PAPER *Special Section on Discrete Mathematics and Its Applications*

Construction of Voronoi Diagram on the Upper Half-Plane

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SUMMARY The Voronoi diagram is the most fundamental and useful concept in computational geometry. To understand impacts of non-Euclidean geometry on computational geometry, this paper investigates the Voronoi diagram in hyperbolic space. We first present characterizations of this diagram by means of the Euclidean Voronoi diagram, and based on them propose efficient algorithms to construct it. Some applications are also mentioned.

key words: *geodesic, Riemannian geometry, Riemannian metric, upper half-plane*

1. Introduction

In this paper, we are going to generalize Voronoi diagrams in the Euclidean space $\mathbf{R}^2$ [3], [13], [14] into the upper half-plane, which is a 2 dimensional manifold with negative constant curvature.

Computational geometry has been mainly studied in the Euclidean space $\mathbf{R}^n$. Most of efficient combinatorial algorithms have been found by using nice properties of $\mathbf{R}^n$.

From the view point of the differential geometry, the Euclidean space $\mathbf{R}^n$ is flat i.e. its curvature is 0. So, it is a natural problem to generalize algorithms in $\mathbf{R}^n$ into the Riemannian manifolds with a non-zero curvature. We will focus on generalizing algorithms of constructing Voronoi diagrams to the upper half-plane as a first step. We can get efficient algorithms by virtue of nice properties of the upper half-plane. These results will be explained in Sects. 2–4. We also emphasize that our problem is not only natural, but also is expected to be important in applications in the last section.

2. Upper Half-Plane

Let $\mathbf{H} = \{(x, y) \mid x, y \in \mathbf{R}, y > 0\}$ be the Riemannian manifold with Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. This manifold is called the *upper half-plane*, a space with negative constant curvature (see [9] Chapter V. 3).

We will summarize well-known facts on the upper half-plane in this section (see e.g. [12] Chapter6 for details). Firstly consider a *geodesic* which is like a line in the Euclidean plane. A geodesic is expressed as

$$(x-p)^2 + y^2 = r^2 \quad (p, r \in \mathbf{R}, r > 0)$$

or

$$x = c \quad (c \in \mathbf{R}).$$

Geodesics are basic building blocks for computational geometry on the upper half-plane. For a given two set of two points, a geodesic is uniquely determined and gives minimum length. The distance of two points is naturally induced from the metric of $\mathbf{H}$; consider two points $a_1(x_1, y_1)$, $a_2(x_2, y_2) \in \mathbf{H}$, the distance of the two points $d(a_1, a_2)$ can be expressed as

$$\begin{aligned} d(a_1, a_2) &= \int_{\text{the geodesic that connects } a_1 \text{ and } a_2} ds \\ &= \left| \log \frac{A + \sqrt{A^2 - 4{y_1}^2{y_2}^2}}{A - \sqrt{A^2 - 4{y_1}^2{y_2}^2}} \right| \end{aligned}$$

where $A = (x_1 - x_2)^2 + ({y_1}^2 + {y_2}^2)$. This distance is called *hyperbolic distance*.

For a geodesic C, each of the two connected components C^+ and C^- of $\mathbf{H} \setminus C$ is called *half-space*.

Lemma 1: Half-space is convex set; if $a_1, a_2 \in C^+$, then $C_{a_1,a_2} \subset C^+$, where C_{a_1,a_2} is the geodesic segment that connects the two points a_1 and a_2.

Proof: Suppose that C and C_{a_1,a_2} are on circles C' and C'_{a_1,a_2} in the Euclidean plane respectively. The two circles intersect at most 2 points and the centers of the circles C', C'_{a_1,a_2} are on the x-axis. Therefore, one of the intersection is in the upper half-plane and the other is in the lower half-plane. Hence, if $a_1, a_2 \in C^+$, then C_{a_1,a_2} is in C^+.

To finish the proof, we need to consider the case in which C is on $x = c$ can be proved in a similar way. □

3. Definition of Voronoi Diagram in the Upper Half-Plane

Suppose that the set of n points $P = \{a_1, \cdots, a_n\} \in \mathbf{H}$ are given.

The *hyperbolic Voronoi polygon* $\mathrm{Vor}(a_i)$ for P is defined as follows:

$$\mathrm{Vor}(a_i) = \{x \in \mathbf{H} \mid d(x, a_i) \leq d(x, a_j)\ \forall j \neq i\}.$$

The hyperbolic Voronoi polygons for P partition $\mathbf{H}$, which is called *Voronoi diagram in the upper half-plane.*

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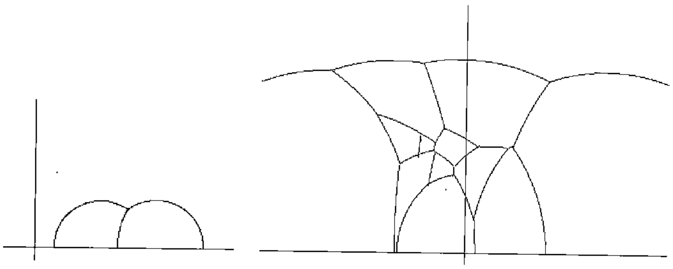

Fig. 1 Example of Voronoi diagram on **H**.

Vertices of hyperbolic Voronoi polygons are called *hyperbolic Voronoi points* and boundaries of Voronoi polygons are called *hyperbolic Voronoi edges*.

The point set of the same distance from two points is called the *perpendicular bisector*. When $P = \{a_1, a_2\}$, the perpendicular bisector gives the Voronoi diagram of P. In general, any hyperbolic Voronoi edge is the part of a perpendicular bisector. A perpendicular bisector in **H** has the following remarkable property.

Lemma 2: A perpendicular bisector in **H** is geodesic. For two points $a_1(x_1, y_1)$, $a_2(x_2, y_2)$ in **H**, the perpendicular bisector can be expressed as

$$x = \frac{x_1 + x_2}{2}$$

when $y_1 = y_2$, and

$$(x-p)^2 + y^2 = y_1 y_2 \left\{ \left(\frac{x_1 - x_2}{y_1 - y_2} \right)^2 + 1 \right\}$$

when $y_1 \neq y_2$, where $p = \dfrac{x_1 y_2 - y_1 x_2}{y_2 - y_1}$.

Proof: Any point $X(x, y)$ on the perpendicular bisector satisfies the equation: $d(X, a_1) = d(X, a_2)$. Solving the equation, we have the conclusion. □

Lemma 3: For any point $a_1(x_1, y_1) \in$ **H**, the set of points of which the hyperbolic distance from the point a_1 is a constant is expressed in the same manner equation of circle in the Euclidean plane. If the hyperbolic distance between a_1 and any point on the circle is r, then the set can be expressed as

$$(x-x_1)^2 + \left(y - \frac{e^r + e^{-r}}{2} y_1 \right)^2 = \left(\frac{e^r - e^{-r}}{2} y_1 \right)^2 \quad (1)$$

where e is the base of the natural logarithm.

Moreover the point a_1 is always in the interior of (1).

Proof: Any point $X(x, y)$ in the set satisfies the equation $d(X, a_1) = r$. Solving the equation, we have the conclusion.

The proof of the latter is as follow:

$$(x_1 - x_1)^2 + \left(y_1 - \frac{e^r + e^{-r}}{2} y_1 \right)^2 - \left(\frac{e^r - e^{-r}}{2} y_1 \right)^2$$
$$= -(e^{r/2} - e^{-r/2})^2 \cdot {y_1}^2 \leq 0$$

If $r = 0$ then the circle is a point circle and coincide with the point a_1. The circle becomes larger monotonously and continuously when r increases. Thus the point a_1 is always contained in the circle. □

[**Remark**]

- The Euclidean center of the circle is different from the point a_1. The point a_1 is called *hyperbolic center*.
- Let $X(x_1, y_1)$ be a hyperbolic Voronoi point for a point set P and, the hyperbolic distance between (x_1, y_1) and nearest point in P is r then the point $(x_1, (e^r + e^{-r})y_1/2)$ is always Euclidean Voronoi point. In addition, the sets of nearest point of P in the Euclidean plane and in the upper half-plane are the same.

Given three distinct points $a_i(x_i, y_i)$ $(i = 1, 2, 3)$, the circle which passes through all three points a_i is defined by

$$\begin{vmatrix} 1 & x_1 & y_1 & {x_1}^2 + {y_1}^2 \\ 1 & x_2 & y_2 & {x_2}^2 + {y_2}^2 \\ 1 & x_3 & y_3 & {x_3}^2 + {y_3}^2 \\ 1 & x & y & x^2 + y^2 \end{vmatrix} = 0$$

in the upper half-plane and the Euclidean plane in both cases.

We expand the determinant:

$$\alpha(x^2 + y^2) - \beta y + \gamma x - \delta = 0 \quad (\alpha, \beta, \gamma, \delta \in \mathbf{R}), \quad (2)$$

where $\alpha, \beta, \gamma, \delta$ are expressed in term of x_i and y_i. The Eqs. (2) and (1) belong to the same class of curve of second order. If the condition of the following lemma holds, we can get the equidistance point from the three points a_i by equating each of the coefficients of the two equations.

Lemma 4: For given three distinct points in **H**, the set of points which has the same distance from three points is, if exist, a point. The point exists if and only if

$$\alpha \neq 0, \left(\frac{-\gamma}{2\alpha}, \frac{\beta}{2\alpha} \right) \in \mathbf{H} \quad \text{and} \quad 4\alpha\delta + \gamma^2 < 0.$$

[**Remark**] The point which is equidistance from given distinct three points certainly exists in $\mathbf{R}^2$. But in **H** the equidistance point does not always exist for three points.

4. Property and Construction of the Voronoi Diagram in the Upper Half-Plane

We shall show some properties of two different algorithm in this section. One algorithm is generalization of Delaunay regular triangulation which is main result Theorem 1. The other is based on incremental method in $\mathbf{R}^2$.

4.1 The Method of Delaunay Regular Triangulation

Before going to the discussion in $\mathbf{H}$, we first reexmine on the method of Delaunay regular triangulation in $\mathbf{R}^2$ to value our idea clear.

A point set $S = \{(x_i, y_i) | x_i, y_i \in \mathbf{R}, i = 1, \ldots, n\}$ in $\mathbf{R}^2$ is given. For any real valued function $w(x, y)$, we define *the lift map* by

$$\psi : (x, y) \longmapsto (x, y, w(x, y)) \in \mathbf{R}^3.$$

Consider the lower convex hull of $\psi(S)$, a structure consisting of faces whose outward normal has negative z coordinate. We project the lower convex hull to $\mathbf{R}^2$ by *the projection map.*

$$\varphi : (x, y, z) \longmapsto (x, y) \in \mathbf{R}^2.$$

The image induces a polyhedral division of S. If the division is triangulation, it is called the *regular triangulation* with the weight function $w(x, y)$. It is well-known that the weight function $w(x, y) = x^2 + y^2$ induces the Delaunay regular triangulation of S. The Voronoi diagram can be obtained as the dual graph of the Delaunay triangulation ([3] Chapter 13, [7] Chapter 5.2).

In order to generalize this algorithm to the upper half-plane, we need to find an appropriate space where the given points should be lifted. The space is the 3-dimensional upper half-space defined below and a lift map and a projection map will be given in (3) and (4). We note that this idea is suggested in ([4]; §1.5)

Let $\mathbf{H}^3 = \{(x, y, z) \mid x, y, z \in \mathbf{R}, z > 0\}$ be the Riemannian manifold with Riemannian metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$. This manifold is called the *upper half-space*.

A submanifold N of a Riemannian manifold M is said to be *totally geodesic at a point x of N* if, for every $X \in T_x(N)$, the geodesic $\tau = x_t$ of M determined by (x, X) lies in N for small values of the parameter t. If N is totally geodesic at every point of N, it is called a *totally geodesic submanifold* of M (see [9] p.180).

A totally geodesic submanifold of $\mathbf{H}^3$ is called *hyperbolic plane* in this paper.

Lemma 5: Any *hyperbolic plane* is expressed as

$$(x - p)^2 + (y - q)^2 + z^2 = r^2 \quad (p, q, r \in \mathbf{R})$$

or

$$px + qy + r = 0 \quad (p, q, r \in \mathbf{R}).$$

This *lemma* can be proved by solving an ordinary differential equation satisfied by geodesics (see [9] Chapter III.7).

[**Remark**] In the upper half-space *hyperbolic plane* can be *naturally* defined, but in the general Riemannian manifold a kind of *plane* can not always be defined.

For a plane F, each of the two connected component $\mathbf{H}^3 \setminus F$ is called *half-space*. Suppose n-points set $P = \{a_1, \cdots, a_n\}$ is given in $\mathbf{H}^3$. Then *convex hull* of P is defined as the intersection of all half-spaces that contain P and we denote the convex hull of P by conv(P).

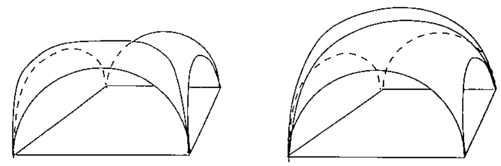

Fig. 2 A lower-hull (left) and a upper-hull (right) of 4 points convex hull.

Moreover a *hyperbolic upper-hull* of conv(P) is the face set of the convex hull whose elements contain P with the domain

$$(x - p)^2 + (y - q)^2 + z^2 < r^2 \quad (p, q, r \in \mathbf{R}),$$

and a *hyperbolic lower-hull* of conv(P) is with the domain that is expressed as

$$(x - p)^2 + (y - q)^2 + z^2 > r^2 \quad (p, q, r \in \mathbf{R}),$$

(see Fig. 2).

We have to find the lift map and the projection map. Put $\overline{\mathbf{H}^3} = \mathbf{H}^3 \cup \{z = 0\}$. The projection map and the lift map $\varphi : \overline{\mathbf{H}^3} \to \mathbf{H}$, $\psi : \mathbf{H} \to \overline{\mathbf{H}^3}$ are defined as follows:

$$\varphi : (x, y, z) \longmapsto (x, \sqrt{y^2 + z^2}) \tag{3}$$
$$\psi : (x, y) \longmapsto (x, y, 0). \tag{4}$$

Lemma 6: Any geodesic on $\mathbf{H}^3$ is projected down to a geodesic on $\mathbf{H}$ by φ.

Proof: A geodesic on $\mathbf{H}^3$ is expressed as

$$\begin{cases} (x - p)^2 + (y - q)^2 + z^2 = r^2 \\ p'(x - p) + q'(y - q) \quad = 0 \end{cases} (p, q, r, p', q' \in \mathbf{R})$$

or

$$x = p, y = q \quad (p, q \in \mathbf{R}).$$

We can show the conclusion by explicitly calculating the image by φ. □

Lemma 7: For a point set $S = \{a_i(x_i, y_i, 0) \mid y_i > 0, i = 1, \ldots, n\} \in \overline{\mathbf{H}^3}$, we can construct the hyperbolic lower-hull of conv(P) in $O(n \log n)$-time.

Proof: Consider a face F which is contained in the hyperbolic lower-hull of S. This face is a part of sphere and all points of S is contained in the exterior of F, where the exterior of F is the domain which can be expressed as $(x - p)^2 + (y - q)^2 + z^2 > r^2$.

Moreover the boundary of the sphere is a circle which contain distinct three points of S. Hence, the boundary is a circumcircle whose interior does not contain the point of S. Therefore, there is one-to-one correspondence between the edge of the lower-hull and the Euclidean Delaunay triangulation of S. If the Delaunay triangulation is made in $O(n \log n)$-time, then the edge of the lower-hull is also made in $O(n \log n)$-time.

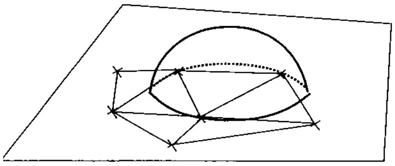

Fig. 3 A *hyperbolic plane* whose boundary is a circumcircle and a Delaunay triangulation on $\mathbf{R}^2$.

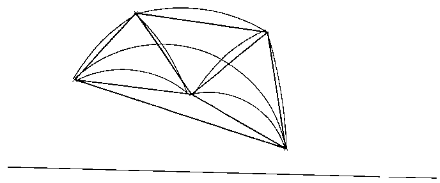

Fig. 4 A case of edges intersect.

The fact is well-known that the Delaunay triangulation is constructed in $O(n \log n)$-time (see [3] Chapter 13). □

Thus we can immediately prove the next property of Voronoi diagram in the upper half-plane.

Theorem 1: The Delaunay graph on the upper half-plane is a subgraph of Delaunay triangulation on the Euclidean plane, where the Delaunay graph is a graph $G(S, E)$, where $E = \{\overline{a_i, a_j}|$ The perpendicular bisector of a_i, a_j is contained in hyperbolic Voronoi diagram of $S\}$.

Lemma 8: The geodesic segments that are mapped by φ the edges of the hyperbolic lower-hull in $\mathbf{H}^3$ do not intersect other edges except the edges that are not contained in the Delaunay graph.

Proof: When a geodesic on $\mathbf{H}^3$ maps to $\mathbf{H}$ by φ, the image is a geodesic segment (from lemma 6). The end points which are contained in the plane $z=0$ map by φ

$$(x, y, 0) \longrightarrow (x, y).$$

Consider $z = 0$ and $\mathbf{H}$ are identified (see Fig. 3), the points are fixed point. By two end points, the geodesic on $\mathbf{H}$ is uniquely decided. Thus the edges of the triangle are not intersected by the convexity of half-space(Lemma 1).

However there is a class of triangles whose longest edge intersects other edge (see Fig. 4). The triangle of the class is satisfied next condition.

(Condition) for a triangle $a_1a_2a_3$, a_2a_3 is longest edge by hyperbolic distance and a_1 is contained in the domain that surrounded by Euclidean segment a_2a_3 and geodesic segment a_2a_3.

For this triangle, the vertex sequence is clockwise $a_1a_2a_3$ in the Euclidean plane, but the sequence is clockwise $a_1a_3a_2$ in the upper half-plane. The longest edge of this triangle does not intersect other edge in the Euclidean plane, but in the upper half-plane there are some case the edge intersect other edge (see Fig. 4).

However these edges is not contained in Delaunay graph. Thus the edges is not intersected. □

Consequently we can obtain the following algorithm by Theorem 1 and Lemma 8.

Algorithm(Construction of Voronoi Diagram)

1. Regard the point set S as in $\mathbf{R}^2$ and compute the Delaunay triangulation of S in the Euclidean plane.
2. Remove the edge that is not contained in the Delaunay graph in $\mathbf{H}$.
3. Construct the Voronoi diagram of S in $\mathbf{H}$ from the Delaunay graph.

[**Remark**] Although the Delaunay triangulation and the Delaunay graph always have the same structure on $\mathbf{R}^2$, the Delaunay triangulation and the Delaunay graph do not always agree on $\mathbf{H}$. This is why we need the step 2.

Theorem 2: Voronoi diagram on $\mathbf{H}$ is constructed in $O(n \log n)$-time by using the above algorithm.

Proof: The Delaunay triangulation of S in $\mathbf{R}^2$ is constructed in $O(n \log n)$. The 3rd step is also done in $O(n)$.

The 2nd step can be done in $O(n)$. In fact, we may apply the following procedure to all the faces of the Delaunay triangulation.

Procedure(Remove the edge of the triangle $a_1a_2a_3$)
1. Compute the point O that has the same hyperbolic distance from a_1, a_2 and a_3.
2. If there does not exist O, then remove the longest edge in the triangle.

This Procedure is performed in $O(1)$-time. Since we apply this Procedure to the all triangles, this stage finishes in $O(n)$-time. □

The algorithm and the theorem were already given in [2]. Here, we could naturally rediscover their algorithm as a natural generalization of the method of Delaunay regular triangulation.

We have shown a method to construct the Voronoi diagram by a machine which can calculate *real number* without computational error. However, the construction can be done on a machine which can correctly calculate only *integers*. Our theorem below is a generalization of the theorem by [14] Sect. 3.5 in $\mathbf{R}^2$.

Theorem 3: For $P = \{(x_i, y_i) \mid x_i, y_i \in \mathbf{Q}, i = 1, \cdots, n\}$, the Voronoi diagram is constructed by only integer calculation.

Proof: In the algorithm, we used real number calculation. We are going to explain how we can replace the real arithmetic by the integer arithmetic.

First, we need to detect whether two geodesics intersect or not. The existence of the intersection is equivalent to the existence of the hyperbolic equidistant point from given three points. Therefore, we can use Lemma 4 and the condition can be checked by only integer arithmetic when the three points are rational.

Secondly, we need to check whether a geodesic segment and a geodesic intersect. The geodesic segment used in the algorithm has two end points which are hyperbolic Voronoi points. When a hyperbolic Voronoi point is denoted by (x, y), we can short that

$$(x, y^2) = \left(-\frac{\gamma}{2\alpha}, -\frac{\gamma^2 + 4\alpha\delta}{4\alpha^2}\right) \in \mathbf{Q}^2.$$

Consider a function f which is defined as

$$f(x, y) = (x - p)^2 + y^2 - r^2 \; (p, r \in \mathbf{Q}, (x, y) \in \mathbf{Q}^2)$$

and assume that the geodesic is given by $f(x, y) = 0$. For given two end points $b_1(x_1, {y_1}^2), b_2(x_2, {y_2}^2) \in \mathbf{Q}^2$, we can check whether the geodesic $f(x, y) = 0$ and the geodesic $\overline{b_1 b_2}$ intersect or not as follows:

$f(b_1) \cdot f(b_2) = 0$
$\Longrightarrow b_1$ or b_2 is contained in geodesic.

$f(b_1) \cdot f(b_2) > 0$
$\Longrightarrow$ the geodesic and the part of geodesic do not intersect.

$f(b_1) \cdot f(b_2) < 0$
$\Longrightarrow$ the geodesic and the part of geodesic intersect.

Finally, we need to find the hyperbolic Voronoi polygon that contains a given point by only integer calculation. This check can be done by the same method of the previous one. □

4.2 Algorithm for Adding a New Site

We shall show that Voronoi diagram in the upper half-plane can be updated in $O(i)$-time when a new point is added. The algorithm is based on the incremental method, which very much like one proposed initially by [5] in constructing the Voronoi diagram in the Euclidean geometry, but our algorithm is different in some points. We shall review the algorithm in the Euclidean plane first, and then explain the difference.

The algorithm in [5] works as follows. For a given set $S = \{a_1, \cdots, a_i\}$ of i distinct points, we sort them by their (x, y)-coordinate with the x-coordinate as the first key. The algorithm proceeds incrementally; we first construct Vor (S_2) and next add the point a_3 to Vor (S_2), which updates Vor (S_2) to Vor (S_3). We repeat adding the points $a_4, a_5, \ldots$, and updating Vor (S_3), Vor (S_4), ... and finally get the Vor (S_i)

Let us look closely at the stage of making the Vor (S_{i+1}) from the Vor (S_i). This algorithm consists of the following two phases.

Algorithm(making Vor (S_{i+1}) from Vor (S_i))

Phase 1 (Nearest neighbor search): Among the sites $a_1, \ldots, a_i$ of the diagram Vor (S_i), find the nearest, say $a_{N(i+1)}$, to a_{i+1}.

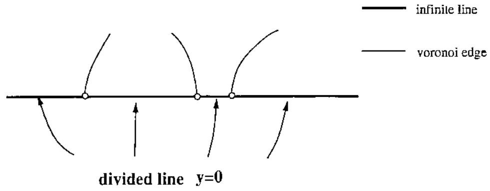

Fig. 5 A part of Voronoi diagram around infinite line.

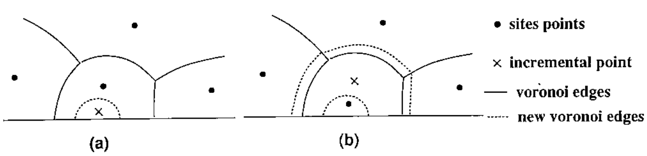

Fig. 6 Case1, only an infinite edge is found.

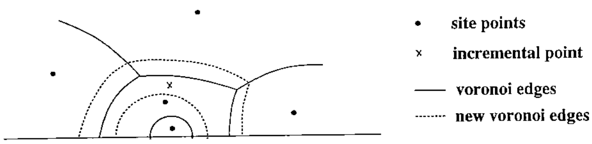

Fig. 7 Case2, two infinite edges are found.

Phase 2 (Local modification): Starting with the perpendicular bisector of the line segment $a_{i+1}a_{N(i+1)}$, find the point of intersection of the bisector with a boundary edge of $\mathrm{Vor}_i(a_{N(i+1)})$ and determine the neighboring region $\mathrm{Vor}_i(P_{N_1(i+1)})$ which lies on the other side of the edge; then draw the perpendicular bisector of $a_{i+1}a_{N_1(i+1)}$ and find its intersection with a boundary edge of $\mathrm{Vor}_i(P_{N_1(i+1)})$ together with the neighboring region $\mathrm{Vor}_i(P_{N_2(i+1)})$; ...; repeating around in this way, create the region of a_{i+1} to obtain Vor (S_{i+1})

Let us turn to the upper half-plane **H** and modify the algorithm above for **H**. The key idea to generalize the algorithm in $\mathbf{R}^2$ is to regard the line $y = 0$ as a phantom Voronoi edge. This edge is called *the infinite edge* (see Fig. 5).

In the Phase 2 of the above algorithm, we search the hyperbolic Voronoi edge that intersects the perpendicular bisector. When the edge is not infinite edge, we may use the Euclidean algorithm. If the edge is infinite line, then call the Procedure below.

Procedure(the case that the edge is infinite edge)
Search a new edge that intersects the perpendicular bisector.
IF (((1) the new edge is also infinite edge) or ((2) the new edge is not found))
THEN use the method below.
ELSE use the new edge and apply the Euclidean algorithm with regarding the edge as starting edge.

We now consider about the cases (1) and (2). There are two cases in this algorithm.

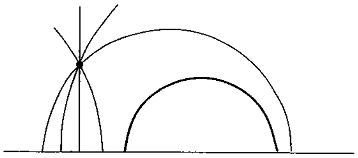

Fig. 8 Many parallel line in the upper half-plane.

Firstly we consider the case (1). This case is divided into the pattern of Fig. 6 (a) and that of Fig. 6 (b). In the case of (a) we may locally change the Voronoi diagram, so we treat only the incremental point and the nearest point. However, in the case (b), not only the points, but also other edges have to be dealt with. We have to repeat the operation of Phase 2 for the all edges of the hyperbolic Voronoi polygon of the nearest point, where the each of starting and ending edge of algorithm is the left and the right infinite edge of the found edge.

Secondly, we consider the case (2) (see Fig. 7). In this case we repeat Phase 2 from left infinite edge to right infinite edge. By this Procedure we can get the incremented Voronoi diagram on $\mathbf{H}$.

One of the significant combinatorial difference between the geometry in the upper half-plane and the Euclidean geometry is the existence of more than one lines that pass through a given point and parallel to a given line (see Fig. 8). Thus we cannot dissolve the many degenerate case in the upper half-plane by the *symbolic perturbation* (see [3]). So we have to use the above Procedure.

Theorem 4: Voronoi diagram is updated in $O(i)$-time by using this algorithm when a new cite is added, where i is the number of points.

Proof: Phase 1 of the above algorithm can be done in $O(i)$-time. Phase 2 can be constructed in the linear time with respect to the number of the hyperbolic Voronoi edges. Therefore, we may prove that the number of the hyperbolic Voronoi edges is at most $O(i)$. Consider the dual graph of the Voronoi diagram. The number of the edge of this graph is same with hyperbolic Voronoi edge. The graph is a planar graph and the vertices are $O(i)$. Therefore, the number of the edges is also $O(i)$. □

5. Applications

5.1 Voronoi Diagram on the Pseudo-Sphere

For a domain $D = \{(u,v) \mid 0 \leqq u \leqq 2\pi, v \geqq 1\} \subset \mathbf{H}$ we define the function $f : D \mapsto \mathbf{R}^3$ as follows:

$$f((u,v)) = \left(\frac{1}{v}\cos u, \frac{1}{v}\sin u, \log(v+w) - \frac{w}{v}\right)$$

where $w = \sqrt{v^2-1}$. The image $f(D)$ is a surface on $\mathbf{R}^3$ which is called a *pseudo-sphere*. The metric in $\mathbf{R}^3$

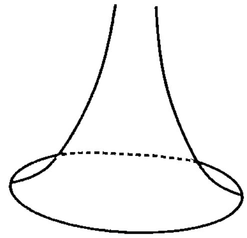

Fig. 9 Pseudosphere in $\mathbf{R}^3$.

naturally induces the metric on the pseudo-sphere. The domain $D \subset \mathbf{H}$ and $f(D)$ is isomorphic as the Riemannian manifold. Hence, our Voronoi diagram on $\mathbf{H}$ gives the Voronoi diagram on the pseudo-sphere in $\mathbf{R}^3$

5.2 Information Geometry

Regard the space of the normal distributions as a 2-dimensional Riemannian manifold with Fisher metric, where the Fisher metric is defined by the following matrix

$$g_{ij} = \int \frac{\partial}{\partial \xi^i} \log p(x;\xi) \cdot \frac{\partial}{\partial \xi^j} \log p(x;\xi) \cdot p(x;\xi) dx$$

and

$$p(x;\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$
$$\xi = [\mu, \sigma],\ \sigma > 0.$$

The space of the normal distributions $\{(\mu,\sigma) | \sigma > 0\}$ has negative constant curvature (see [1]). In the space, the distance of 2 points measures a similarity of two probability distributions. For given normal distributions $S = \{(\mu_i, \sigma_i) | 1 \leqq i \leqq n\}$, we can construct the Voronoi diagram with the Fisher metric in $O(n \log n)$-time. If we are given a normal distribution (μ, σ), we can find the nearest distribution of (μ, σ) in S by the Voronoi diagram.

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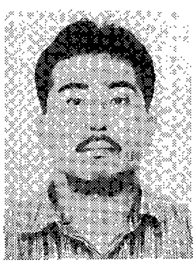

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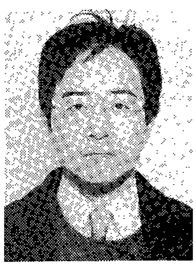

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