$$\lambda = e^{-\frac{1}{2\sigma_0^2}\sum(x_i-\mu_0)^2+\frac{1}{2\sigma_0^2}\sum(x_i-\bar{x})^2} \leq k$$

Taking natural logarithm

$$-\frac{1}{2\sigma_0^2}\sum(x_i-\bar{x})^2+\frac{1}{2\sigma_0^2}\sum(x_i-\mu_0)^2 \leq \ln k$$

Simplifying further we have

$$\frac{1}{2\sigma_0^2}n(\bar{x}-\mu_0)^2 \leq \ln k+\frac{1}{2\sigma_0^2}\sum(x_i-\bar{x})^2$$

Then

$$\frac{n(\bar{x}-\mu_0)^2}{\sigma_0^2} < k$$

Let $Z = \frac{\bar{x}-\mu_0}{\frac{\sigma_0}{\sqrt{n}}} \sim N(0,1)$

Hence

$$\lambda < c \Rightarrow Z^2 > k$$

or

$$P[\lambda < c|H_0] = \alpha \Rightarrow P[|Z| \geq \sqrt{k}] = \alpha$$

The test now is to compute

$$Z = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma_0}$$

and reject H_0 when

$$|Z| \geq \sqrt{k}$$

where k is such that

$$P[|Z| \geq \sqrt{k}] = \alpha \Rightarrow P[-\sqrt{k} \leq Z \leq \sqrt{k}] = 1-\alpha$$

Testing the Variance of Normal Population

Let $x_1, x_2, ..., x_n$ be a random sample from a normal variable X.

Test the hypothesis

$$H_0 : \sigma^2 = \sigma_0^2 \;\; vs \;\; H_a : \sigma^2 \neq \sigma_0^2$$

μ is unknown. The parameter spaces are

$$\Omega = \{(\mu,\sigma) : -\infty < \mu < \infty, \sigma > 0\}$$

$$\Omega_0 = \{(\mu,\sigma) : -\infty < \mu < \infty, \sigma = \sigma_0\}$$

$$L(X, \mu, \sigma) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

The maximum likelihood estimates under Ω are

$$\hat{\mu} = \bar{x}, \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Hence

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]^{\frac{n}{2}} e^{-\frac{n}{2}}$$

Under Ω_0 the maximum likelihood estimates are

$$\hat{\mu} = \bar{x}, \hat{\sigma^2} = \sigma_0^2$$

Hence

$$L(\hat{\Omega_0}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{2\sigma_0^2}\right) e^{-\frac{1}{2\sigma_0^2}\sum(x_i-\bar{x})^2}$$

$$\lambda = \frac{\hat{\Omega}_0}{\hat{\Omega}} = \left[\frac{\sum(x_i - \bar{x})}{n\sigma_0^2}\right] e^{-\frac{1}{2\sigma_0^2}\sum(x_i-\bar{x})^2 + \frac{n}{2}}$$

$$\lambda = \left(\frac{U}{n}\right)^{\frac{n}{2}} e^{-\frac{1}{2}(U-n)}$$

where $U = \frac{\sum(x_i-\bar{x})^2}{\sigma_0^2}$

The test is to compute

$$U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2}$$

and reject H_0 when $0 < U < b$ or $U > b$ where the constants a and b are such that

$$P[0 < U < a] \;\; or \;\; U > b] = \alpha$$

When H_0 is true the statistic

$$U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2}$$

has χ^2 distribution with $n-1$ degrees of freedom

a and b are chosen such that

$$P[0 < \chi^2_{n-1} < a] = \frac{\alpha}{2} = P[\chi^2_{(n-1)} > b]$$

Consider the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

It is possible to write the variable U in terms of S as

$$U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2(n-1)}{\sigma_0^2(n-1)} = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$$

Example

In a random sample of the weights of 19 babies of certain age. The standard deviation was found to be 2.5kgs. Test the hypothesis that

$$H_0 : \sigma = 3 \ \ vs \ \ H_1 : \sigma \neq 3$$

at $\alpha = 0.05$

Solution

The statistic is

$$U = \frac{(n-1)S^2}{\sigma_0^2} = \frac{18 \times 2.5^2}{3^2} = 12.5$$

The critical value is

$$\chi^2_{0.025,18} = 31.53 = b$$

and

$$\chi^2_{0.975,18} = 8.23 = a$$

$U = 12.5$ lies in the acceptance region therefore we do not reject H_0 at 5% level of significance.