# Adjoint actions of nilpotent elements in the 2D superconformal algebra $\mathfrak{sconf}(N)$

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**Abstract**

The goal of this paper is to investigate the nilpotence variety of the 2D superconformal Lie superalgebra $\mathfrak{sconf}(N)$. We begin by giving an explicit characterization of this variety. We will proceed to analyze the group actions of $\mathrm{SO}(N)$ and $\mathrm{SL}(2)$ on this algebra, and then we will examine how the kernels and images of the adjoint actions associated to nilpotent elements vary under these group actions.

## Contents

# 1 Introduction

## 1.1 Opening

Conformal geometry, particularly in two dimensions, is a rich and deep subject, with connections and applications to analysis, algebra, and physics. In this paper, we will explore some features of a supersymmetric extension of conformal symmetry. This extension is inspired by questions in physics, but in this paper we will focus exclusively on algebra. Our main object of study is a Lie superalgebra that we denote $\mathfrak{sconf}(N)$ (see section 2.4 for the definition and our conventions) which extends the ordinary conformal Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, which the reader might recognize as the Lie algebra of the group $\mathrm{SL}(2, \mathbb{C})$ of Möbius transformations. This Lie superalgebra is also known, to super aficionados, as $\mathfrak{osp}(N|2)$. Here $N$ is a positive integer that indicates how many extra "supersymmetries" appear in the superconformal geometry.

Note that throughout this paper, we work exclusively over the complex numbers, so every Lie group is a complex algebraic group and every Lie algebra is a complex Lie algebra.

In a Lie superalgebra, one can ask to classify the odd elements that are *nilpotent*: for us, $Q \in \mathfrak{sconf}(N)_1$ such that $[Q, Q] = 0$. The space of such elements is called the *nilpotence variety*, and it is the topic of section 3. We show it is preserved by the adjoint action of the group $\mathrm{SL}(2) \times \mathrm{SO}(N)$, whose Lie algebra is the even component $\mathfrak{sconf}(N)_0$ of our Lie superalgebra. Our first main result is a classification of the orbits, proved in section 4.

**Theorem 1.1.** *For $N \geq 3$, there is a single nonzero* $\mathrm{SL}(2) \times \mathrm{SO}(N)$*-orbit in the nilpotence variety.*

We also examine the cases of $N = 1$, where there is only a zero orbit, and $N = 2$, where there are two nonzero orbits.

Another focus in this paper is on how a nilpotent element acts on the Lie superalgebra. In particular, given a nilpotent element $Q$, we explore the question of which elements $X \in \mathfrak{sl}(2) \subset \mathfrak{sconf}(N)_0$ satisfy $[Q, X] = 0$ and $X = [Q, Q']$ for some $Q' \in \mathfrak{sconf}(N)_1$. In other words, we study the kernel and image of the adjoint action $[Q, -]$. Our second main result completely characterizes these spaces, and it is the content of section 5.

**Theorem 1.2.** *For any $Q \in \mathrm{Nilp}(N)$, the $\mathfrak{sl}(2)$-component of the image and kernel of the adjoint action $[Q, -]$ are equal and one-dimensional. Furthermore, the collection of these subalgebras can be given an explicit parametrization in terms of a single variable $r \in \mathbb{CP}^1$.*

There results are of particular interest in the study of two-dimensional superconformal field theory, which motivated Chris Elliott and Owen Gwilliam to suggest this research to us.

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# 2 Review of Lie (super)algebras

## 2.1 Lie algebras

A Lie algebra is a vector space $L$ equipped with a bilinear map $[-,-] : L \times L \to L$ called its Lie bracket, which satisfies several axioms we will not list here. The algebras we will be working with are generally going to be (isomorphic to) subalgebras of the general linear algebra, defined as follows:

**Definition 2.1.** The *general linear algebra* $\mathfrak{gl}(n)$ is the set of all $n \times n$ matrices (over some field) with the bracket operation $[A, B] \coloneqq AB - BA$, which is called the "commutator bracket".

**Definition 2.2.** The *special linear algebra* $\mathfrak{sl}(n)$ is defined as

$$\mathfrak{sl}(n) \coloneqq \{m \in \mathfrak{gl}(n) : \mathrm{Tr}(m) = 0\}$$

As a subalgebra of $\mathfrak{gl}(n)$, it inherits the commutator bracket.

**Definition 2.3.** The *special orthogonal algebra* $\mathfrak{so}(n)$ is defined as

$$\mathfrak{so}(n) \coloneqq \{m \in \mathfrak{gl}(n) : m^T = -m\}$$

i.e. the set of skew-symmetric $n \times n$ matrices. It has a canonical basis $\{T_{ij}\}$ for $1 \leq i < j \leq n$, where $T_{ij}$ has a 1 in the $(i, j)$th place and a $-1$ in the $(j, i)$th place. We use the convention that $T_{ji}$ means $-T_{ij}$ and $T_{ii} = 0$.

## 2.2 Ideals and homomorphisms

**Definition 2.4.** An *ideal* $I$ of a Lie algebra $L$ is a vector subspace of $L$ with the property that $[x, l] \in I$ whenever $x \in I$ and $l \in L$. Note that all ideals are both left and right ideals since $[l, x] = -[x, l]$ is also in any vector subspace containing $[x, l]$.

**Example 2.5.** The *derived algebra* $L' \coloneqq [L, L]$ of a Lie algebra $L$ is an ideal of $L$ (in the case of $L = \mathfrak{gl}(n)$, $L' = \mathfrak{sl}(n)$).

**Example 2.6.** The *center* $Z(L) \coloneqq \{z \in L : \forall x \in L, [x, z] = 0\}$ of a Lie algebra $L$ is the ideal of zero divisors with respect to the bracket. For $L = \mathfrak{gl}(n)$, this corresponds to the matrices that commute with all others in regular matrix multiplication, so it coincides with the center $Z(\mathrm{GL}(N))$ of the corresponding Lie group.

**Definition 2.7.** A *Lie algebra homomorphism* $\phi : L \to K$ is a linear map satisfying $[\phi(x), \phi(y)] = \phi([x, y])$ for all $x, y \in L$, i.e. a homomorphism that preserves both vector space structure and bracket structure.

## 2.3 Lie superalgebras

We will be working with a kind of extension of the concept of a Lie algebra, by adding a notion of parity.

**Definition 2.8.** A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. When $x \in V_i$, we write that its *parity* $|x|= i$.

**Definition 2.9.** A *Lie superalgebra* is a super vector space $L = L_0 \oplus L_1$ equipped with a Lie super bracket operation $[-, -] : L \times L \to L$, defined analogously to that of regular Lie algebras, but modified to "respect parity": it must satisfy

- $[x, y] = -(-1)^{|x||y|}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x.z]]$

Importantly, the bracket of odd elements is commutative, not anticommutative, so in general, $[x, x] \neq 0$ when $x \in L_1$.

## 2.4 Definition of $\mathfrak{sconf}(N)$

We will fix a basis $\{L_{-1}, L_0, L_1\}$ for $\mathfrak{sl}(2)$ as follows:

$$L_{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ L_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix},\ L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which, the reader can check, satisfies the identity $[L_m, L_n] = (m - n)L_{m+n}$.

Using $\mathfrak{sl}(2)$ and $\mathfrak{so}(N)$ from Definition 2.3, we can now define our central object of study: the 2D superconformal Lie superalgebra $\mathfrak{sconf}(N)$.

**Definition 2.10.** Let $N \geq 1$. The 2D superconformal algebra $\mathfrak{sconf}(N)$ is the Lie superalgebra with even part $\mathfrak{sl}(2) \oplus \mathfrak{so}(N)$ and odd part $\mathbb{C}^{2\times N}$. We adopt a basis $\{G^i_r : r \in \{-1/2, 1/2\}, 1 \leq i \leq N\}$ for $C^{2\times N}$. The bracket operations within each of $\mathfrak{sl}(2)$ and $\mathfrak{so}(N)$ are inherited from their definitions, but we will now define the rest.

**Remark 2.11.** for brevity, we will often denote $G^i_{1/2}$ or $G^i_{-1/2}$ by simply $G^i_+$ or $G^i_-$ and likewise for labeled coefficients.

The bracket of an element of $\mathfrak{sl}(2)$ with one of $\mathfrak{so}(N)$, $[L_m, T_{ij}] \coloneqq 0$ is defined to be zero. Between even and odd elements, we define

$$\left[G^k_r, L_m\right] \coloneqq \left(r - \frac{m}{2}\right) G^k_{m+r}$$

and

$$\left[G^k_r, T_{ij}\right] \coloneqq \delta^k_i G^j_r - \delta^k_j G^i_r$$

as well as

$$[G^i_r, G^j_s] \coloneqq 2L_{r+s}\delta^i_j - (r-s)T_ij$$

the bracket between purely odd elements.

We will frequently refer to odd elements in a matrix form, rather than using the basis elements: we will represent the element $Q = a^+_1 G^1_+ + a^+_2 G^2_+ + \cdots + a^+_N G^N_+ + a^-_1 G^1_- + a^-_2 G^2_- + \cdots + a^-_N G^N_-$ as

$$Q = \begin{bmatrix} a^+_1 & a^+_2 & \cdots & a^+_N \\ a^-_1 & a^-_2 & \cdots & a^-_N \end{bmatrix}$$

From this point of view, it will be clear how the Lie groups associated to the even sector $\mathfrak{sconf}(N)_0$ act on the odd sector $\mathfrak{sconf}(N)_1$.

The above algebra provides a generalization of the usual 2D conformal algebra $\mathfrak{sl}(2)$ by the addition of a number $N$ of "supersymmetries", extra symmetries of space that prove useful to physicists when they appear in a theory. However, we will not focus much on the physical relevance of these topics.

**Remark 2.12.** This algebra $\mathfrak{sconf}(N)$ is isomorphic to the classical simple Lie superalgebra $\mathfrak{osp}(N|2) \subset \mathfrak{gl}(N|2)$. The matrix $Q$ describing an odd element of $\mathfrak{sconf}(N)$ corresponds to the odd element $\begin{bmatrix} 0 & JQ \\ Q^T & 0 \end{bmatrix}$ where $J$ is a symplectic form.

# 3 The nilpotence variety $\mathrm{Nilp}(N)$

Here we define the subset of the algebra that we're most interested in: the so-called "nilpotence variety".

**Definition 3.1.** Let $N \geq 1$. We call an odd element $Q \in \mathfrak{sconf}(N)_1$ *nilpotent* if $[Q, Q] = 0$. The collection of all such elements forms a variety called the *nilpotence variety* $\mathrm{Nilp}(N)$.

## 3.1 Defining Equations for $\mathrm{Nilp}(N)$

We would like to find the explicit polynomials for which $\mathrm{Nilp}(N)$ is the zero set. Let $Q \in \mathfrak{sconf}(N)_1$ be non-zero and assume $[Q, Q] = 0$. Write

$$Q = \sum_{i=1}^{N} a^i_{1/2} G^i_{1/2} + a^i_{-1/2} G^i_{-1/2}$$

We then explicitly compute the bracket $[Q, Q]$ using the formulas given in section 2.4 and set each coefficient equal to 0. Applying bilinearity, we get

$$\begin{aligned}[Q, Q] = \sum_{i=1}^{N} \sum_{j=1}^{N} & a^i_{1/2} a^j_{1/2} [G^i_{1/2}, G^j_{1/2}] + a^i_{1/2} a^j_{-1/2} [G^i_{1/2}, G^j_{-1/2}] \\ & + a^i_{-1/2} a^j_{1/2} [G^i_{-1/2}, G^j_{1/2}] + a^i_{-1/2} a^j_{-1/2} [G^i_{-1/2}, G^j_{-1/2}]\end{aligned}$$

Now we benefit from distinguishing the cases $i = j$ and $i \neq j$. For $i = j$, the terms have only an $\mathfrak{sl}(2)$ component:

$$\sum_{i=1}^{N} 2(a^i_{1/2})^2 L_1 + 4a^i_{1/2} a^i_{-1/2} L_0 + 2(a^i_{-1/2})^2 L_{-1} = 0$$

Similarly, in the $i \neq j$ case, each term has only a $\mathfrak{so}(N)$ component, so for every pair $(i, j)$ with $1 \leq i < j \leq N$, we get

$$(2a^i_{-1/2} a^j_{1/2} - 2a^i_{1/2} a^j_{-1/2}) T_{ij} = 0$$

We then set each basis element to 0 to get a system of equations. From the three $\mathfrak{sl}(2)$ basis elements, we get

$$\begin{aligned}\sum_{i=1}^{N}(a^i_{1/2})^2 &= 0\\ \sum_{i=1}^{N} a^i_{1/2}a^i_{-1/2} &= 0\\ \sum_{i=1}^{N}(a^i_{-1/2})^2 &= 0\end{aligned} \tag{3.1}$$

and for every $\mathfrak{so}(N)$ basis element $T_{ij}$ with $1 \leq i < j \leq N$, we get another equation

$$a^i_{1/2}a^j_{-1/2} = a^i_{-1/2}a^j_{1/2} \tag{3.2}$$

By dividing out by the $a_{-1/2}$ factors on both sides, we get $a^i_{1/2}/a^i_{-1/2} = a^j_{1/2}/a^j_{-1/2}$ for all $i, j$, so it follows that all these ratios are equal. This only works when none of the $a_{-1/2}$ terms are zero, but we can alleviate this problem by recognizing that the equation is homogeneous and writing

$$r \coloneqq [a^i_{1/2} : a^i_{-1/2}] = [a^j_{1/2} : a^j_{-1/2}]$$

in homogeneous coordinates for every $i, j$. This is well-defined as long as $Q \neq 0$. Therefore, $r \in \mathbb{CP}^1$ is a constant associated to each non-zero nilpotent element. For brevity, we will denote the unique point at infinity by $r = \infty$ (which just means that the $a_{1/2}$ coefficients are all 0), and otherwise write $r \in \mathbb{C}$. Note that the three equations in 3.1 above are simply related by powers of this scalar $r$.

By writing $Q$ in matrix form, we can give an elegant statement of these conditions. Write

$$Q = \begin{bmatrix} a^1_+ & a^2_+ & \dots & a^N_+ \\ a^1_- & a^2_- & \dots & a^N_- \end{bmatrix} = \begin{bmatrix} \mathbf{a}_+ \\ \mathbf{a}_- \end{bmatrix}$$

Now, the conditions from 3.1 can be expressed simply as $Q^TQ = 0$, and the condition from 3.2 is equivalent to writing $Q = \begin{bmatrix} \mathbf{a}_+ & r\mathbf{a}_+ \end{bmatrix}^T$ or equivalently $\mathrm{rank}(Q) \leq 1$. In summary, we have the following characterization of $\mathrm{Nilp}(N)$ for $N \geq 1$:

$$\mathrm{Nilp}(N) = \{Q \in \mathbb{C}^{2\times N} : Q^TQ = 0, \mathrm{rank}(Q) \leq 1\}$$

# 4 Group actions of $\mathrm{SL}(2)$ and $\mathrm{SO}(N)$ on $\mathrm{Nilp}(N)$

The Lie groups $\mathrm{SL}(2)$ and $\mathrm{SO}(N)$ associated to the even part $\mathfrak{sconf}(N)_0$ have a canonical group action on $\mathfrak{sconf}(N)$. On the even part $\mathfrak{sconf}(N)_0$, this action is given by conjugation in the relevant component:

$$(g,h)\cdot(S,T) \coloneqq (gSg^{-1}, hTh^{-1})$$

for $(g,h) \in \mathrm{SL}(2)\times\mathrm{SO}(N)$ and $(S,T) \in \mathfrak{sl}(2)\times\mathfrak{so}(N) = \mathfrak{sconf}_0$. On the odd part, this action is given simply by left and right multiplication compatible with the dimensions of odd elements:

$$(g,h)\cdot R = gRh$$

for $R \in \mathbb{C}^{2\times N} = \mathfrak{sconf}(N)_1$.

An important fact is that we can view this action as an action on the nilpotence variety $\mathrm{Nilp}(N)$.

**Lemma 4.1.** *The image of the group action of* $\mathrm{SL}(2)\times\mathrm{SO}(N)$ *on* $\mathrm{Nilp}(N)$ *is contained in* $\mathrm{Nilp}(N)$*. Hence, this action restricts to a well-defined action on the nilpotence variety* $\mathrm{Nilp}(N)$*.*

*Proof.* To see this, take $g \in \mathrm{SL}(2)$, $h \in \mathrm{SO}(N)$ and $Q \in \mathrm{Nilp}(N)$. Then, $hh^T = \mathrm{Id}_N$, $Q$ satisfies $QQ^T = 0$, and $\mathrm{rank}(Q) \leq 1$. Now notice that

$$(gQh)(gQh)^T = gQhh^TQ^Tg^T = gQQ^Tg^T = g0g^T = 0$$

and $g,h$ having full rank implies that $\mathrm{rank}(gQh) \leq 1$, so $gQh \in \mathrm{Nilp}(N)$. $\square$

## 4.1 Families of $\mathrm{SO}(N)$ orbits

The $\mathrm{SL}(2)\times\mathrm{SO}(N)$ action on $\mathrm{Nilp}(N)$ gives a partition of $\mathrm{Nilp}(N)$ into orbits. To sufficiently characterize the nilpotence variety we would like to find all of the orbits and a representative member from each orbit. We will first focus on doing this for the action of $\mathrm{SO}(N)$ on $\mathrm{Nilp}(N)$.

#### 4.1.1 Case $N \geq 3$

**Theorem 4.2.** *When $N \geq 3$, every non-zero nilpotent element $Q \in \mathrm{Nilp}(N)$ lives inside the $\mathrm{SO}(N)$ orbit of a unique member of the family of nilpotent elements $Q_r = \begin{bmatrix} 1 & i & 0 & \dots \\ r & ri & 0 & \dots \end{bmatrix}$ for $r \in \mathbb{C}$, and $Q_\infty = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & i & 0 & \dots \end{bmatrix}$ for the case when $r = \infty$.*

**Remark 4.3.** From here on we will abuse notation by saying $r \in \mathbb{CP}^1$ where $r = \infty$ means that $Q = \begin{bmatrix} \mathbf{c}_+ & \mathbf{c}_- \end{bmatrix}^T$ with $\mathbf{c}_+ = 0$ and $\mathbf{c}_- \neq 0$. In these cases, no ambiguity is introduced, as one can simply exchange $c_-$ and $c_+$ in calculation and set $r = 0$. In other matrices parametrized by $r$, the same process of zeroing out non $r$ terms and setting $r$ to 1 will be done as well.

*Proof.* Let $Q = \begin{bmatrix} \mathbf{c}_+ \\ r\mathbf{c}_+ \end{bmatrix}$ be nilpotent, with $\mathbf{c}_+ \in \mathbb{C}^N$ satisfying $\mathbf{c}_+^T\mathbf{c}_+ = 0$ and $r \in \mathbb{CP}^1$. Now take a simple representative $Q_r = \begin{bmatrix} 1 & i & 0 & \dots \\ r & ri & 0 & \dots \end{bmatrix}$ and let $h \in \mathrm{SO}(N)$. By the definition of $\mathrm{SO}(N)$, we can write $h = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_n \end{bmatrix}^T$ where $\mathbf{h}_j \in \mathbb{C}^N$ form an orthonormal basis (where by "normal", we mean the sum of squares $\mathbf{h}_j^T\mathbf{h}_j = 1$), and $\det(h) = 1$. The multiplication $Q_r h$ then simplifies to

$$Q_r h = \begin{bmatrix} \mathbf{h}_1 + i\mathbf{h}_2 \\ r(\mathbf{h}_1 + i\mathbf{h}_2) \end{bmatrix}$$

We can now see our goal: given that $\mathbf{c}_+^T\mathbf{c}_+ = 0$ and $\mathbf{c}_+ \neq 0$, find $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{C}^N$ such that $\mathbf{c}_+ = \mathbf{h}_1 + i\mathbf{h}_2$, $\mathbf{h}_1^T\mathbf{h}_1 = \mathbf{h}_2^T\mathbf{h}_2 = 0$, and $\mathbf{h}_1^T\mathbf{h}_2 = 0$. Since $\mathbf{c}_+ \neq 0$, we can choose $\mathbf{h}_1$ such that $\mathbf{h}_1^T\mathbf{h}_1 = \mathbf{h}_1^T\mathbf{c}_+ = 1$. Then set $\mathbf{h}_2 = i(\mathbf{c}_+ - \mathbf{h}_1)$, so that $\mathbf{c}_+ = \mathbf{h}_1 + i\mathbf{h}_2$. Now we just check:

$$\mathbf{h}_1^T\mathbf{h}_2 = \mathbf{h}_1^T i(\mathbf{c}_+ - \mathbf{h}_1) = i(\mathbf{h}_1^T\mathbf{c}_+ - \mathbf{h}_1^T\mathbf{h}_1) = 0$$

and

$$\begin{aligned} \mathbf{h}_2^T\mathbf{h}_2 &= i(\mathbf{c}_+ - \mathbf{h}_1)^T i(\mathbf{c}_+ - \mathbf{h}_1) \\ &= -(\mathbf{c}_+^T\mathbf{c}_+ + \mathbf{h}_1^T\mathbf{h}_1 - 2\mathbf{h}_1^T\mathbf{c}_+) = -(1-2) = 1 \end{aligned}$$

Hence, we have our first two columns. To build the rest of the matrix $h$, we can use the Gram-Schmidt process to extend $\mathbf{h}_1$ and $\mathbf{h}_2$ to a full "orthonormal basis". If $\det(h) = -1$ after doing this, simply negate one of the column vectors $h_j$ for $j \geq 3$ so that $\det(h) = 1$. Thus we have found $h \in \mathrm{SO}(N)$ such that $Q_r h = Q$, so $Q$ is in the $\mathrm{SO}(N)$-orbit of $Q_r$. $\square$

Note that since $\mathrm{SO}(N)$ acts linearly from the left, $\mathbf{c}_+ = r\mathbf{c}_-$ implies that $\mathbf{c}_+ h = (r\mathbf{c}_-)h = r(\mathbf{c}_- h)$, so $r$ is constant over each orbit. Therefore, the family $\{Q_r\}$ gives a bijective parametrization of the orbits.

#### 4.1.2 Case $N = 2$

**Proposition 4.4.** *Every non-zero nilpotent element of* $\mathfrak{sconf}(2)$ *lives in a unique* $\mathrm{SO}(2)$ *orbit from one of the two families of representative elements* $Q_r = \begin{bmatrix} 1 & i \\ r & ri \end{bmatrix}$ *or* $Q'_r = \begin{bmatrix} 1 & -i \\ r & -ir \end{bmatrix}$ *for* $r \in \mathbb{CP}^1$

*Proof.* The proof proceeds analogously to the previous one, except we only need $\mathbf{c}_+ = \mathbf{h}_1 \pm \mathbf{h}_2$. These are the same constraint up to complex conjugation, so the other equations we require still hold. Now, when we solve for $\mathbf{h}_1$ and $\mathbf{h}_2$, we get two possible solutions, say $h = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 \end{bmatrix}^T$ and $h = \begin{bmatrix} \mathbf{h}_1 & -\mathbf{h}_2 \end{bmatrix}^T$. One of these will have $\det(h) = 1$, so by choosing that solution, the theorem follows. $\square$

Note that $Q_r$ and $Q'_r$ now describe distinct orbits, as $\mathrm{Nilp}(2)$ is the union of two one-dimensional linear subspaces, one of which contains $Q_r$, and the other of which contains $Q'_r$.

#### 4.1.3 Case $N = 1$

**Proposition 4.5.** $\mathrm{Nilp}(1) = \{0\}$, *i.e. there are no nonzero nilpotent elements in* $\mathfrak{sconf}(1)$. *Hence, the only* $\mathrm{SO}(N)$ *orbit of* $\mathrm{Nilp}(1)$ *is* $\{0\}$.

*Proof.* Any odd element $Q \in \mathfrak{sconf}(1)_1$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ $a, b \in \mathbb{C}$. If $Q$ is nilpotent, then $Q^T Q = 0 \implies a^2 = b^2 = 0 \implies a = b = 0$, i.e. $Q = 0$. $\square$

## 4.2 Families of $\mathrm{SL}(2) \times \mathrm{SO}(N)$ orbits

Now that we've characterized the $\mathrm{SO}(N)$ orbits as described in the previous section, we can see that when we add in the $\mathrm{SL}(2)$ action, all of the representative elements are in the same orbit.

**Theorem 4.6.** *Let $N \geq 3$. Every nonzero nilpotent element of $\mathfrak{sconf}(N)$ lives in the same $\mathrm{SL}(2) \times \mathrm{SO}(N)$ orbit.*

*Proof.* First we note that, for any family of $\mathrm{SO}(N)$ orbits, if there is some collection of representative elements, all of which are in the same $\mathrm{SL}(2)$ orbit, then the entire family is one large $\mathrm{SL}(2) \times \mathrm{SO}(N)$ orbit by composition of the actions.

Now we can simply show that for any $Q_r = \begin{bmatrix} 1 & i & 0 & \dots \\ r & ri & 0 & \dots \end{bmatrix}, r \in \mathbb{CP}^1$, and $Q^* = \begin{bmatrix} 1 & i & 0 & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix}$, there exists a $g \in \mathrm{SL}(2)$ such that $gQ^* = Q'$. Namely, $g = \begin{bmatrix} 1 & -r^{-1} \\ r & 0 \end{bmatrix}$ (where $g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ when $r = \infty$ and $g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ when $r = 0$).

Hence all nonzero $\mathrm{SO}(N)$ orbits of $\mathrm{Nilp}(N)$, and therefore all nonzero nilpotent elements of $\mathfrak{sconf}(N)$, lie in the same $\mathrm{SL}(2) \times \mathrm{SO}(N)$ orbit. $\square$

**Remark 4.7.** Every nonzero nilpotent element of $\mathfrak{sconf}(2)$ is in the orbit of

$$Q^* = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \text{ or } Q^{*\prime} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

which aren't in the same $\mathrm{SL}(2) \times \mathrm{SO}(2)$ orbit, so the two families of $\mathrm{SO}(2)$ orbits described in section 4.1.2 collapse into two distinct orbits under the whole group action.

# 5 The adjoint action of a nilpotent element

Each element of $\mathfrak{sconf}(N)$, in particular each nilpotent element $Q \in \mathrm{Nilp}(N)$, induces a linear map $[Q, -] : \mathfrak{sconf}(N) \to \mathfrak{sconf}(N)$ denoted $(\mathrm{ad}\, Q)$. Since $Q$ is odd and hence switches the parity of an element multiplied with it, we can decompose this as two maps $(\mathrm{ad}\, Q)_0 : \mathfrak{sconf}(N)_0 \to \mathfrak{sconf}(N)_1$ and $(\mathrm{ad}\, Q)_1 : \mathfrak{sconf}(N)_1 \to \mathfrak{sconf}(N)_0$. We are interested in both $\ker(\mathrm{ad}\, Q)_0$ and

$\mathrm{im}(\mathrm{ad}\, Q)_1$. These are both Lie algebra ideals of $\mathfrak{sconf}(N)_0$, the latter being so due to the super Jacobi identity.

We will henceforth denote these subspaces as $\mathfrak{z}_Q = \ker(\mathrm{ad}\, Q)_0$ and $\mathfrak{b}_Q = \mathrm{im}(\mathrm{ad}\, Q)_1$. Also using the super Jacobi identity, $Q \in \mathrm{Nilp}(N)$ implies that $\mathfrak{b}_Q$ is an ideal of $\mathfrak{z}_Q$, and it is worth noting that the quotient algebra $\mathfrak{z}_Q/\mathfrak{b}_Q$ is of particular interest in mathematical physics.

We are particularly interested in the $\mathfrak{sl}(2)$ component of these ideals, which we know is invariant under the action of $\mathrm{SO}(N)$. Therefore, we will only consider representatives of the orbits of $\mathrm{Nilp}(N)$ under the action of $\mathrm{SO}(N)$.

## 5.1 Lemmas for computing $\mathfrak{z}_q \cap \mathfrak{sl}(2)$ and $\mathfrak{b}_q \cap \mathfrak{sl}(2)$

Our primary goal when calculating $\mathfrak{z}_Q$ and $\mathfrak{b}_Q$ is to see how they intersect with just the space of conformal symmetries $\mathfrak{sl}(2) \subset \mathfrak{sconf}(N)_0$. To classify all potential subspaces we can generate with arbitrary $Q \in \mathrm{Nilp}(N)$ we note these three facts:

**Lemma 5.1.** *For any $Q \in \mathrm{Nilp}(N)$ and $h \in \mathrm{SO}(N)$ we have the equivalences $\mathfrak{z}_Q \cap \mathfrak{sl}(2) = \mathfrak{z}_{Qh} \cap \mathfrak{sl}(2)$ and $\mathfrak{b}_Q \cap \mathfrak{sl}(2) = \mathfrak{b}_{Qh} \cap \mathfrak{sl}(2)$.*

**Lemma 5.2.** *For any $Q = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$ and any $g \in \mathrm{SL}(2)$, we have that $gQ = aQ'$ where $a \in \mathbb{C}$ and $Q' = \begin{bmatrix} c_+ \\ rc_- \end{bmatrix}$ for some $r \in \mathbb{CP}^1$.*

**Corollary 5.3.** *For any $Q$, $g$, and $Q'$ defined identically as in the preceding lemma, $\mathfrak{z}_{gQ} = \mathfrak{z}_{Q'}$ and $\mathfrak{b}_{gQ} = \mathfrak{b}_{Q'}$.*

Generally this means, because $\mathrm{Nilp}(N)$ is just the zero vector plus a single $\mathrm{SL}(2) \times \mathrm{SO}(N)$ orbit, the family of possible subspace intersections $\mathfrak{z}_Q \cap \mathfrak{sl}(2)$ and $\mathfrak{b}_Q \cap \mathfrak{sl}(2)$ are parameterized only by this variable $r \in \mathbb{CP}^1$ (which is the ratio between the spin up and spin down components of $Q$). Because of this, when calculating these spaces, we will use exclusively $Q_r$'s as defined below.

## 5.2 Image of $(\operatorname{ad} Q)_1$

Let $Q_r = \begin{bmatrix} 1 & i & 0\dots \\ r & ri & 0\dots \end{bmatrix} \in \mathbb{C}^{2\times N}$. We can write

$$(\operatorname{ad} Q_r)_1 = \operatorname{ad} G_+^1 + r \operatorname{ad} G_-^1 + i \operatorname{ad} G_+^2 + ri \operatorname{ad} G_-^2$$

Now, calculating the matrix form for each of these (term-by-term using the formulas from section 2.4) and summing gives

$$\mathcal{Q}_1 = \begin{array}{c} \\ {\scriptstyle L_{-1}} \\ {\scriptstyle L_0} \\ {\scriptstyle L_1} \\ {\scriptstyle T_{1,2}} \\ {\scriptstyle T_{1,3\dots N}} \\ {\scriptstyle T_{2,3\dots N}} \\ {\scriptstyle T_{3,4\dots N-1,N}} \end{array} \begin{array}{c} \begin{array}{cccccc} {\scriptstyle G_+^1} & {\scriptstyle G_+^2} & {\scriptstyle G_+^{3\dots N}} & {\scriptstyle G_-^1} & {\scriptstyle G_-^2} & {\scriptstyle G_-^{3\dots N}} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 2r & 2ir & 0 \\ 2r & 2ir & 0 & 2 & 2i & 0 \\ 2 & 2i & 0 & 0 & 0 & 0 \\ -ir & r & 0 & i & -1 & 0 \\ 0 & 0 & r\mathrm{Id}_{N-2} & 0 & 0 & -\mathrm{Id}_{N-2} \\ 0 & 0 & ri(\mathrm{Id}_{n-2}) & 0 & 0 & -i(\mathrm{Id}_{n-2}) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The desired $\operatorname{im}(\operatorname{ad} Q_r)_1$ is then simply the column space of this matrix. First, we note that the $G_\pm^2$ columns are simply $i$ times the $G_\pm^1$ columns, and hence can be ignored. Similarly, each of the $G_+^i$ columns for $3 \le i \le N$ is just $-r$ times the $G_-^i$ column. The rest of the columns are linearly independent.

### 5.2.1 Finding a basis for $\mathfrak{b}_Q \cap \mathfrak{sl}(2)$

Let $H$ be an odd element, and $Q$ be a nilpotent element, so therefore.

$$H = \sum_{i=1}^{N} b_{1/2}^i G_{1/2}^i + b_{-1/2}^i G_{-1/2}^i$$

$$Q = \sum_{i=1}^{N} a_{1/2}^i G_{1/2}^i + a_{-1/2}^i G_{-1/2}^i$$

Where $a$ satisfies equations (3.2) and (3.1), and $b \in \mathbb{C}^{2\times N}$

By definition of $\mathfrak{b}_q = \operatorname{im}(\operatorname{ad} Q)_1$, every element in this subspace is given by some $[Q, H]$. Expanding out this bracket, we get.

$$[Q,H] = \sum_{i=1}^{N}\sum_{j=1}^{N} a^i_{1/2}b^j_{1/2}[G^i_{1/2},G^j_{1/2}] + a^i_{1/2}b^j_{-1/2}[G^i_{1/2},G^j_{-1/2}]$$
$$+ a^i_{-1/2}b^j_{1/2}[G^i_{-1/2},G^j_{1/2}] + a^i_{-1/2}b^j_{-1/2}[G^i_{-1/2},G^j_{-1/2}]$$

Then, computing the basis brackets, we have

$$[Q,H] = \sum_{i=1}^{N} 2a^i_{1/2}b^i_{1/2}L_1 + 2a^i_{1/2}b^i_{-1/2}L_0 + 2a^i_{-1/2}b^i_{1/2}L_0 + 2a^i_{-1/2}b^i_{-1/2}L_{-1}$$
$$+ \sum_{i\neq j}^{N} 2a^i_{1/2}b^j_{-1/2}T_{ij} - 2a^i_{-1/2}b^j_{1/2}T_{ij} =$$
$$\sum_{i=1}^{N} 2a^i_{1/2}b^i_{1/2}L_1 + (2a^i_{1/2}b^i_{-1/2} + 2a^i_{-1/2}b^i_{1/2})L_0 + 2a^i_{-1/2}b^i_{-1/2}L_{-1}$$
$$+ \sum_{i\neq j}^{N}(2a^i_{1/2}b^j_{-1/2} - 2a^i_{-1/2}b^j_{1/2})T_{ij}$$

Taking an arbitrary $H$ this describes the entire subspace $\mathrm{im}(\mathrm{ad}\, Q)_1 = \mathfrak{b}_Q$. To intersect this space with $\mathfrak{sl}(2)$, we must determine for what values $a, b$ the basis elements $T_{ij}$ disappear from the image. For this to happen, it follows from above that

$$\forall i \neq j : a^i_+ b^j_- = a^i_- b^j_+$$

But because $Q \in \mathrm{Nilp}(N)$, by equation (3.2) this simplifies to

$$b^j_- = r b^j_+ \tag{5.1}$$

Because of lemma 5.3 we restrict

$$Q = Q_r = \begin{bmatrix} 1 & i & \ldots & 0 \\ r & ri & \ldots & 0 \end{bmatrix}$$

Plugging in $Q_r$ to the general formula for $\mathfrak{b}_{Q_r} \cap \mathfrak{sl}(2)$ giving

$$2(b^1_+ + b^2_+ + ib^1_+ + ib^2_+)L_1 + 2(rb^1_- + rb^2_- + irb^1_- + irb^2_-)L_{-1} +$$
$$2(b^1_- + b^2_- + ib^1_- + ib^2_- + rb^1_+ + rb^2_+ + irb^1_+ + irb^2_+)L_0 =$$
$$2zL_1 + 2zr^2L_{-1} + 4zrL_0$$

by collecting $z = b^1_+ + b^2_+ + ib^1_+ + ib^2_+$ into one complex variable and simplifying using (5.1). Because $z$ is free this gives the one dimensional subspace

$$\mathfrak{b}_Q \cap \mathfrak{sl}(2) = \langle L_1 + 2rL_0 + r^2 L_{-1} \rangle$$

## 5.3 Kernel of $(\operatorname{ad} Q)_0$

### 5.3.1 Finding a basis for $\mathfrak{z}_Q \cap \mathfrak{sl}(2)$

The definition of $\ker(\operatorname{ad} Q)_0$ is $[Q, J]_s = 0$, where $J$ is an even element. $J$ is therefore defined by

$$J = \sum_{m=\pm 1,0} l_m L_m + \sum_{1 \le i < j \le N} T_{ij}\omega_{ij}$$

and $Q$ is a nilpotent element. We then bracket $[Q, J] = 0$

$$\sum_{i=1}^{N} \sum_{m=\pm 1,0} a^i_{1/2} l_m [G^i_{1/2}, L_m] + a^i_{-1/2} l_m [G^i_{-1/2}, L_m] + \sum_{i=1}^{N} \sum_{1 \le i < j \le N} a^i_{1/2}\omega_{ij} [G^i_{1/2}, T_{ij}] + a^i_{-1/2}\omega_{ij} [G^i_{-1/2}, T_{ij}] = 0$$

We then compute the bracket, while setting the coefficient matrix of $T_{ij}$ to 0, to calculate $\mathfrak{z}_Q \cap \mathfrak{sl}(2)$. From computing the bracket of each basis element, we get

$$\sum_{i=1}^{N} -a^i_{-1/2} l_1 G^i_{1/2} + a^i_{1/2} l_{-1} G^i_{-1/2} - \frac{1}{2} a^i_{-1/2} l_0 G^i_{-1/2} + \frac{1}{2} a^i_{1/2} l_0 G^i_{1/2} = 0$$

Grouping like terms gives

$$\sum_{i=1}^{N} (-a^i_{-1/2} l_1 + \frac{1}{2} a^i_{1/2} l_0) G^i_{1/2} + (a^i_{1/2} l_{-1} - \frac{1}{2} a^i_{-1/2} l_0) G^i_{-1/2} = 0$$

Setting each coefficient of each basis element $G$ to 0, we get a general description of these subspaces as a system of equations in terms of $a^s_r$:

$$\sum_{i=1}^{N} -a^i_{-1/2} l_1 + \frac{1}{2} a^i_{1/2} l_0 = 0, \quad \sum_{i=1}^{N} a^i_{1/2} l_{-1} - \frac{1}{2} a^i_{-1/2} l_0 = 0$$

because $\mathfrak{z}_Q \cap \mathfrak{sl}(2)$ is unchanged by the action of $\mathrm{SO}(N)$ on $Q$, we can calculate the family of subspaces using the representative elements from the families of $\mathrm{SO}(N)$ orbits

$$Q = Q_r = \begin{bmatrix} 1 & i & \dots & 0 \\ r & ri & \dots & 0 \end{bmatrix}$$

Plugging these coefficients into the system of equations. We get

$$-(r+ri)l_1 + \frac{1}{2}(1+i)l_0 = 0, \quad (1+i)l_{-1} - \frac{1}{2}(r+ri)l_0 = 0 \implies$$
$$-rl_1 + \frac{1}{2}l_0 = 0 \quad l_{-1} - r\frac{1}{2}l_0 = 0$$

which defines the one dimensional subspace of $\mathfrak{sl}(2)$ given by

$$\mathfrak{z}_{Q_r} \cap \mathfrak{sl}(2) = \langle r^2 L_{-1} + 2rL_0 + L_1 \rangle$$

Note that $\mathfrak{b}_Q$ is an ideal of $\mathfrak{z}_Q$ so we know that the one is a subspace of the other, but for the intersection with $\mathfrak{sl}(2)$, the subspace relationship is trivial because

$$\mathfrak{z}_{Q_r} \cap \mathfrak{sl}(2) = \mathfrak{b}_{Q_r} \cap \mathfrak{sl}(2)$$

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