A Review on Quantum Scrambling within Classically Simulable Circuit Models

Samuel A. Hopkins[1*]
[1] *H. H. Wills Physics Laboratory, University of Bristol, Bristol, BS8 1TL, UK**
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CONTENTS

I. INTRODUCTION

Understanding the evolution of quantum many-body systems presents challenging problems and has provided insightful results in condensed matter physics and quantum information [1]. Studying the dynamics of such systems is fundamentally a computational challenge due to the exponential growth of the Hilbert space with the number of qubits. However, quantum dynamics can be efficiently simulated in atypical quantum circuit models, which will be analysed in hopes of understanding the spreading and scrambling of encoded local information, a process known as quantum scrambling. More precisely, quantum scrambling describes the process in which local information encoded by a simple product operator, becomes 'scrambled' by a unitary time evolution amongst the large number of degrees of freedom in the system, such that, the operator becomes a highly complicated sum of product operators. In recent years, the study of this scrambling process has yielded insightful results and new perspectives. These include new insights into black hole information and the AdS/CFT correspondence [2–9], quantum chaos, [10, 11] and hydrodynamics in many-body systems [12–15].

In order to investigate this phenomenon, we turn to simulable circuit models to gain a deeper understandingin hopes of applying any findings to generic cases of unitary evolution in many-body systems. As previously mentioned, some families of circuits are able to be efficiently simulated with polynomial effort on a classical computer. Two well-studied circuit models take the principal interest of this review, namely Clifford circuits and non-interacting fermi circuits. Clifford circuits and the Clifford group have played a key role in the study fault-tolerant quantum computing and, more recently, many-body physics via random Clifford unitaries [16, 17]. This is due to their efficient simulability on classical computers, with no constraints on the amount of entanglement, allowing for the dynamics of large numbers of qubits to be studied via stabilizer states [18].

Valiant [19] presented a new class of quantum circuits that can be simulated in polynomial time. This class of circuits was later identified and mapped onto a non-interacting fermion system, where the Hamiltonian describing the dynamics is quadratic in creation and annihilation operators. This particular fermion system was shown to be classically simulable by DiVincenzo and Terhal [20] as an extension of Valiant's findings and hence forms a good candidate to study scrambling phenomena.

With the use of recently developed techniques, the key objective of this project is to examine local operator spreading and scrambling in models where the dynamics are analytically and numerically tractable. This is not possible in the generic unitary evolution of many-body systems due to an exponential growth of the Hilbert space with system size. By simulating the dynamics of scrambling in simulable circuit models, a picture of unitary evolution can be constructed, in hopes of gaining an understanding of dynamics of generic unitary evolution

* Email: cn19407@bristol.ac.uk
Supervisor: Stephen R. Clark
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whilst examining the differences between the two cases.

II. BACKGROUND THEORY

A. Quantum Information and Computing

Quantum Information and Computation is built upon the concept of quantum bits (qubits for short), represented as a linear combination of states in the standard basis, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α, β are complex probability amplitudes and the vectors, $|0\rangle$, $|1\rangle$ are the computational basis states that form the standard basis.

This is extended to multi-party or composite systems of n qubits via the tensor product. The total Hilbert space, $\mathcal{H}$ of a many-body system is defined as the tensor product of n subsystem Hilbert Spaces,

$$\mathcal{H} = \otimes_n \mathcal{H}_n = \mathcal{H}_{n-1} \otimes \mathcal{H}_{n-2} \otimes \cdots \otimes \mathcal{H}_0 \tag{1}$$

The computational basis states of this system, are tensor products of qubit states, often written as a string [21],

$$|x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_n\rangle \equiv |x_1 x_2 ... x_n\rangle.$$

The evolution and dynamics of many-body systems can be represented via quantum circuits, constructed from a set of quantum logic gates acting upon the qubits of a system. Analogous to a classical computer which is comprised of logic gates that act upon bit-strings of information. In contrast, quantum logic gates are linear operators acting on qubits, often represented in matrix form. This allows for the decomposition of a unitary evolution into a sequence of linear transformations. A common practice is to create diagrams of such evolutions, with each quantum gate having their own symbol, analogous to circuit diagrams in classical computation, allowing the creation of complicated quantum circuitry that can be directly mapped to a sequence of linear operators acting on a one or more qubits. Some example gate symbols can be seen in Fig. 1.

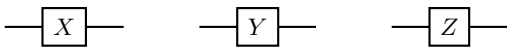

FIG. 1: Gate symbols for the Pauli operators in quantum circuits.

The gates shown in Fig. 1 are known as the Pauli operators, equivalent to the set of Pauli matrices, $P \equiv \{X, Y, Z\}$ for which X, Y and Z are defined in their matrix representation as;

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

These gates are all one-qubit gates, as they only act upon a single qubit. Together withthe Identity operator, I, the Pauli matrices form an algebra . Satisfying the following relations:

$$XY = iZ \qquad YZ = iX \qquad ZX = iY \tag{3}$$

$$YX = -iZ \qquad ZY = -iX \qquad XZ = -iY \tag{4}$$

$$X^2 = Y^2 = Z^2 = I \tag{5}$$

Notably, the set of Pauli matrices and the identity form the Pauli group, $\mathcal{P}_n$, defined as the 4^n n-qubit tensor products of the Pauli matrices (2) and the Identity matrix, I, with multiplicative factors, ± 1 and $\pm i$ to ensure a legitimate group is formed. For clarity, consider the Pauli group on 1-qubit, $\mathcal{P}_1$;

$$\mathcal{P}_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX \pm Y, \pm iY, \pm Z, \pm iZ\}, \tag{6}$$

From this, another group of interest can be defined, namely the Clifford group, $\mathcal{C}_n$, defined as a subset of unitary operators that normalise the Pauli group [22]. The elements of this group are the Hadamard, Controlled-Not and Phase operators.

The Hadamard, H, maps computational basis states to a superposition of computational basis states, written explicitly in it's action;

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$
$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or in matrix form;

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Controlled-Not ($CNOT$) is a two-qubit gate. One qubit acts as a control for an operation to be perfomed on the other qubit. It's matrix representation is

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

The Phase operator, denoted S is defined as,

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix}$$

The CNOT operator is often used to an generate entangled state. One such state is the maximally entangled 2-qubit state, called a Bell state, $|\mathbf{\Phi}^+\rangle_=(|00\rangle +$

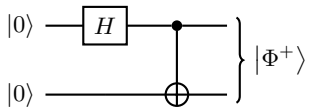

FIG. 2: Preparation of a Bell state from $|0\rangle$ using a Hadamard and CNOT.

$|11\rangle)/\sqrt{2}$. This is prepared from a $|00\rangle$ state, by applying a Hadamard to the first qubit, and subsequently a Controlled-Not gate:

$$H \otimes I|00\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle \tag{8}$$

$$CNOT\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{9}$$

The corresponding circuit representation of this preparation is given in Fig. 2.

The output Bell state, cannot be written in product form. That is, the state cannot be written as,

$$\begin{aligned}|\mathbf{\Phi}^+\rangle =& [\alpha_0|0\rangle + \beta_0|1\rangle] \otimes [\alpha_1|0\rangle + \beta_1|1\rangle] \\ =& \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle\end{aligned}$$

since the α_0 or β_1 must be zero in order to ensure the $|01\rangle$, $|10\rangle$ vanish. However, this would make the coefficients of the $|00\rangle$ or $|11\rangle$ terms zero, breaking the equality. Thus, $|\mathbf{\Phi}^+\rangle$ cannot be written in product form and is said to be entangled. This defines a general condition for a arbitrary state to be entangled [23].

III. CLASSICALLY SIMULABLE QUANTUM CIRCUITS

A. Clifford Circuits and Stabilizer Formalism

Quantum circuits comprised of only quantum gates from the Clifford group are known as *Clifford Circuits*. This family of circuits are of considerable interest due to their classical simulability and ability to showcase complex phenomena [24], that is, circuits of this construction can be efficiently simulated on a classical computer with polynomial effort via the Gottesman-Knill theorem. For this reason, Clifford gates do not form a set of universal quantum gates, meaning a universal quantum computer cannot be constructed using only Clifford unitaries. A set of universal quantum gates allows for any unitary operation to be approximated to arbitrary accuracy by a quantum circuit, constructed using only the original set of gates.

1. Stabilizer Formalism

An arbitrary, pure quantum state, $|\psi\rangle$ is *stabilized* by a unitary operator, M if $|\psi\rangle$ is an eigenvector of M, with eigenvalue 1, satisfying:

$$M|\psi\rangle = |\psi\rangle \tag{10}$$

It will be convinient to utilise the language of stabilizers, to define an initial state by the operators that stabilize it [25]. This can be seen from considering the Pauli matrices, and the unique states they stabilize. In the one-qubit case these are the $+1$ eigenstates of the pauli matrices (omitting normalisation factors);

$$X(|0\rangle + |1\rangle) = |0\rangle + |1\rangle \tag{11}$$

$$Y(|0\rangle + i|1\rangle) = |0\rangle + i|1\rangle \tag{12}$$

$$Z|0\rangle = |0\rangle \tag{13}$$

If given a group or subgroup of unitaries, $\mathcal{U}$, the vector space, V_n, of n qubit states is stabilized by $\mathcal{U}$ if every element of V_n is stable under action from every element of $\mathcal{U}$. This description is more appealing, as we can exploit mathematical techniques from group theory to describe quantum states and vector spaces. In particular, a group $\mathcal{U}$ can be described using it's generators. In general, a set of elements $g_1, \ldots, g_d$ of a group, $\mathcal{G}$, generate the group if every element of $\mathcal{G}$ can be written as a product of elements from the set of generators, $g_1, \ldots, g_d$, such that $\mathcal{G}$ is expressed as $\mathcal{G} \coloneqq \langle g_1, \ldots, g_d\rangle$ [26].

This allows for the description of a quantum state, and subsequently it's dynamics, in terms of the generators of a stabilizer group. To see how the dynamics of a state are represented in terms of generators, consider a stabilizer state under the action of an arbitrary unitary operator:

$$UM|\psi\rangle = UMU^\dagger U|\psi\rangle \tag{14}$$

The state $|\psi\rangle$ is an eigenvector of M if and only if, $U|\psi\rangle$ is an eigenvector of $UMU^\dagger$. Thus, the application of an unitary operator transforms $M \to UMU^\dagger$. Moreover, if the state $|\psi\rangle$ is stabilized by M, then the evolved state $U|\psi\rangle$ will be stabilized by $UMU^\dagger$. If M is an element of a stabilizer group $\mathcal{S}$ such that $M_1, \ldots, M_l$ generate $\mathcal{S}$, then $UM_1U^\dagger, \ldots, UM_lU^\dagger$ must generate $U\mathcal{S}U^\dagger$. This implies that to compute the dynamics of a stabilizer, only the transformation of the generators needs to be considered [27]. It is because of this, Clifford circuits are able to be efficiently classically simulated via the Gottesman-Knill theorem:

Theorem 1 (Gottesman-Knill Theorem [28])**.** *Given an n qubit state $|\psi\rangle$, the following statements are equivalent:*

- $|\psi\rangle$ *can be obtained from* $|0\rangle^{\otimes n}$ *by CNOT, Hadamard and phase gates only.*
- $|\psi\rangle$ *is stabilized by exactly* 2^n *Pauli operators*
- $|\psi\rangle$ *can be uniquely identified by the group of Pauli operators that stabilize* $|\psi\rangle$*.*

Evolution via unitaries from the Clifford group, take an appealing form when acting on simple Pauli operators that act as stabilizers. In particular, Clifford unitaries map pauli operators to products of pauli operators, and since any product of Pauli operators can be written in terms of only products of X and Z, a generic Pauli string may be written as,

$$S \propto X_1^{v_{1_x}} Z_1^{v_{1_z}} \dots X_N^{v_{N_x}} Z_N^{v_{N_z}} \tag{15}$$

and can be specified by a binary vector,

$$\vec{v} = (v_{1_x}, v_{1_z}, \dots, v_{N_x}, v_{N_z}). \tag{16}$$

Thus, conjugation by Clifford unitaries on S, corresponds to local updates in $\vec{v}$.

The operation of Clifford unitaries on Pauli operators by conjuagtion is shown in Table I.

Clifford Unitary	Operation
CNOT	$X_1 \to X_1X_2$ $X_2 \to X_2$ $Z_1 \to Z_1$ $Z_2 \to Z_1Z_2$
H	$X \to Z$ $Z \to X$
S	$X \to X$ $Z \to -Z$

TABLE I: Action of Unitary operators from the Clifford group on Pauli operators by conjuagation.

B. Non-interacting Fermi Circuits

1. Fermionic Fock Space

Another class of circuits that are classically simulable, are known as non-interacting or free fermi circuits. These circuits are derived from a fermionic Fock space formalism, where the basic units of information are local fermionic modes (LFMs). Each mode can be occupied or unoccupied in the same fashion as qubits, where a mode j is associated with an occupation number, $n_j = 0$ (unoccupied mode) or $n_j = 1$ (occupied mode). The state space of a many-fermion system is the fermionic Fock space, $\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ correspond to subspaces of even and odd number of particles respectively. The natural basis vectors for a Fock space of m LFMs are Fock states $|n_0, \dots, n_{m-1}\rangle$, where each element is the previously mentioned occupation number, n_j at each site [29]. This system is similar to the qubit set-up as the Hilbert space for m LFMs is identified with the Hilbert space of m qubits such that

$$|n_0, \dots, n_{m-1}\rangle \equiv |n_0\rangle \otimes |n_1\rangle \otimes \dots \otimes |n_{m-1}\rangle, \quad n_j = 0, 1$$

where the RHS of the expression descibes a system of m qubits. This allows a mapping of a qubit system (spin 1/2 system) onto a fermionic system via a Jordan-Wigner Transformation, such that the representation and dynamics of a quantum state are described using creation and annihilation operators from second quantization. The creation operator, $a_j^\dagger$ creates a fermion at mode j (if the mode is unoccupied), whilst the annihilation operator, a_j removes a fermion at site j if the mode is occupied. In the case of fermions, these operators obey canonical anti-commutation relations:

$$\{a_i, a_j\} \equiv \{a_i^\dagger, a_j^\dagger\} = 0 \qquad \{a_i, a_j^\dagger\} = \delta_{ij} I \tag{17}$$

The occupation number, n_j is then the eigenvalue of the number operator, $\hat{n}_j = a_j^\dagger a_j$ The creation and annihilation operators act on Fock states in the following way:

$$a_j^\dagger |n_0, \dots, n_j, \dots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s} \delta_{n_j,1} |n_0, \dots, n_j', \dots, n_{m-1}\rangle$$
$$a_j |n_0, \dots, n_j, \dots, n_{m-1}\rangle = (-1)^{\sum_{s=0}^{j-1} n_s} \delta_{n_j,0} |n_0, \dots, n_j', \dots, n_{m-1}\rangle$$

Where n_j' is the updated occupation number at the jth site. To prepare an arbitrary Fock state, $|\mathbf{n}\rangle$, from a vacuum, $|\mathbf{0}\rangle = |0, 0, \dots, 0\rangle$, with fermions occupying arbitrary positions, creation operators are labeled to act

on specified modes:

$$|\mathbf{n}\rangle = a^\dagger_{i_1} a^\dagger_{i_2} \dots a^\dagger_{i_m} |\mathbf{0}\rangle \tag{18}$$

2. Fermionic Circuits

A fermionic circuit is constructed from a sequence of elementary gates that mediate an interaction between modes j and k, written as $U = e^{iH_g}$. Where H_g is a general gate Hamiltonian:

$$H_g = b_{jj} a^\dagger_j a_j + b_{kk} a^\dagger_k a_k + b^*_{jk} a^\dagger_k a_j, \tag{19}$$

where b_{jj}, b_{jk}, b_{kk} are complex coefficients that form a hermitian matrix, $\mathbf{b}$. The circuit is constructed by defining the sequence of gates as $U = U_{(n)} \dots U_2 U_1$, such that the number of gates, n is no more than polynomial in n in order to be simulated classically. Then considering the evolution of Fock state in a circuit, given that the circuit preserves the number of fermions ($U|\mathbf{0}\rangle = |\mathbf{0}\rangle$), the state evolves as:

$$U a^\dagger_j |\mathbf{0}\rangle = U a^\dagger_j U^\dagger U |\mathbf{0}\rangle = U a^\dagger_j U^\dagger |\mathbf{0}\rangle \tag{20}$$

Where U acts by conjugation as

$$U a^\dagger_j U^\dagger = \sum_s B_{is} a^\dagger_s \tag{21}$$

and the matrix $B = \exp(i\mathbf{b})$ is defined from the previous coefficient matrix in Eq. (19). To compute the final state of the system, only the matrix B needs evaluation, which is classically simulable if U contains a polynomial number of gates [20]. More generally, if U acts on an arbitrary Fock state, $|\mathbf{n}\rangle$, it's action is written as:

$$U|\mathbf{n}\rangle = U a^\dagger_{i_1} a^\dagger_{i_2} \dots a^\dagger_{i_m} |\mathbf{0}\rangle \tag{22}$$

$$= \sum_{j_1 \dots j_k} B_{i_1,j_1} \dots B_{i_k,j_k} a^\dagger_{j_1} a^\dagger_{j_2} \dots a^\dagger_{j_k} |\mathbf{0}\rangle \tag{23}$$

Some simpler operators that act on Fock states, are the spin 1/2 Pauli operators, that are mapped onto a non-interacting fermion system via the Jordan Wigner Transformation (JWT). In particular, the set of Pauli operators that act on the jth mode, $\{X_j, Y_j, Z_j\}$, can be expressed in terms of the creation and annihlation operators by first defining the operators $\sigma^\pm_j = \frac{1}{2}(X_j \pm iY_j)$. Then in terms of the creation and annihilation operators:

$$\sigma^+_j = e^{i\pi \sum_{m=0}^{j-1} a^\dagger_m a_m} a^\dagger_j \tag{24}$$

$$\sigma^-_j = e^{i\pi \sum_{m=0}^{j-1} a^\dagger_m a_m} a_j \tag{25}$$

The inverse transformation can also be made, to define the Majorana fermion operators that effectively split each LFM into halves [30];

$$c_{2j} = a_j + a^\dagger_j \qquad c_{2j+1} = \frac{a_j - a^\dagger_j}{i} \tag{26}$$

Majorana fermion operators satisfy the anti-commutation relation;

$$\{c_j, c_k\} = 2\delta_{jk} \tag{27}$$

and therefore can be expressed in terms of Pauli operators,

$$c_{2j} = X_j \prod_{k=0}^{j-1} Z_k \qquad c_{2j+1} = Y_j \prod_{k=0}^{j-1} Z_k \tag{28}$$

where the term $\prod_{k=0}^{j-1} Z_k$ is called a Jordan Wigner string.

IV. OPERATOR SCRAMBLING

A. Operator Spreading

As previously mentioned, quantum scrambling is the process in which local information encoded by a simple product operator is rapidly spread over a large number of degrees of freedom via a unitary time evolution. The information becomes highly non-local and the initial simple product operator becomes a complicated sum of product operators. To give picture of operator spreading, consider a local operator, O_j, which has support on either one site or a finite region of sites, so that it acts like the Identity on sites without support. The simplest local operator would be a Pauli operator acting on a single site, j, e.g $O_j \equiv X_j$. Then the evolution of O_j, is given, in the heisenberg picture, by $O_j(t) = U^\dagger(t) O_j U(t)$, where $U(t)$ is some unitary time evolution operator. Following this evolution, the local operator with minimal support, O_j has evolved to $O_j(t)$ with support over a large region of sites [12]. Operators that grow in this way, will spread ballistically [31–33] and are often characterised by the out-of-time ordered correlator (OTOC) [34] and the square-commutator[35]. This also gives an intuitive picture of operator spreading, with the squared commutator defined as

$$C(t) = \langle [O(t), W_i][O(t), W_i]^\dagger \rangle \tag{29}$$

where W_i is a static local operator at site i [36]. Then at $t = 0$, $O(t)$ acts on a single site or a finite region of

sites, such that it commutes with the static operator, W_i and $C(t) = 0$. Once the operator spreads, and becomes more non-local, the commutator increases as it's support overlaps with V_i.

B. Entanglement Growth in Operator Space

Under a generic unitary evolution, we expect an initially simple product operator, for example, a Pauli string, $O = P_1, \ldots, P_n$ to evolve into a linear superposition of Pauli strings [37, 38]. This entanglement growth corresponds to an exponential increase in the possible number of Pauli strings, requiring the use of stabilizer states from Clifford circuits. However, since Clifford unitaries only map products of Pauli operators to products of Pauli operators, a Pauli string will remain unentangled [39]. Despite this, the scrambling behaviour can be recovered, by the construction of *super-Clifford operators*, whilst remaining classically simulable as shown by Blake and Linden [40].

1. Blake and Linden's Construction

Blake and Linden introduce a family of circuits that exhibit scrambling on the space spanned by Pauli operators. They present a gate-set of 'super-Clifford operators' that generate a near-maximal amount of operator entanglement within the Pauli operator space, called super-Clifford circuits, when under time-evolution. These super-Clifford operators remain classically simulable, via an extension of stabilisers to operator space, called 'super-stabilisers'. Super-Clifford operators 'act' on Pauli operators, via conjugation in the Heisenberg picture. The first super-Clifford operator in the Gate set is denoted as **Z.H**, which is identified with a on Pauli operators conjugation by a Phase gate, T;

$$T^\dagger X T = \frac{X - Y}{\sqrt{2}} \qquad T^\dagger Y T = \frac{X + Y}{\sqrt{2}} \tag{30}$$

Following this, the operators, X and Y can be changed to a state-like representation, with X denoted as $[\mathbf{0}\rangle$ and Y denoted as $[\mathbf{1}\rangle$. Then the action of **Z.H** can be written as,

$$\mathbf{Z.H}[\mathbf{0}\rangle = \frac{[\mathbf{0}\rangle - [\mathbf{1}\rangle}{\sqrt{2}} \tag{31}$$

$$\mathbf{Z.H}[\mathbf{1}\rangle = \frac{[\mathbf{0}\rangle + [\mathbf{1}\rangle}{\sqrt{2}} \tag{32}$$

The second gate in the set of super-Clifford operators, is the **SWAP** gate,

$$\mathbf{SWAP}[\mathbf{01}\rangle = [\mathbf{10}\rangle \tag{33}$$

formed from the regular 2-qubit SWAP gate, that swaps two nearest neighbour qubits. It conjugates Pauli operators in the following way:

$$\mathrm{SWAP}^\dagger X_1 Y_2 \mathrm{SWAP} = Y_1 X_2 \tag{34}$$

The third gate, denoted **C3**, acts as a combination of controlled-**Y** super-operators,

$$\mathbf{C3}[\mathbf{000}\rangle = \mathbf{CY}_{12}\mathbf{CY}_{13}[\mathbf{000}\rangle = [\mathbf{000}\rangle \tag{35}$$

$$\mathbf{C3}[\mathbf{100}\rangle = \mathbf{CY}_{12}\mathbf{CY}_{13}[\mathbf{100}\rangle = -[\mathbf{111}\rangle \tag{36}$$

These three super-operators form the gate set $\{\mathbf{SWAP}, \mathbf{Z.H}, \mathbf{C3}\}$, which generates entanglement through unitary evolution in operator space, as this gate set maps Pauli strings to a linear superposition of Pauli strings. Even though the a simple string of Pauli operators can evolve into a sum of potentially exponential operator strings, the dynamics can be computed classically by extending the formalism of stabilizer states to operator space. Following this, Blake and Linden showed that the gate set was capable of generating near-maximal amounts of entanglement (slightly less than the Page value [41]) among Pauli strings on a chain of 120 qubits, quantified by the von Neumann entropy. This was only shown for operators with global support, and therefore shows no notion of operator spreading or entanglement growth in local operators.

2. Scrambling in Free-Fermi Circuits

As previously mentioned, a spin 1/2 or qubit, system can be mapped onto a fermionic system via a Jordan Wigner transformation, with the introduction of Majorana fermion operators. The evolution of Majorana fermion Operators has been shown to exhibit operator entanglement growth by Prosen and Pižorn [42] by mapping an Ising spin-1/2 system onto a Majorana fermion representation. Further efforts from now on will be directed towards simulating non-interacting fermion systems and computing the entanglement entropy so that the dynamics of local operator spreading and entanglement growth can be fully investigated.

V. FURTHER DIRECTION

Further research will be directed towards free-fermion systems and how to simulate them. This will be car-

ried out by writing a python program to simulate random unitary circuits and computing the entanglement entropy. We will first reproduce Blake and Linden's results for the $N = 120$ qubits case of super-Clifford evolution, and then go on to simulate free-fermion circuits to confirm a non-trivial operator scrambling. Following this, the idea of combining Clifford and non-interacting fermion circuits will be explored to investigate if any amplification of the scrambling occurs [43], along with relaxing the condition of nearest-neighbour interactions, by using long-ranged gates.

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