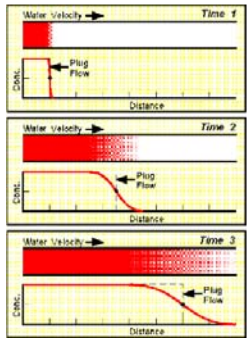
General Transport Equation:
Application to Special Cases



Environmental Hydraulics

The General Transport Equation

Advection-Diffusion (AD) Equation

3-D:

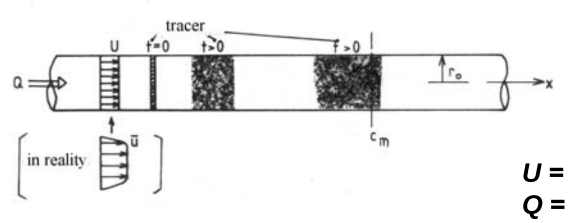
$$\boxed{\frac{\partial \bar{c}}{\partial t}} + \boxed{\bar{u}\frac{\partial \bar{c}}{\partial x} + \bar{v}\frac{\partial \bar{c}}{\partial y} + \bar{w}\frac{\partial \bar{c}}{\partial z}} = \boxed{\frac{\partial}{\partial x}(D_x \frac{\partial \bar{c}}{\partial x}) + \frac{\partial}{\partial y}(D_y \frac{\partial \bar{c}}{\partial y}) + \frac{\partial}{\partial z}(D_z \frac{\partial \bar{c}}{\partial z})}$$

Change in concentration with time at x,y,z

Change due to advection

Change due to diffusion

Transport of Tracer in a Pipe



$U = Q/A$,
Q = flow rate
A = cross sectional area

1D approach:

$$\frac{\partial c_m}{\partial t} + U\frac{\partial c_m}{\partial x} = \frac{\partial}{\partial x}\left(E_d \frac{\partial c_m}{\partial x}\right)$$

change in concentration with time | **advection** | **diffusion (dispersion)**

Dispersion Coefficient

Taylor (1954) for turbulent pipe flow:

$$E_d = 10.1 r_0 u_*$$

Elder (1959) for flow in a wide channel:

$$E_d = 5.9 H u_*$$

Fischer (1966) for natural rivers:

$$50 u_* R < E_d < 700 u_* R$$

Comparison between Mixing Coefficients

molecular diffusion	**10^{-7} (heat)** **10^{-9} (dissolved material)**
turbulent diffusion	**10^{-4} – 10^{-0}**
longitudinal dispersion	**10^{-3} – 10^{2}**

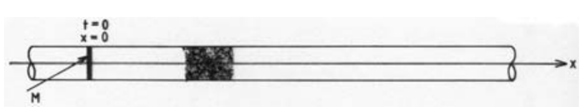
(unit: m^2/s)

Example I: Spreading of Tracer in a Pipe

Example: Consider a pipe with the diameter *D*=0.10 m. A certain amount of tracer is introduced homogeneously across the pipe cross section at *x*=0 at time *t*=0.

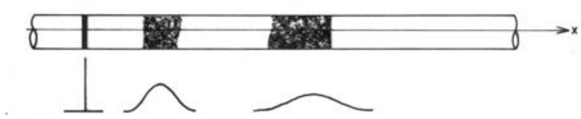
a) spreading of the tracer?
b) maximum concentration 1 km downstream of the injection point? (added amount of tracer is M=1 gram; assume that *U*=1 m/s and that the pipe wall is smooth)

$$\frac{\partial c_m}{\partial t} + U\frac{\partial c_m}{\partial x} = E_d \frac{\partial^2 c_m}{\partial x^2}$$



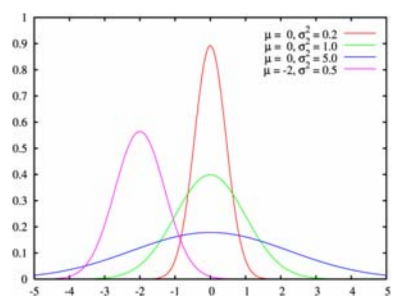
Solution:

$$c_m(x,t) = \frac{M}{A\rho\sqrt{4\pi E_d t}} \cdot \exp\left(-\frac{(x-Ut)^2}{4E_d t}\right)$$



Gaussian (normal) distribution

Normal Distribution



$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Comparison with previous solution $\longrightarrow$

$$\mu = Ut$$
$$\sigma = \sqrt{2E_d t}$$

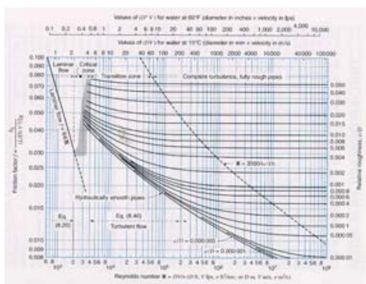
Estimate of E_d:

Taylor =>

$$E_d = 10.1 r_0 u_*$$

What is u_*?

The Reynolds number: $\mathrm{Re} = \frac{1 \cdot 0.1}{10^{-6}} = 10^5$



Smooth pipe flow gives that the frictional coefficient is *f*=0.018 according to Moody's diagram

Shear stress:

$$\frac{\tau_0}{\rho} = \frac{f \cdot U^2}{8} = \frac{0.018 \cdot 1^2}{8} = 0.00225$$

E_d = 10.1 0.05 4.75 10^{-2} = 2.38 10^{-2} m/s

Maximum concentration:

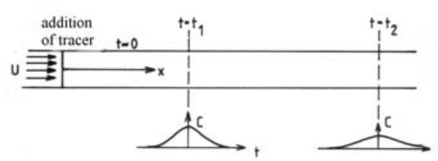
$$c_{\max,1000\,m} = \frac{10^{-3}}{10^{-2}\frac{\pi}{4}10^{3}(4\pi\, 2.38\, 10^{-2}\, 1000)^{0.5}} = 0.7 \cdot 10^{-6} \quad \mathrm{kg/kg} \quad \mathrm{water}$$

Example II: Determine the Dispersion Coefficient

Release tracer and measure *c* at two times (t_1 and t_2)

General solution:

$$c = \text{const} \cdot \exp\left(-\frac{(x - U \cdot t)^2}{4 \cdot E_d \cdot t}\right)$$



x=Ut

Measure standard deviation σ in *c*. From Gaussian distribution:

$$\sigma^2 = 2 E_d t$$

(difficult to measure *c* in space at a specific time)

Time rate of change in σ:

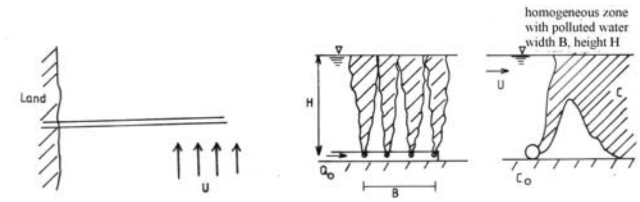
$$\Delta\sigma^2 = 2 E_d \Delta t$$

$$\sigma_2^2 - \sigma_1^2 = 2 E_d (t_2 - t_1)$$

$$\longrightarrow \quad E_d = \frac{1}{2} \frac{\sigma_2^2 - \sigma_1^2}{t_2 - t_1}$$

Example III: Spreading in the Far Field Zone

Example from Trelleborg



Discharge of treated municipal wastewater from a diffuser

Initial dilution of 50 – 100 times
(buoyant jet theory)

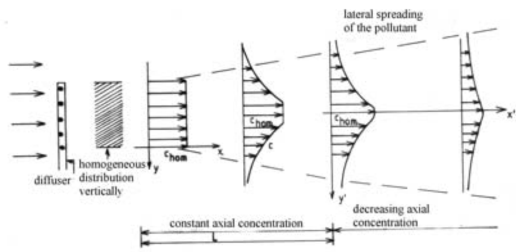
Assume complete vertical mixing downstream
the diffuser. Dilution of wastewater:

$$S = \frac{c_0}{c_{\text{hom}}} = \frac{U \cdot H \cdot B}{Q_0}$$

Dilutionen increases with:

- velocity in receiving water
- water depth
- length of diffuser

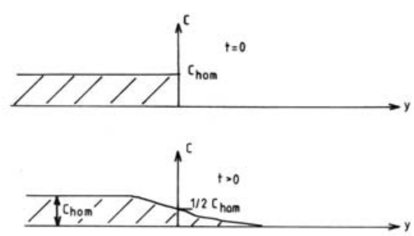
Far-field mixing and dilution



Uniform distribution after near-field mixing followed by spreading at the edges of the advected wastewater

Model of spreading (compare AD equation):

$$U\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial y^2}$$



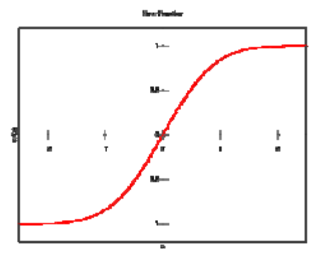
semi-infinite discharge

Solution:

$$\frac{c}{c_{\text{hom}}} = \frac{1}{2}\left(1 - \text{erf}\left(\frac{y}{\sqrt{4Dx/U}}\right)\right)$$

Definition of error function:

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z \exp\left(-\xi^2\right)d\xi$$



y=-B/2, x=L =>

erf(-z) = -erf(z)

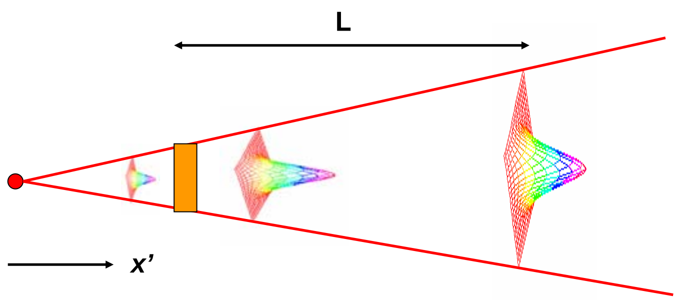
$$\frac{c}{c_{\text{hom}}} = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{B/2}{\sqrt{4DL/U}}\right)\right)$$

Centerline concentration affected after c/c_{hom}=0.99

Lateral spreading after decreasing centerline concentration:

$$U\frac{\partial c}{\partial x'} = D\left(\frac{\partial^2 c}{\partial x'^2}+\frac{\partial^2 c}{\partial y'^2}\right)$$

(*x'* and *y'* taken from location of imaginary source)

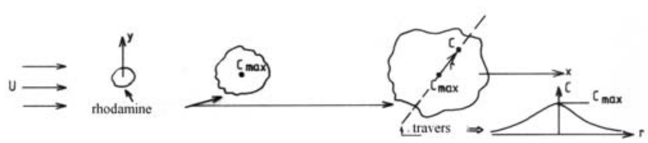


Solution:

$$\frac{c}{c_{\text{hom}}} = B\sqrt{\frac{U}{4D\pi x'}}\exp\left(-\frac{y'^2 U}{x'^2 4D}\right)$$

Knowledge of *D* necessary.

Estimate *D* based on field measurements with tracer:



rhodamine

Spreading of injected tracer cloud (2D in space):

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2} + D\frac{\partial^2 c}{\partial y^2}$$

Solution:

$$c = \frac{M_0}{4\pi D t}\exp\left(-\frac{(x-Ut)^2 + y^2}{4Dt}\right)$$

Moving with the tracer:

$$c = \frac{M_0}{4\pi D t}\exp(-\frac{r^2}{4Dt}) \qquad r^2 = x^2 + y^2$$

***r*=0 => c_{max}**

***r*=r_o => c_o**

Solve for *D*:

$$D = \frac{r_o^2}{4t\ln(c_{\max}/c_o)}$$

Numerical example:

A diffuser with five nozzles at a distance of 25 m in a water depth of 6 m.

Current parallel to the coast (at least 0.05 m/s).

Turbulent diffusion coefficient *D*=0.054 m^2/s.

Wastewater discharge of Q_o=0.2 m^3/s and initial dilution at least 50 times (c_o=1).

Concentration after homogenization:

$$c_{\text{hom}} = c_o \frac{Q_o}{UHB} = 1.0 \cdot \frac{0.2}{0.05 \cdot 6 \cdot 100} = 0.00667$$

Corresponds to a dilution of *s*=150

Reduction of centerline concentration (1%):

$$0.99 = \left(1 + \text{erf}\left(\frac{50}{\sqrt{4 \cdot 0.054 \cdot \frac{L}{0.05}}} \right) \right) \cdot 0.5$$

$$\text{erf}\left(\frac{24}{\sqrt{L}} \right) = 0.98 \longrightarrow$$ ***L* ≈ 214 m**

Spreading after centerline concentration affected

Location of imaginary source (y'=0 and c/c_{hom}=0.99):

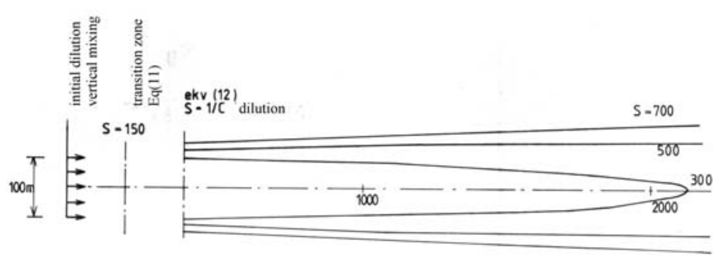
$$0.99 = 100 \cdot \sqrt{\frac{0.05}{4 \cdot 0.054 \pi x'}} \longrightarrow x' = 740 \text{ m}$$

Concentration 2000 m downstream the diffuser:

$$\frac{c_{center}}{c_{\mathrm{hom}}} = 100 \cdot \sqrt{\frac{0.05}{4 \cdot 0.054 \pi (2000 + 740 - 214)}} = 0.54$$

Total dilution from nozzle: S=150/0.54=300 times

Iso-Concentration Lines Downstream Diffuser



Zones:

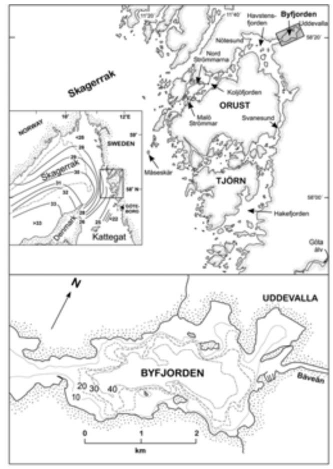
- **near-field**
- **transition**
- **far-field**

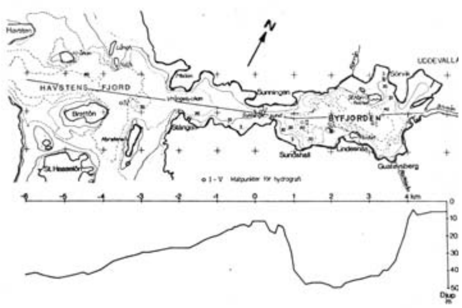
Example IV: Water Quality of Byfjorden

Complex fjordsystem

Polluted from municipal discharge, industry etc

Studies on water exchange and mixing processes



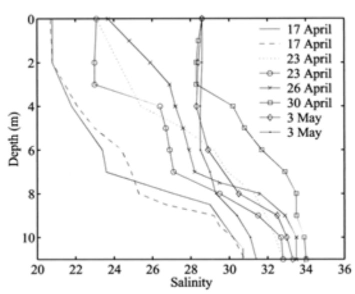


Discharge point for treated wastewater from Uddevalla

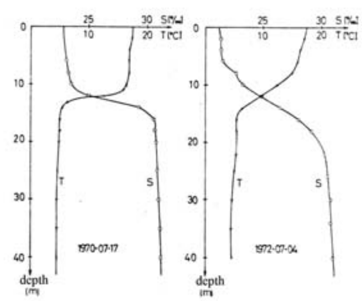
Two alternatives:

• discharge into surface water

• discharge into deep water



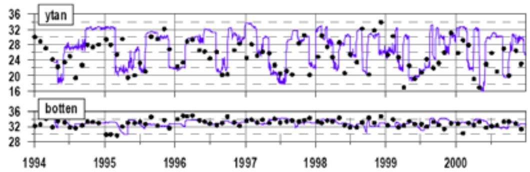
Salinity profiles at the sill



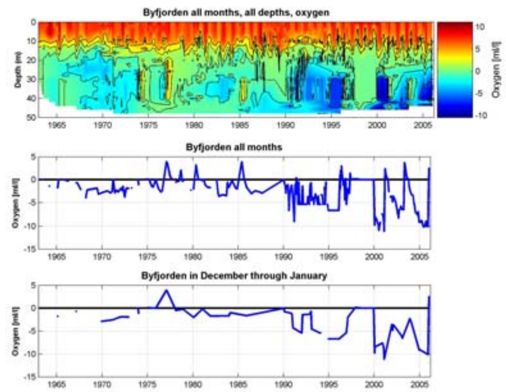
stratification

(salinity)

Salinity



Oxygen



Estimate spreading in the surface layer (1D model; integration over cross section):

$$\bar{u}\frac{d\bar{c}}{dx} = \frac{1}{A}\frac{d}{dx}\left(AD\frac{d\bar{c}}{dx}\right) - Q$$

degradation $Q = k\bar{c}$

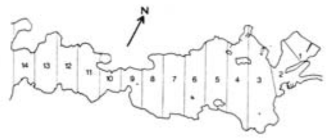
dispersion

Advection (fresh-water flow)

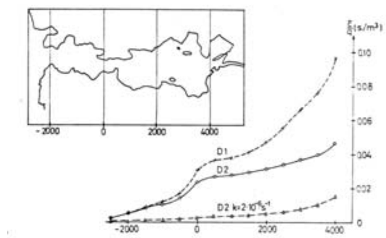
No water exchange explicitly included.

D **determined experimentally**

Numerical solution



Divide area into boxes



Steady-state concentration distribution (differen values on *D* and degradation)