W203 Statistics for Data Science
Unit 3 Homework: Probability Theory

Answer Key

1 Gas Station Analytics

At a certain gas station, 40% of customers use regular gas (event R), 35% use mid-grade (event M), and 25% use premium (event P). Of the customers that use regular gas, 30% fill their tanks (Event F). Of the customers that use mid-grade gas, 60% fill their tanks, while of those that use premium, 50% fill their tanks. Assume that each customer is drawn independently from the entire pool of customers.

(a.) What is the probability that the next customer will request regular gas and fill the tank?
(b.) What is the probability that the next customer will fill the tank?
(c.) Given that the next customer fills the tank, what is the conditional probability that they use regular gas?

Solution 1.a.
Given that a customer can only take regular gas, mid-graded gas or premium gas during a visit to the gas station, event R, event M and event P are mutually exclusive events. And also, since each customer must choose at least one type of the gas to fill, event R, event M and event P are exhaustive events.

$\Pr(R) = 0.4$
$\Pr(M) = 0.35$
$\Pr(P) = 0.25$

$\Pr(F \mid R) = 0.3$
$\Pr(F \mid M) = 0.6$
$\Pr(F \mid P) = 0.5$

Probability that the next customer will request regular gas and fill the tank = $\Pr(R \cap F)$
= $\Pr(R) \cdot \Pr(F \mid R) = 0.4 \cdot 0.3 = 0.12$

Solution 1.b.
Probability that next customer will fill the tank = $\Pr(F)$

$= \Pr(R) \cdot \Pr(F \mid R) + \Pr(M) \cdot \Pr(F \mid M) + \Pr(P) \cdot \Pr(F \mid P)$
$= 0.4 \cdot 0.3 + 0.35 \cdot 0.6 + 0.25 \cdot 0.5 = 0.12 + 0.21 + 0.125 = 0.455$

Solution 1.c.
Probability that the next customer uses regular gas given that he/she fills the tank = $\Pr(R \mid F)$
$= \frac{\Pr(F \mid R) \cdot \Pr(R)}{\Pr(F)} = \frac{0.3 \cdot 0.4}{0.455} \approx 0.264$

2 The Toy Bin

In a collection of toys, 1/2 are red, 1/2 are waterproof, and 1/3 are cool. 1/4 are red and waterproof. 1/6 are red and cool. 1/6 are waterproof and cool. 1/6 are neither red, waterproof, nor cool. Each toy has an equal chance of being selected.

(a.) Draw an area diagram to represent these events.
(b.) What is the probability of getting a red, waterproof, cool toy?
(c.) You pull out a toy at random and you observe only the color, noting that it is red. Conditional on just this information, what is the probability that the toy is not cool?
(d.) Given that a randomly selected toy is red or waterproof, what is the probability that it is cool?

Solution 2.a. and 2.b.
We define event of selecting a red toy to be R, event of selecting a waterproof toy to be W and event of selecting a cool toy to be C.

Let $x = \Pr(R \cap W \cap C) \implies$

$\Pr(R \cap W \cap !C) = \frac{1}{4} - x$

$\Pr(C \cap W \cap !R) = \frac{1}{6} - x$

$\Pr(R \cap C \cap !W) = \frac{1}{6} - x$

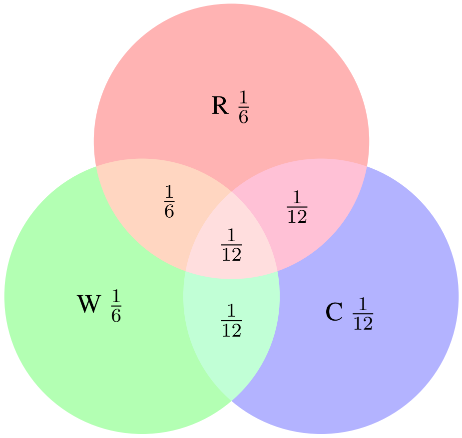
$\Pr(R \cap !W \cap !C) = \frac{1}{12} + x$

$\Pr(W \cap !R \cap !C) = \frac{1}{12} + x$

$\Pr(C \cap !W \cap !R) = x$

$\Pr(R \cup W \cup C) = 1 - \Pr(!R \cap !W \cap !C) = \frac{5}{6}$
$= x + \frac{1}{4} - x + \frac{1}{6} - x + \frac{1}{6} - x + \frac{1}{12} + x + \frac{1}{12} + x + x = \frac{3}{4} + x$

$\implies \Pr(R \cap W \cap C) = \frac{1}{12}$



$\Pr(!R \cap !W \cap !C) = 1/6$

Solution 2.c.
Probability that the toy is not cool given that it is red = $\Pr(!C \mid R)$
= $\frac{\Pr(!C \cap R)}{\Pr(R)}$ = $\frac{\Pr(R) - \Pr(C \cap R)}{\Pr(R)}$ = $\frac{1/2 - 1/6}{1/2}$ = $\frac{2}{3}$

Solution 2.d.
$\Pr(C \mid R \cup W)$ = $\frac{\Pr(C \cap (R \cup W))}{\Pr(R \cup W)}$ = $\frac{\Pr((C \cap R) \cup (C \cap W))}{\Pr(R \cup W)}$ = $\frac{\Pr(C \cap R) + \Pr(C \cap W) - \Pr(R \cap W \cap C)}{\Pr(R) + \Pr(W) - \Pr(R \cap W)}$
= $\frac{1/6 + 1/6 - 1/12}{1/2 + 1/2 - 1/4}$
= $\frac{1}{3}$

3 On the Overlap of Two Events

Suppose for events A and B, $\Pr(A)$ = 1/2, $\Pr(B)$ = 2/3, but we have no more information about the events.

(a.) What are the maximum and minimum possible values for $\Pr(A \cap B)$?
(b.) What are the maximum and minimum possible values for $\Pr(A \mid B)$?

Solution 3.a.
Given that $\Pr(A) + \Pr(B) > 1$, it is impossible that event A and event B are mutually exclusive. That means $\Pr(A \cap B) > 0$. The minimum $\Pr(A \cap B)$ happens when $\Pr(A \cup B) = 1$. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 1/2 + 2/3 - \Pr(A \cap B) = 1$. This gives the minimum value of $\Pr(A \cap B)$ to be $\frac{1}{6}$.

The maximum $\Pr(A \cap B)$ happens when all of A events happen with B event together.
$\Pr(A) = \Pr(A \cup B) = 1/2$

$\implies \frac{1}{6} \leq \Pr(A \cap B) \leq \frac{1}{2}$

Solution 3.b.
$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \implies \frac{1/6}{2/3} \leq \frac{\Pr(A \cap B)}{\Pr(B)} \leq \frac{1/6}{2/3} \implies \frac{1}{4} \leq \frac{\Pr(A \cap B)}{\Pr(B)} \leq \frac{3}{4}$

4 Cant Please Everyone!

Among Berkeley students who have completed w203, 3/4 like statistics. Among Berkeley students who have not completed w203, only 1/4 like statistics. Assume that only 1 out of 100 Berkeley students completes w203. Given that a Berkeley student likes statistics, what is the probability that they have completed w203?

Solution 4.
$\Pr\left(stat \mid W203\right)$ = 3/4
$\Pr\left(stat \mid !W203\right)$ = 1/4
$\Pr\left(W203\right)$ = 1/100
$\Pr\left(!W203\right)$ = 99/100
$\Pr\left(stat \cap W203\right) = \Pr\left(stat \mid W203\right) \cdot \Pr\left(W203\right)$ = 3/4 $\cdot$ 1/100 = 3/400

$\Pr\left(stat \cap !W203\right) = \Pr\left(stat \mid !W203\right) \cdot \Pr\left(!W203\right)$ = 1/4 $\cdot$ 99/100 = 99/400

$\Pr\left(W203 \mid stat\right) = \frac{\Pr(stat \cap W203)}{\Pr(stat)} = \frac{\Pr(stat \cap W203)}{\Pr(stat \cap W203)+\Pr(stat \cap !W203)} = \frac{3/400}{3/400+99/400}$ = 1/34