Quantum Scrambling project Oct 22

Pauli operators:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$Z|0\rangle = |0\rangle \quad Z|1\rangle = -|1\rangle$ defines basis $\{|0\rangle, |1\rangle\}$

so $Z = |0\rangle\langle 0| - |1\rangle\langle 1| \qquad Y = iXZ$

$\sigma^-|0\rangle = |1\rangle \qquad \sigma^+|1\rangle = |0\rangle$

$\therefore \quad \sigma^- = |1\rangle\langle 0| \qquad \sigma^+ = |0\rangle\langle 1| \qquad X = \sigma^+ + \sigma^-$

$\sigma^- Z = \sigma^- \qquad Z\sigma^- = -\sigma^-$

$\sigma^+ Z = -\sigma^+ \qquad Z\sigma^+ = \sigma^+$

Clifford gates:

These are unitaries which map products of Paulis to products of Paulis, so (ignoring overall phase)

$$P = \prod_{j=1}^{N} (X_j)^{x_j} \prod_{j=1}^{N} (Z_j)^{z_j} \qquad \underline{x}, \underline{z} \text{ binary strings}$$

$$UPU^\dagger \mapsto P'$$

Key gates: single-qubit $UPU^\dagger = P'$

U	P	P'
X	X	X
	Z	$-Z$
Y	X	$-X$
	Z	$-Z$

U	P	P'
Z	X	$-X$
	Z	Z

Paulis are hermitian unitary operators $U^{-1} = U^\dagger = U$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad H^2 = \mathbb{1}$$

U	P	P'
S	X	Y
	Z	Z
H	X	Z
	Z	X
CX	$X \otimes \mathbb{1}$	$X \otimes X$
	$\mathbb{1} \otimes X$	$\mathbb{1} \otimes X$
	$Z \otimes \mathbb{1}$	$Z \otimes \mathbb{1}$
	$\mathbb{1} \otimes Z$	$Z \otimes Z$

Two-qubit gate:

$$CX = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad CX^{\dagger} = CX$$

Any circuit comprising these gates is a Clifford unitary.

Stabilizer states

An operator A is a "stabilizer" of a state $|\psi\rangle$ if:

$$A|\psi\rangle = |\psi\rangle \qquad \text{i.e. } +1 \text{ eigenstate of } A$$

Example:

$$Z|0\rangle = |0\rangle \qquad X|+\rangle = |+\rangle \qquad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

If $B|\psi\rangle = |\psi\rangle$ as well then:

$$AB|\psi\rangle = |\psi\rangle \text{ and } BA|\psi\rangle = |\psi\rangle \Rightarrow [A,B]|\psi\rangle = 0$$

Restrict stabilizers to products of Paulis P then for two stabilizers A, B we have that

$$[A,B] = \alpha P \quad \text{so} \quad [A,B]|\psi\rangle = 0$$

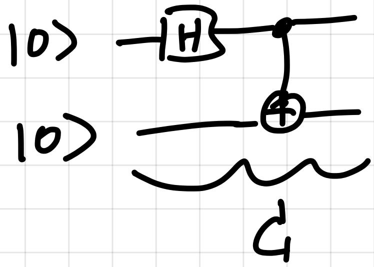
is only satisfied if $\alpha = 0$.

An n-qubit state is stabilized by n independent generators that commute. Example:

$$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$Z \otimes Z\,|\Psi_{00}\rangle = |\Psi_{00}\rangle \qquad X \otimes X\,|\Psi_{00}\rangle = |\Psi_{00}\rangle$$

We can generate $|\Psi_{00}\rangle$ from $|00\rangle$ as:



$|\Psi_{00}\rangle$ $\qquad G\,|00\rangle = |\Psi_{00}\rangle$

$|00\rangle$ is stabilized by $\mathbb{1} \otimes Z$ and $Z \otimes \mathbb{1}$ and the circuit above is clifford. So generally:

$S\,|\Psi\rangle = |\Psi\rangle$ then

$$G|\Psi\rangle = G S |\Psi\rangle = G S G^\dagger G |\Psi\rangle$$

so $G|\Psi\rangle$ is stabilized by $G S G^\dagger$. Here:

$$G(\mathbb{1} \otimes Z)G^\dagger = CX(H \otimes \mathbb{1})(\mathbb{1} \otimes Z)(H \otimes \mathbb{1})CX$$

$$= CX(\mathbb{1} \otimes Z)CX = Z \otimes Z$$

$$G(Z \otimes \mathbb{1})G^\dagger = CX(X \otimes \mathbb{1})CX = X \otimes X$$

In general a random circuit of CX and H gates acting on any qubit configuration state will generate a highly entangled state.

Let us now denote multiqubit operators as:

$$X_j = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underset{j}{X} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

Operator evolution

Alternative viewpoint:

$$|\underline{q}\rangle = \underbrace{(\sigma_1^-)^{q_1}(\sigma_2^-)^{q_2}\cdots(\sigma_N^-)^{q_N}}_{c^\dagger_{\underline{q}}}|00\cdots0\rangle$$

$\underline{q} = \{0,1\}^N$ binary string.

Here we have an operator $c^\dagger_{\underline{q}}$ creating $|\underline{q}\rangle$ from a fixed reference state $|00\cdots0\rangle$.

Now $\quad c^\dagger_{\underline{q}} + c_{\underline{q}} = \prod_{j=1}^N (X_j)^{q_j} \quad \equiv$ string of Pauli X's

$c_{\underline{q}}|00\cdots0\rangle = 0 \quad$ so:

$$|\underline{q}\rangle = \prod_{j=1}^N (X_j)^{q_j}|00\cdots0\rangle$$

Now suppose that we unitarily evolve $|\underline{q}\rangle$ as

$$U|\underline{q}\rangle = U c^\dagger_{\underline{q}} U^\dagger U|00\cdots0\rangle$$

and that $U|00\cdots0\rangle = |00\cdots0\rangle$. Then all the complexity of $U|\underline{q}\rangle$ is encoded in

$$U c^\dagger_{q} U^\dagger$$

Can we concoct a U that creates a highly entangled state $U|\underline{q}\rangle$ which is classically simulable?

Clifford circuits don't work!! why?

$\mathcal{U} c_{\underline{q}}^\dagger \mathcal{U}^\dagger$ is effectively $\mathcal{U} \prod_{j=1}^{N} (X_j)^{a_j} \mathcal{U}^\dagger$ so

is a product of Paulis, hence will only ever evolve into another product (no entanglement in operator space).

Worse still only Clifford circuits with no H gates obey $\mathcal{U}|00\cdots0\rangle = |00\cdots0\rangle$, which are essentially classical.

We can't use stabilizer states directly to map to a non-trivial operator evolution.

<u>Fermionic encoding</u>

we can map qubits to spinless fermions via a Jordan Wigner transform. specifically

Fermions creation: $c_j^\dagger$

annihilation: c_j

anticommutativity $\{c_j, c_j^\dagger\} = \mathbb{1} = c_j c_j^\dagger + c_j^\dagger c_j = \mathbb{1}$

$\{c_j, c_k^\dagger\} = \delta_{jk}\mathbb{1}$ so $c_j c_k^\dagger = -c_k^\dagger c_j \quad j \neq k$

$\{c_j^\dagger, c_k^\dagger\} = 0 \quad \therefore \quad c_j^\dagger c_k^\dagger = -c_k^\dagger c_j^\dagger$ any j, k

so if $j = k$ $(c_j^\dagger)^2 = 0$ $\{c_j, c_k\} = 0$ similarly.

Build basis Fock states from $|vac\rangle$ as:

$$|\underline{n}\rangle = (c_1^\dagger)^{n_1} (c_2^\dagger)^{n_2} \cdots (c_N^\dagger)^{n_N} |vac\rangle$$

mode ordering is important.

$c_j^\dagger c_j$ is the number operator since:

$$c_j^\dagger c_j |\underline{n}\rangle = (c_1^\dagger)^{n_1} \cdots c_j^\dagger c_j (c_j^\dagger)^{n_j} \cdots (c_N^\dagger)^{n_N} |vac\rangle$$

commutes to jth place

$$c_j^\dagger c_j (c_j^\dagger)^{n_j} = \begin{cases} c_j^\dagger c_j & n_j = 0 \\ c_j^\dagger c_j c_j^\dagger = c_j^\dagger (1 - c_j^\dagger c_j) & n_j = 1 \\ \quad = c_j^\dagger & \end{cases}$$

$$= n_j (c_j^\dagger)^{n_j}$$

so $c_j^\dagger c_j |\underline{n}\rangle = n_j |\underline{n}\rangle$

Note that $(c_j^\dagger c_j)^2 = c_j^\dagger c_j c_j^\dagger c_j = c_j^\dagger (1 - c_j^\dagger c_j) c_j$
$$= c_j^\dagger c_j$$

Unitary evolution is built from Hamiltonians involving $c_j^\dagger$, c_j's which map mode operators.

Example:

$$H = \epsilon c_j^\dagger c_j$$

$$u = e^{-iHt} = e^{-i\epsilon t c_j^\dagger c_j}$$
$$= \mathbb{1} - i\epsilon t (c_j^\dagger c_j) + \frac{(-i\epsilon t)^2}{2!} (c_j^\dagger c_j)^2 + \ldots$$
$$= \mathbb{1} + \left(-i\epsilon t + \frac{(-i\epsilon t)^2}{2!} + \ldots\right) c_j^\dagger c_j$$
$$= \mathbb{1} + (e^{-i\epsilon t} - 1) c_j^\dagger c_j$$

so

$$u c_j^\dagger u^\dagger = (\mathbb{1} + (e^{-i\epsilon t} - 1) c_j^\dagger c_j) c_j^\dagger (\mathbb{1} + (e^{i\epsilon t} - 1) c_j^\dagger c_j)$$
$$= c_j^\dagger + (e^{-i\epsilon t} - 1) c_j^\dagger = e^{-i\epsilon t} c_j^\dagger$$

Now locally σ_j^- and σ_j^+ obey the correct anticommutation relations:

$$\sigma_j^- \sigma_j^+ + \sigma_j^+ \sigma_j^- = |1\rangle\langle 0| \cdot |0\rangle\langle 1| + |0\rangle\langle 1| \cdot |1\rangle\langle 0|$$
$$= |1\rangle\langle 1| + |0\rangle\langle 0| = \mathbb{1}$$

so we might identify:

$$c_j^\dagger \sim \sigma_j^- \quad \text{since} \quad \sigma_j^- |0\rangle = |1\rangle$$
$$c_j \sim \sigma_j^+ \quad \text{since} \quad c_j |1\rangle = |0\rangle$$

but these operators don't anticommute on different sites, e.g.

$$c_j^\dagger c_k^\dagger = -c_k^\dagger c_j^\dagger \qquad j \neq k$$

but

$$\sigma_j^- \sigma_k^- = \sigma_k^- \sigma_j^-$$

This is fixed by adding a JW string so:

$$c_j^\dagger = \left(\prod_{k=1}^{j-1} Z_k \right) \sigma_j^-$$

since $Z\sigma^- + \sigma^- Z = 0$.

We could map a qubit state directly to a fermionic Fock state as:

$$|\tilde{\underline{q}}\rangle = (c_1^\dagger)^{q_1} (c_2^\dagger)^{q_2} \cdots (c_N^\dagger)^{q_N} |\mathrm{vac}\rangle$$

since $|\mathrm{vac}\rangle = |00\cdots 0\rangle$ and then:

$$|\tilde{\underline{q}}\rangle = (\sigma_1^-)^{q_1} (Z_1 \sigma_2^-)^{q_2} \cdots (Z_1 Z_2 \cdots Z_{N-1} \sigma_N^-)^{q_N} |00\cdots 0\rangle$$

but we can apply each JWT (due to the ordering we chose) to $|00\cdots0\rangle$ since $Z_j|00\cdots0\rangle = |00\cdots0\rangle$ then:

$$|\underline{\tilde{q}}\rangle = (\sigma_1^-)^{q_1}(\sigma_2^-)^{q_2}\cdots(\sigma_N^-)^{q_N}|00\cdots0\rangle$$

and we see that $|\underline{\hat{q}}\rangle \equiv |\underline{q}\rangle$

The problem with this is approach is two-fold:

(i) $|\underline{q}\rangle$ has no fixed fermion number

(ii) local rotations of a qubit are now highly non-local fermionic operators

Suppose we wanted to Hadamard qubit j then we want a unitary which does the following:

$$U c^\dagger|vac\rangle = \frac{1}{\sqrt{2}}(\mathbb{1} - c^\dagger)|vac\rangle$$

$$U|vac\rangle = \frac{1}{\sqrt{2}}(\mathbb{1} + c^\dagger)|vac\rangle$$

$$U = \frac{1}{\sqrt{2}}(c^\dagger + c + cc^\dagger - c^\dagger c) = \frac{1}{\sqrt{2}}(c^\dagger + c + \mathbb{1} - 2c^\dagger c)$$

$$UU^\dagger = \frac{1}{2}(c^\dagger + c + \mathbb{1} - 2c^\dagger c)(c^\dagger + c + \mathbb{1} - 2c^\dagger c)$$

$$= \frac{1}{2}(c^\dagger c + c^\dagger + cc^\dagger + c - 2cc^\dagger c + c^\dagger + c + \mathbb{1} - \cancel{2c^\dagger c} - 2c^\dagger cc^\dagger$$
$$- \cancel{2c^\dagger c} + \cancel{4c^\dagger c})$$

$$= \frac{1}{2}(2\mathbb{1} + 2c^\dagger + 2c - 2cc^\dagger c - 2c^\dagger cc^\dagger)$$

$$= \frac{1}{2}(2\mathbb{1} + 2c^\dagger + 2c - 2(1 - c^\dagger c)c - 2(1 - cc^\dagger)c^\dagger))$$

$$= \frac{1}{2}(2\mathbb{1} + \cancel{2c^\dagger} + \cancel{2c} - \cancel{2c} - \cancel{2c^\dagger}) = \mathbb{1}$$

$$U c^\dagger |vac\rangle = \frac{1}{\sqrt{2}}(c^\dagger + c + cc^\dagger - c^\dagger c)\, c^\dagger |vac\rangle$$
$$= \frac{1}{\sqrt{2}}(cc^\dagger - c^\dagger c c^\dagger)|vac\rangle$$
$$= \frac{1}{\sqrt{2}}(1 - c^\dagger c - c^\dagger(1 - c^\dagger c))\,|vac\rangle$$
$$= \frac{1}{\sqrt{2}}(1 - c^\dagger)\,|vac\rangle$$

$$U|vac\rangle = \frac{1}{\sqrt{2}}(c^\dagger + c + cc^\dagger - c^\dagger c)\,|vac\rangle$$
$$= \frac{1}{\sqrt{2}}(c^\dagger + cc^\dagger)\,|vac\rangle$$
$$= \frac{1}{\sqrt{2}}(c^\dagger + 1 - c^\dagger c)\,|vac\rangle$$
$$= \frac{1}{\sqrt{2}}(1 + c^\dagger)\,|vac\rangle$$

This suggests that for many-fermion modes we can use:
$$U_j = \frac{1}{\sqrt{2}}(c_j^\dagger + c_j + c_j c_j^\dagger - c_j^\dagger c_j)$$

<u>BUT</u>, this doesn't work since $c_j^\dagger + c_j$ part anticommutes past $c_k^\dagger$'s making $|\underline{q}\rangle$ with $k<j$ while $c_j c_j^\dagger - c_j^\dagger c_j$ commutes. For example:

$$U_2 c_1^\dagger |vac\rangle = \frac{1}{\sqrt{2}}(c_2^\dagger + c_2 + c_2 c_2^\dagger - c_2^\dagger c_2)\, c_1^\dagger\, |vac\rangle$$
$$= c_1^\dagger \frac{1}{\sqrt{2}}(-c_2^\dagger - c_2 + c_2 c_2^\dagger - c_2^\dagger c_2)|vac\rangle$$
$$= c_1^\dagger \frac{1}{\sqrt{2}}(-c_2^\dagger + c_2 c_2^\dagger)|vac\rangle$$

$$= c_1^\dagger \frac{1}{\sqrt{2}}\left(-c_2^\dagger + (1 - c_2^\dagger c_2)\right)|vac\rangle$$

$$= c_1^\dagger \frac{1}{\sqrt{2}}(1 - c_2^\dagger)|vac\rangle \neq c_1^\dagger \frac{1}{\sqrt{2}}(1 + c_2^\dagger)|vac\rangle$$

This means we need to remove the JWT in the fermionic operator as:

$$U_j = \frac{1}{\sqrt{2}}\Big(\underbrace{\prod_{k=1}^{j-1} e^{-i\pi c_k^\dagger c_k}}_{\Downarrow\ \prod_{k=1}^{j-1}(1 - 2c_k^\dagger c_k)} (c_j^\dagger + c_j) + c_j c_j^\dagger - c_j^\dagger c_j \Big)$$

This is highly non-local.

Converting this into the qubit picture we get:

$$U_j = \frac{1}{\sqrt{2}}(\sigma_j^- + \sigma_j^+ + \sigma_j^+\sigma_j^- - \sigma_j^-\sigma_j^+)$$

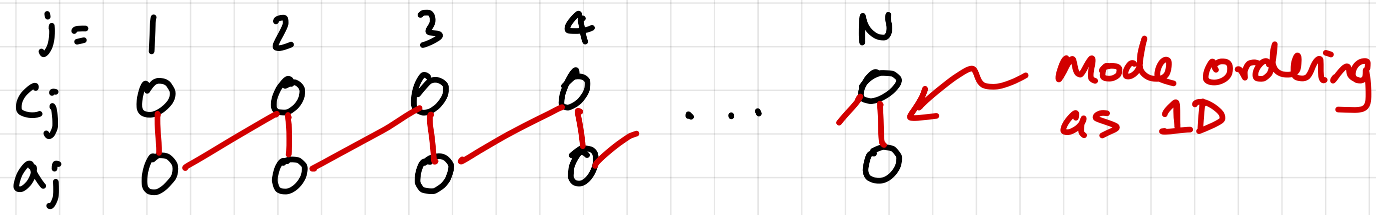
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_j = H_j \quad \text{which is local}$$

Not have a fixed Fermion number (or at least a fixed number parity) is undesirable.

Better to use a mapping of qubit states to fermions that is indirect, e.g. encode logical qubits as pairs of fermionic modes:

Instead of $\circ = |0\rangle$ $\bullet = |1\rangle$ use $\substack{\circ \\ \bullet} = |0\rangle$ $\substack{\bullet \\ \circ} = |1\rangle$

Now both logical qubits have one fermion.



Instead of : $|\underline{q}\rangle = (c_1^\dagger)^{q_1}(c_2^\dagger)^{q_2}\cdots(c_N^\dagger)^{q_N}|vac\rangle$

we now have:

$$|\underline{q}\rangle = (c_1^\dagger)^{q_1}(a_1^\dagger)^{1-q_1}(c_2^\dagger)^{q_2}(a_2^\dagger)^{1-q_2}\cdots(c_N^\dagger)^{q_N}(a_N^\dagger)^{1-q_N}|vac\rangle$$

There are precisely N fermions in $|\underline{q}\rangle$ distributed over 2N modes

$$|0101\rangle = \left|\begin{smallmatrix}\circ & \bullet & \circ & \bullet \\ \bullet & \circ & \bullet & \circ\end{smallmatrix}\right\rangle$$

The qubit encoding restricts each pair to having precisely one fermion. We don't do any operation that violates this, like creating:

$$\left|\begin{smallmatrix}\circ & \bullet & \circ & \bullet \\ \bullet & \circ & \bullet & \circ\end{smallmatrix}\right\rangle \mapsto \left|\begin{smallmatrix}\circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \circ & \circ\end{smallmatrix}\right\rangle \quad \times$$

Single logical qubit gates are now local in terms of fermions and physical qubits:

$$Z_j^L = \mathbb{1} - 2c_j^\dagger c_j$$

hermitian/unitary

$$Z_j^L c_j^\dagger Z_j^L = -c_j^\dagger$$
$$Z_j^L a_j^\dagger Z_j^L = a_j^\dagger$$

logical Z is a single mode/physical qubit operator. It simply introduces a −1 if mode c_j is occupied.

On the j^{th} pair of physical qubits:

$$Z_j^L = Z_{2j-1} \otimes \mathbb{1}_{2j} \qquad Z_j^L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{0} & 0 \\ 0 & \boxed{0} & \boxed{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{2j-1,2j}$$

$$\begin{array}{ccccccc} 1 & 2 & & j & & N \\ \boxed{\begin{matrix} O_1 \\ O_2 \end{matrix}} & \boxed{\begin{matrix} O_3 \\ O_4 \end{matrix}} & \cdots & \boxed{\begin{matrix} O_{2j-1} \\ O_{2j} \end{matrix}} & \cdots & \boxed{\begin{matrix} O_7 \\ O_8 \end{matrix}} \end{array}$$

$$Z_j^L = e^{i\pi c_j^\dagger c_j} = \mathbb{1} + (e^{i\pi} - 1) c_j^\dagger c_j = \mathbb{1} - 2 c_j^\dagger c_j$$

Logical X is a fermionic-SWAP gate on the pair of physical qubits:

$$X_j^L = e^{i\frac{\pi}{2}(c_j^\dagger - a_j^\dagger)(c_j - a_j)} = e^{i\frac{\pi}{2}\overbrace{(c_j^\dagger c_j + a_j^\dagger a_j - a_j^\dagger c_j - c_j^\dagger a_j)}^{\mathbb{1} - X_j^L}}$$

$$= 1 - c_j^\dagger c_j - a_j^\dagger a_j + c_j^\dagger a_j + a_j^\dagger c_j$$

This gives

$$X_j^L c_j^\dagger X_j^L = a_j^\dagger$$

hermitian/unitary

check:

$$X_j^L a_j^\dagger X_j^L = c_j^\dagger$$

$$X_j^L c_j^\dagger = (1 - c_j^\dagger c_j - a_j^\dagger a_j + c_j^\dagger a_j + a_j^\dagger c_j)\, c_j^\dagger =$$

$$= c_j^\dagger - c_j^\dagger c_j c_j^\dagger - a_j^\dagger a_j c_j^\dagger + c_j^\dagger a_j c_j^\dagger + a_j^\dagger c_j c_j^\dagger$$

$$= \cancel{c_j^\dagger} - \cancel{c_j^\dagger}(1 - \cancel{c_j^\dagger} c_j) - a_j^\dagger a_j c_j^\dagger - \cancel{c_j^\dagger c_j^\dagger} a_j + a_j^\dagger c_j c_j^\dagger$$

$$= -a_j^\dagger a_j c_j^\dagger + a_j^\dagger c_j c_j^\dagger$$

so $X_j^L c_j^\dagger X_j^L = \ldots$

$$(-a_j^\dagger a_j c_j^\dagger + a_j^\dagger c_j c_j^\dagger)(1 - c_j^\dagger c_j - a_j^\dagger a_j + c_j^\dagger a_j + a_j^\dagger c_j)$$

$$= \cancel{-a_j^\dagger a_j c_j^\dagger} + a_j^\dagger c_j c_j^\dagger + \cancel{a_j^\dagger a_j c_j^\dagger} - a_j^\dagger a_j c_j^\dagger a_j^\dagger c_j$$

$$= a_j^\dagger c_j c_j^\dagger + a_j^\dagger a_j a_j^\dagger c_j^\dagger c_j$$

$$= a_j^\dagger c_j c_j^\dagger + a_j^\dagger (1 - a_j^\dagger a_j) c_j^\dagger c_j$$

$$= a_j^\dagger (c_j c_j^\dagger + c_j^\dagger c_j) = a_j^\dagger \quad \text{as required}$$

In terms of physical qubits we have:

$$X_j^L = 1 - c_j^\dagger c_j - a_j^\dagger a_j + c_j^\dagger a_j + a_j^\dagger c_j$$

$$= \mathbb{1} - \tfrac{1}{2}(\mathbb{1} - Z_{2j-1}) - \tfrac{1}{2}(\mathbb{1} - Z_{2j})$$

$$+ \sigma_{2j-1}^- Z_{2j-1} \sigma_{2j}^+ + Z_{2j-1} \sigma_{2j}^- \sigma_{2j-1}^+$$

$$= \tfrac{1}{2}(Z_{2j-1} + Z_{2j}) + \sigma_{2j-1}^- \sigma_{2j}^+ + \sigma_{2j-1}^+ \sigma_{2j}^-$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{2j-1,2j}$$

Fermionic swap due to the -1 sign

For a qubit $X = e^{\pm i \frac{\pi}{2}(X - \mathbb{1})}$ consistent with above.

Logical Hadamard

$$H_j^L c_j^\dagger H_j^L = \frac{1}{\sqrt{2}}\left(a_j^\dagger - c_j^\dagger\right)$$

$$H_j^L a_j^\dagger H_j^L = \frac{1}{\sqrt{2}}\left(a_j^\dagger + c_j^\dagger\right)$$

with H_j^L hermitian/unitary.

For a qubit $H = e^{i\frac{\pi}{2}\left(\frac{1}{\sqrt{2}}(X+Z) - \mathbb{1}\right)}$

For fermions we have:

$$H_j^L = e^{i\frac{\pi}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)c_j^\dagger c_j + \left(1-\frac{1}{\sqrt{2}}\right)a_j^\dagger a_j - \frac{1}{\sqrt{2}}c_j^\dagger a_j - \frac{1}{\sqrt{2}}a_j^\dagger c_j\right]}$$

$$= \mathbb{1} - \left(1+\frac{1}{\sqrt{2}}\right)c_j^\dagger c_j - \left(1-\frac{1}{\sqrt{2}}\right)a_j^\dagger a_j + \frac{1}{\sqrt{2}}c_j^\dagger a_j + \frac{1}{\sqrt{2}}a_j^\dagger c_j$$

In terms of physical qubits:

$$H_j^L = \mathbb{1} - \left(1+\frac{1}{\sqrt{2}}\right)\sigma_{2j-1}^- \sigma_{2j-1}^+ - \left(1-\frac{1}{\sqrt{2}}\right)\sigma_{2j}^-\sigma_{2j}^+ + \frac{1}{\sqrt{2}}\sigma_{2j-1}^-\sigma_{2j}^+ + \frac{1}{\sqrt{2}}\sigma_{2j-1}^+\sigma_{2j}^-$$

$$H_j^L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{2j-1,2j}$$

$\Uparrow$ NOT Clifford gate

This is also called a fermionic W-gate

We can check this works in terms of the physical qubits by computing

$$H^L(\sigma^- \otimes \mathbb{1})H^L = \frac{1}{\sqrt{2}}(Z \otimes \sigma^- - \sigma^- \otimes \mathbb{1})$$

$$H^L(Z \otimes \sigma^-)H^L = \frac{1}{\sqrt{2}}(Z \otimes \sigma^- + \sigma^- \otimes \mathbb{1})$$

Reproducing:
$$H_j^L c_j^\dagger H_j^L = \frac{1}{\sqrt{2}}(a_j^\dagger - c_j^\dagger)$$
$$H_j^L a_j^\dagger H_j^L = \frac{1}{\sqrt{2}}(a_j^\dagger + c_j^\dagger)$$

Logical CZ gate

$$CZ_{jj+1} = e^{i\pi c_j^\dagger c_j c_{j+1}^\dagger c_{j+1}} = \mathbb{1} - 2c_j^\dagger c_j c_{j+1}^\dagger c_{j+1}$$

$$CZ_{jj+1}\, c_j^\dagger\, CZ_{jj+1}$$
$$= (\mathbb{1} - 2c_j^\dagger c_j c_{j+1}^\dagger c_{j+1})\, c_j^\dagger\, (\mathbb{1} - 2c_j^\dagger c_j c_{j+1}^\dagger c_{j+1})$$
$$= c_j^\dagger - 2c_j^\dagger c_j c_{j+1}^\dagger c_{j+1} c_j^\dagger = c_j^\dagger(1 - 2c_{j+1}^\dagger c_{j+1})$$
$$= c_j^\dagger e^{i\pi c_{j+1}^\dagger c_{j+1}}$$

$$CZ_{jj+1}\, c_{j+1}^\dagger\, CZ_{jj+1} = c_{j+1}^\dagger e^{i\pi c_j^\dagger c_j}$$

So

$$CZ_{jj+1}\, c_j^\dagger c_{j+1}^\dagger\, CZ_{jj+1} = CZ_{jj+1}\, c_j^\dagger\, CZ_{jj+1}\, CZ_{jj+1}\, c_{j+1}^\dagger\, CZ_{jj+1}$$
$$= c_j^\dagger e^{i\pi c_{j+1}^\dagger c_{j+1}}\, c_{j+1}^\dagger e^{i\pi c_j^\dagger c_j}$$

noting $\quad c^\dagger c\, c^\dagger = c^\dagger, \quad c^\dagger c^\dagger c = 0$

$$= e^{i\pi} c_j^\dagger c_{j+1}^\dagger$$

Given $\quad c_j \; O \quad O \; c_{j+1}$

$\qquad\quad a_j \; O \quad O \; a_{j+1}$

$$CZ_{jj+1}\, a_j^\dagger a_{j+1}^\dagger\, CZ_{jj+1} = a_j^\dagger a_{j+1}^\dagger$$

$$CZ_{jj+1}\, c_j^\dagger a_{j+1}^\dagger\, CZ_{jj+1} = e^{i\pi c_{j+1}^\dagger c_{j+1}}\, c_j^\dagger a_{j+1}^\dagger$$

$$CZ_{jj+1}\, a_j^\dagger c_{j+1}^\dagger\, CZ_{jj+1} = e^{i\pi c_j^\dagger c_j}\, a_j^\dagger c_{j+1}^\dagger$$

$$CZ_{jj+1}\, c_j^\dagger c_{j+1}^\dagger\, CZ_{jj+1} = e^{i\pi}\, c_j^\dagger c_{j+1}^\dagger$$

Acting on $|vac\rangle$ this gives:

$$|00\rangle_L \rightarrow |00\rangle_L$$
$$|01\rangle_L \rightarrow |01\rangle_L$$
$$|10\rangle_L \rightarrow |10\rangle_L$$
$$|11\rangle_L \rightarrow -|11\rangle_L$$

In terms of operators we pickup a parity contribution.

For physical qubits:

j o—o $j+2$

$j+1$ o $\quad$ o $j+3$

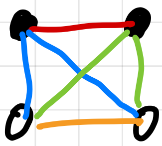
$$CZ_{jj+1} = e^{i\frac{\pi}{4}(\mathbb{1} - Z_j)(\mathbb{1} - Z_{j+2})}$$

$$CZ\,(Z \otimes \sigma^-)\,CZ = \mathbb{1} \otimes \sigma^-$$

$$CZ\,(\sigma^- \otimes \mathbb{1})\,CZ = \sigma^- \otimes Z$$

$$CZ\,(\sigma^- \otimes \sigma^-)\,CZ = -\sigma^- \otimes \sigma^-$$

$$e^{i\pi c^\dagger c} = Z$$



write $n_j = c_j^\dagger c_j$ $\quad \tilde{n}_j = a_j^\dagger a_j$

Consider:

$$U = e^{i\pi n_j n_{j+1}(1-\tilde{n}_j)(1-\tilde{n}_{j+1})}$$

$$U c_j^\dagger U^\dagger = e^{i\pi n_j Q} c_j^\dagger e^{-i\pi n_j Q}$$

$[n_j, Q] = 0$ with $Q = n_{j+1}(1-\tilde{n}_j)(1-\tilde{n}_{j+1})$

$$e^{i\pi n_j Q} = \mathbb{1} + (i\pi Q) n_j + \frac{(i\pi Q)^2}{2!} n_j + \cdots$$

$$= \mathbb{1} + (e^{i\pi Q} - \mathbb{1}) n_j$$

$$\left(\mathbb{1} + (e^{i\pi Q} - \mathbb{1}) n_j\right) c_j^\dagger \left(\mathbb{1} + (e^{-i\pi Q} - \mathbb{1}) n_j\right)$$

$$= c_j^\dagger + (e^{i\pi Q} - \mathbb{1}) c_j^\dagger = e^{i\pi Q} c_j^\dagger$$

$c_j^\dagger n_j = 0$
$n_j c_j^\dagger = c_j^\dagger$

Likewise

$$e^{i\pi(1-\tilde{n}_j)Q'} a_j^\dagger e^{-i\pi(1-\tilde{n}_j)Q'}$$

$Q' = n_j n_{j+1}(1-\tilde{n}_{j+1})$

$$\left(\mathbb{1} + (e^{i\pi Q'} - \mathbb{1})(1-\tilde{n}_j)\right) a_j^\dagger \left(\mathbb{1} + (e^{-i\pi Q'} - \mathbb{1})(1-\tilde{n}_j)\right)$$

$(1-\tilde{n}_j) a_j^\dagger = 0$ $\quad a_j^\dagger (1-\tilde{n}_j) = \tilde{a}_j$ $\quad$ so

$$= e^{-i\pi Q'} a_j^\dagger$$

so:

$$U c_j^\dagger a_j^\dagger U^\dagger = U c_j^\dagger U^\dagger U a_j^\dagger U^\dagger$$

$$= e^{i\pi Q} c_j^{\dagger} e^{-i\pi Q'} a_j^{\dagger}$$

since $[Q, c_j^{\dagger}] = 0$

$[Q, Q'] = 0$

$Q^2 = Q$, $Q'^2 = Q'$

$$= \left(c_j^{\dagger} e^{-i\pi Q'}\right)\left(e^{i\pi Q} a_j^{\dagger}\right)$$

$$= c_j^{\dagger}(\mathbb{1} - 2Q')(\mathbb{1} - 2Q)\, a_j^{\dagger}$$

$$= c_j^{\dagger}\left(\mathbb{1} - 2n_j \cancel{n_{j+1}(1-\tilde{n}_{j+1})}\right)\left(\mathbb{1} - 2n_{j+1}\cancel{(1-\tilde{n}_j)(1-\tilde{n}_{j+1})}\right) a_j^{\dagger}$$

$$= c_j^{\dagger} a_j^{\dagger}$$

$$u\, c_j^{\dagger} c_{j+1}^{\dagger}\, u^{\dagger} = e^{i\pi Q} c_j^{\dagger} e^{i\pi Q''} c_{j+1}^{\dagger}$$

$$= \left(c_j^{\dagger} e^{i\pi Q''}\right)\left(e^{i\pi Q} c_{j+1}^{\dagger}\right)$$

$$= c_j^{\dagger}\left(\mathbb{1} - \cancel{2n_j(1-\tilde{n}_j)(1-\tilde{n}_{j+1})}\right)\left(\mathbb{1} - 2n_{j+1}(1-\tilde{n}_j)(1-\tilde{n}_{j+1})\right) c_{j+1}^{\dagger}$$

$$= c_j^{\dagger}\left(\mathbb{1} - 2n_{j+1}(1-\tilde{n}_j)(1-\tilde{n}_{j+1})\right) c_{j+1}^{\dagger}$$

$$= c_j^{\dagger} c_{j+1}^{\dagger} - 2 c_j^{\dagger} c_{j+1}^{\dagger} (1-\tilde{n}_j)(1-\tilde{n}_{j+1})$$

$$= \left(1 - 2(1-\tilde{n}_j)(1-\tilde{n}_{j+1})\right) c_j^{\dagger} c_{j+1}^{\dagger}$$

$$= e^{i\pi(1-\tilde{n}_j)(1-\tilde{n}_{j+1})} c_j^{\dagger} c_{j+1}^{\dagger}$$

Doesn't work!! → we can't find a unitary which maps mode operators identically to states.

We want

$$U|\underline{q}\rangle = U c_{\underline{q}}^{\dagger} U^{\dagger} |00\cdots 0\rangle = \sum_{\underline{p}} \psi_{\underline{p}} |\underline{p}\rangle$$

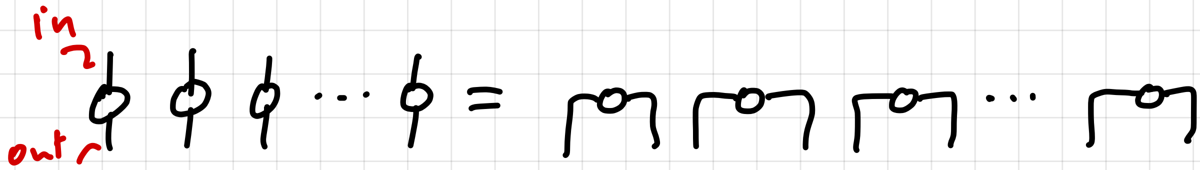
$$= \sum_{\underline{p}} \psi_{\underline{p}} c_{\underline{p}}^{\dagger} |00\cdots 0\rangle$$

but this doesn't imply the operator eq.

$$U c_{\underline{q}}^{\dagger} U^{\dagger} = \sum_{\underline{p}} \psi_{\underline{p}} c_{\underline{p}}^{\dagger}$$

since it is fixed by $|00\cdots 0\rangle$. A special case where this is true is for non-interacting fermionic circuits.

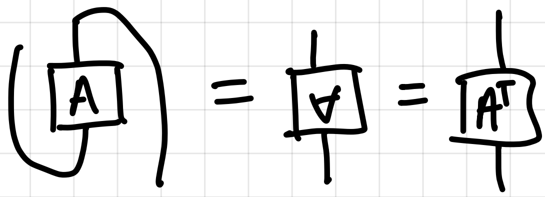
Can view an operator as a state in a doubled system:

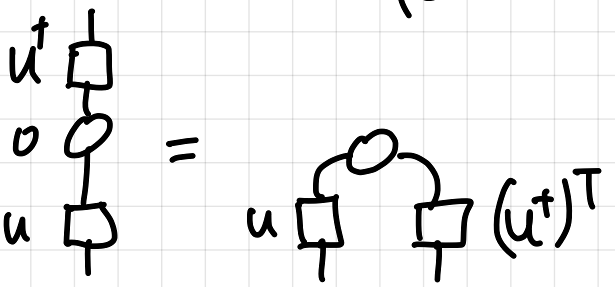


Eg. $$\hat{O} = \sum_{ij} O_{ij} |i\rangle\langle j| \Rightarrow |O\rangle\rangle = \sum_{ij} O_{ij} |i\rangle \otimes |j\rangle$$

$$U\hat{O}U^{\dagger} = \sum_{ij} O_{ij} U|i\rangle\langle j|U^{\dagger}$$

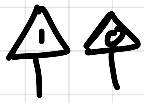
$$= (U \otimes U^{*}) |O\rangle\rangle$$

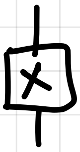
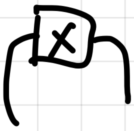




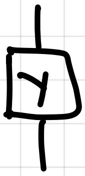
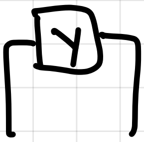
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^{+} = |0\rangle\langle 1|$$

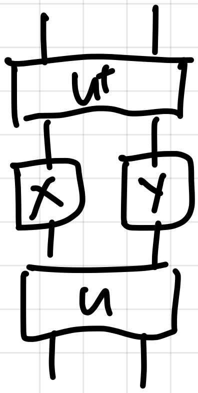
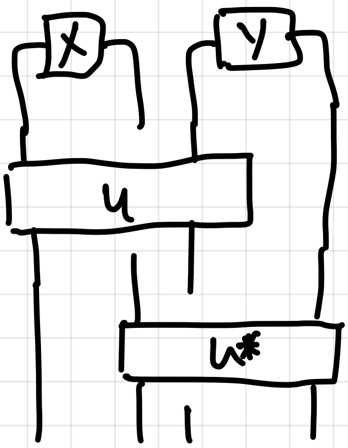
$|0\rangle_L =$  $\simeq \sigma^+$ $\quad |1\rangle_L =$  $\simeq \sigma^-$

 $=$  $\simeq \frac{1}{\sqrt{2}}\Big($  $\Big)$

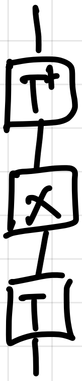
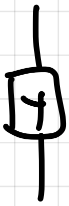
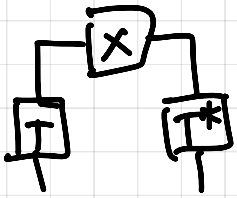
$$= \frac{1}{\sqrt{2}}\left(|0\rangle_L + |1\rangle_L\right) = |+\rangle_L$$

 $=$  $\simeq \frac{-i}{\sqrt{2}}\left(|0\rangle_L - |1\rangle_L\right) = -i|-\rangle_L$

circuit has form:

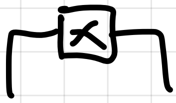
 $\Rightarrow$ 

we can't apply gates between the qubit pairs making up encoding.

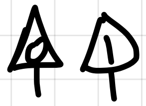
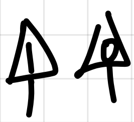
 $=$  $+$  $=$ 

Blake and Linden essentially uses $|\pm\rangle_L$ as logical qubit states and applies stabilizer gates within this encoding.

Their qubit states are entangled:

 $\simeq \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \simeq |\tilde{0}\rangle_L$

 $\simeq \frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle) \simeq |\tilde{1}\rangle_L$

 $= |0\rangle_L$  $= |1\rangle_L$ $\Rightarrow$ $X_L = X \otimes X$

$Z_L = S \otimes S^*$ up to phase i

But it is not possible to find $U \otimes U^*$ which implements H_L in this encoding.

In contrast for the entangled encoding:

$$Z_L = X \otimes X$$

$$(X \otimes X)\frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{-i}{\sqrt{2}}(|10\rangle - |01\rangle) = -|\tilde{1}\rangle_L$$

$$X_L = XS \otimes S^* X \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$(XS \otimes XS^*)\frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{-i}{\sqrt{2}}(i|10\rangle - (-i)|01\rangle)$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\tilde{0}\rangle_L$$

but $H_L = XT \otimes XT^*$ $\qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

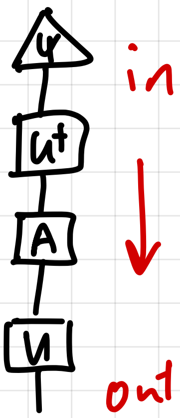
$$T \otimes T^* \frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{-i}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}|01\rangle - e^{i\frac{\pi}{4}}|10\rangle\right)$$

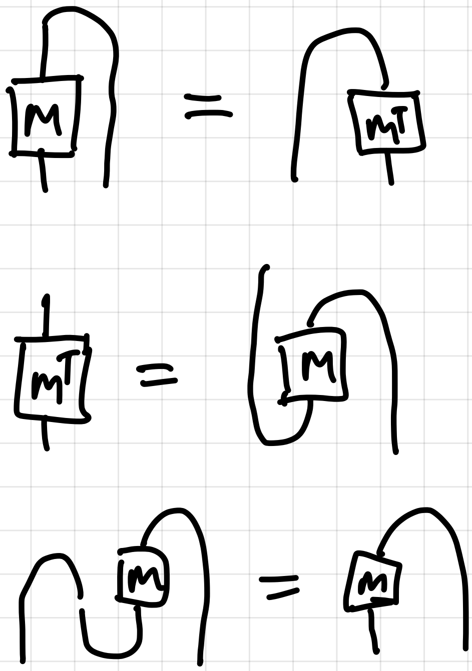
$e^{\pm i\frac{\pi}{4}} = \frac{(1 \pm i)}{\sqrt{2}}$ so

$$T\otimes T^*|\tilde{1}\rangle_L = \frac{-i}{2}\Big((1-i)|01\rangle - (1+i)|10\rangle\Big)$$
$$= \frac{1}{2}\Big(-i|01\rangle + i|10\rangle - |01\rangle - |10\rangle\Big)$$
$$= \frac{1}{\sqrt{2}}\Big(|\tilde{1}\rangle_L - |\tilde{0}\rangle_L\Big)$$

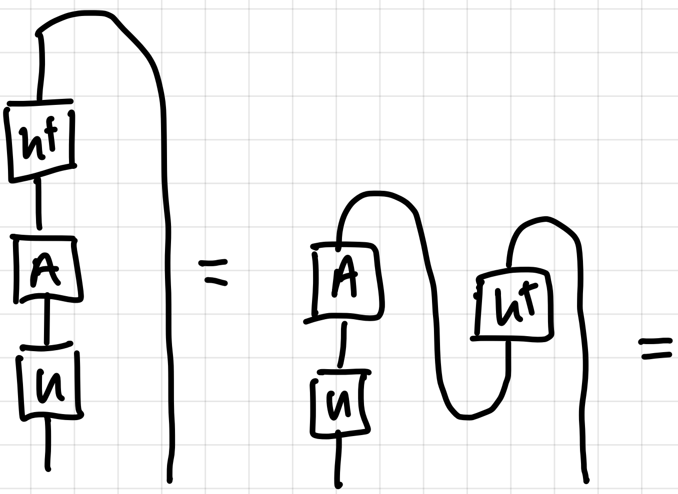
$$T\otimes T^*|\tilde{0}\rangle_L = T\otimes T^* \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)$$
$$= \frac{1}{\sqrt{2}}\Big(e^{-i\frac{\pi}{4}}|01\rangle + e^{i\frac{\pi}{4}}|10\rangle\Big)$$
$$= \frac{1}{2}\Big((1-i)|01\rangle + (1+i)|10\rangle\Big)$$
$$= \frac{1}{\sqrt{2}}\Big(|0\rangle_L + |1\rangle_L\Big)$$

Application of $X\otimes X$ inverts $T\otimes T^*|\tilde{1}\rangle_L$ case.

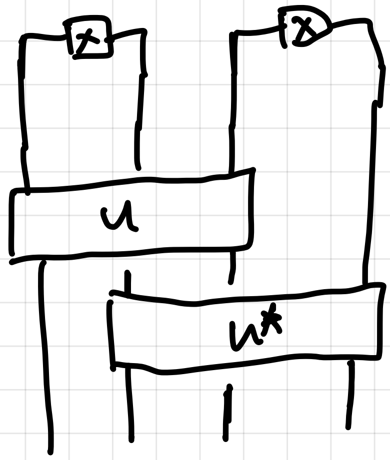
$UAU^\dagger|\psi\rangle =$ 



$\therefore$



Two logical qubit gates



Must have this form $U \otimes U^*$ between pairs.

U must be a particle number preserving unitary:

$$U = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & \gamma & 0 \\ 0 & \delta & \varepsilon & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix}$$

Does not mix $|00\rangle$ or $|11\rangle$

Only mixes $|01\rangle$, $|10\rangle$

Dual qubit encoding

Do we really need fermions? Suppose we encode of logical qubit as:

$$|1\rangle_L = |1\rangle_c \otimes |0\rangle_a = |10\rangle_{ca}$$
$$|0\rangle_L = |0\rangle_c \otimes |1\rangle_a = |01\rangle_{ca}$$

Then

$$\sigma_L^- = |1\rangle\langle 0|_L = |10\rangle\langle 01| = |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$
$$= \sigma_c^- \otimes \sigma_a^+$$
$$\sigma_L^+ = \sigma_c^+ \otimes \sigma_a^-$$

$$|\underline{q}\rangle_L = (\sigma_{L1}^-)^{q_1} (\sigma_{L2}^-)^{q_2} \cdots (\sigma_{LN}^-)^{q_N} |00\cdots 0\rangle_L$$

$|00\cdots 0\rangle_L = |0101\cdots 01\rangle$ Logical zero state

$$|\underline{q}\rangle = (\sigma_{c1}^- \otimes \sigma_{a1}^+)^{q_1} (\sigma_{c2}^- \otimes \sigma_{a2}^+)^{q_2} \cdots (\sigma_{cN}^- \otimes \sigma_{aN}^+)^{q_N} |0101\cdots 01\rangle$$

$$Z_L = Z_c \otimes \mathbb{1}_a \qquad Z_L|1\rangle_L = -|1\rangle_L$$
$$Z_L|0\rangle_L = |0\rangle_L$$

$$X_L = \text{SWAP} \qquad \text{SWAP}\,|10\rangle = |01\rangle \qquad X_L|1\rangle_L = |0\rangle_L$$
$$\text{SWAP}\,|01\rangle = |10\rangle \qquad X_L|0\rangle_L = |1\rangle_L$$

$$H_L = W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$W|10\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$W|0\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$$
$$W|1\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L - |1\rangle_L)$$

$$CX_L = CSWAP_{c_1 c_2 a_2} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 0 & 1 & \\ &&&&& 1 & 0 & \\ &&&&&&& 1 \end{bmatrix}$$

qubit a_1 untouched

$CSWAP_{c_1 c_2 a_2} |11\underline{0}10\rangle \mapsto |11\underline{0}01\rangle$ $\quad |1\rangle_{L_1} |1\rangle_{L_2} \to |1\rangle_{L_1} |0\rangle_{L_2}$

$CSWAP_{c_1 c_2 a_2} |11\underline{0}01\rangle \mapsto |11\underline{0}10\rangle$ $\quad |1\rangle_{L_1} |0\rangle_{L_2} \to |1\rangle_{L_1} |1\rangle_{L_2}$

$$|\underline{q}\rangle = (\sigma_{c_1}^- \otimes \sigma_{a_1}^+)^{q_1} \sigma_{a_1}^- (\sigma_{c_2}^- \otimes \sigma_{a_2}^+)^{q_2} \sigma_{a_2}^-$$

$$\cdots (\sigma_{cN}^- \otimes \sigma_{aN}^+)^{q_N} \sigma_{aN}^- |0000\cdots 00\rangle$$

Note W and CSWAP gates don't change the physical zero state $|0000\cdots 00\rangle$.

$$U|\underline{q}\rangle = U\Big[(\sigma_{c_1}^- \otimes \sigma_{a_1}^+)^{q_1} \sigma_{a_1}^- (\sigma_{c_2}^- \otimes \sigma_{a_2}^+)^{q_2} \sigma_{a_2}^-$$

$$\cdots (\sigma_{cN}^- \otimes \sigma_{aN}^+)^{q_N} \sigma_{aN}^- \Big] U^\dagger |0000\cdots 00\rangle$$

The logical stabilizer state is now encoded in the complexity of this operator $U[\cdots]U^\dagger$.

Now we can rewrite:

$$|\underline{q}\rangle = (\sigma_{c_1}^-)^{q_1} \otimes (\sigma_{a_1}^-)^{1-q_1} \otimes (\sigma_{c_2})^{q_2} \otimes (\sigma_{a_2}^-)^{1-q_2}$$

$$\cdots (\sigma_{cN}^-)^{q_N} (\sigma_{aN}^-)^{1-q_N} |0000\cdots 00\rangle$$

since

$$(\sigma_a^+)^{q}\sigma_a^- = \begin{cases} \sigma_a^- & q=0 \\ \sigma_a^+\sigma_a^- = |0\rangle\langle 0| & q=1 \end{cases}$$

and since we act on $|0\rangle$ we can replace $\sigma_a^+\sigma_a^-$ with $\mathbb{1}_a$ with the same effect.

compute evolution of operators:

$$W|10\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$W(\sigma_c^- \otimes \mathbb{1}_a)W^\dagger W|00\rangle = W(\sigma_c^- \otimes \mathbb{1}_a)W^\dagger|00\rangle$$

but

$$W(\sigma_c^- \otimes \mathbb{1}_a)W^\dagger = \frac{1}{\sqrt{2}}\left(\mathbb{1}_c \otimes \sigma_a^- - \sigma_c^- \otimes Z_a\right)$$

$$W(\mathbb{1}_c \otimes \sigma_a^-)W^\dagger = \frac{1}{\sqrt{2}}\left(Z_c \otimes \sigma_a^- + \sigma_c^- \otimes \mathbb{1}_a\right)$$

The presence of Z's has no effect on the action on $|00\rangle$, but complicates the operator evolution.

Consider CX_L:

$$CSWAP\left(\sigma_{c_1}^- \otimes \mathbb{1}_{c_2} \otimes \sigma_{a_2}^-\right)CSWAP^\dagger$$

$$= \sigma_{c_1}^- \otimes \left(\sigma_{c_2}^- \otimes \underbrace{\sigma_{a_2}^+\sigma_{a_2}^-}_{|0\rangle\langle 0|} + \underbrace{\sigma_{c_2}^-\sigma_{c_2}^+}_{|1\rangle\langle 1|} \otimes \sigma_{a_2}^-\right)$$

$$\mathrm{CSWAP}\left(\sigma_{c_1}^- \otimes \sigma_{c_2}^- \otimes \mathbb{1}_a\right)\mathrm{CSWAP}^\dagger$$

$$= \sigma_{c_1}^- \otimes \left(\sigma_{c_2}^- \otimes \underbrace{\sigma_{a_2}^- \sigma_{a_2}^+}_{|1\rangle\langle 1|} + \underbrace{\sigma_{c_2}^+ \sigma_{c_2}^-}_{|0\rangle\langle 0|} \otimes \sigma_{a_2}^-\right)$$

The CX_L gate thus gives a more complicated operator evolution than we might anticipate by creating new terms which vanish when acting on $|10000\cdots00\rangle$