2. Bra-ket Manipulations and Normalization

a) $|x\rangle = c \cdot \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

$\langle x| = c^* \cdot (1-2i, 1)$

$\langle x|x\rangle = 1$

$= c^* c \, (1-2i \quad 1) \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

$= c^2 \left((1-2i)(1+2i) + 1(1) \right)$

$= c^2 (5+1) = 6c^2$

$6c^2 = 1, \quad c^2 = 1/6 \rightarrow c = \frac{1}{\sqrt{6}}$

b) compute: $\langle x| \cdot |R\rangle$

$\left(\frac{1}{\sqrt{6}} \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} \right)^\dagger \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ RHCP

$\langle x| = |x\rangle^\dagger = \left(\frac{1}{\sqrt{6}} \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} \right)^\dagger = \frac{1}{\sqrt{6}} \left((1+2i)^* \; 1^* \right) = \frac{1}{\sqrt{6}} (1-2i, 1)$

$\langle x|R\rangle = \frac{1}{\sqrt{6}} (1-2i, 1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$= \frac{1}{\sqrt{12}} \left[(1-2i, 1) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \right.$

$= \frac{1}{\sqrt{2}} \left[(1-2i)(1) + (1)(i) \right] = \frac{1}{\sqrt{12}} [1-2i+i]$

$= \frac{1}{\sqrt{12}} (1-i)$

$\langle x|R\rangle = \frac{1-i}{\sqrt{12}}$

we computed $\langle x|R\rangle$. next check $\langle R|x\rangle$

c) $\langle R| = |R\rangle^{\dagger} = \frac{1}{\sqrt{2}}(1, -i)$

$\langle R|x\rangle = \frac{1}{\sqrt{2}}(1, -i) \cdot \frac{1}{\sqrt{6}}\begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

$= \frac{1+i}{\sqrt{12}}$

$\langle R|x\rangle = (\langle x|R\rangle)^{*}$ complex conjugates

and $|R\rangle\langle R|x\rangle$ = projection of $|x\rangle$ onto direction of $|R\rangle$

↳ scalar multiple of a vector $|R\rangle$

inner product
complex scalar

d) Pauli Matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

← horizontal linear polarized

$\sigma_z|H\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |H\rangle$

$\sigma_x|D\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|D\rangle = |D\rangle$

$\sigma_y|R\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \cdot i \\ i \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}} = |R\rangle$

e) each of the above states $|H\rangle$, $|D\rangle$, and $|R\rangle$ are the Pauli matrix eigenvalues of σ_z, σ_x, and σ_y.

<u>eigenvalues</u>

$\det(\sigma_x - \lambda I) = \lambda^2 - 1 = 0, \ \lambda = \pm 1$

$\det(\sigma_y - \lambda I) = \lambda^2 - 1 = 0, \ \lambda = \pm 1$

σ_z is diagonal, so $\lambda = \pm 1$

f) $\sigma_x |R\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \cdot i + 1 \cdot 1 \\ 1 \cdot i + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$

$\sigma_x \sigma_y |H\rangle$ matrix operations right to left

$\sigma_y |H\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$

$\downarrow$

$\sigma_x (\ ) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot i \\ 1 \cdot 0 + 0 \cdot i \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$

$\sigma_x \sigma_y |H\rangle = i \cdot |H\rangle$

find scalar a and Pauli matrix σ_i : $\sigma_x \sigma_y = a \cdot \sigma_i$

$\sigma_x \sigma_y |H\rangle = i \, |H\rangle = i \cdot \sigma_z \, |H\rangle$ eigenvector

$\sigma_x \sigma_y = i \cdot \sigma_z \ \rightarrow \ a = i$

$\sigma_i = \sigma_z$

g) $\sigma_y \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i\cdot 1 & 0\cdot 1 \\ i\cdot 1 & i\cdot 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$\sigma_y \sigma_x = -i\sigma_z$

$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = i\sigma_z - -i\sigma_z = 2i\sigma_z$

Not commutative

h) $\sigma_x \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0+1\cdot 1 & 0\cdot 1+1\cdot 0 \\ 1\cdot 0+0\cdot 1 & 1\cdot 1+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

identity matrix when applying same Pauli matrix twice

i) general commutation: $[\sigma_i, \sigma_j] = 2i\sigma_\ell$

Google says: $2i\,\epsilon_{ijk}\sigma_k$
and Wikipedia link in pset

$[\sigma_x, \sigma_y] = 2i\sigma_z$ etc

j) unitary matrix M: $M^\dagger M = MM^\dagger = I$

$\sigma_x^\dagger = \sigma_x^T = \sigma_x$

$\sigma_x^\dagger \sigma_x = \sigma_x^2 = I$ ✓

k) $R(\theta) = e^{i\theta\sigma_z}|2\rangle$

$\exp(i\theta\sigma_x) = I + (i\theta\sigma_x) + \frac{(i\theta\sigma_x)^2}{2!} + \frac{(i\theta\sigma_x)^3}{3!} + \ldots$

$\sigma_x^n = \sigma_x$ | x = odd

$\sigma_x^n = I$ | x = even

$\exp(i\theta\sigma_x) = \cos\theta \cdot I + i\sin(\theta)\sigma_x = R_x(-2\theta)$

L rotated ↺ in x by -2θ

l) continue to rotate ↺ about x but $|D\rangle$ is already on x-axis, so it would be fixed

m) Yes, unitary: $M = \exp(i\theta\sigma_x)$

$$M^\dagger = M^{-1}$$

$$\begin{pmatrix} \cos(-\theta) & i\sin(-\theta) \\ i\sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

Problem 3 - Quantum Gates

a basic rotation gate composition → any single qubit gate

3a) $U = e^{i\phi} R_z(\gamma) R_x(\beta) R_z(\alpha)$

$R_z(\theta) = e^{-i\theta Z/2}$
$R_x(\theta) = e^{-i\theta X/2}$
$e^{i\phi}$ global phase that doesn't affect meas. outcomes

a matrix is unitary if $U^{+}U = I$
↳ preserves the norm

$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad |a|^2 + |c|^2 = 1$ → 1 constraint
$|b|^2 + |d|^2 = 1$ → 1 constraint
$a \cdot b + c \cdot d = 0$ → 2 constraint

→ 8 vars
4 complex numbers but 4 constraints → $8 - 4 = 4$ dof
$\phi, \alpha, \beta, \gamma$ → 4 real parameters
same as # of degrees of freedom in general 2x2 U

Now: show that this decomposition can match any unitary matrix

1. $R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$ phase diff btwn $|0\rangle$ and $|1\rangle$

2. $R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$

$U = e^{i\phi} R_z(\gamma) R_x(\beta) R_z(\alpha)$
rotate about z (phase), rotate about x (amplitudes), rotate again around z, global phase

Question 5 - BB84 Protocols

a) determine X_HWP_B angle settings for Bob

Alice wants to produce H, V, D, A from $f_{HWP}(\theta)$

$$|H\rangle = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{aligned} |H\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \\ |D\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ |V\rangle &= \begin{pmatrix}0\\1\end{pmatrix} \\ |A\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\end{aligned}$$

$$x_H = 0°$$

$$x_V = 45°$$

$$x_D = 22.5° \rightarrow 2x_A = 45°$$

$$x_A = -22.5° \rightarrow \cos(2x_A) = \frac{1}{\sqrt{2}}$$

b) Alice will transmit:

VDH AAV HDV DDV

c)

Bit #	Alice Basis	Bob Basis	Same Basis	P(1)
1	+	x	No	0.5
2	x	x	Yes	0
3	+	+	Yes	0
4	x	+	No	0.5
5	x	x	Yes	1
6	+	+	Yes	1
7	+	x	No	0.5
8	x	+	No	0.5
9	+	+	Yes	1
10	x	x	Yes	0
11	x	+	No	0.5
12	+	x	No	0.5

d) 001110

e) 110 1111 011 11

f) VVH VVA AAD VVV A

based on Eve's corresponding Random Basis

g) w/ same basis: 000110101011

unmatching bases

h) 101 111

i) Alice: full sequence
Alice and Bob bases

if Eve measured photons, Alice & Bob will detect her presence

↳ when Eve intercepts a photon, she has to randomly choose a basis

↳ if $\text{Basis}_{\text{Eve}} = \text{Basis}_{\text{Alice}}$ → correct bit

≠ basis → collapse quantum state

If A and B publicly compare key bits:

basis choices and bits should match

Eve's interference causes a bit error rate

change result: 1) Eve DOESN'T measure qubits
2) eve lucky guesses every time