Numerical Methods

PDE 4 Worksheet – Navier stokes

December 12, 2017

Today's exercise is not for the light-hearted – we are going to study the flow of blood through a dog's artery! But calm down, in the end it's all just mathematics.

Consider the flow of blood through a rigid, long cylindrical blood vessel, illustrated in Figure 1. Flow is only in the axial direction ($v_r = v_\theta = 0$) and is fully developed (v_z is a function of r only). As a result, only the z-component of the Navier-Stokes equations is nonzero. The non-vanishing terms are given by the formula

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right). \tag{1}$$

Assume $R = 2.7\,\mathrm{mm}$ is the radius of the vessel, $\rho = 1\,\mathrm{g/cm^3}$ the density of blood, and $\mu = 12\,\mathrm{Pa\,s}$ the viscosity of blood. Further assume the heart can exert a maximum pressure gradient of $|\frac{\partial p}{\partial z}| \leq A = 5000\,\mathrm{N/m}$.[1]

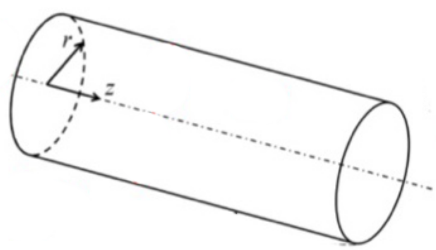
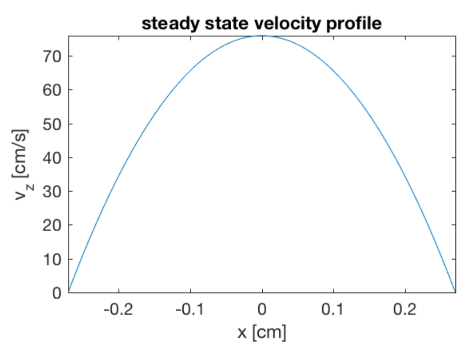


Figure 1: A long cylindrical vessel.

1. First, we are interested in the maximum possible flow rate in the blood vessel. Assume a constant pressure gradient of $\frac{\partial p}{\partial z} = A$ and no-slip boundary conditions, $v_z(R) = 0$. Then solve equation (1) using `pdepe` until a time where the flow becomes steady.[2] Use $v_z = 0$ at $t = 0$ as initial conditions.[3] You should get the following steady state velocity profile.[4]



2. Compute the Reynolds number, $Re = \frac{\rho v_z^{\max} R}{\mu}$, to confirm the assumption that the flow is laminar.

3. Find the volumetric flow rate $Q_{\max} = \int_0^R 2\pi r v_z \, dr$ in steady state[5].

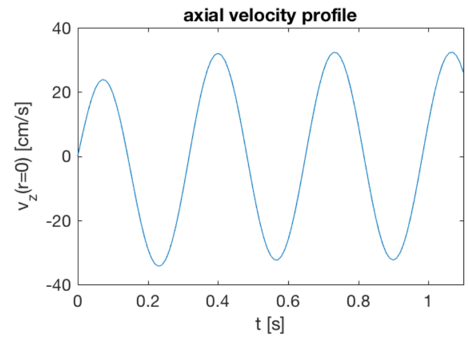
[1] It's best if you first convert all parameter values to SI-units.
[2] This should happen at $t \approx 1$.
[3] Don't forget to use cylindrical coordinates ($m = 1$). Use no flux boundary conditions in the center $\partial v_z / \partial r = 0$ at $r = 0$.
[4] Instead of plotting $v_z(r)$ using `plot(r,v)`, I plotted $v_z(x)$ using `plot([r(end:-1:1) r],[v(end:-1:1) v])`.
[5] $Q_{\max} \approx 8.7\,\mathrm{cm^3/s}$

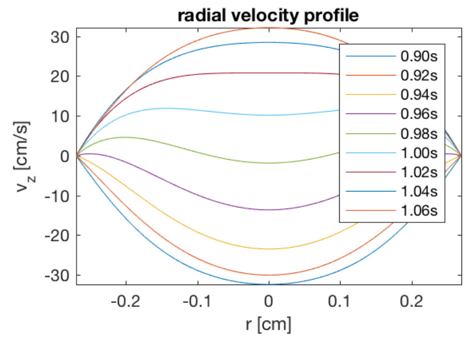
4. In reality, of course, the heart does not pump blood at a fixed rate. Rather, the pressure gradient oscillates in time with a frequency ω and can be represented as

$$-\frac{\partial p}{\partial z} = A\cos(\omega t).$$

Assume a heart rate of $3\,\mathrm{Hz}$ ($\omega = 6\pi\,\mathrm{rad/s}$). Then solve (1) for $0 < t < 1.1\,\mathrm{s}$. Plot the axial velocity profile $v_z(t, r)$ at $r = 0$. The blood now flows periodically forward and backward, as shown in the figure below.

5. The axial velocity reaches a minimum at $t \approx 0.9$ and a maximum at $t \approx 1.06$. Plot several radial velocity profiles to get an idea of the flow behaviour. The flow profile is now quite complicated and far from parabolic.





6. However, this still doesn't reflect realistic blood flow, as the heart valves close to impede back flow. This can be modelled by assuming that the pressure gradient remains negative

$$-\frac{\partial p}{\partial z} = \max(0, A\cos(\omega t)).$$

Now the axial velocity profile $v_z(t, r)$ should show only forward flow, as in the figure below.

7. Plot the behaviour of the volumetric flow rate, $Q = \int_0^R 2\pi r v_z\,dr$, as a function of time, and compare this value with the maximum volumetric flow rate obtained in question 3.