Numerical Methods

PDE 3 – Worksheet

December 4, 2017

1 Penetration of heat into a wall

We will now look at a simple problem of penetration theory, discussed in § 7.3.3 in the reader.

Assume we have an infinitely stretched plate of width $2L$, as shown in the figure below. The initial temperature is uniform and given by T_0. At time $t = 0$, the temperature of the wall of the body at $x = \pm L$ is instantaneously raised to $T_1 > T_0$.

Figure 7.8 from the Transport Phenomena reader

We assume that the heat transfer coefficient λ, bulk density ρ and thermal capacity C of the material is constant, and no source term is present.

1. Find the heat profile $T(t, x)$ after $t_{max} = 6\,\mathrm{h}$, using `pdepe`. For this, you have to solve the heat equation

$$\frac{\partial \rho C T}{\partial t} = \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x})$$

   on $-L < x < L$ and $0 < t < t_{max}$ for initial and boundary conditions $T(0, x) = T_0$, $T(t, \pm L) = T_1$.

   Assume the plate is $L = 0.1\,\mathrm{m}$ thick, made from concrete, and the temperature is raised from $T_0 = 20\,°\mathrm{C}$, $T_1 = 30\,°\mathrm{C}$. You may use $C = 1000\,\mathrm{J/kg/K}$, $\rho = 2000\,\mathrm{kg/m^3}$, $\lambda = 1.35\,\mathrm{W/m/K}$.

2. Plot the temperature profile for intermediate time steps. Confirm that your solution looks similar to Figure 7.8.

3. Note that the problem is symmetric, i.e. $T(t, x) = T(t, -x)$. Therefore, the temperature gradient is zero in the center of the body. Thus you can reduce the problem to solving the heat equation on $0 < x < L$ with van Neumann boundary conditions at $x = 0$, $\frac{dT}{dx}(t, 0) = 0$.

   Solve this new initial-boundary value problem to find the heat profile $T(t, x)$ after $t_{max} = 6\,\mathrm{h}$. Plot the solution together with the solution of question 1 and confirm that they give the same result.

4. In the Transport Phenomena course you have derived that the mean temperature $T_m(t)$ and the temperature at the centre of the body $T_G(t)$ can be found by using Figure 7.9. where the nondimensionalised values of t, T_M and T_G are plotted:

$$F_0 = \frac{at}{L^2}, \quad Y_M = \frac{T_1 - T_M(t)}{T_1 - T_0}, \quad Y_G = \frac{T_1 - T_G(t)}{T_1 - T_0}. \tag{1}$$

   where $a = \frac{\lambda}{\rho C}$ is the thermal diffusivity. We are now going to reproduce Figure 7.9 using numerical methods:

5. Calculate the mean temperature T_M and the temperature at the center of the body, T_G. Then nondimensionalise time and temperature according to Equation (1) and plot the values Y_G, Y_M as a function of F_0. Compare your solution to Figure 7.9.

6. *Challenge*: The exact solution for the temperature profile (for a flat plate) is given by equation (7.73) as an infinite series in the Reader. Evaluate only the first term in the infinite series at $\phi = 0$ and compare with Y_G. Evaluate only the first term in the infinite series and average numerically over $0 \leq \phi \leq L$ and compare with Y_M. The approximation should be very accurate for large F_0.

7. So far, we have studied heat penetration for a flat plate. Now do the same for a cylindrical body and a spherical body of radius L[1] Compare the resulting temperature profiles at a fixed time, e.g. $t = 1\,\mathrm{h}$. Which body heats quickest?

8. Plot Y_G and Y_M for the cylindrical body, and add it to the plot in equation 5.[2].

[1] to change the geometry from planar to cylindrical/spherical in `pdepe`, set the parameter $m = 1$, respectively $m = 2$. The coordinate x is now the radial coordinate.

[2] The mean temperature in a cylinder of radius L is given by $\int_0^L x^m T \,\mathrm{dx} / \int_0^L x^m \,\mathrm{dx}$