Quantum Scrambling Review*

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Abstract Goes Here...

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INTRODUCTION

Understanding the evolution of quantum many-body systems presents challenging problems and has provided insightful results into diverse areas. Studying the dynamics of such systems is fundamentally a computational challenge due to the exponential growth of the Hilbert space with the number of qubits. However, quantum dynamics can be efficiently simulated in particular quantum circuits which provide minimal models for a wide range of complex phenomena. Some examples of such quantum circuits are Clifford circuits and non-interacting fermi circuits, which will be of principal interest in this review. These circuits will form the playground in which to study the phenomena known as *Quantum Scrambling*. Quantum Scrambling describes the process in local information encoded by a simple product operator becomes 'scrambled' by a unitary time evolution amongst the large number of degrees of freedom in the system, such that the operator becomes a highly complicated sum of product operators.

BACKGROUND THEORY

Quantum Information and Computing

Quantum Information is built upon the concept of quantum bits (qubits for short), represented as a linear combination of states in the standard basis, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α, β are complex probability amplitudes and the vectors, $|0\rangle$, $|1\rangle$ are the computational basis states that form the standard basis.

This is easily extended to multi-party or composite systems of n qubits. The total Hilbert space, $\mathcal{H}$ of such a system is defined as the tensor product of n subsystem Hilbert Spaces,

$$\mathcal{H} = \otimes_n \mathcal{H}_n = \mathcal{H}_{n-1} \otimes \mathcal{H}_{n-2} \otimes \cdots \otimes \mathcal{H}_0 \tag{1}$$

Then we may write the computational basis states of this system, as strings of qubits, using the tensor product [1];

$$|x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_n\rangle \equiv |x_1 x_2 ... x_n\rangle.$$

The evolution and dynamics of such systems can be represented via quantum circuits, made up of quantum logic gates acting upon the qubits of a system. Analogous to a classical computer which is comprised of logic gates that act upon bit-strings of information. In contrast, quantum logic gates are linear operators acting on qubits, often represented in matrix form. This allows us to decompose a complicated unitary evolution into a sequence of linear transformations. A common practice is to create diagrams of such evolutions, with each quantum gate having their own symbol, analogous to circuit diagrams in classical computation, allowing the creation of complicated quantum circuitry that can be directly mapped to a sequence of linear operators acting on a one or more qubits. Some example gate symbols can be seen in Fig. 1.

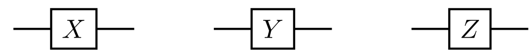

FIG. 1: Gate symbols for the Pauli operators in quantum circuits.

The gates shown in Fig. 1 are known as the Pauli operators, equivalent to the set of Pauli matrices, $P \equiv \{X, Y, Z\}$ for which X, Y and Z are defined in their matrix representation as;

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

These gates are all one-qubit gates, as they only act upon a single qubit. The simplest gate is that of the Identity operator (or matrix), I, from which, an algebra can be formed with the Pauli matrices. Satisfying the following relations:

$$XY = iZ \qquad YZ = iX \qquad ZX = iY \tag{3}$$

$$YX = -iZ \qquad ZY = -iX \qquad XZ = -iY \tag{4}$$

$$X^2 = Y^2 = Z^2 = I \tag{5}$$

Notably, the set of Pauli matrices and the identity form the Pauli group, $\mathcal{P}_n$, defined as the 4^n n-qubit tensor products of the Pauli matrices (2) and the Identity matrix, I, with multiplicative factors, ± 1 and $\pm i$ to ensure a legitimate group is formed. For clarity, consider the Pauli group on 1-qubit, $\mathcal{P}_1$;

$$\mathcal{P}_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX \pm Y, \pm iY, \pm Z, \pm iZ\}, \tag{6}$$

From this, another group of interest can be defined, namely the Clifford group, $\mathcal{C}_n$, defined as a subset of unitary operators that normalise the Pauli group

[2]. The elements of this group are the Hadamard, Controlled-Not and Phase operators.

The Hadamard, H, maps computational basis states to a superposition of computational basis states, written explicitly in it's action;

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$
$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or in matrix form;

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Controlled-Not ($CNOT$) is a two-qubit gate. One qubit acts as a control for an operation to be perfomed on the other qubit. It's matrix representation is

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

The Phase operator, denoted S is defined as,

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix}$$

The CNOT operator is often used to create entanglement in circuits to generate states that cannot be written in product form. For example, if we prepare a Bell state $|\mathbf{\Phi}^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. We cannot write this as

$$\begin{aligned} |\mathbf{\Phi}^+\rangle &= [\alpha_0|0\rangle + \beta_0|1\rangle] \otimes [\alpha_1|0\rangle + \beta_1|1\rangle] \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \end{aligned}$$

since the α_0 or β_1 must be zero in order to ensure the $|01\rangle$, $|10\rangle$ vanish. However, this would make the coefficients of the $|00\rangle$ or $|11\rangle$ terms zero, breaking the equality. Thus, $|\mathbf{\Phi}^+\rangle$ cannot be written in product form and is said to be entangled. This defines a general condition for a arbitrary state to be entangled. [3]

CLASSICALLY SIMULABLE QUANTUM CIRCUITS

Clifford Circuits and Stabilizer Formalism

Quantum circuits comprised of only quantum gates from the Clifford group are known as *Clifford Circuits*. This family of circuits are of considerable interest due to their classical simulability, that is, circuits of this construction can be efficiently simulated on a classical computer with polynomial effort via the Gottesman-Knill theorem. For this reason, Clifford gates do not form a set of universal quantum gates, meaning a universal quantum computer cannot be constructed using a set of Clifford gates. A set of universal quantum gates allows for any unitary operation to be approximated to arbitrary accuracy by a quantum circuit, constructed using only the original set of gates. Clifford circuits are also known as Stabilizer circuits.

Stabilizer Formalism. An arbitrary, pure quantum state, $|\psi\rangle$ is *stabilized* by a unitary operator, S if $|\psi\rangle$ is an eigenvector of S, with eigenvalue 1, satisfying:

$$S|\psi\rangle = |\psi\rangle \tag{8}$$

The key idea here is that the quantum state can be described in terms of the unitaries that stabilize it, instead of the state itself. This is due to $|\psi\rangle$ being the unique state that is stabilized by S. [4] This can be seen from considering the Pauli matrices, and the unique states they stabilize. In the one-qubit case these are the $+1$ eigenstates of the pauli matrices (omitting normalisation factors);

$$X(|0\rangle + |1\rangle) = |0\rangle + |1\rangle \tag{9}$$

$$Y(|0\rangle + i|1\rangle) = |0\rangle + i|1\rangle \tag{10}$$

$$Z|0\rangle = |0\rangle \tag{11}$$

If given a group or subgroup of unitaries, $\mathcal{U}$, the vector space, V, of n qubit states is stabilized by $\mathcal{U}$ if every element of V is stable under action from every element of $\mathcal{U}$. This description is more appealing, as we can exploit mathematical techniques from group theory to describe quantum states and vector spaces. More precisely, the group $\mathcal{U}$ can be described using it's generators. In general, a set of elements $g_1, \dots, g_d$ of a group, $\mathcal{G}$, generate the group if every element of $\mathcal{G}$ can be written as a product of elements from the set of generators, $g_1, \dots, g_d$, such that $\mathcal{G} := \langle g_1, \dots, g_d\rangle$. For example, consider the group $\mathcal{U} \equiv \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$. $\mathcal{U}$ can be compactly written as $\mathcal{U} = \langle Z_1Z_2, Z_2Z_3\rangle$, by recognising $(Z_1Z_2)^2 = I$ and $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$.

This allows for the description of a quantum state, and subsequently it's dynamics, in terms of the generators of a stabilizer group. To see how the dynamics of a state are represented in terms of generators, consider a stabilizer state under the action of an arbitrary unitary operator:

$$UM|\psi\rangle = UMU^\dagger U|\psi\rangle \tag{12}$$

The state $|\psi\rangle$ is an eigenvector of M if and only if, $U|\psi\rangle$ is an eigenvector of $UMU^\dagger$. Thus, the application of an unitary operator transforms $M \to UMU^\dagger$. Moreover, if the state $|\psi\rangle$ is stabilized by M, then the evolved state $U|\psi\rangle$ will be stabilized by $UMU^\dagger$. If M is an element of a stabilizer group $\mathcal{S}$ such that $M_1, \dots, M_l$ generate $\mathcal{S}$, then $UM_1U^\dagger, \dots, UM_lU^\dagger$ must generate $U\mathcal{S}U^\dagger$. This implies that to compute the dynamics of a stabilizer, only the transformation of the generators needs to be considered [5]. It is because of this, Clifford circuits are able to be efficiently classically simulated via the Gottesman-Knill theorem.

Theorem 1 (Gottesman-Knill Theorem [6])**.** *Given an n qubit state $|\psi\rangle$, the following statements are equivalent:*

- *$|\psi\rangle$ can be obtained from $|0\rangle^{\otimes n}$ by CNOT, Hadamard and phase gates only.*
- *$|\psi\rangle$ is stabilized by exactly 2^n Pauli operators*
- *$|\psi\rangle$ can be uniquely identified by the group of Pauli operators that stabilize $|\psi\rangle$.*

Free-Fermion Circuits

Another class of circuits that are classically simulable, are known as free-fermion circuits. These circuits were built from a special set of 2 qubit gates, acting on nearest neighbouring qubits, first introduced by Valiant [7] and later mapped onto a non-interacting fermion system by DiVincenzo and Terhal [8].

The fermion system is setup as n qubit computational basis states, $|x_0, \ldots, x_{n-1}\rangle$ which are identified with n local fermionic modes (LFM's), each of which having an occupation number $n_j = 0, 1$ at the j-th mode. $n_j = 0$ corresponds to an unoccupied mode, while $n_j = 1$ corresponds to an occupied mode. The mapping of a qubit system (spin 1/2 system) onto a fermionic system is done via a Jordan-Wigner Transformation, such that the representation and dynamics of a quantum state are described using creation and annihilation operators from second quantization. The creation operator, $a_j^\dagger$ creates a fermion at mode j (if the mode is unoccupied), whilst the annihilation operator, a_j removes a fermion at site j if the mode is occupied. In the case of fermions, these operators obey canonical anti-commutation relations:

$$\{a_j, a_k\} \equiv \{a_j^\dagger, a_k^\dagger\} = 0 \qquad \{a_j, a_k^\dagger\} = \delta_{jk} I \tag{13}$$

The Hilbert space of such a system is called a Fock space, for which the basis vectors are Fock states, $|n_1, n_2, \ldots n_l\rangle$. The elements of which, are the occupation number at each mode. The creation and annihilation operators act on Fock states in the following way:

$$a_j^\dagger|\mathbf{n}\rangle = (-1)^{\sum_{m=0}^{j-1}}\delta_{n_j,1}|\mathbf{n}'\rangle$$
$$a_j|\mathbf{n}\rangle = (-1)^{\sum_{m=0}^{j-1}}\delta_{n_j,0}|\mathbf{n}'\rangle$$

Where $|\mathbf{n}\rangle = |n_1, \ldots, n_j, \ldots, n_l\rangle$ and $|\mathbf{n}'\rangle = |n_1, \ldots, n_j', \ldots, n_l\rangle$ includes the updated occupation number at site j. Analagous to stabilizer states, an arbitrary Fock state, $|\mathbf{n}\rangle$, can be created from a vacuum state via creation operators,

$$|\mathbf{n}\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots(a_l^\dagger)^{n_l}|\mathbf{0}\rangle \tag{14}$$

where $|\mathbf{0}\rangle = |0, 0, \ldots, 0\rangle$ is a vacuum state of size l. Then a number operator can be defined, $\hat{n}_j = c_j^\dagger c_j$, such that it's action on a Fock state returns the occupation number on site j. As previously mentioned, the Pauli operators can be mapped onto a non-interacting fermion system via the Jordan-Wigner transformation (JWT). In particular, the set of Pauli operators that act on the jth site, $\{X_j, Y_j, Z_j\}$, can be expressed in terms of the creation and annihlation operators by first defining the operators $\sigma_j^\pm = \frac{1}{2}(X_j \pm iY_j)$. Then in terms of the creation and annihilation operators:

$$\sigma_j^+ = e^{i\pi\sum_{m=0}^{j-1} a_m^\dagger a_m} a_j^\dagger \tag{15}$$
$$\sigma_j^- = e^{i\pi\sum_{m=0}^{j-1} a_m^\dagger a_m} a_j \tag{16}$$

OPERATOR SCRAMBLING

Operator Entanglement

One of the key papers originally investigated, was that of Blake and linden's where they showed

FURTHER RESEARCH

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