Notes on Morita categories

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Fall 2024

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1 Monoidal categories, monoids, and modules

1.1 Monoidal categories

Definition 1.1. A *monoidal category* $\mathsf{C}^{\otimes}$ is a category C with a notion of "tensor product" given by a functor $\otimes\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ and a "unit object" $1 \in \mathsf{C}$, together with three natural isomorphisms:

- the *associator* $\alpha\colon (-\otimes-)\otimes - \Rightarrow -\otimes(-\otimes-)$
- the *left unitor* $\lambda\colon 1\otimes - \Rightarrow \mathrm{id}_{\mathsf{C}}$
- the *right unitor* $\rho\colon -\otimes 1 \Rightarrow \mathrm{id}_{\mathsf{C}}$

satisfying the following coherence conditions: the triangle identity

$$\begin{tikzcd}
(x\otimes 1)\otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr, "\rho_x\otimes\mathrm{id}_y"'] & & x\otimes(1\otimes y) \arrow[dl, "\mathrm{id}_x\otimes\lambda_y"] \\
& x\otimes y &
\end{tikzcd}$$

and the pentagon identity

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dl, "\alpha_{w,x,y\otimes z}"'] & \\
w \otimes (x \otimes (y \otimes z)) & & ((w \otimes x) \otimes y) \otimes z \arrow[ul, "\alpha_{w\otimes x,y,z}"'] \arrow[d, "\alpha_{w,x,y}\otimes \mathrm{id}_z"] \\
w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_w \otimes \alpha_{x,y,z}"] & & (w \otimes (x \otimes y)) \otimes z \arrow[ll, "\alpha_{w,x\otimes y,z}"]
\end{tikzcd}$$

Claim 1.1. The following form (non-strict) monoidal categories:

- Sets with cartesian product
- Sets with disjoint union
- R-modules with tensor product

Proof. We'll write down the unit object and the natural transformations, and then naturality and the fact that the identities hold will be clear by looking at the diagrams.

For $\mathsf{Set}^{\times}$, the unit is $\{*\}$, the associator $\alpha_{X,Y,Z}$ maps $((x,y),z) \mapsto (x,(y,z))$, and the unitors map $\lambda_X\colon (*,x) \mapsto x$ and $\rho_X\colon (x,*) \mapsto x$.

For $\mathsf{Set}^{\amalg}$, let's fix the notation/construction that $X \amalg Y = \{(x,0)\colon x \in X\} \cup \{(y,1)\colon y \in Y\}$. The unit is $\emptyset$, and this is actually strict with this construction, meaning the unitors can both be just the identity. The associator $\alpha_{X,Y,Z}$ maps $((x,0),0) \mapsto (x,0)$, $((y,1),0) \mapsto ((y,0),1)$, and $(z,1) \mapsto ((z,1),1)$.

For $\mathsf{Mod}_R^{\otimes}$, the unit is R, the associator maps $\alpha_{M,N,P}\colon (m\otimes n)\otimes p \mapsto m\otimes(n\otimes p)$, and the unitors map $\lambda_M\colon r \otimes m \mapsto rm$ and $\rho_M\colon m \otimes r \mapsto rm$. These are all valid morphisms since they all come from (compositions of) R-bilinear maps, and also are all clearly isomorphisms.

Definition 1.2. A *strict monoidal category* is a monoidal category in which the associator and unitors all consist of identity morphisms, i.e. a category C, tensor product functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and unit object $1 \in \mathsf{C}$, such that $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ and $1 \otimes x = x = x \otimes 1$ for all objects $x, y, z \in \mathsf{C}$, and $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $1 \otimes f = f = f \otimes 1$ for all morphisms f, g, h in C (strict associativity and unitality on morphisms is equivalent to naturality of the associator and unitors).

Claim 1.2. Every monoid M can be seen

- as a strict monoidal category M in which the objects are the elements of the monoids
- as a strict monoidal category BM in which there is a single object.

Proof. Let M be a monoid, and view it as a discrete category with objects $m \in M$. The product in M is strictly associative and has a unit $1 \in M$, so taking this extra data makes M a strict monoidal category in a canonical way. Strict associativity and unitality on morphisms holds since the only morphisms are the identities.

Alternatively, let BM be a category with one object $*$ with endomorphisms $m \in M$. This is fine since M has associative multiplication with a unit. We must of course take $1 = *$, and we define a functor $\otimes\colon BM \times BM \to BM$ the only way we can on objects, and on morphisms, by the product in M (composition in BM), $(m, n) \mapsto mn$. We have strict associativity and unitality on objects, since there's one object, and on morphisms since they hold in the monoid M.

Definition 1.3. The *(augmented) simplex category* Δ is the category whose objects are the finite ordinals $[n] = \{0, 1, \dots, n\}$ (including $[-1] = \emptyset$) and whose morphisms are monotonic functions. *Ordinal sum* is the operation $\oplus$ given by reindexing of the disjoint union with the dictionary order, or explicitly, that which sends $([m], [n]) \mapsto [m + n + 1]$) and $(f\colon [m_1] \to [n_1], g\colon [m_2] \to [n_2]) \mapsto f \oplus g\colon [m_1 + m_2 + 1] \to [n_1 + n_2 + 1]$ given by

$$(f \oplus g)(k) = \begin{cases} f(k) & 0 \le k \le m_1 \\ g(k - (m_1 + 1)) + n_1 + 1 & m_1 + 1 \le k \le m_1 + m_2 + 1 \end{cases}$$

Claim 1.3. The simplex category Δ is a strict monoidal category with respect to ordinal sum.

Proof. We can take $[-1]$ as the unit, since $[n] \oplus [-1] = [n - 1 + 1] = [n] = [-1 + n + 1] = [-1] \oplus [n]$, and $\mathrm{id}_1\colon \emptyset \to \emptyset$ is the empty function, so clearly from the definition $\mathrm{id}_1 \oplus f = f = f \oplus \mathrm{id}_1$.

For $\ell, m, n \in \mathbb{N} \cup \{-1\}$, we have

$$\begin{aligned} ([\ell] \oplus [m]) \oplus [n] &= [\ell + m + 1] \oplus n \\ &= [\ell + m + 1 + n + 1] \\ &= [\ell + m + n + 1 + 1] \\ &= [\ell] \oplus [m + n + 1] \\ &= [\ell] \oplus ([m] \oplus [n]) \end{aligned}$$

so ordinal sum is strictly associative on objects. Now, instead of writing out all the arithmetic and worrying about indices, we can see that $\oplus$ is strictly associative on morphisms as follows: we can interpret a morphism $f\colon m \to n$ as its graph, a subset of the $m \times n$ grid, and in this setting, $f \oplus g$ corresponds to placing the graph of g immediately above and to the right of that of f, so that the corners of the grids are adjacent, thus forming a new graph of a monotonic function. This justaposition is clearly associative, meaning that $\Delta^{\oplus}$ is a strict monoidal category.

Claim 1.4. The category $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ (in which the objects are the extended real numbers, and there is a unique morphism from x to x' if and only if $x \leq x'$) is a strict monoidal category with respect to the function max (or min).

Proof. It's clear that max and min are both well-defined functors $\mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$. We have a strict unit: $-\infty$ in the case of max, or ∞ in the case of min, and its clear that max and min are both strictly associative on objects and morphisms. Therefore, $\mathbb{R}^*$ is a strict monoidal category with respect to either max or min.

Definition 1.4. A *lax monoidal functor* is a functor $F\colon \mathsf{C}^{\otimes} \to \mathsf{D}^{\boxtimes}$ between monoidal categories, together with a morphism $\epsilon\colon 1_{\mathsf{D}} \to F(1_{\mathsf{C}})$ and a natural transformation $\mu_{x,y}\colon F(x) \boxtimes F(y) \to F(x \otimes y)$ satisfying associativity: for all objects $x, y, z \in \mathsf{C}$, the diagram

$$\begin{tikzcd}
(F(x) \boxtimes F(y)) \boxtimes F(z) \arrow[r, "\alpha^{\mathsf{D}}_{F(x),F(y),F(z)}"] \arrow[d, "\mu_{x,y} \boxtimes \mathrm{id}_{F(z)}"] & F(x) \boxtimes (F(y) \boxtimes F(z)) \arrow[d, "\mathrm{id}_{F(x)} \boxtimes \mu_{y,z}"] \\
F(x \otimes y) \boxtimes F(z) \arrow[d, "\mu_{x \otimes y, z}"] & F(x) \boxtimes F(y \otimes z) \arrow[d, "\mu_{x, y \otimes z}"] \\
F((x \otimes y) \otimes z) \arrow[r, "F(\alpha^{\mathsf{C}}_{x,y,z})"] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

commutes, and unitality: for all objects $x \in \mathsf{C}$, the diagrams

$$\begin{tikzcd}
1_{\mathsf{D}} \boxtimes F(x) \arrow[r, "\epsilon \boxtimes \mathrm{id}_{F(x)}"] \arrow[d, "\lambda^{\mathsf{D}}_x"] & F(1_{\mathsf{C}}) \boxtimes F(x) \arrow[d, "\mu_{1_{\mathsf{C}},x}"] \\
F(x) & F(1_{\mathsf{C}} \otimes x) \arrow[l, "F(\lambda^{\mathsf{C}}_x)"]
\end{tikzcd}$$

and

$$\begin{tikzcd} F(x) \boxtimes 1_{\mathsf{D}} \arrow[r, "\mathrm{id}_{F(x)} \boxtimes \epsilon"] \arrow[d, "\rho^{\mathsf{D}}_x"] & F(x) \boxtimes F(1_{\mathsf{C}}) \arrow[d, "\mu_{x,1_{\mathsf{C}}}"] \\ F(x) & F(x \otimes 1_{\mathsf{C}}) \arrow[l, "F(\rho^{\mathsf{C}}_x)"] \end{tikzcd}$$

commute. A *(strong) monoidal functor* is one in which ϵ and μ are isomorphisms, and a *strict monoidal functor* is one in which ϵ and μ are identities.

Definition 1.5. A *monoidal transformation* is a natural transformation $\eta\colon F \Rightarrow G$ between two parallel monoidal functors $F, G\colon \mathsf{C}^{\otimes} \to \mathsf{D}^{\boxtimes}$ such that the diagrams

$$\begin{tikzcd} F(1_{\mathsf{C}}) \arrow[rr, "\eta_{1_{\mathsf{C}}}"] \arrow[dr, "\epsilon_F"] & & G(1_{\mathsf{C}}) \arrow[dl, "\epsilon_G"] \\ & 1_{\mathsf{D}} & \end{tikzcd}$$

and

$$\begin{tikzcd} F(x) \boxtimes F(y) \arrow[r, "\eta_x \boxtimes \eta_y"] \arrow[d, "\mu^F_{x,y}"] & G(x) \boxtimes G(y) \arrow[d, "\mu^G_{x,y}"] \\ F(x \otimes y) \arrow[r, "\eta_{x \otimes y}"] & G(x \otimes y) \end{tikzcd}$$

commute. A *monoidal equivalence* is an equivalence of monoidal categories $F\colon \mathsf{C}^{\otimes} \leftrightarrows \mathsf{D}^{\boxtimes} :G$ through monoidal functors F, G with monoidal natural isomorphsims $\eta\colon FG \Rightarrow \mathrm{id}_{\mathsf{D}}$ and $\nu\colon GF \Rightarrow \mathrm{id}_{\mathsf{C}}$.

Claim 1.5. Every monoidal category is strictly monoidally equivalent to a strict monoidal category.

Proof. Let $\mathsf{C}^{\otimes}$ be a monoidal category. Choose a skeleton S of C and an equivalence $F\colon \mathsf{C} \to S$ (relative to the inclusion $I\colon S \hookrightarrow \mathsf{C}$) with $FI = \mathrm{id}_S$ and a natural isomorphism $\eta\colon IF \Rightarrow \mathrm{id}_{\mathsf{C}}$.

Define a product $\boxtimes\colon S \times S \to S$ by $x \boxtimes y = F(x \otimes y)$ (on objects and morphisms), and a unit object for $S^{\boxtimes}$ given by $F(1_{\mathsf{C}})$. We also have an associator, also by composing the source and target functors with F. As for the unitors, let $x \in S$ and choose $y \in S$ with $F(y) = x$. Then

$$1_S \boxtimes x = F(1_{\mathsf{C}}) \boxtimes x = F(F(1_{\mathsf{C}}) \otimes x) = F(F(1_C \otimes y)) = F(1_C \otimes y) \xrightarrow{F(\lambda_y)} F(y) = x$$

so we can just define the left and right unitors by composing with F as well. All the axioms of a monoidal category now follow immediately from functoriality of F.

For monoidality of F, the morphism $\epsilon\colon 1_S = F(1_{\mathsf{C}})$ is just the identity, and so is $\mu_{x,y}\colon F(x) \boxtimes F(y) = F(x \otimes y)$. The rest of the axioms for monoidality of F and η are immediate to verify by looking at the diagrams, by functoriality of F and our definitions. Therefore, $\mathsf{C}^{\otimes}$ is strictly monoidally equivalent to the strict monoidal category $S^{\boxtimes}$.

Claim 1.6. Every complete category is a monoidal category with respect to the categorical product.

Proof. Let C be a complete category. Since it's complete, all products exist, so $\times\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a well-defined functor, and we take the terminal object $*$ to be the unit. Given objects x, y, z, let's construct the associator $\alpha_{x,y,z}$. We have a diagram of projections

$$\begin{tikzcd}
(x \times y) \times z \arrow[rr] \arrow[d] & & z \\
x \times y \arrow[r] \arrow[d] & y & y \times z \arrow[l] \arrow[u] \\
x & & x \times (y \times z) \arrow[ll] \arrow[u]
\end{tikzcd}$$

We have spans $y \leftarrow (x \times y) \times z \to z$ and $x \leftarrow x \times (y \times z) \to y$, so by the universal property of products, we have a unique morphism $f\colon (x \times y) \times z \to y \times z$ and one $g\colon x \times (y \times z) \to x \times y$. Now we repeat this with the spans $x \leftarrow (x \times y) \times z \xrightarrow{f} y \times z$ and $x \times y \xleftarrow{g} x \times (y \times z) \to z$, thus obtaining morphisms $\alpha_{x,y,z}\colon (x \times y) \times z \to x \times (y \times z)$ and $\beta\colon x \times (y \times z) \to (x \times y) \times z$. Since these morphisms are the unique ones along the diagonal above such that the above diagram commutes, their composites must both be the respective identites, so indeed $\alpha_{x,y,z}$ is an isomorphism with $\alpha^{-1}_{x,y,z} = \beta$.

To verify naturality, given morphisms $f\colon x \to x'$, $g\colon y \to y'$, $h\colon z \to z'$, we can just construct the diagram

$$\begin{tikzcd}
& (x' \times y') \times z' \arrow[rrr] \arrow[dd] \arrow[ddddrrr, dashed] & & & z' \\
(x \times y) \times z \arrow[ur] \arrow[rrr] \arrow[dd] \arrow[ddddrrr, dashed] & & & z \arrow[ur] & \\
& x' \times y' \arrow[r] \arrow[dd] & y' & & y' \times z' \arrow[ll] \arrow[uu] \\
x \times y \arrow[ur] \arrow[rr] \arrow[dd] & & y & y \times z \arrow[l] \arrow[ur] \arrow[uu] & \\
& x' & & & x' \times (y' \times z') \arrow[lll] \arrow[uu] \\
x \arrow[ur] & & & x \times (y \times z) \arrow[lll] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

in which the dashed arrows are our associators and the diagonal arrows are the (products of) f, g, h. The front-facing polygons commute by construction, and the side-facing and up-facing ones commute by the universal property of products (applied to the span $x' \leftarrow x \times y \rightarrow y'$ in diagrams such as the following, so that $f \times g$ is the unique morphism making the diagram commute)

$$\begin{tikzcd}
x \arrow[d, "f"] & x \times y \arrow[l] \arrow[r] \arrow[d, dashed, "f \times g"] & y \arrow[d, "g"] \\
x' & x' \times y' \arrow[l] \arrow[r] & y'
\end{tikzcd}$$

Thus α is a natural transformation.

For the unitors, let $\lambda_x \colon * \times x \to x$ and $\rho_x \colon x \times * \to x$ be the projections. We'll prove λ is a natural isomorphism; the proof for ρ is entirely analogous. Applying the universal property of products to the span $* \leftarrow x \rightarrow x$, there is a unique morphism $x \to * \times x$ making the following diagram commute:

$$\begin{tikzcd}
* \times x \arrow[r, shift left] \arrow[d] & x \arrow[l, dashed, shift left] \arrow[dl] \\
*
\end{tikzcd}$$

so this morphism is an inverse to the projection, meaning λ_x is an isomorphism. Furthermore, given $f \colon x \to x'$, the universal property tells us that there's a unique morphism $* \times x \to * \times x'$ making the following diagram commute, which must be $\mathrm{id}_* \times f$:

$$\begin{tikzcd} & * \times x \arrow[r] \arrow[dl] \arrow[d, dashed, "\mathrm{id}_* \times f"] & x \arrow[d, "f"] \\ * & * \times x' \arrow[l] \arrow[r] & x' \end{tikzcd}$$

Thus, λ is a natural isomorphism (similarly for ρ). We leave it to the reader with a large blackboard to check the triangle and pentagon identities.

Claim 1.7. Every cocomplete category is a monoidal category with respect to the categorical coproduct.

Proof. The proof is exactly dual to that of Claim 1.6.

Claim 1.8. The dual C^{op} of a monoidal category $\mathsf{C}^{\otimes}$ is a monoidal category.

Proof. We still have the unit object $1 \in \mathsf{C}$, and a tensor product $\otimes^{\mathrm{op}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$. If we take the duals of the inverses of the three natural isomorphisms, we obtain such in the dual category, of the same type. It's clear that the identity diagrams will still commute, so C^{op} is a monoidal category with this structure.

Claim 1.9. The product of two monoidal categories $\mathsf{C}^{\otimes}, \mathsf{D}^{\boxtimes}$ is a monoidal category.

Proof. The product of the units $1_{\mathsf{C}} \times 1_{\mathsf{D}}$ will be the unit object, and there's an operation $\otimes \times \boxtimes \colon (\mathsf{C} \times \mathsf{D}) \times (\mathsf{C} \times \mathsf{D}) \to \mathsf{C} \times \mathsf{D}$. With some appropriate commutation, the products of the unitors and associators from C and D yield the unitors and associators for $\mathsf{C} \times \mathsf{D}$, and the identity diagrams will commute, as products of the identity diagrams in C and D.

1.2 Monoids

Definition 1.6. Let $\mathsf{C}^{\otimes}$ be a monoidal category. A *monoid in* $\mathsf{C}^{\otimes}$ consists of an object $A \in \mathsf{C}$, a "multiplication" morphism $\mu \colon A \otimes A \to A$, and a "unit" morphism $\eta \colon 1 \to A$, such that the following diagram commutes:

$$\begin{tikzcd} 1 \otimes A \arrow[r, "\eta \otimes \mathrm{id}_A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes 1 \arrow[l, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd}$$

i.e. multiplication is unital, and this diagram also commutes:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha_{A,A,A}"] \arrow[d, "\mu \otimes \mathrm{id}_A"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{id}_A \otimes \mu"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

i.e. multiplication is associative. A *monoid homomorphism* between monoids (A, μ_A, η_A) and (B, μ_B, η_B) in $\mathsf{C}^{\otimes}$ is a morphism $\varphi\colon A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\varphi \otimes \varphi"] \arrow[d, "\mu_A"'] & & B \otimes B \arrow[d, "\mu_B"] \\
A \arrow[rr, "\varphi"] & & B \\
& 1 \arrow[ul, "\eta_A"] \arrow[ur, "\eta_B"'] &
\end{tikzcd}$$

i.e., φ commutes with the monoid multiplications and units.

Claim 1.10. The monoidal unit 1 in a monoidal category $\mathsf{C}^{\otimes}$ admits a canonical monoidal structure.

Proof. It follows from the monoidal category axioms that $\lambda_1 = \rho_1$ (see Theorem 3′ of [1]). Thus, take $\eta = \mathrm{id}\colon 1 \to 1$ and $\mu = \lambda_1 = \rho_1\colon 1 \otimes 1 \to 1$. By this definition, unitality is immediate, and since μ is an isomorphism and $A = 1$, the associativity diagram collapses into the triangle identity, meaning 1 with $\eta = \mathrm{id}_1$ and $\mu = \lambda_1 = \rho_1$ is a monoid in $C^{\otimes}$.

Claim 1.11. The monoids in $\mathsf{Set}^{\times}$ are precisely ordinary monoids.

Proof. The definition clearly reduces to a unital associative binary operation on a set, which is just an ordinary monoid.

Claim 1.12. The monoids in Ab are precisely ordinary rings.

Proof. Follows from Claim 1.14: Since $\mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$, the scalar multiplication is redundant, and we obtain the structure for an ordinary unital ring.

Claim 1.13. The monoids in $\mathsf{Set}^{\amalg}$ are precisely ordinary sets. More generally, the monoids in $\mathsf{C}^{\amalg}$ are precisely objects of C (for C a cocomplete category).

Proof. Let $0 \in \mathsf{C}$ be the unit object (initial object in C) and let A be a monoid in C with unit $\eta\colon 0 \to A$ and multiplication $\mu\colon A \amalg A \to A$. Note that since

0 is initial, η is specified uniquely by A. Compare the unitality diagram with the universal property of coproducts. The unitors are defined as the inverses of the inclusions of A, so we see that in the unitality diagram, we have the two inclusions of A into $A \amalg A$, and implicitly, two maps $A \to A$, both the identity. Hence, by the universal property, μ is forced to be the fold map $\mathrm{id}_A \amalg \mathrm{id}_A : A \amalg A \to A$, and this works for any object A. It's also quite clearly associative, so each object of $\mathsf{C}^{\amalg}$ has exactly one monoid structure, with multiplication given by the fold map. In particular, monoids in $\mathsf{Set}^{\amalg}$ are just sets.

Claim 1.14. The monoids in Mod_R are precisely ordinary unital R-algebras.

Proof. Similarly, a monoid in Mod_R is a set with addition, scalar multiplication by R, and an associative unital multiplication, which is just a unital R-algebra.

Question 1.1. If A is a monoid, what are the monoids in the discrete monoidal category A? What about in BA?

Answer. A monoid a in A must have a morphism $1 \to a$, which is only possible if $a = 1$, and then $1 \cdot 1 = 1$, so we have only one choice: $\eta = \mu = \mathrm{id}_1$. This is clearly associative and unital: the arrows in the diagrams are all id_1. Hence there's just the "trivial monoid".

A monoid structure on the unique object $* \in BA$ consists of two morphisms $\eta\colon * \to *$ and $\mu\colon * \otimes * = * \to *$, i.e. two monoid elements $\eta, \mu \in A$. The unitality diagram reduces to $\mu\eta = 1$, and associativity reduces to the tautology $\mu^2 = \mu^2$. Hence, the data of a monoid in BA is a "split pair" in A; a factorization of the unit.

Question 1.2. What can we say about the monoids in Δ?

Answer. Δ is kind of like finite sets with coproduct, except we take into account order. Let μ, η be a monoid structure on $[n] \in \Delta$. For the same reason as with coproducts, η carries no information, and unitality reduces to the assertion that μ is the fold map. But this is only monotone in the case that $n = 0$. Thus, there is just one nontrivial monoid: the object $[0] = \{0\}$ with constant multiplication $\mu\colon [0] \oplus [0] = [1] \to [0]$. (This is clearly associative.)

Question 1.3. What can we say about the monoids in $\mathbb{R} \cup \{\pm\infty\}$?

Answer. In $\mathbb{R} \cup \{\infty\}$ with min as the tensor product, the only object with a morphism from ∞ is ∞ itself, which only has the canonical monoidal structure of Claim 1.10 since it's terminal. Hence, there's just the trivial monoid on ∞.

In $\mathbb{R} \cup \{-\infty\}$ with max as the tensor product, we have a unique choice for every object x (real number or $-\infty$): η is the unique morphism $-\infty \to x$ and $\mu = \mathrm{id}_x \colon \max(x, x) = x \to x$. Unitality and associativity are immediate since the diagrams just consist of id_x. Hence, there's one uninteresting monoid structure on every real number x and on $-\infty$.

Question 1.4. What are monoids in $\mathsf{Cat}^{\times}$?

Answer. It's a category C with a strictly associative and unital multiplication bifunctor $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$. This should sound familiar: it's just a strict monoidal category.

1.3 Modules over monoids

Definition 1.7. Let (A, η, μ) be a monoid in a monoidal category $\mathsf{C}^{\otimes}$. A *(left) module over* A is an object $M \in \mathsf{C}$ together with an "action" morphism $\nu \colon A \otimes M \to M$ such that the following diagrams commute:

$$\begin{tikzcd} 1 \otimes M \arrow[rr, "\eta \otimes \mathrm{id}_M"] \arrow[dr, "\lambda_M"'] & & A \otimes M \arrow[dl, "\nu"] \\ & M & \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes A) \otimes M \arrow[rr, "\alpha_{A,A,M}"] \arrow[d, "\mu \otimes \mathrm{id}_M"'] & & A \otimes (A \otimes M) \arrow[d, "\mathrm{id}_A \otimes \nu"] \\ A \otimes M \arrow[r, "\nu"'] & M & A \otimes M \arrow[l, "\nu"] \end{tikzcd}$$

(unitality and associativity). An *A-module homomorphism* between A-modules (M, ν_M) and (N, ν_N) is a morphism $f \colon M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "\mathrm{id}_A \otimes f"] \arrow[d, "\nu_M"] & A \otimes N \arrow[d, "\nu_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

that is, f commutes with the actions of A on M and N. Right A-modules and their morphisms are defined analogously. Given monoids (A, η_A, μ_A)

and (B, η_B, μ_B) in $\mathsf{C}^\otimes$, an (A, B)-*bimodule* is an object M with a left action of A and a right action of B, such that the two actions commute:

$$\begin{tikzcd}
(A \otimes M) \otimes B \arrow[rr, "\alpha"] \arrow[d, "\nu_A \otimes B"'] & & A \otimes (M \otimes B) \arrow[d, "A \otimes \nu_B"] \\
M \otimes B \arrow[r, "\nu_B"] & M & A \otimes M \arrow[l, "\nu_A"']
\end{tikzcd}$$

and an (A, B)-*bimodule homomorphism* is a morphism that's simultaneously a left A-module homomorphism and a right B-module homomorphism.

Example 1.3.1. A module M over the unit 1 (with the canonical monoid structure) in a monoidal category $\mathsf{C}^\otimes$ is just an object of C.

Proof. Module unitality is equivalent to $\nu = \lambda_M$ since $\eta = \mathrm{id}_1$, and then associativity is equivalent to the triangle identity. So each object has a unique 1-module structure, with action given by the left unitor.

Example 1.3.2. A module M over a monoid A in $\mathsf{Set}^\times$ is an M-set in the ordinary sense.

Proof. This is clear; an A-module structure on a set M is just a monoid homomorphism $A \to \mathrm{End}(M)$.

Example 1.3.3. A module M over a monoid (R-algebra) A in Mod_R is an A-module in the usual sense of "module over an R-algebra". In particular, for $\mathsf{Mod}_\mathbb{Z} = \mathsf{Ab}$, A is just a ring, and we recover the usual notion of "module over a ring A".

Proof. This is also clear.

Example 1.3.4. A module M over a monoid (object) A in $\mathsf{C}^\amalg$ is just a morphism $A \to M$.

Proof. Let $\nu\colon A \amalg M \to M$ be the action. Since $\lambda_M = \iota_M^{-1}$, where $\iota_M\colon M \to 1 \amalg M$ is the inclusion, unitality says that $\iota_M^{-1} = \nu(\eta \amalg \mathrm{id}_M)$, i.e. $\nu(\eta \amalg \mathrm{id}_M)\iota_M = \mathrm{id}_M$, i.e. $\nu\iota'_M = \mathrm{id}_M$ where $\iota'_M\colon M \to M \amalg A$ is the inclusion. That is, all the information of ν is in its A-component $\nu\iota_A$. It's easy to check that associativity holds in general for this, so $\nu\iota_A$ is an arbitrary morphism $A \to M$.

Example 1.3.5. Let M be a monoid. In the discrete monoidal category M, the modules over the unique monoid 1_M are precisely the elements of M. In BM, a module over the monoid given by the pair of elements $\mu, \eta \in M$ with $\mu\eta = 1$ is an element $\nu \in M$ such that $\nu\eta = 1$ and $\nu\mu = \nu^2$.

Proof. For the first part, a module over 1_M must have action map given by the unitor (which in this case is just the identity), since the $\eta \otimes \mathrm{id}$ arrow in the unitality diagram collapses to an identity. This also satisfies associativity since all the arrows are identities, so each object has a unique 1_M-module structure on the left and right.

For the second, the action morphism $\nu\colon * \otimes * \to *$ is just an element of M. Since $\lambda = \alpha = \mathrm{id}$, unitality reduces to $\nu\eta = 1$, and associativity reduces to $\nu\eta = \nu^2$.

Example 1.3.6. A module over the unique nontrivial monoid $[0]$ in Δ is just an ordinal $[n] \in \Delta$.

Proof. Just like with coproduct, unitality gives us $\nu\iota_n = \mathrm{id}_{[n]}$, where $\iota_n\colon [n] \to [0] \oplus [n] = [n+1]$ is the inclusion given by the $+1$ map. With this in mind, there's only one possibility for $\nu(0)$ that keeps ν monotone, that is, $\nu(0) = 0$. Now it's easy to check that this is associative. Thus any object $[n] \in \Delta$ has a unique $[0]$-module structure given by $\nu(0) = 0$, $\nu(k+1) = k$ $(k \in [n])$.

Example 1.3.7. A module over ∞ in $\mathbb{R} \cup \{\infty\}$ is just an object ($x \in \mathbb{R}$ or ∞). A module over $x \in \mathbb{R}$ in $\mathbb{R} \cup \{-\infty\}$ is a real number $y \geq x$.

Proof. The first part follows from Claim 1.10. For the second, we need a morphism $\nu\colon x \otimes y = \max(x, y) \to y$, which exists iff $y \geq x$. If this is the case, the morphism is just id_y, and the diagrams commute since the only morphism involved is id_y.

Example 1.3.8. A module D over a strict monoidal category $\mathsf{C}^{\otimes}$ in $\mathsf{Cat}^{\times}$ is a strict monoidal category action: a functor $F\colon \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ such that $F(1_{\mathsf{C}}, -) = \mathrm{id}_{\mathsf{D}}$ and $F(c \otimes c', d) = F(c, F(c', d))$.

Claim 1.15. Given any monoid A, we can endow A with a somewhat canonical (A, A)-bimodule structure.

Proof. We can take the left and right structures $\nu_\ell, \nu_r\colon A \otimes A \to A$ to both be μ. Associativity and unitality are both immediate.

Claim 1.16. Given $\varphi\colon A \to B$ a map of monoids, we get an (A, B)-bimodule structure on B.

Proof. We can define the right action of B on B by μ_B as above, and the left action $\nu_A\colon A \otimes B \to B$ we define as the composite $\mu_B(\varphi \otimes \mathrm{id}_B)\colon A \otimes B \to B \otimes B \to B$. We must verify that the action of A is unital and associative.

The unitality triangle breaks up as follows:

$$
\begin{tikzcd}
 & 1 \otimes B \arrow[dl, "\lambda_B"'] \arrow[d, "\eta_B \otimes \mathrm{id}_B"] \arrow[dr, "\eta_A \otimes \mathrm{id}_B"] & \\
B & B \otimes B \arrow[l, "\mu_B"] & A \otimes B \arrow[l, "\varphi \otimes \mathrm{id}_B"]
\end{tikzcd}
$$

The left triangle commutes since it expresses unitality for the monoid B, and the right triangle commutes since it is obtained by tensoring the unitality diagram for the monoid homomorphism φ with B. The associativity diagram breaks up like so:

$$
\begin{tikzcd}
(A \otimes A) \otimes B \arrow[rrrr, "\alpha_{A,A,B}"] \arrow[dd, "\mu_A \otimes \mathrm{id}_B"'] \arrow[dr, "(\varphi \otimes \varphi) \otimes \mathrm{id}_B"] & & & & A \otimes (A \otimes B) \arrow[dl, "\varphi \otimes (\varphi \otimes \mathrm{id}_B)"'] \arrow[d, "\mathrm{id}_A \otimes (\varphi \otimes \mathrm{id}_B)"] \\
 & (B \otimes B) \otimes B \arrow[rr, "\alpha_{B,B,B}"] \arrow[d, "\mu_B \otimes \mathrm{id}_B"'] & & B \otimes (B \otimes B) \arrow[d, "\mathrm{id}_B \otimes \mu_B"] & A \otimes (B \otimes B) \arrow[l, "\varphi \otimes \mathrm{id}_{B \otimes B}"] \arrow[d, "\mathrm{id}_A \otimes \mu_B"] \\
A \otimes B \arrow[r, "\varphi \otimes \mathrm{id}_B"'] & B \otimes B \arrow[r, "\mu_B"'] & B & B \otimes B \arrow[l, "\mu_B"] & A \otimes B \arrow[l, "\varphi \otimes \mathrm{id}_B"]
\end{tikzcd}
$$

The leftmost square commutes since it's obtained by tensoring the square which says φ is a homomorphism with B on the right. The upper trapezoid commutees by naturality of α. The lower middle square is associativity for B as a monoid. The triangle and square on the far right both commute by functoriality of $\otimes$. Therefore, ν_A is a right action of A on B, so B is an (A, B)-bimodule with this structure.

1.4 (co)Equalizers

Definition 1.8. Given two parallel morphisms $f, g\colon x \rightrightarrows y$ in a category C, their *(co)equalizer* is the (co)limit of the diagram $x \rightrightarrows y$. That is, their equalizer is the universal morphism $e\colon w \to x$ into x such that $fe = ge$, and their coequalizer is the universal morphism $h\colon y \to z$ out of y such that $hf = hg$.

Claim 1.17. In Ab, the (co)equalizer of $f\colon A \to B$ with 0 is the (co)kernel of f (with the inclusion or quotient).

Proof. If $g\colon Z \to A$ is any morphism such that $fg = 0$, then $\operatorname{im} g \subseteq \ker f$, so g factors through the inclusion of $\ker f$ into A. Similarly, if $g\colon B \to C$ is any morphism such that $gf = 0$, then $\operatorname{im} f \subseteq \ker g$, so g factors through the quotient onto $\operatorname{coker} f$.

Claim 1.18. Let C be one of the usual concrete category examples, such as Set, Group, Top, etc. The equalizer of $f, g\colon X \rightrightarrows Y$ is the subset of X of elements x such that $f(x) = g(x)$.

Proof. Let $E := \{x \in X : f(x) = g(x)\} \subseteq X$. Any other function $e\colon W \to X$ with $f(e(w)) = g(e(w))$ for all $w \in W$ has $e(w) \in E$, and hence factors through the inclusion of E into X. Also, E is an actual object in all the examples we mentioned. If this isn't true in some other example, if the colimit exists, it is the minimal object whose underlying set contains E.

Claim 1.19. The coequalizer of $f, g\colon A \rightrightarrows B$ in Set or Top is the quotient Q of B by the equivalence relation generated by $f(x) \sim g(x)$.

Proof. If $h\colon B \to C$ is a map such that $hf = hg$, then indeed $h(f(x)) = h(g(x))$ for all $x \in A$. Hence, it factors through the quotient $q\colon B \to Q$.

Claim 1.20. The coequalizer of $f, g\colon G \rightrightarrows H$ in Group is the quotient of H by the normal subgroup generated by the elements $f(x)g(x)^{-1}$.

Proof. The proof is pretty much the same as the previous three.

Question 1.5. Do (co)equalizers exist in our weird examples: posets, discrete categories, BA for A a monoid?

Answer. Two parallel morphisms in $\mathbb{R}$ (or any poset category) must be equal, hence their coequalizer is the identity on the codomain (the larger element). In a discrete category, two parallel morphisms are both the identity on some object, so the coequalizer is just that same identity. In BA, I'm not sure how else to characterize it; given two elements $a, b \in A$, it's the universal element c such that $ca = cb$. In many monoids, for many pairs, this won't exist: take any distinct elements in any monoid with left cancellation, such as $0, 1 \in \mathbb{N}$. But in, say, a field after forgetting addition, 0 is the coequalizer of any pair of elements, since it's the only element for which cancellation fails.

Claim 1.21. The coequalizer of the two functors $0, 1\colon \mathbb{1} \to \mathbb{2}$ is the monoid $B\mathbb{N}$ of the natural numbers under addition, with the map taking the non-identity morphism $0 \to 1$ to $1 \in \mathbb{N}$

Proof. Any functor $F\colon \mathbb{2} \to \mathsf{C}$ with $F0 = F1$ has 1 object in its image, $F(0) = F(1)$, so it factors through a 1-object category. A fortiori, it factors through $B\mathbb{N}$, since $B\mathbb{N}$ is the free monoid on one generator, meaning it admits a homomorphism to any monoid carrying the generator to any specified element, in particular, the endomorphism monoid of $F(0)$ and the element $F(0 \to 1)$.

1.5 Tensor product of modules over monoids

Definition 1.9. Let $\mathsf{C}^{\otimes}$ be a complete monoidal category, or at least one in which the following coequalizer always exists. Let A be a monoid in $\mathsf{C}^{\otimes}$, and let M be an right A-module and N be a left A-module. The *tensor product* $M \otimes_A N$ of M with N is the coequalizer of the parallel pair

$$\begin{tikzcd}
M \otimes (A \otimes N) \arrow[drr, "\mathrm{id}_M \otimes \nu_N"] & & \\
(M \otimes A) \otimes N \arrow[u, "\alpha_{M,A,N}"] \arrow[rr, "\nu_M \otimes \mathrm{id}_N"'] & & M \otimes N
\end{tikzcd}$$

Claim 1.22. Let A, B, C be monoids in $\mathsf{C}^{\otimes}$, and assume $\otimes$ commutes with coequalizers in both components. If M is an (A, B)-bimodule and N is a (B, C)-bimodule, we obtain a canonical (A, C)-bimodule structure on $M \otimes_B N$.

Proof. We'll construct the left A-module structure; the right C-module structure is analogous. We have a canonical right B-module structure on $A \otimes M$ with multiplication given by the composite $(A \otimes M) \otimes B \xrightarrow{\sim} A \otimes (M \otimes B) \to A \otimes M$. With this structure in mind, consider the following diagram:

$$\begin{tikzcd}
M \otimes B \otimes N \arrow[r, shift left] \arrow[r, shift right] & M \otimes N \arrow[r] & M \otimes_B N \\
(A \otimes M) \otimes B \otimes N \arrow[r, shift left] \arrow[r, shift right] & (A \otimes M) \otimes N \arrow[r] \arrow[u, "\ell_A"'] \arrow[d, "\alpha"] & (A \otimes M) \otimes_B N \arrow[u, dashed] \arrow[d, dashed, shift left] \\
A \otimes (M \otimes B \otimes N) \arrow[r, shift left] \arrow[r, shift right] & A \otimes (M \otimes N) \arrow[r] & A \otimes (M \otimes_B N) \arrow[u, "\star", shift left]
\end{tikzcd}$$

The first two rows are a priori coequalizers, but not the third. The third row is a coequalizer diagram by our assumption that $\otimes$ commutes with coequalizers. To verify unitality, we must establish the commutativity of the following diagram:

$$\begin{tikzcd}
1 \otimes (M \otimes_B N) \arrow[rr, "\eta\otimes(M\otimes_B N)"] \arrow[dr, "\lambda"'] & & A \otimes (M \otimes_B N) \arrow[dl, "\nu"] \\
& M \otimes_B N &
\end{tikzcd}$$

Since taking colimits is functorial, it is enough to establish a "commutative prism" in the pre-tensor setting. Modulo our isomorphism ($\star$) and associators (a bunch of extra commutative squares that would just make the diagram look horrible), the situation is

$$\begin{tikzcd}
1 \otimes (M \otimes_B N) \arrow[rr, "\eta\otimes(M\otimes_B N)"] \arrow[dr, "\lambda"'] & & A \otimes (M \otimes_B N) \arrow[dl, "\nu"] \\
& M \otimes_B N & \\
1 \otimes M \otimes N \arrow[uu] \arrow[rr, "\eta\otimes M\otimes N"] \arrow[dr, "\lambda"'] & & A \otimes M \otimes N \arrow[uu] \arrow[dl, "\nu\otimes N"] \\
& M \otimes N \arrow[uu] & \\
1 \otimes M \otimes B \otimes N \arrow[uu, shift left] \arrow[uu, shift right] \arrow[rr, "\eta\otimes M\otimes B\otimes N"] \arrow[dr, "\lambda"'] & & A \otimes M \otimes B \otimes N \arrow[uu, shift left] \arrow[uu, shift right] \arrow[dl, "\nu\otimes B\otimes N"] \\
& M \otimes B \otimes N \arrow[uu, shift left] \arrow[uu, shift right] &
\end{tikzcd}$$

The blue arrows correspond to B acting from the right on M, and red to B acting from the left on N. The vertical arrows from coequalizer diagrams. The ν on top is induced by the maps involving ν below, by definition. For the λ on top to be induced by the corresponding maps below, we need the colimit map $1 \otimes M \otimes N \to 1 \otimes (M \otimes_B N)$ to equal the map $1 \otimes (\otimes_B)$ where $\otimes_B$ denotes the tensor product map $M \otimes N \to M \otimes_B N$, which leads me to believe that we probably just want the monoidal product to commute with at least coequalizers. (Then it follows from naturality of λ.) The map involving η on top is induced by those below simply by functoriality of $\otimes$.

Now we must argue for commutativity of the blue and red squares on the bottom, and the triangles. The triangles commute since they're functorial images of the unitality triangle for the A-module M (the bottom one through $- \otimes (B \otimes N)$, and the middle through $- \otimes N$). The squares involving λ both commute by naturality of λ. The blue squares involving ν and η commute since M is an (A, B)-bimodule (after applying the functor $- \otimes N$), and the red ones commute by bifunctoriality of $\otimes$ (the orthogonal maps are in different components). Therefore, all faces the prism on the bottom commute, so the squares constitute maps between coequalizer diagrams fitting into a commutative triangle, meaning the induced triangle at the top is commutative, as desired.

We set up a similar argument for associativity (again, modulo associators and $(\star)$ isomorphisms so we can drop all brackets):

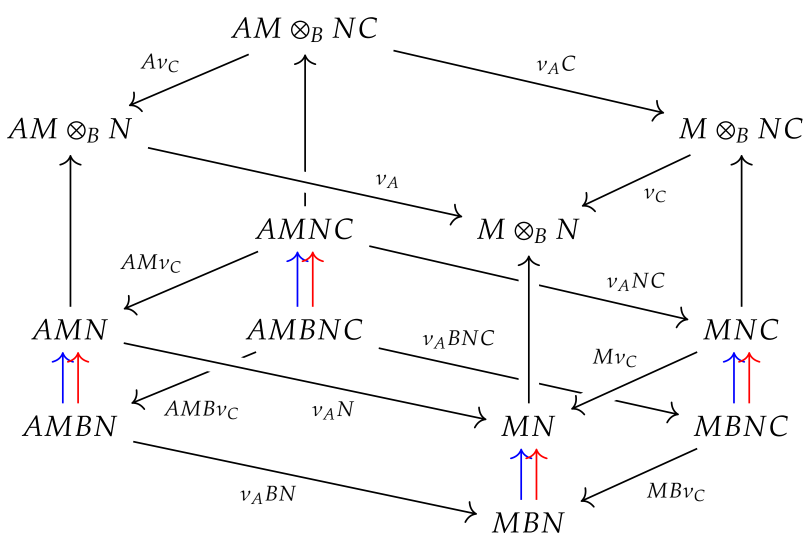
$$\begin{tikzcd}
 & AAM \otimes_B N \arrow[dl, "\mu M \otimes_B N"'] \arrow[drrr, "A\nu"] & & & \\
AM \otimes_B N \arrow[drrr, "\nu"] & & & & AM \otimes_B N \arrow[dl, "\nu"] \\
 & AAMN \arrow[uu] \arrow[dl, "\mu MN"'] \arrow[drrr, "A\nu N"] & & M \otimes_B N & \\
AMN \arrow[uu] \arrow[drrr, "\nu N"] & AAMBN \arrow[u, shift left, color=red] \arrow[u, shift right, color=blue] \arrow[dl, "\mu MBN"'] \arrow[drrr, "A\nu BN"] & & & AMN \arrow[uu] \arrow[dl, "\nu N"'] \\
AMBN \arrow[u, shift left, color=red] \arrow[u, shift right, color=blue] \arrow[drrr, "\nu BN"'] & & & MN \arrow[uu] & AMBN \arrow[u, shift left, color=red] \arrow[u, shift right, color=blue] \arrow[dl, "\nu BN"] \\
 & & & MBN \arrow[u, shift left, color=red] \arrow[u, shift right, color=blue] &
\end{tikzcd}$$

(I've dropped the $\otimes$ for monoidal product since it was getting cumbersome; can add them later.) Again the vertical arrows form coequalizer diagrams and we want to establish that the top square is induced by the bottom prism, and that the bottom prism commutes, so that the top square commutes. The ν morphisms on top are induced by those below by definition of ν, and the $A\nu$ face on the back is just the composition of the front face with the functor $A-$, so that's induced as well (so long as $- \otimes -$ commutes with coequalizers). The μ face commutes by functoriality of $- \otimes -$, so indeed the μ map up top is induced by the ones below.

The horizontal squares in the prism commute since they are obtained by applying the functors $-N$ and $-BN$ to the associativity square for M as an A-module. The blue ν squares commute since they are obtained by applying

$-N$ to the bimodule condition square for ${}_AM_B$, and the red ones commute since $- \otimes -$ is a bifunctor. The $A\nu$ squares follow from the previous sentence by applying the functor $A-$. Both the blue and red μ squares commute since $- \otimes -$ is a bifunctor. Therefore, the prism on the bottom commutes, so the induced square above commutes as desired, meaning we have associativity for the left A-module $M \otimes_B N$.

The same arguments (mirrored) apply to show that $M \otimes_B N$ is a left C-module. Now to show that it's an (A, C)-bimodule, we must show that these actions commute, using a similar argument one more time:



The ν_A and ν_C maps are induced by the squares below them by definition, and the other ones follow by applying the functors $A-$ and $-C$. The horizontal squares in the prism on the bottom commute by bifunctoriality of $- \otimes -$ (ν_A and ν_C are orthogonal). The rest of the squares on the bottom commute by either bifunctoriality of $- \otimes -$ or by taking the monoidal product with the bimodule condition square for ${}_AM_B$ or ${}_BN_C$. Therefore, $M \otimes_B N$ is an (A, C)-bimodule, as desired.

Question 1.6. What does the tensor product look like in our examples?

Answer. In $\mathsf{Ab}^{\otimes}$, of course we will obtain the usual definition of tensor product of modules over rings.

In $\mathsf{C}^{\amalg}$, where our monoids are objects and our modules are morphisms, our bimodules are thus cospans, and the tensor product of two cospans

$A \to M \leftarrow B \to N \leftarrow C$ is the pushout cospan

$$\begin{tikzcd}
 & & M \amalg_B N & & \\
 & M \arrow[ur] & & N \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C \arrow[ul]
\end{tikzcd}$$

This is because our coequalizer is of the two morphisms $B \rightrightarrows M \amalg N$, which is just the cospan $M \leftarrow B \to N$, so the tensor product is the universal cocone over this cospan, which is the pushout.

In $\mathbb{R} \cup \{-\infty\}$, the tensor product of modules $y, z \geq x$, over x is the coequalizer of two copies of the identity morphism $\max(y, x, z) \to \max(y, z)$, which is just $\max(y, z)$. If these are bimodules, of course their maximum is a bimodule in the obvious way.

Remark 1.1. In a cocartesian monoidal category $\mathsf{C}^{\amalg}$, the property that we required in the proof of Claim 1.22 always holds. For a quick way to see this, note that the coproduct is a colimit, and so are coequalizers, and colimits commute with each other, so for $A \in \mathsf{C}$, the functor $A \amalg -$ preserves coequalizers, and hence we obtain an inverse for our map $(A \amalg M) \amalg_B N \to A \amalg (M \amalg_B N)$. More explicitly, we can construct this map using the universal property of coproducts, and our knowledge that the tensor product here is the pushout cospan. To specify a map out of $A \amalg (M \amalg_B N)$, it suffices to specify a map out of A and maps out of M, N that complete a commutative square with the span $M \leftarrow B \to N$. Indeed, we have all these maps, as the tensor product map $(A \amalg M) \amalg N \to (A \amalg M) \amalg_B N$ (precomposed with the associator, or its inverse), and it's easy to see by definition that these complete a commutative square with $M \leftarrow B \to N$, and hence their coproduct factors uniquely through the pushout, yielding our desired inverse map.

Remark 1.2. The aforementioned condition holds trivially for a discrete monoidal category A. In a one-element monoidal category BA, the tensor product functor $* \otimes -$ is the identity, so certainly it preserves coproducts. And in any case, a bimodule structure there is just given by two commuting invertible elements of A, and the tensor product map for $*$ viewed as a left and right module over an invertible element μ (which are the same) is just the coequalizer of μ and itself, which is the identity.

2 The Morita (2-)category

2.1 The Morita category of a strict monoidal category

Claim 2.1. Let $\mathsf{C}^{\otimes}$ be a strict monoidal category with all tensor products, such that $\otimes$ commutes with coequalizers. Then there exists a category $m(\mathsf{C}^{\otimes})$ whose objects are monoids in $\mathsf{C}^{\otimes}$, morphisms $A \to B$ are bimodules ${}_AM_B$, and composition of morphisms $A \to B \to C$ is the tensor product ${}_AM \otimes_B N_C$.

Proof. We know all that stuff exists, so it remains to check unitality and associativity. If A is a monoid, we can view it in a canonical way as a bimodule ${}_AA_A$ over itself, with both actions given by multiplication. (Indeed, this is the construction from Claim 1.16 performed on the identity id_A.) Now, given a bimodule ${}_AM_B$, we'd like to know the coequalizer of the left and right multiplication maps $A \otimes A \otimes M \to A \otimes M$. But recall the "associativity" diagram for the left A-module M: it says precisely that $\nu\colon A \otimes M \to M$ is a cone over those two multiplication maps. Our assumption that $\otimes$ commutes with coequalizers implies that up to canonical isomorphism, ν is the coequalizer, and hence $A \otimes_A M = M$. We must also verify that the left and right actions are as desired: up to our chosen isomorphisms, the diagram defining the left action of A on $A \otimes_A M = M$ is

$$\begin{tikzcd} AAAM \arrow[r, shift left] \arrow[r, shift right] & AAM \arrow[r, "A\nu"] \arrow[d, "\mu M"] & AA \otimes_A M = AM \arrow[d, dashed] \\ AAM \arrow[r, shift left] \arrow[r, shift right] & AM \arrow[r, "\nu"] & A \otimes_A M = M \end{tikzcd}$$

By our assumption, the top row is a coequalizer diagram, so there exists a unique dashed arrow on the right completing that square, which we can recognize as the action $\nu\colon AM \to M$ in the associativity diagram for ${}_AM$, and this arrow is also by definition the action of A on $A \otimes_A M$. Thus the left actions are the same. By writing down the same diagram for the right action of B, the proof that the right action is what we want goes the same way, except instead of associativity of ${}_AM$ we use commutativity of the actions of ${}_AM_B$. Therefore, $m(\mathsf{C}^{\otimes})$ has all identities with respect to tensor product composition.

Turning now to associativity, let A, B, C, D be monoids in $\mathsf{C}^{\otimes}$ and let ${}_AM_B$, ${}_BN_C$, ${}_CP_D$ be three bimodules. We aim to show that ${}_A(M \otimes_B N) \otimes_C P_D =$

${}_A M \otimes_B (N \otimes_C P)_D$. Consider the following diagram:

$$
\begin{tikzcd}
 & & MBNCP \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & & \\
 & MNCP \arrow[dl] \arrow[dr, shift left] \arrow[dr, shift right] & & MBNP \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dr] & \\
M \otimes_B NCP \arrow[dr, dashed, shift left] \arrow[dr, dashed, shift right] & & MNP \arrow[dl] \arrow[dr] & & MBN \otimes_C P \arrow[dl, dashed, shift left] \arrow[dl, dashed, shift right] \\
 & M \otimes_B NP \arrow[dr] & & MN \otimes_C P \arrow[dl] & \\
 & & M \otimes_B N \otimes_C P & &
\end{tikzcd}
$$

The two compositions $MNCP \to M \otimes_B NP$ are both cones over the parallel pair $MBNCP \rightrightarrows MNCP$, so they both factor uniquely through $M \otimes_B NCP$, and so indeed they must be equal to the parallel pair $M \otimes_B NCP \rightrightarrows M \otimes_B NP$ corresponding to the actions of C on N and P, since those make the corresponding squares commute. The same is true on the right side of the diagram. Thus, at the bottom of the diagram we have $(M \otimes_B N) \otimes_C P$ and $M \otimes_B (N \otimes_C P)$. The Fubini theorem for colimits states that these are both equal to the colimit of the double square at the top, since it's a diagram over the product of two parallel pairs, so we can compute the coequalizers in either order. Therefore, we have $(M \otimes_B N) \otimes_C P = M \otimes_B (N \otimes_C P)$, so our composition is strictly associative, and therefore $m(\mathsf{C}^{\otimes})$ is a category.

Remark 2.1. This construction fails to yield a category if $\mathsf{C}^{\otimes}$ is only a non-strict monoidal category: strictly speaking, we choose representatives of the isomorphism classes of coequalizers to define the tensor product, but it could happen that we are forced to choose them inconsistently. This can be fixed if we take our morphisms to be isomorphism classes of modules instead.

Remark 2.2. A cocartesian monoidal category $\mathsf{C}^{\amalg}$ satisfies ($\star'$): given a monoid (object) $A \in \mathsf{C}$ and a module (morphism) $\nu \colon A \to M$, the monoid (and hence A-module) structure on A is just given by the identity $\mathrm{id}_A \colon A \to$

A, and the tensor product $A \otimes_A M$ is just given by the pushout of the cospan $A \leftarrow A \rightarrow M$. It is immediate to see that this pushout is the span $\nu\colon A \rightarrow M \leftarrow M : \mathrm{id}_M$. Hence, the canonical map $A \amalg_A M = M \rightarrow M$ is just the identity id_M, an isomorphism as desired.

2.2 The Morita 2-category of a strict monoidal category

Claim 2.2. Let $\mathsf{C}^{\otimes}$ be a strict monoidal category with all tensor products, such that $\otimes$ commutes with coequalizers. Then there exists a (strict) 2-category $m(\mathsf{C}^{\otimes})$ whose objects are monoids in $\mathsf{C}^{\otimes}$, morphisms $A \rightarrow B$ are bimodules ${}_AM_B$, 2-morphisms ${}_AM_B \Rightarrow {}_AN_B$ are bimodule homomorphisms, composition of morphisms $A \rightarrow B \rightarrow C$ is the tensor product ${}_AM \otimes_B N_C$, horizontal composition of 2-morphisms is the bimodule homomorphism induced between tensor products, and vertical composition is composition in C.

Proof. All that remains to be shown is that (A, B)-bimodules with their homomorphisms form a category and tensor product is bifunctorial. Looking back at the definition of bimodule homomorphism, it's easy to see that the identity map id_M of a module M is a module homomorphism (functoriality of $\otimes$), and the composition of two module homomorphisms still commutes with the actions (we can glue the commutativity diagrams together horizontally), and composition of morphisms is already strictly associative. Thus (A, B)-bimodules with their homomorphisms do form a category. Bifunctoriality of tensor product follows from functoriality of colimits. Therefore, $m(\mathsf{C}^{\otimes})$ as defined above forms a strict 2-category.

3 The Morita ∞-category

3.1 The $(\infty, 1)$-category

Definition 3.1. Let $C^{\otimes}$ be a monoidal category. Define a truncated simplicial set $X^t \in \mathsf{sSet}_{\leq 3}$ as follows:

$$X_0^t = \{\text{monoids } A\}$$

$$X_1^t = \{\text{bimodules } {}_AM_B\}$$

$$X_2^t = \left\{ \begin{tikzcd} & B \arrow[dr, "N"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "M"] \arrow[rr, "P"'] & {} & C \end{tikzcd} \quad \begin{array}{c} \text{monoids } A, B, C, \\ \text{bimodules } {}_AM_B, {}_BN_C, {}_AN_C, \\ \text{and a bimodule homomorphism} \\ \alpha \colon {}_A(M \otimes N)_C \to {}_AP_C \text{ which} \\ \text{coequalizes } M \otimes B \otimes N \rightrightarrows M \otimes N \end{array} \right\}$$

$$X_3^t = \left\{ \begin{tikzcd} B \arrow[r, "N"] & C \arrow[d, "P"] \\ A \arrow[u, "M"] \arrow[ur, "R"] \arrow[r, "Q"'] & D \arrow[ul, phantom, "\Searrow\alpha", near end] \arrow[ul, phantom, "\Swarrow\beta", near start] \end{tikzcd} \quad \begin{tikzcd} B \arrow[r, "N"] \arrow[dr, "S"] & C \arrow[d, "P"] \\ A \arrow[u, "M"] \arrow[r, "Q"'] & D \arrow[ul, phantom, "\Swarrow\gamma", near end] \arrow[ul, phantom, "\Searrow\delta", near start] \end{tikzcd} \quad \begin{array}{c} \text{tetrahedra with labels as} \\ \text{above such that the} \\ \text{compositions of bimodule} \\ \text{homomorphisms agree} \end{array} \right\}$$

where $\otimes$ denotes the *monoidal* product, and for X_3^t we mean all tuples of monoids $A, \ldots, D$, bimodules $M, \ldots, S$, and bimodule homomorphisms $\alpha, \ldots, \delta$ as in the diagrams, such that each bimodule homomorphism is a cone over the appropriate coequalizer diagram, and the two compositions $\beta \circ (\alpha \otimes P)$ and $\delta \circ (M \otimes \gamma) \colon {}_A(M \otimes N \otimes P)_D \to {}_AQ_D$ agree. The face and degeneracy maps are given in the obvious way from the diagrams.

Now extend X^t to a simplicial set $X \coloneqq \operatorname{cosk}_3 X^t$ via the coskeletal extension construction, defined as the right adjoint to the truncation functor $\mathsf{sSet} \to \mathsf{sSet}_{\leq 3}$ given by precomposition with the inclusion $\Delta_{\leq 3} \hookrightarrow \Delta$. This just fills in all possible simplices in dimensions higher than 3. If X is a quasicategory, we call X the *Morita $(\infty,1)$-category of* $C^{\otimes}$, and we denote this again by $m(C^{\otimes})$.

Claim 3.1. X is a quasicategory whenever $C^{\otimes}$ has all coequalizers and $\otimes$ commutes with colimits.

Claim 3.2. The nerve of $m(C^{\otimes})$ is just the ordinary Morita 1-category of $C^{\otimes}$ we constructed earlier.

Claim 3.3. The construction m is functorial: a monoidal functor $F \colon C^{\otimes} \to D^{\boxtimes}$ induces a map of simplicial sets $m(F) \colon m(C^{\otimes}) \to m(D^{\boxtimes})$, which becomes a quasifunctor when $m(C^{\otimes})$ and $m(D^{\boxtimes})$ are quasicategories.

References

[1] Max Kelly. "On MacLane's Conditions for Coherence of Natural Associativities, Commutativities, etc." In: *Journal of Algebra* 1 (1964), pp. 397–402. DOI: 10.1016/0021-8693%2864%2990018-3.