Quantum Scrambling

Samuel A. Hopkins[1*]
[1] *H. H. Wills Physics Laboratory, University of Bristol, Bristol, BS8 1TL, UK**
(Dated: April 18, 2023)

Abstract goes here, state your claim!

* Email: cn19407@bristol.ac.uk
Supervisor: Stephen R. Clark
Word Count: TBA

CONTENTS

I. INTRODUCTION

Many-body systems and their dynamics play a central role in our understanding of modern physics. The dynamics of quantum many-body systems applies to a wide variety of fields in contemporary physics. Fields such as, quantum computation and information, modern condensed matter theory, quantum gravity.

II. INTRODUCTION

Understanding the evolution of quantum many-body systems presents challenging problems and has provided insightful results in condensed matter physics and quantum information [**?**]. Studying the dynamics of such systems is fundamentally a computational challenge due to the exponential growth of the Hilbert space with the number of qubits. However, quantum dynamics can be efficiently simulated in atypical quantum circuit models, which will be analysed in hopes of understanding the spreading and scrambling of encoded local information, a process known as quantum scrambling. More precisely, quantum scrambling describes the process in which local information encoded by a simple product operator, becomes 'scrambled' by a unitary time evolution amongst the large number of degrees of freedom in the system, such that, the operator becomes a highly complicated sum of product operators. In recent years, the study of this scrambling process has yielded insightful results and new perspectives. These include new insights into black hole information and the AdS/CFT correspondence [**? ? ? ? ? ? ? ? ?**], quantum chaos, [**? ?**] and hydrodynamics in many-body systems [1**? ? ?**].

In order to investigate this phenomenon, we turn to simulable circuit models to gain a deeper understanding in hopes of applying any findings to generic cases of unitary evolution in many-body systems. As previously mentioned, some families of circuits are able to be efficiently simulated with polynomial effort on a classical computer. Two well-studied circuit models take the principal interest of this review, namely Clifford circuits and non-interacting fermi circuits. Clifford circuits and the Clifford group have played a key role in the study fault-tolerant quantum computing and, more recently, many-body physics via random Clifford unitaries [**? ?**]. This is due to their efficient simulability on classical computers, with no constraints on the amount of entanglement, allowing for the dynamics of large numbers of qubits to be studied via stabilizer states [**?**].

Valiant [**?**] presented a new class of quantum circuits that can be simulated in polynomial time. This class of circuits was later identified and mapped onto a non-interacting fermion system, where the Hamiltonian describing the dynamics is quadratic in creation a bnd annihilation operators. This particular fermion system was shown to be classically simulable by DiVincenzo and Terhal [**?**] as an extension of Valiant's findings and hence forms a good candidate to study scrambling phenomena.

With the use of recently developed techniques, the key objective of this project is to examine local operator spreading and scrambling in models where the dynamics are analytically and numerically tractable. This is not possible in the generic unitary evolution of many-body systems due to an exponential growth of the Hilbert space with system size. By simulating the dynamics of scrambling in simulable circuit models, a picture of unitary evolution can be constructed, in hopes of gaining an understanding of dynamics of generic unitary evolution whilst examining the differences between the two cases.

III. QUBIT SYSTEMS

A. Qubit Systems

1. Quantum Bits

In the theory of quantum computation, a quantum bit (qubit) is a spin-$\frac{1}{2}$ particle or a two-level system, in which a single bit of information can be encoded. Unlike it's classical counterpart, where a bit occupies a binary state of 0 or 1, a qubit exists in a linear superposition of quantum states, expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex probability amplitudes that satisfy $|\alpha|^2 + |\beta|^2 = 1$. The states $|0\rangle$ and $|1\rangle$ form an orthonormal basis in the simplest Hilbert space, $\mathbb{C}^2$, and are known as computational basis states. To extend this description to a system with

2 or more qubits, the use of the tensor product is required. For example, consider two subsystems A and B, with their respective Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$ such that they each describe a single qubit. The total Hilbert space, $\mathcal{H}_{AB}$ for the two-qubit, is constructed from $\mathcal{H}_A$ and $\mathcal{H}_B$, as

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_A. \tag{1}$$

To generalise, the Hilbert space of an n qubit system is written as,

$$\mathcal{H} = \mathcal{H}^{\otimes n} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n. \tag{2}$$

The many-qubit states that span $\mathcal{H}$ are constructed identically, and are often expressed as a binary strings for a given configuration,

$$|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \equiv |x_1 x_2 \ldots x_n\rangle. \tag{3}$$

2. Quantum Circuits

The overarching aim of this project focuses on the *interesting phenomena* that can be found in the evolution of quantum many-body systems, such as (2). The dynamics of such evolutions are described by the time-evolution operator, which that maps an initial configuration, $|\psi_0\rangle$ to a time-evolved configuration $|\psi(t)\rangle$ as follows,

$$|\psi(t)\rangle = U|\psi_0\rangle. \tag{4}$$

Where U is an arbitrary matrix from the unitary group, $U(2^N)$, acting on the total Hilbert space and satisfying $UU^\dagger = U^\dagger U = I$. Conviniently, unitary evolution may be deconstructed into a sequence of linear transformations acting on finite subregions of the Hilbert space, represented as a quantum circuit. The quantum circuit is constructed to act on a set of qubits, called a register, with each time-step corresponding to a specific action by a quantum logic gate. This is analogous to classical computation, where circuits are comprised of logic gates acting on bit-strings of information. In contrast, quantum logic gates are linear operators that have a distinct matrix representation [2].

Each notable quantum logic gate has a specified gate symbol, as can be seen in Fig. 1, allowing the creation of complicated diagrammatic quantum circuitry that can be directly mapped to simple matrix manipulations. *maybe talk about benefits on computation*

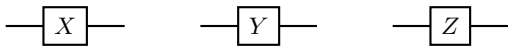

FIG. 1: Gate symbols for the Pauli operators in quantum circuits.

The gates shown in Fig. 1 are the Pauli operators, equivalent to the set of Pauli matrices, $P \equiv \{X, Y, Z\}$ for which X, Y and Z are defined in their matrix representation as

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{5}$$

These gates are all one-qubit gates, as they only act upon a single qubit. Together with the Identity operator, I, the Pauli matrices form an algebra, satisfying the following relations:

$$XY = iZ, \qquad YZ = iX, \qquad ZX = iY, \tag{6}$$

$$YX = -iZ, \qquad ZY = -iX, \qquad XZ = -iY, \tag{7}$$

$$X^2 = Y^2 = Z^2 = I. \tag{8}$$

The set of Pauli matrices and the identity form the Pauli group, $\mathcal{P}_n$, defined as the 4^n n-qubit tensor products of the Pauli matrices (5) and the Identity matrix, I, with multiplicative factors, ± 1 and $\pm i$ to ensure a legitimate group is formed under multiplication. For clarity, consider the Pauli group on 1-qubit, $\mathcal{P}_1$:

$$\mathcal{P}_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX \pm Y, \pm iY, \pm Z, \pm iZ\}. \tag{9}$$

From this, another group of interest can be defined, namely the Clifford group, $\mathcal{C}_n$, defined as a subset of unitary operators that normalise the Pauli group *INSERT CLIFFORD GROUP DEFINITION*. Notable elements of this group are the Hadamard, Controlled-Not and Phase operators.

The Hadamard operator, H maps computational basis states to a superposition of computational basis states, written explicitly in it's action as,

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad\qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or in matrix form,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{10}$$

Controlled-NOT, $CNOT_{AB}$, is a controlled two-qubit gate. The first qubit, A acts as a 'control' for an operation to be performed on the target qubit, B. It's matrix representation is,

$$CNOT_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Phase operator, denoted R is defined as,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix}.$$

IV. FERMIONIC SYSTEMS

A. Fermionic Systems

The familiar qubit system may be mapped onto a system of identical particles (fermions), such that the overall many body state describing the system, is invariant under particle exchange. This is performed via a Jordan-Wigner transformation, which maps any local spin-model to a local fermionic model. To gain an understanding of how this can be carried out and why it is relevant, it will be useful to introduce the core concepts and language from *Second Quantization*.

1. Second Quantization and Indistinguishable Particles

The wave function for a system of N identical particles is $\psi(x_1, x_2, \ldots x_N)$, where a particle is specified by it's position vector, $\vec{x}$. For bosonic systems, the wavefunction is symmetric under particle exhchange, while fermionic systems present anti-symmetric wavefunctions under particle exchange.

The construction of the n particle state via the extension of the single particle wavefunction, as described in (3), leads to a redundancy in it's description of a many-body state and an unnecessarily large Hilbert space.

A more efficient appraoch to describing many-body states is the formalism of second quantisation.Instead of describing states with Slater determinants, a *Fock state* presents an elegant and convinient basis in which to work

in. The Fock state of a many-body system is represented in an occupancy number basis, written as $|n_1, n_2, \ldots, n_L\rangle$, where n_i is the occupation number. For wavefunctions that are symmetric under particle exchange, i.e bosonic systems, the occupancy can take any real non-negative integer. For wavefunctions that are anti-symmetric under particle exchange (fermionic systems), the occupancy number is either 0 or 1 such that the state satisfies the Pauli exclusion principle.

To preserve the symmetric properties of Fock states, second quantization introduces fermionic creation and annihilation operators. The creation operator, $a_i^\dagger$ creates a particle at site i if unoccupied, and the annihilation operator, a_i, removes a particle at site i if occupied. More formally, it's action on a Fock state may be written as,

$$a_j^\dagger|n_0, \ldots, n_j, \ldots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s}(1 - n_j)|n_0, \ldots, n_j - 1, \ldots, n_{m-1}\rangle, \tag{11}$$

$$a_j|n_0, \ldots, n_j, \ldots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s} n_j|n_0, \ldots, n_j - 1, \ldots, n_{m-1}\rangle. \tag{12}$$

Using these operators, an arbitrary Fock state, $|\psi_F\rangle$ may be constructed from a set of creation operators acting on the vacuum,

$$|\psi_f\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \ldots (a_L^\dagger)^{n_L}|\mathbf{0}\rangle. \tag{13}$$

The fermionic creation and annihilation operators obey crucial anti-commutation relations, constructed to

$$\{a_j, a_k\} \equiv \{a_j^\dagger, a_k^\dagger\} = 0, \qquad \{a_j, a_k^\dagger\} = \delta_{jk} I \tag{14}$$

As an alternative description, the the *Majorana operators* may be defined as $c_{2k} = a_k + a_k^\dagger$ and $c_{2k+1} = i(a_k^\dagger - a_k)$ for $k = 1, \ldots, N$ such that they satisfy $\{c_j, c_k\} = 2\delta_{jk}$. From this, the Jordan-Wigner transformation may be defined, which maps a system of N spin-$\frac{1}{2}$ particles onto a system of N 'spinless' fermions as by defining the Majorana operators onto Pauli spin operators:

$$c \tag{15}$$

Such a mapping allows the Hilbert space of N qubits to be identified with the Hilbert space of N local fermionic modes (LFM's).

V. SCRAMBLING DYNAMICS

A. Random Unitary Circuits

To study the generic quantum many-body systems, random unitary circuits provide a minimally structured model that can emulate the required dynamics of generic or 'chaotic' unitary evolutions. *Include some motivations and works, this section will focus on literature findings to motivate why we didnt feel the need to explore more circuit desgins and operator spreading*

In the study of operator hydrodynamics The set-up is a chain of L spins, labelled as $q = 0, 1, \ldots, L-1$ each with their respective local dimension h and is subject to a random unitary circuit $\mathcal{U}$. The total circuit is constructed per timestep, by acting with a layer of 2-qubit unitaries on evenly-bonded spins, followed by a layer of unitaries applied to odd-bonded spins. More formally, each timestep in the circuit corresponds to an action by $U = U_{\text{even}}U_{\text{odd}}$, where U_{even} is given by the tensor product of individual 2-qubit unitaries $U_{q,q+1}$ applied to all evenly bonded sites, $q, q+1$ for q even, and U_{odd} is given by the tensor product of individual 2-qubit unitaries $U_{q,q+1}$ applied to all odd bonded sites, $q, q+1$ for q odd. Each 2-qubit unitary acts on nearest neighbour spins, and is drawn from the uniform (Haar) probability distribution on the unitary group $U(4)$ or from a specific subgroup [3].

* Image* The structure of this model can be seen in FIG.

To extract solvable dynamics and utilise well-estabilished measures, information about the state of the system is encoded as strings of operators. A convinient basis to work with in qubit systems is found in the set of Pauli

matrices. *insert weighted strings* For example, a system with $L = 5$ spins could be represented by the operator string $\mathcal{O} = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_i \otimes \mathbb{1} \otimes \mathbb{1}$, where σ_i is an arbitrary Pauli matrix.An operator of this form is *local* operator, as it acts non-trivially on a single site. For a system subject to generic unitary evolution , we expect that information encoded by an initially simple product operator to spread over the large number of degrees of freedom becoming highly complicated sum of global operators, such that they have support over the entire system. This process is known as Quantum Scrambling, and can be regarded as the combined notion of *Operator Spreading* and a growth in *Operator complexity.*

B. Operator Spreading

To give a picture of operator spreading, this section will primarily focus on the evolution of Pauli strings. Starting from some initially local product operator, such as $\mathcal{O} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. This system will evolve via $|\psi(t)\rangle = U|\psi_0\rangle$ and the operator string will evolve via the Heisenberg evolution of operators,

$$\mathcal{O}(t) = U(t)\mathcal{O}U^\dagger(t). \tag{16}$$

* Relevant work*

Following this evolution, the local operator with minimal support, O_j has evolved to $O_j(t)$ with support over a large region of sites [1]. Operators that grow in this way, will spread ballistically [4–6] and are often characterised by the out-of-time ordered correlator (OTOC) [7] and the square-commutator[8]. This also gives an intuitive picture of operator spreading, with the squared commutator defined as *Image here, sites, with support growing to overlap*

$$C(t) = \langle [O(t), W_i][O(t), W_i]^\dagger \rangle \tag{17}$$

where W_i is a static local operator at site i [9]. Then at $t = 0$, $O(t)$ acts on a single site or a finite region of sites, such that it commutes with the static operator, W_i and $C(t) = 0$. Once the operator spreads, and becomes more non-local, the commutator increases as it's support overlaps with V_i.

C. Operator Complexity and Entanglement Entropy

The key component in quantum supremacy lies in a quantum algorithm's ability to utilise the abudant *entanglement* within a system. In simple bipartite systems, where $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, the state of this system, $|\Psi_{AB}\rangle \in \mathcal{H}_{AB}$ is said to be entangled if and only if the state cannot be written in product form, $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, where $|\Psi_A\rangle$ and $|\Psi_B\rangle$ are the two vectors corresponding to the Hilbert spaces of each subsystem. If the state $|\Psi_{AB}\rangle$ is not entangled, then it is a product state and is said to be separable.

1. *Entanglement In Qubit Systems*

Qubit systems provide the simplest decription of entangled states in bipartite systems. For a system of two qubits, there exist 4 specific maximally entangled configurations, called Bell states:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

We can verify that the state, $|\Phi^+\rangle$ is an entangled state by writing,

$$\begin{aligned} |\Phi^+\rangle =& \left[\alpha_0|0\rangle + \beta_0|1\rangle\right] \otimes \left[\alpha_1|0\rangle + \beta_1|1\rangle\right], \\ =& \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \end{aligned}$$

We require that the α_0 or β_1 terms must be zero in order to ensure the $|01\rangle$, $|10\rangle$ vanish. However, this would make the coefficients of the $|00\rangle$ or $|11\rangle$ terms zero, breaking the equality. Thus, $|\mathbf{\Phi}^+\rangle$ cannot be written in product form and is said to be entangled.

For bipartite systems with a greater number of particles, this definition cannot be used to detect entanglement between states. Instead, entanglement measures provide a useful tool for detecting and verifying the entanglement between states and in the case of distinguishable particles, one such entanglement measure is the *Schmidt Rank*, derived from the *Schmidt Decomposition*. This measure provides a nice introduction to how entanglement measures are constructed in bipartite systems, but are not fit for the purposes of this project. A full description can be found in Appendix (X).

2. Entanglement In Fermionic Systems

3. Entanglement Measures

VI. SIMULATING A QUANTUM CIRCUIT

A. Stabilizer Circuits

B. Blake and Linden's Construction

C. Free Fermion Circuits

VII. RESULTS

A. Stabilizer Circuits

B. Non-interacting Fermion Circuits

1. Non-Number Conserving Gates

2. Number-Conserving Gates

VIII. DISCUSSION

IX. CONCLUSION

---

[1] V. Khemani, A. Vishwanath, and D. A. Huse, "Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws," *Physical Review X*, vol. 8, sep 2018.

[2] Any linear map between two finite dimensional vector spaces, in this case finite dimensional Hilbert spaces, may be represented as a matrix.

[3] N. Hunter-Jones, "Operator growth in random quantum circuits with symmetry," 2018.

[4] D. A. Roberts, D. Stanford, and L. Susskind, "Localized shocks," *Journal of High Energy Physics*, vol. 2015, mar 2015.

[5] E. H. Lieb and D. W. Robinson, "The finite group velocity of quantum spin systems," *Commun. Math. Phys.*, vol. 28, pp. 251–257, 1972.

[6] T. Schuster and N. Y. Yao, "Operator growth in open quantum systems," 2022.

[7] S. Xu and B. Swingle, "Scrambling Dynamics and Out-of-Time Ordered Correlators in Quantum Many-Body Systems: a Tutorial," feb 2022.

[8] M. Blake, H. Lee, and H. Liu, "A quantum hydrodynamical description for scrambling and many-body chaos," *Journal of High Energy Physics*, vol. 2018, oct 2018.

[9] X. Chen and T. Zhou, "Operator scrambling and quantum chaos," 2018.