ is \ \ true\right] = \alpha$$

Example

$X \sim N(\mu, \sigma^2)$ where $\mu \& \sigma^2$ are unknown. Test the hypothesis

$$H_0 : \mu = \mu_0 \ \ vs \ \ H_1 : \mu \neq \mu_0$$

Solution

- Let $x_1, x_2, ..., x_n$ be a random sample of size n from X then

$$L(X, \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}} \sum_{i=1}^{n}(x_i - \mu)^2$$

- In

$$\Omega = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma > 0\}$$

$$\Omega_0 = \{(\mu, \sigma) : \mu = \mu_0, \sigma > 0\}$$

- $L(\hat{\Omega}) = \max_{\mu,\sigma^2 \in \Omega} L(X, \mu, \sigma^2)$

-

$$\ln L(\mu, \sigma^2) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2$$

- $$\frac{\partial l}{\partial \mu} = 0 \to 2\sum_{i=1}^{n} \frac{(x_i - \mu)}{2\sigma^2} 0 \to \hat{\mu} = \bar{x}$$

- Further
$$\frac{\partial \ln L}{\partial \sigma^2} = 0 \to -(\frac{n}{2})\frac{1}{\sigma^2} + \frac{\sum_{i=1}^{2}(x_i - \mu)^2}{2(\sigma^2)^2}$$

$$
\begin{aligned}
L(\hat{\Omega}) &= \left(\frac{1}{2\pi\hat{\sigma^2}}\right) e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2} \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]^{\frac{n}{2}} e^{-\frac{-\sum_{i=1}^{n}(x_i-\bar{x})^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}} \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum(x_i-\bar{x})^2}\right]^{\frac{n}{2}} e^{-\frac{n}{2}}
\end{aligned} \tag{1}
$$

- Under Ω_0 the maximum likelihood estimates $\mu\&\sigma^2$ are

$$\hat{\mu} = \mu_0, \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

$$\begin{aligned} L(\hat{\Omega}) &= \left(\frac{1}{2\pi\hat{\sigma^2}}\right) e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2} \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum_{i=1}^{n}(x_i-\mu_0)^2}\right]^{\frac{n}{2}} e^{-\frac{-\sum_{i=1}^{n}(x_i-\mu_0)^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2}} \qquad (2) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum(x_i-\mu_0)^2}\right]^{\frac{n}{2}} e^{-\frac{n}{2}} \end{aligned}$$

- Therefore the likelihood ratio criterion is given by

$$\begin{aligned}
\lambda &= \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} \leq k \\
&= \frac{\left[\frac{n}{\sum_{i=1}^{n}(x_i-\mu_0)^2}\right]^{\frac{n}{2}}}{\left[\frac{n}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]^{\frac{n}{2}}} \leq k \qquad (3) \\
&= \left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\mu_0)^2}\right]^{\frac{n}{2}} \leq k
\end{aligned}$$

- The test is to reject H_0 when $\lambda \leq c$ Where c is such that

$$P[\lambda \leq c] = \alpha$$

- But

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \mu_0)^2 &= \sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \mu_0)^2 \\ &= \sum(x_i - \bar{x})^2 + 2(\bar{x} - \mu_0)\sum(x_i - \bar{x}) + n(\bar{x} - \mu_0)^2 \\ &= \sum(X_i - \bar{X})^2 + n(\bar{x} - \mu_0)^2\end{aligned} \tag{4}$$

- So

$$\lambda = \left[\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2} \right]^{\frac{n}{2}}$$

- $$\lambda = \left[\frac{1}{1 + \frac{\sum (\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}} \right]^{\frac{n}{2}}$$

- $$T = \frac{\sqrt{n}(\bar{x} - \mu_0)/\sigma}{\left[\frac{\sum (x_i - \bar{x})^2}{\sigma^2 (n-1)} \right]^{\frac{1}{2}}} = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}}$$

- The variable has

Thank You!