Quantum Scrambling

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Abstract goes here, state your claim!

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I. INTRODUCTION

Many-body systems and their dynamics play a central role in our understanding of modern physics. The dynamics of quantum many-body systems applies to a wide variety of fields in contemporary physics. Fields such as, quantum computation and information, modern condensed matter theory, quantum gravity.

II. QUBIT SYSTEMS

A. Quantum Bits

In the theory of quantum computation, a quantum bit (qubit) is a spin-$\frac{1}{2}$ particle or a two-level system, in which a single bit of information can be encoded. Unlike it's classical counterpart, where a bit occupies a binary state of 0 or 1, a qubit exists in a linear superposition of quantum states, expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex probability amplitudes. The states $|0\rangle$ and $|1\rangle$ form an orthonormal basis in the simplest Hilbert space, $\mathbb{C}^2$, and are known as computational basis states. To extend this description to a system with 2 or more qubits, the use of the tensor product is required. For example, consider two subsystems A and B, with their respective Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$ such that they each describe a single qubit. The total Hilbert space, $\mathcal{H}_{AB}$ for the two-qubit, is constructed from $\mathcal{H}_A$ and $\mathcal{H}_B$, as

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_A. \tag{1}$$

To generalise, the Hilbert space of an n qubit system is written as,

$$\mathcal{H} = \mathcal{H}^{\otimes n} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n. \tag{2}$$

The many-qubit states that span $\mathcal{H}$ are constructed identically, and are often expressed as a binary strings for a given configuration,

$$|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \equiv |x_1 x_2 \ldots x_n\rangle. \tag{3}$$

B. Quantum Circuits

The overarching aim of this *report* focuses on how many body systems, e.g an n qubit system, as described above, evolves in time. The evolution of a quantum system is described by a unitary transformation, that maps an initial configuration, $|\psi_0\rangle$ to a time-evolved configuration $|\psi\rangle$ as follows,

$$|\psi\rangle = U|\psi_0\rangle. \tag{4}$$

Where U is some unitary operator, $UU^\dagger = U^\dagger U = I$. In the context of qubit systems, unitary evolution may be deconstructed into a sequence of linear transformations acting on a finite subregion of the Hilbert space. This results in an intuitive description of many-body dynamics, where evolutions are represented as a circuit diagrams, with each time step in the unitary evolution corresponding to a quantum logic gate action upon a set of qubits. This is analogous to classical computation, where circuits are comprised of logic gates acting on bit-strings of information. In contrast, quantum logic gates are linear operators that have a distinct matrix representation [1].

Each quantum logic gate has a specified gate symbol, as can be seen in Fig. 1, allowing the creation of complicated quantum circuitry that can be directly mapped to simple matrix manipulations.

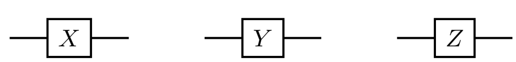

FIG. 1: Gate symbols for the Pauli operators in quantum circuits.

The gates shown in Fig. 1 are the Pauli operators, equivalent to the set of Pauli matrices, $P \equiv \{X, Y, Z\}$ for which X, Y and Z are defined in their matrix representation as

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{5}$$

These gates are all one-qubit gates, as they only act upon a single qubit. Together with the Identity operator, I, the Pauli matrices form an algebra, such that they satisfy the following relations:

$$XY = iZ, \qquad YZ = iX, \qquad ZX = iY, \tag{6}$$
$$YX = -iZ, \qquad ZY = -iX, \qquad XZ = -iY, \tag{7}$$

$$X^2 = Y^2 = Z^2 = I. \tag{8}$$

The set of Pauli matrices and the identity form the Pauli group, $\mathcal{P}_n$, defined as the 4^n n-qubit tensor products of the Pauli matrices (5) and the Identity matrix, I, with multiplicative factors, ± 1 and $\pm i$ to ensure a legitimate group is formed under multiplication. For clarity, consider the Pauli group on 1-qubit, $\mathcal{P}_1$:

$$\mathcal{P}_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX \pm Y, \pm iY, \pm Z, \pm iZ\}. \tag{9}$$

From this, another group of interest can be defined, namely the Clifford group, $\mathcal{C}_n$, defined as a subset of unitary operators that normalise the Pauli group *INSERT CLIFFORD GROUP DEFINITION*. Notable elements of this group are the Hadamard, Controlled-Not and Phase operators.

The Hadamard operator, H maps computational basis states to a superposition of computational basis states, written explicitly in it's action as,

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or in matrix form,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{10}$$

Controlled-NOT, $CNOT_{AB}$, is a controlled two-qubit gate. The first qubit, A acts as a 'control' for an operation to be performed on the target qubit, B. It's matrix representation is,

$$CNOT_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Phase operator, denoted R is defined as,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix}.$$

C. Entanglement in Qubit Systems

The CNOT operator is often used to an generate entangled state. One such state is the maximally entangled 2-qubit state, called a Bell state, $|\mathbf{\Phi}^+\rangle_= (|00\rangle + |11\rangle)/\sqrt{2}$. This is prepared from a $|00\rangle$ state, by applying a Hadamard to the first qubit, and subsequently a Controlled-Not gate:

I. $H \otimes I|00\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle$

II. $CNOT \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$

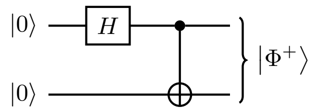

FIG. 2: Preparation of a Bell state from $|0\rangle$ using a Hadamard and CNOT.

The corresponding circuit representation of this preparation is given in Fig. 2.

The output Bell state, cannot be written in product form. That is, the state cannot be written as,

$$
\begin{aligned}
|\boldsymbol{\Phi}^{+}\rangle &= [\alpha_0|0\rangle + \beta_0|1\rangle] \otimes [\alpha_1|0\rangle + \beta_1|1\rangle] \\
&= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle
\end{aligned}
$$

since the α_0 or β_1 must be zero in order to ensure the $|01\rangle$, $|10\rangle$ vanish. However, this would make the coefficients of the $|00\rangle$ or $|11\rangle$ terms zero, breaking the equality. Thus, $|\boldsymbol{\Phi}^{+}\rangle$ cannot be written in product form and is said to be entangled. This defines a general condition for a arbitrary state to be entangled [2].

III. FERMIONIC SYSTEMS

The familiar qubit system may be mapped onto a system of identical particles (fermions), such that the overall many body state describing the system, is invariant under particle exchange. This is performed via a Jordan-Wigner transformation, which maps any local spin-model to a local fermionic model. To gain an understanding of how this can be carried out and why it is relevant, it will be useful to introduce the core concepts and language from *Second Quantization*.

A. Second Quantization and Indistinguishable Particles

A key distinction between classical and quantum mechanics, lies in the notion of the indistinguishability of particles. In quantum mechanics, particles are treated as identical and the exhanging of particles leaves the quantum many-body state unchanged, as opposed to classical mechanics where permutations on the set of particles' position vectors results in a new many-body state.

The construction of the n particle state via the extension of the single particle wavefunction, as described in (3), leads to a redundancy in it's description of a many-body state and an unnecessarily large Hilbert space. Second quantization remedies this redundancy by only considering the number of particles in each state, resulting in an efficient formalism to describe many-body systems.

In second quantization, the state of a many-body system is represented in an occupancy number basis, known as Fock states. For a given configuration, a Fock state may be written as $|n_1, n_2, \ldots, n_L\rangle$, where n_i is the occupation number. For fermions (corresponding to wavefunctions that are anti-symmetric under particle exchange), the max occupancy of a given site, i is $n_i = 1$. To preserve the symmetric properties of Fock states, second quantization introduces fermionic creation and annihilation operators. The creation operator, $a_i^\dagger$ creates a particle at site i if unoccupied, and the annihilation operator, a_i, removes a particle at site i if occupied. More formally, it's action on a Fock state may be written as,

$$
\begin{aligned}
a_j^\dagger|n_0, \ldots, n_j, \ldots, n_{m-1}\rangle &= (-1)^{\sum_{s=0}^{j-1} n_s}(1-n_j)|n_0, \ldots, n_j - 1, \ldots, n_{m-1}\rangle, \\
a_j|n_0, \ldots, n_j, \ldots, n_{m-1}\rangle &= (-1)^{\sum_{s=0}^{j-1} n_s} n_j|n_0, \ldots, n_j - 1, \ldots, n_{m-1}\rangle,
\end{aligned}
$$

B. Entanglement in Fermionic Systems

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[1] Any linear map between two finite dimensional vector spaces, in this case finite dimensional Hilbert spaces, may be represented as a matrix.

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition.* Cambridge University Press, 2010.