A Review on Quantum Operator Scrambling within Classically Simulable Circuit Models

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(Dated: November 22, 2022)

Abstract Goes Here...

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I. INTRODUCTION

Understanding the evolution of quantum many-body systems presents challenging problems and has provided insightful results in condensed matter physics and quantum information [1]. Studying the dynamics of such systems is fundamentally a computational challenge due to the exponential growth of the Hilbert space with the number of qubits. However, quantum dynamics can be efficiently simulated in atypical quantum circuit models, which will be studied in hopes of understanding the spreading and scrambling of encoded local information. This process is known as quantum scrambling. More precisely, quantum scrambling describes the process in which local information encoded by a simple product operator, becomes 'scrambled' by a unitary time evolution amongst the large number of degrees of freedom in the system, such that, the operator becomes a highly complicated sum of product operators with no decoding protocol to recover the information. In recent years, the study of this scrambling process has yielded insightful results and new perspectives. These include new insights into black hole information and the AdS/CFT correspondence [2–7], quantum chaos, [8] and hydrodynamics in many-body systems [9–13].

In order to investigate this phenomenon, we turn to toy circuit models that can be analytically and numerical tractable, so that we can obtain a deeper understanding to apply to the generic cases of unitary evolution in many-body systems. As previously mentioned, some families of circuits are able to be efficiently simulated with polynomial effort on a classical computer. Two circuit models take the principal interest of this review, namely Clifford circuits and non-interacting fermi circuits. Clifford circuits and the Clifford group have played a key role in the study fault-tolerant quantum computing and, more recently, many-body physics via random Clifford unitaries [14, 15]. This is due to their efficient simulability on classical computers, with no constraints on the amount of entanglement, allowing for the dynamics of large numbers of qubits to be studied via stabilizer states.

Valiant [16] presented a new class of quantum circuits that can be simulated in polynomial time. This class of circuits was later identified and mapped onto a non-interacting fermion system, where the Hamiltonian describing the dynamics is quadratic in creation and annihilation operators. This particular fermion system was shown to be classically simulable by DiVincenzo and Terhal [17] as an extension of Valiant's findings. This model has also shown to exhibit

II. BACKGROUND THEORY

A. Quantum Information and Computing

Quantum Information is built upon the concept of quantum bits (qubits for short), represented as a linear combination of states in the standard basis, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α, β are complex probability amplitudes and the vectors, $|0\rangle$, $|1\rangle$ are the computational basis states that form the standard basis.

This is easily extended to multi-party or composite systems of n qubits. The total Hilbert space, $\mathcal{H}$ of such a system is defined as the tensor product of n subsystem

Hilbert Spaces,

$$\mathcal{H} = \otimes_n \mathcal{H}_n = \mathcal{H}_{n-1} \otimes \mathcal{H}_{n-2} \otimes \cdots \otimes \mathcal{H}_0 \tag{1}$$

Then we may write the computational basis states of this system, as strings of qubits, using the tensor product [18];

$$|x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_n\rangle \equiv |x_1 x_2 ... x_n\rangle.$$

The evolution and dynamics of such systems can be represented via quantum circuits, made up of quantum logic gates acting upon the qubits of a system. Analogous to a classical computer which is comprised of logic gates that act upon bit-strings of information. In contrast, quantum logic gates are linear operators acting on qubits, often represented in matrix form. This allows us to decompose a complicated unitary evolution into a sequence of linear transformations. A common practice is to create diagrams of such evolutions, with each quantum gate having their own symbol, analogous to circuit diagrams in classical computation, allowing the creation of complicated quantum circuitry that can be directly mapped to a sequence of linear operators acting on a one or more qubits. Some example gate symbols can be seen in Fig. 1.

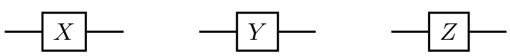

FIG. 1: Gate symbols for the Pauli operators in quantum circuits.

The gates shown in Fig. 1 are known as the Pauli operators, equivalent to the set of Pauli matrices, $P \equiv \{X, Y, Z\}$ for which X, Y and Z are defined in their matrix representation as;

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

These gates are all one-qubit gates, as they only act upon a single qubit. The simplest gate is that of the Identity operator (or matrix), I, from which, an algebra can be formed with the Pauli matrices. Satisfying the following relations:

$$XY = iZ \qquad YZ = iX \qquad ZX = iY \tag{3}$$

$$YX = -iZ \qquad ZY = -iX \qquad XZ = -iY \tag{4}$$

$$X^2 = Y^2 = Z^2 = I \tag{5}$$

Notably, the set of Pauli matrices and the identity form the Pauli group, $\mathcal{P}_n$, defined as the 4^n n-qubit tensor products of the Pauli matrices (2) and the Identity matrix, I, with multiplicative factors, ± 1 and $\pm i$ to ensure a legitimate group is formed. For clarity, consider the Pauli group on 1-qubit, $\mathcal{P}_1$;

$$\mathcal{P}_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX \pm Y, \pm iY, \pm Z, \pm iZ\}, \tag{6}$$

From this, another group of interest can be defined, namely the Clifford group, $\mathcal{C}_n$, defined as a subset of unitary operators that normalise the Pauli group [19]. The elements of this group are the Hadamard, Controlled-Not and Phase operators.

The Hadamard, H, maps computational basis states to a superposition of computational basis states, written explicitly in it's action;

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$
$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or in matrix form;

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Controlled-Not ($CNOT$) is a two-qubit gate. One qubit acts as a control for an operation to be perfomed on the other qubit. It's matrix representation is

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

The Phase operator, denoted S is defined as,

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix}$$

The CNOT operator is often used to create entanglement in circuits to generate states that cannot be written in product form. For example, if we prepare a Bell state $|\mathbf{\Phi}^+\rangle_= \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. We cannot write this as

$$\begin{aligned} |\mathbf{\Phi}^+\rangle &= [\alpha_0|0\rangle + \beta_0|1\rangle] \otimes [\alpha_1|0\rangle + \beta_1|1\rangle] \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \end{aligned}$$

since the α_0 or β_1 must be zero in order to ensure the $|01\rangle$, $|10\rangle$ vanish. However, this would make the coefficients of the $|00\rangle$ or $|11\rangle$ terms zero, breaking the equality. Thus, $|\mathbf{\Phi}^+\rangle$ cannot be written in product form and is said to be entangled. This defines a general condition for a arbitrary state to be entangled. [20]

III. CLASSICALLY SIMULABLE QUANTUM CIRCUITS

A. Clifford Circuits and Stabilizer Formalism

Quantum circuits comprised of only quantum gates from the Clifford group are known as *Clifford Circuits*. This family of circuits are of considerable interest due to their classical simulability, that is, circuits of this construction can be efficiently simulated on a classical computer with polynomial effort via the Gottesman-Knill theorem. For this reason, Clifford gates do not form a set of universal quantum gates, meaning a universal quantum computer cannot be constructed using a set of Clifford gates. A set of universal quantum gates allows for any unitary operation to be approximated to arbitrary accuracy by a quantum circuit, constructed using only the original set of gates. Clifford circuits are also known as Stabilizer circuits.

Stabilizer Formalism. An arbitrary, pure quantum state, $|\psi\rangle$ is *stabilized* by a unitary operator, S if $|\psi\rangle$ is an eigenvector of S, with eigenvalue 1, satisfying:

$$S|\psi\rangle = |\psi\rangle \tag{8}$$

The key idea here is that the quantum state can be described in terms of the unitaries that stabilize it, instead of the state itself. This is due to $|\psi\rangle$ being the unique state that is stabilized by S. [21] This can be seen from considering the Pauli matrices, and the unique states they stabilize. In the one-qubit case these are the +1 eigenstates of the pauli matrices (omitting normalisation factors);

$$X(|0\rangle + |1\rangle) = |0\rangle + |1\rangle \tag{9}$$
$$Y(|0\rangle + i|1\rangle) = |0\rangle + i|1\rangle \tag{10}$$
$$Z|0\rangle = |0\rangle \tag{11}$$

If given a group or subgroup of unitaries, $\mathcal{U}$, the vector space, V, of n qubit states is stabilized by $\mathcal{U}$ if every element of V is stable under action from every element of $\mathcal{U}$. This description is more appealing, as we can exploit mathematical techniques from group theory to describe quantum states and vector spaces. More precisely, the group $\mathcal{U}$ can be described using it's generators. In general, a set of elements $g_1, \ldots, g_d$ of a group, $\mathcal{G}$, generate the group if every element of $\mathcal{G}$ can be written as a product of elements from the set of generators, $g_1, \ldots, g_d$, such that $\mathcal{G} \coloneqq \langle g_1, \ldots, g_d\rangle$. For example, consider the group $\mathcal{U} \equiv \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$. $\mathcal{U}$ can be compactly written as $\mathcal{U} = \langle Z_1Z_2, Z_2Z_3\rangle$, by recognising $(Z_1Z_2)^2 = I$ and $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$.

This allows for the description of a quantum state, and subsequently it's dynamics, in terms of the generators of a stabilizer group. To see how the dynamics of a state are represented in terms of generators, consider a stabilizer state under the action of an arbitrary unitary operator:

$$UM|\psi\rangle = UMU^\dagger U|\psi\rangle \tag{12}$$

The state $|\psi\rangle$ is an eigenvector of M if and only if, $U|\psi\rangle$ is an eigenvector of $UMU^\dagger$. Thus, the application of an unitary operator transforms $M \to UMU^\dagger$. Moreover, if the state $|\psi\rangle$ is stabilized by M, then the evolved state $U|\psi\rangle$ will be stabilized by $UMU^\dagger$. If M is an element of a stabilizer group $\mathcal{S}$ such that $M_1, \ldots, M_l$ generate $\mathcal{S}$, then $UM_1U^\dagger, \ldots, UM_lU^\dagger$ must generate $U\mathcal{S}U^\dagger$. This implies that to compute the dynamics of a stabilizer, only the transformation of the generators needs to be considered [22]. It is because of this, Clifford circuits are able to be efficiently classically simulated via the Gottesman-Knill theorem.

Theorem 1 (Gottesman-Knill Theorem [23])**.** *Given an n qubit state $|\psi\rangle$, the following statements are equivalent:*

- *$|\psi\rangle$ can be obtained from $|0\rangle^{\otimes n}$ by CNOT, Hadamard and phase gates only.*
- *$|\psi\rangle$ is stabilized by exactly 2^n Pauli operators*
- *$|\psi\rangle$ can be uniquely identified by the group of Pauli operators that stabilize $|\psi\rangle$.*

B. Non-interacting Fermi Circuits

1. Fermionic Fock Space

Another class of circuits that are classically simulable, are known as non-interacting or free fermi circuits. These circuits are built from a fermionic Fock space formalism, where the basic units of information are local fermionic modes (LFMs). Each mode can be occupied or unoccupied in the same fashion as qubits, where a mode j is associated with an occupation number, $n_j = 0$ (unoccupied mode) or $n_j = 1$ (occupied mode). The state space of a many-fermion system is the fermionic Fock space, $\mathcal{F} = \mathcal{H}_0 + \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ correspond to subspaces of even and odd number of particles respectively. The natural basis vectors for a Fock space of m LFMs are Fock states $|n_0, \ldots, n_{m-1}\rangle$, where each element is the previously mentioned occupation number, n_j at each site [24]. This system is similar to the qubit set-up as the Hilbert space for m LFMs is identified with the Hilbert space of m qubits such that

$$|n_1, \ldots, n_{k-1}\rangle \equiv |n_0\rangle \otimes |n_1\rangle \otimes \cdots \otimes |n_{m-1}\rangle, \quad n_j = 0, 1$$

where the RHS of the expression descibes a system of m qubits. This allows a mapping of a qubit system (spin 1/2 system) onto a fermionic system via a Jordan-Wigner Transformation, such that the representation and dynamics of a quantum state are described using creation and annihilation operators from second quantization. The creation operator, $a_j^\dagger$ creates a fermion at mode j (if the mode is unoccupied), whilst the annihilation operator, a_j removes a fermion at site j if the mode is occupied. In the case of fermions, these operators obey canonical anti-commutation relations:

$$\{a_i, a_j\} \equiv \{a_i^\dagger, a_j^\dagger\} = 0 \qquad \{a_i, a_j^\dagger\} = \delta_{ij} I \tag{13}$$

The occupation number, n_j is then the eigenvalue of the number operator, $\hat{n}_j = a_j^\dagger a_j$ The creation and annihilation operators act on Fock states in the following way:

$$a_j^\dagger |n_0, \dots, n_j, \dots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s} \delta_{n_j,1} |n_0, \dots, n_j', \dots, n_{m-1}\rangle$$
$$a_j |n_0, \dots, n_j, \dots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s} \delta_{n_j,0} |n_0, \dots, n_j', \dots, n_{m-1}\rangle$$

Where n_j' is the updated occupation number at the jth site. To prepare an arbitrary Fock state, $|\mathbf{n}\rangle$, from a vacuum, $|\mathbf{0}\rangle = |0, 0, \dots, 0\rangle$, with fermions occupying arbitrary positions, creation operators are labeled to act on specified modes:

$$|\mathbf{n}\rangle = a_{i_1}^\dagger a_{i_2}^\dagger \dots a_{i_m}^\dagger |\mathbf{0}\rangle \tag{14}$$

2. Fermionic Circuits

A fermionic circuit is constructed from a sequence of elementary gates that mediate an interaction between modes j and k, written as $U = e^{iH_g}$.H_g is a general gate Hamiltonian:

$$H_g = b_{jj} a_j^\dagger a_j + b_{kk} a_k^\dagger a_k + b_{jk}^* a_k^\dagger a_j, \tag{15}$$

where b_{jj}, b_{jk}, b_{kk} are complex coefficients that form a hermitian matrix, $\mathbf{b}$. The circuit is constructed by defining the sequence of gates as $U = U_{\text{poly}(n)} \dots U_2 U_1$, such that the number of gates is no more than a polynomial in order to be simulated classically. Then considering the evolution of Fock state in a circuit, given that the circuit preserves the number of fermions ($U|\mathbf{0}\rangle = |\mathbf{0}\rangle$), the state evolves as:

$$U a_j^\dagger |\mathbf{0}\rangle = U a_j^\dagger U^\dagger U |\mathbf{0}\rangle = U a_j^\dagger U^\dagger |\mathbf{0}\rangle \tag{16}$$

Where U acts by conjugation as

$$U a_j^\dagger U^\dagger = \sum_s B_{is} a_s^\dagger \tag{17}$$

and the matrix $B = \exp(i\mathbf{b})$ is defined from the previous coefficient matrix in Eq. (15). To compute the final state of the system, only the matrix B needs evaluation, which is classically simulable if U contains a polynomial number of gates [17]. More generally, if U acts on an arbitrary Fock state, $|\mathbf{n}\rangle$, it's action is written as:

$$U|\mathbf{n}\rangle = U a_{i_1}^\dagger a_{i_2}^\dagger \dots a_{i_m}^\dagger |\mathbf{0}\rangle \tag{18}$$
$$= \sum_{j_1 \dots j_k} V_{i_1,j_1} \dots V_{i_k,j_k} a_{j_1}^\dagger a_{j_2}^\dagger \dots a_{j_k}^\dagger |\mathbf{0}\rangle \tag{19}$$

Some simpler operators that act on Fock states, are the spin 1/2 Pauli operators, that are mapped onto a non-interacting fermion system via the Jordan Wigner Transformation (JWT). In particular, the set of Pauli operators that act on the jth mode, $\{X_j, Y_j, Z_j\}$, can be expressed in terms of the creation and annihlation operators by first defining the operators $\sigma_j^\pm = \frac{1}{2}(X_j \pm iY_j)$. Then in terms of the creation and annihilation operators:

$$\sigma_j^+ = e^{i\pi \sum_{m=0}^{j-1} a_m^\dagger a_m} a_j^\dagger \tag{20}$$
$$\sigma_j^- = e^{i\pi \sum_{m=0}^{j-1} a_m^\dagger a_m} a_j \tag{21}$$

IV. OPERATOR SCRAMBLING

A. Operator Entanglement

One of the key papers originally investigated, was that of Blake and linden's where they showed

V. FURTHER RESEARCH

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