Financial Securities and Markets Final Project

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1 Introduction

In this report, we detail the specification for the pricing model of the equity-rate hybrid exotic with the following payoff.

$$\max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(k' - \frac{L(T,T,T+\Delta)}{L(0,T,T+\Delta)}\right)\right]$$

We have broken the core model into three components: Equity Risk, Rate Risk, and Equity-Rate correlation risk.

- **Equity Risk** is present due to the payoff based on Nikkei-225 index returns at a future time. We can approximate this part of the equation as a vanilla European option with an effective strike of $K \cdot S_0$. However, since Nikkei-225 is in Japanese Yen, it needs to be quantoed to USD. This adds a non-zero drift to the equation, which is stated below.

  $$\frac{dF_t}{dt} = (r_{\text{local}} - \rho \cdot \sigma_S \cdot \sigma_{FX})dt + \sigma_s dW_t$$

  Here F_t is the underlying stock in the quantoed currency and r_{local} refers to the local risk free rate, in the underlying stock's currency. ρ is the correlation between the logarithm of FX exchange returns of USD/JPY and Nikkei-225 returns, σ_s is the volatility of the Nikkei-225 index returns and σ_{FX} is the volatility of the USD/JPY exchange returns.

- **Rate Risk** is due to exposure to LIBOR-3m forward rate. We model this payoff using the standard short-rate model. L(0, T, T + delta) is the current 3m LIBOR rate between time T and T + delta. And we model LIBOR 3m forward rate using the following Hull-White equation. In this report, we will go into the reasoning behind using the model for this payoff.

  $$dr_t = (\theta(t) - \alpha r_t)dt + \sigma dW_t, \quad (\alpha(t) > 0)$$

  The parameter σ in the above equation can be written as a function of σ_P, a, T and δ_T.

  $$\sqrt{S} \cdot \sigma_P = \frac{\sigma}{\alpha} \cdot \left(1 - e^{-\alpha\Delta}\right) \cdot \sqrt{\frac{1 - e^{-2\alpha S}}{2\alpha}}.$$

  σ_P is the market-implied volatility of the underlying rate which not unlike, implied volatility of an equity derivative is a product of market perception and trading of LIBOR, and its volatility. σ is calculated from the values of market observable σ_P by inverting the above formula. Here, α is the rate of mean reversion. The mean-reversion parameter θ is derived from the market-implied term structure of the forward rate. θ has a closed-form solution, the derivation of which we will detail in section 3.

- **Correlation Risk** One of the more complex aspects of this hybrid model is the correlation between two asset classes of risk: Equity and Rates. While there are elaborate methods used industry-wide for this calculation, we have made certain simplifying assumptions (which we will describe in the next section) to make the model parsimonious.

  Also, worth mentioning is the usage of the **SVI model** for interpolating market-implied volatilities required in the project. The parameterization and calibration of the SVI model are done as described in [Gatheral and Jacquier, (3)]. Simply said, the model follows the condition that there is no arbitrage on the continuous volatility surface and applies that by creating an arbitrage-free Butterfly spread.

2 Data

The project uses data from several sources. We have made sure that all the data is either statically available or dynamically queries through the code. Because of this, running the code is easy. Also, for an in-depth understanding of the data usage, please also read Section 3, which describes assumptions and details of the decision-making for this project. Following is the data used and the sources it is acquired from:

- US Treasury yield curve is obtained from the market yield of Treasuries. Adjusting the yield for coupons, in a process called bootstrapping, we get the zero curve for UST. This is already listed in the code for the standard maturity term structure. We have used the US curve for various discounting, analysis, and sanity checks on other rates which are less readily available. Most notably parameterization of the Hull-White model.
- Japanese Yield curve is obtained similarly to the US yield curve, bootstrapping the market data. We use JPY curve as a deterministic rate which serves as the local risk-free rate in the Quanto equation described in Equity Risk of Section 1.
- LIBOR yield curve is obtained from various market data sources and is stored as a part of the jupyter notebook. LIBOR curve is used as the reference forward rate curve for calculating the parameters θ in the Hull-White model.
- LIBOR-3m daily rate data is downloaded from sources and is adjoined to the code in the form of a CSV. This data is primarily important to calculate the historical correlation between LIBOR rate and the Nikkei index. This will be submitted along with the code.
- Nikkei index data is downloaded from Yahoo Finance. This data is used to calculate the correlation between N225 and 3-m LIBOR, as well as between N-225 and USD/JPY FX rate. This data gets queried directly from the code.
- SPX market data is downloaded from Yahoo Finance as well. SPX index data is used as an important benchmark for us to generate various correlations. SPX Options data is also downloaded to calibrate the SVI Vol Surface.
- Final piece of data is the USD/JPY exchange rate which had been downloaded from FRED(Federal Reserve Economic Data). This rate is used for finding the correlation between FX rate and N225 index data.

3 Assumptions, Decisions, and Details

1. One of the most fundamental assumptions of the project is that the Nikkei-225 follows a geometric Brownian Motion. This is because there is no path dependency in the payoff, only the final price, it is safe to take this assumption and treat this as a quantoed European option.
2. Volatility of the Nikkei-225 used is Implied Volatility. This is a better estimate of future volatility than historical volatility, as this factors in market expectations of volatility several years down the line.
3. To calibrate IV using SVI, we could not source option chains of Nikkei-225 with varying strikes, and maturities mapped to market prices. Hence, we used SPX options to get a sense of that relationship. Historically, we find a high correlation between SPX and Nikkei-225, and this goes on to suggest that relative IVs(different strikes and maturities) have a similar relationship. Prof. Javaheri mentioned in class that we could use synthetic data if required, this is a similar application. The FX volatility and LIBOR volatility are scaled versions of the implied vols we obtained in the above process. This scaling is done based on the basis of relative values of volatilities across different asset classes. For example, Indexes have really high volatility(15%-30%), whereas for rates annualized volatility is a much smaller number, 0.5% to 1%. These scalings are done on the basis of realized ATM vols of SPX, LIBOR, and USD/JPY. Typically these scalings are non-linear and should be done with a complex polynomial fitter. I have approximated this by scaling linearly. Due to this results are as expected for reasonable values, but might become unpredictable for very deep out-of-money options.

4. Correlation between USD/JPY and Nikkei is historical. It is hard to find a security available in the market that could help estimate the market-implied correlation between an FX rate and index returns. Hence, this seems like a valid simplifying assumption. The historical correlation of JPY/USD as estimated in the model comes out to be 0.73, which makes sense since in general, a stronger Japanese Yen would also prompt the Japanese stock market to do better than average.

5. As compared to the most popular short-rate models, Hull-White enjoys simplicity in computing the bond prices owing to the linear structure of the drift. The model is very efficient in capturing the dynamics of the term structure. Moreover, it has the mean reversion property, a linear diffusion term, and an affine term structure which further reduces the underlying differential equations to standard equations like the Riccati equation.

6. The payoffs for the Nikkei-225 and LIBOR stochastic differential equations are priced using Monte-Carlo simulations. These simulations are generated from the Euler-Maruyama numerical method. The approximation is very precise for the SDEs employed in this project.

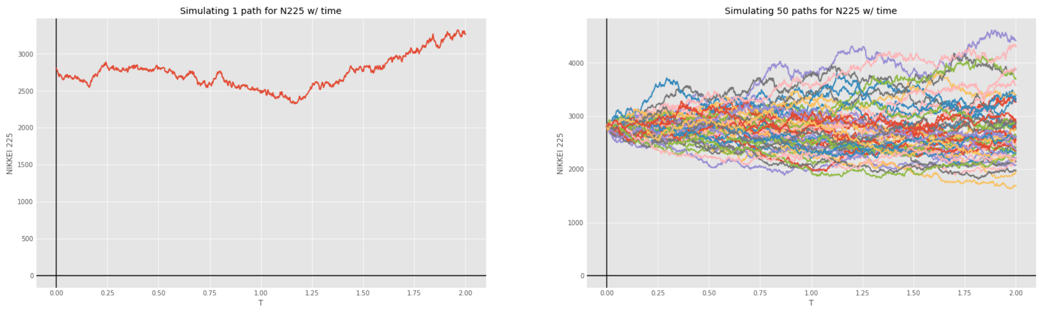


Figure 1: Simulating 1 and 50 paths for the Nikkei-225 with time for $r_{\text{local}} = 0.0004$, $S_0 = 2800$, $K = 1.1$, $\rho = 0.4$, $\sigma_S = \sigma_{FX} = 0.21$.

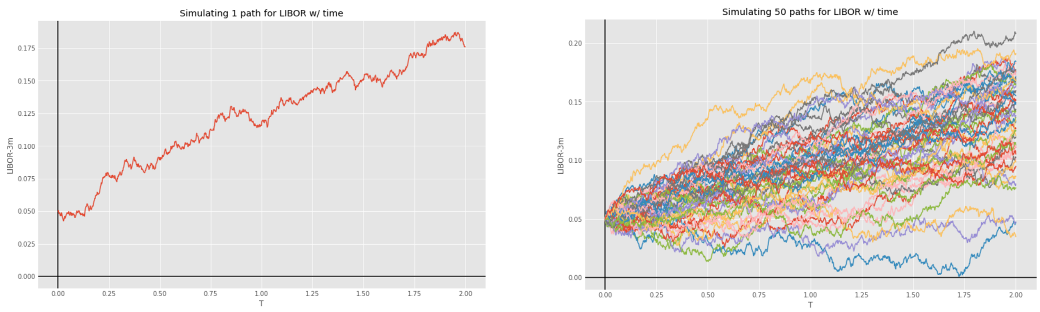
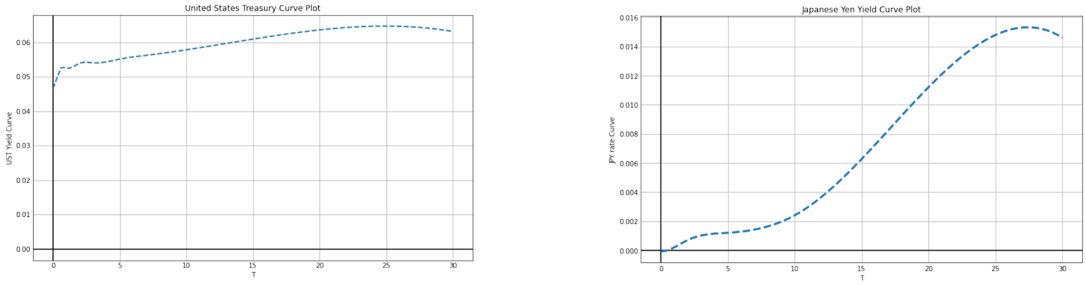


Figure 2: Simulating 1 and 50 paths for LIBOR with time for $\theta = 0.9$, $L_0 = 0.05$, $K' = 1.5$, $\alpha = 0.06$, $\sigma = 0.04$.

7. The Japanese rate taken as the local rate is assumed to be a deterministic rate. The reason for this is the very low correlation between a very static Japanese yield curve and the other much more volatile parameters like Nikkei-225 vol, USD/JPY vol etc. in the drift of the Quanto equation.

8. The shorter maturities on the Japanese rate curve are negative. To avoid any issues in the volatility surface calibration, we have capped the negative values with very small positive values.

9. There are three yield curves used in the process: the US Treasury yield curve, the Japanese yield curve, and LIBOR 3m curve. UST and Japanese yield curves are calculated using their respective sovereign bond yields. All these curves are interpolated using the Cubic Spline methodology. Cubic Spline constructs a third-order polynomial and fits them through existing data points. The function is twice differentiable and the second order of each derivative is set to zero at endpoints. This is one of the most widely used methods for yield curve interpolation.



(a) United States Treasury Curve for $\alpha = 0.03$, $\sigma = 0.2$ and $t = 30$

(b) Japanese Yen Yield Curve for maturities $[0.5, 1, 2, 5, 10, 20, 30]$.

10. There is a possibility to use a simple closed formula for the Quanto equation, but for the sake of simplicity, we generate Monte Carlo simulations.

11. To establish historical correlations, between N225 and USD/JPY rate, we have compared the results going back in history for up to 5 years. There are a few reasons we have chosen this timeline. According to our analysis, the correlations for slightly older timelines do not vary as much. Moreover, we believe this 5-year time frame includes several market regime changes, which make this result itself encompassing and complete. Secondly, we believe that the current market dynamics are very different from the ones in place 10+ years ago, due to several changes in regulation in the past decade, which impacts the behaviors of its participants.

12. The rate of mean reversion in Hull-White, α is assumed to be constant at 0.06. The values for this parameter are to be in the range of lower than 1. This is extremely well documented and cited across research papers.

13. Since only 1-year returns of LIBOR market data are publicly available, we used these to calculate N225 and LIBOR correlations.

14. We run both the model simulations on 1000 paths, and take the rate of change of drift, dt to be equal to 0.001

15. In SVI calibration, we have computed the optimization and stored these parameters since it takes essentially a very long time to run the algorithm for all iterations.

4 The pricing model

4.1 Equity Risk

In the first part of the payoff, $C_T = \max(0, \frac{S(T)}{S(0)} - k)$, the underlying is denominated in JPY but our numeraire or base currency is USD, therefore we use a quantity adjusting option to shield from the fluctuations in both the Nikkei index and the exchange rate between USD and JPY.

To model the quanto option, we account for the stock price in the underlying currency, the exchange rate volatility and a correlation between the two terms introduced. We introduce a model like Black-Scholes but with a drift term that accounts for the quanto adjustment. As described in the introduction, the model is given by $\frac{dF_t}{F_t} = (r_{\text{local}} - \rho \cdot \sigma_S \cdot \sigma_{FX})dt + \sigma_S dW_t$. The term $-\rho\sigma_S\sigma_{FX}$ is the quanto adjustment and it has an opposite effect on the dividends. See (1).

4.2 Interest Rate Risk (LIBOR)

The term structure can be completely determined by the term structure equation given that the drift term, the diffusion term and the market risk are specified. In our model, we obtain the diffusion term using volatility calibration, described in the next section. Moreover, instead of specifying the other two terms, we specify the short rate dynamics under the martingale measure Q.

4.2.1 Hull-White Model

Among the several commonly used martingale models, we specify the Q dynamics of the short rate using the Hull White (extended Vasicek) model. This model offers a number of computational and analytical advantages over the other common models like the CIR, Dothan and Black-Derman-Troy.

- **Linearity**. First, θ is linear in r_t with the assumption that σ is known. This makes computation of the bond prices given by $p(0,T) = \mathbb{E}\left[e^{-\int_0^T r_s ds}\right]$ relatively easy since the the r process turns out to be normal if r_0 is a constant or Gaussian, and the bond price is the expectation of a log-normal variable if the time was discretized. Other models pose a computational problem because the computation of bond prices involves an exponent of a log-normal variable, which is analytically infeasible.

- **Mean Reversion**. As $t \to \infty$, r_t has a limiting Gaussian distribution, which implies that it has the tendency to revert to the mean $\frac{\theta(t)}{a}$. This is important for short-rate models because a low short rate is not desired and when the short rate becomes significantly high, external factors like the central bank could intervene to get it down. Therefore, when $r_t < \frac{\theta(t)}{a}$, the drift is positive and the r increases. Whereas when $r_t > \frac{\theta(t)}{a}$, the drift is negative and r decreases as desired. This is not the case for the Ho Lee model, despite it having linear dynamics.

- The Hull White Model also has a linear diffusion term and offers advantages opposed to square root models which are difficult to study since for existence and uniqueness of SDEs, the diffusion term has to be Lipschitz continuous and clearly $\sqrt{\sigma}$ is not Lipschitz over $\mathbb{R}$.

- **Affine Term Structure**. Since $\mu(t) = \theta(t) - ar_t$ and σ are affine functions of r, the existence of functions $A(t,T)$ and $B(t,T)$ is guaranteed such that the term structure $p(t,T); t \in [0,T], T > 0$ has the form
$$p(t,T) = F(t, r_t; T) = e^{A(t,T) - B(t,T)\cdot r}.$$
Such a form enables us to solve the partial derivatives of F in the general term structure equation and we get a separable differential equation, giving us two separate systems dependent on A and B. This is particularly simple because we get a Riccati equation for the determination of B and substituting B and integrating gives us A. This term structure is again not possessed by Dothan and Black-Derman-Troy.

Finally, Hull-White is preferred over Vasicek despite having similar advantages because it is an extension. One disadvantage of Hull-White is the theoretical possibility of the interest rate being negative, however, this model is widely used in practice.

Parameter estimation. In the Hull-White model, A and B are given by

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\},$$
$$A(t,T) = \int_t^T \left\{\frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T)\right\} ds.$$

The drift parameter θ can be obtained by fitting the theoretical forward rate curve to the observed forward rate curve as the following if $f^*(0,T)$ is the observed forward rate and $g(T) = \frac{\sigma^2}{2}B^2(0,T)$.

$$\theta(T) = f_T^*(0,T) + g'(T) + a\{f^*(0,T) + g(T)\}$$

Note that this requires a choice of a and σ, as can be seen in (2) and (4).

5 SVI Volatility Surface Calibration

In this model, we use three classes of volatilities to model implied volatility: Volatility of Nikkei index, volatility of FX exchange rate, and volatility of LIBOR-3m rate.

We obtain the implied volatility smile from the Stochastic Volatility Inspired (SVI) calibration with the Black-Scholes Model. This model is useful because the implied volatility obtained is absent of a butterfly arbitrage. A Butterfly trading strategy is a 4-options trading strategy that bets on a high volatility structure and the high vol spread is the pnl. This is essential because it ensures the existence of a non-negative probability density, and hence any static arbitrage. This is true because it has been shown that given two volatility smiles at fixed times, t_1 and t_2, then under mild conditions there exists an interpolation such that the implied volatility surface is free of static arbitrage for $t \in (t_1, t_2)$.

Suppose that $w\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes total implied variance where k is the moneyness factor, also defined as $k = \log(\frac{K}{S})$ where K is the strike price and S is the spot price, and t is the maturity. For the butterfly arbitrage, we consider a slice of the implied volatility, i.e. we fix the maturity and therefore denote the volatility surface map as $w(k)$.

Definition 1 (Butterfly Free Arbitrage)**.** If a function $g\colon \mathbb{R} \to \mathbb{R}$ is introduced such that

$$g(k) := \left(\frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{4},$$

then for a fixed t, the volatility surface is free of butterfly arbitrage if and only if:

1. $g(k) \geq 0$
2. $\lim_{k\to\infty} \frac{-k}{\sqrt{w(k)}} + \frac{\sqrt{w(k)}}{2} = -\infty$.

Such a condition ensures that the probability density computed from the call price function of Black-Scholes is non-negative. Note that the model assumes implicitly that the density is well defined in the sense that it integrates to 1.

Because it is commonly used by practitioners and is tractable, we apply the *raw SVI parameterization* which reads as the following for a given parameter set $\chi = \{a, b, \rho, m, \sigma\}$.

$$w(k;\chi) = a + b \cdot \left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}.$$

Here, $a, m \in \mathbb{R}, |\rho| < 1$, $\sigma > 0$ and we get the desirable convexity condition on w if $a + b\sigma\sqrt{1-\rho^2}$ is non negative. The parameters a, m translate the smile vertically and horizontally respectively, b increases the slope, while ρ, σ modify the curvature.

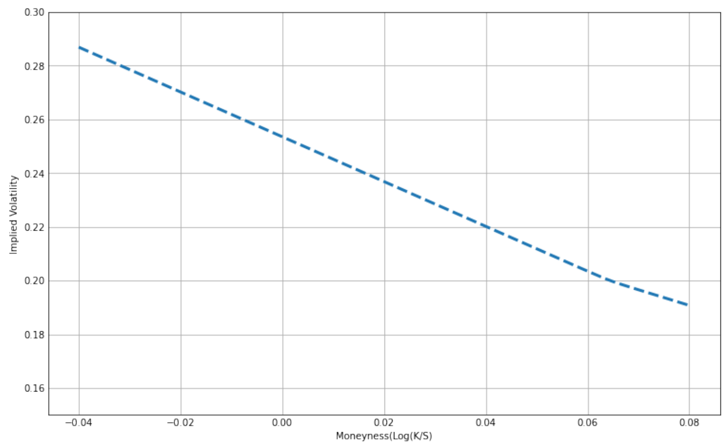


Figure 4: Implied Volatility Curve calibrated on an SPX Call with 6-month maturity

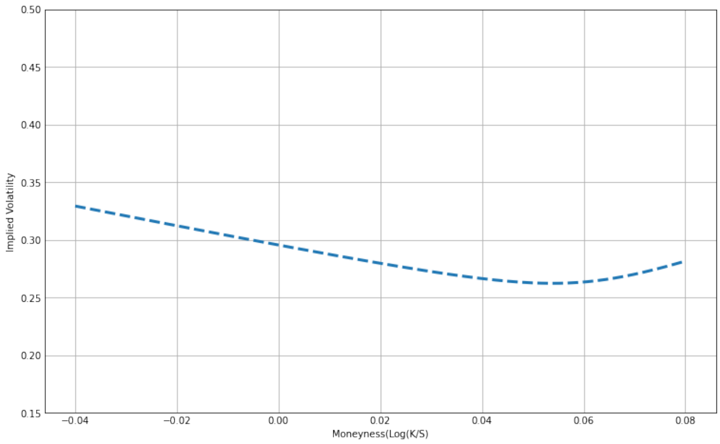


Figure 5: Implied Volatility Curve calibrated on an SPX Put with 6-month maturity

The values of SVI parameters obtained through optimization for the Call above are, a = 0.199, b = 0.1273, ρ = -5.552, m = 0.0644, and σ = 0.002. The values for Put are, a = 0.2381, b = 1.2779, ρ = 0.3069, m = 0.0603, and σ = 0.02.

5.1 Objective Function

We want to define an objective function such that it's minimization would allow us to calibrate the volatility obtained from SVI with the observed implied volatility. Additionally, we want to be devoid of arbitrage. The following objective function is used.

$$J = \min \sum_K \left(\frac{\sigma_{\text{model}}}{\sigma_K} - 1 \right)^2 \cdot \exp(\eta_{\text{butterfly}} + \eta_{\text{envelope}}).$$

Here $\eta_{\text{butterfly}}$ equals 1 if g(x) is negative for a given strike, and zero otherwise. g(x) checks for butterfly arbitrage. η_{envelope} is zero if the adjusted value is within the bid-ask spread and $\eta_{\text{envelope}} = \frac{\sigma - \sigma_{\text{average}}}{\text{spread}}$ otherwise.

5.2 Optimisation Algorithm

We perform the calibration by the Differential Evolution optimization algorithm, which iterates over the parameter space according to metaheuristics, instead of standard gradient descent algorithms which require strong assumptions on the underlying function like continuity.

In this algorithm, each candidate solution is replaced with a potentially better solution (with a lower objective function) based on heuristics determined by a uniformly distributed random number applied to the current candidate solution. The implementation is available from python's `scipy` library as `scipy.optimise.differential_evolution`.

6 Results and Conclusion

The final value of the security comes out to be in the range of [0.10, 0.26] depending on the maturity. These values tie out when we historically backtest the data. These numbers are marked to the current market values of assets, N225 and LIBOR-3m, and assume the values of the other input variables to be reasonable.

Intuitively as well, these figures make sense from the perspective of the payoffs. The payoff unless the values of K and K' are extreme, is going to be a modest ratio of around 1. A product of these payoffs would also be within the low range of 0 and 1.

Implied Volatility curves are smooth and continuous, and yield curves reflect the usual market values really well.

Overall, we can conclude that while there are objectively correct values, the numbers we obtain are sensible.

7 Improvements

- While usage of the Hull-White model provides a great way to incorporate term-structure dynamics, it also makes the model more susceptible to outlier irregularities in the yield curve, like a heavily inverted yield curve. There are processes that are used in industry to make this model lot more robust. These can be implemented to make the model better.
- SVI model can be improved by also adding a check for Calendar Spread arbitrage. In this model, we verify the absence of only Butterfly Spread arbitrage. An addition like this would make the model more resilient to higher maturities.
- There are complex methods used by the industry to measure correlation, such as Copulas. These can be employed to get even deeper and more meaningful relationships between the market factors.

References

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