W203 Lab 2: Probability Theory

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1. a. $P(T|H_k) = P(T \cap H_k) / P(H_k) = P(T) \cdot P(H_k|T) / P(H_k)$

since we know that $P(T) = 0.01$ and $P(H_k|T) = 1$

we can rewrite $P(T|H_k) = \frac{0.01 \cdot 1}{P(H_k)}$,

Moreover, using Law of Total Probability:

$$P(H_k) = P(H_k|T) \cdot P(T) + P(H_k|!T) \cdot P(!T)$$
$$= 1 \cdot 0.01 + (0.5)^k \cdot 0.99$$
$$= 0.01 + (0.5)^k \cdot 0.99$$

So $P(T|H_k) = \boxed{\frac{0.01}{0.01 + (0.5)^k \cdot 0.99}}$

b. $P(T|H_k) > 0.99 \rightarrow \frac{0.01}{0.01+(0.5)^k \cdot 0.99} > 0.99$

$$0.01 > 0.99(0.01 + (0.5)^k 0.99)$$
$$0.01 + (0.5)^k \cdot 0.99 < \frac{1}{99}$$
$$(0.5)^k < (\frac{1}{99} - \frac{1}{100}) / 0.99$$
$$(0.5)^k < \frac{1}{9801}$$
$$k > \log_{0.5}(\frac{1}{9801}) = 13.26$$

So, you'll need to observe 14 heads in a row.

2. a

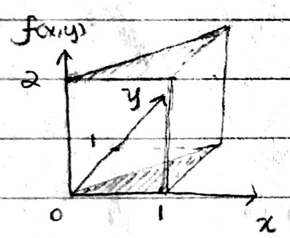
$$b(x;2,\tfrac{3}{4}) = \begin{cases} \binom{2}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{2-x} & x = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

b. $B(x;2,\frac{3}{4}) = \sum_{y=0}^{x} b(y;2,\tfrac{3}{4}) \quad x = 0,1,2$

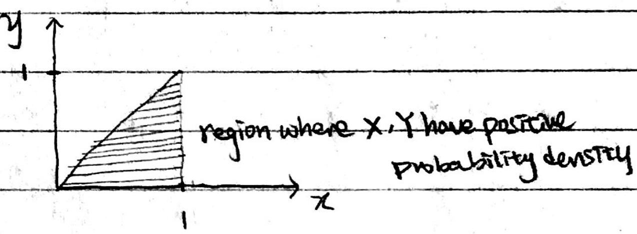
c. $E(X) = n \cdot p = 2 \cdot \frac{3}{4} = \boxed{1.5}$

d. $Var(X) = np(1-p) = 2 \cdot \frac{3}{4} \cdot \frac{1}{4} = \boxed{0.375}$

3. a) 3D:



2D:



b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 2\,dy = 2y\Big|_0^x = 2x$

c) $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2}{3}x^3\Big|_0^1 = \boxed{\frac{2}{3}}$

d) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$ for $y \in (0,x)$

e) $E(Y|X) = \int_0^x y \cdot f_{Y|X}(y|x)\,dy = \int_0^x y \cdot \frac{1}{x}\,dy = \frac{1}{x} \cdot y^2/2\Big|_0^x = \frac{x}{2}$

f) $E(XY) = E(E(XY|X)) = E(X\,E(Y|X)) = E(\frac{x^2}{2}) = \int_{-\infty}^{\infty} \frac{x^2}{2} \cdot f_X(x)\,dx$

$= \int_0^1 \frac{x^2}{2} \cdot 2x\,dx$

$= \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$

g) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 2\,dx = 2-2y$

$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 2y - 2y^2\,dy = y^2 - \frac{2}{3}y^3\Big|_0^1 = \frac{1}{3}$

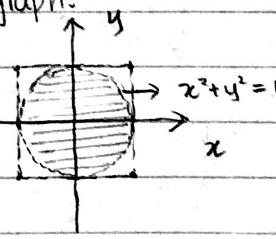
$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$
$$= \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{36}$$

4. a. $E(D_i) = 1 \cdot P[(X_i^2 + Y_i^2) < 1]$

$P[X_i^2 + Y_i^2 < 1]$ for $X_i, Y_i \in [-1,1]$. can be calulated as following:
Since X_i and Y_i are uniformly distributed between $[-1,1]$, the joint pdf $f(X,Y)$ has a evenly distributed density over the 2x2 area where $X \in [-1,1]$ and $Y \in [-1,1]$.
And writing $X^2 + Y^2 = 1$ gives us a circle centered a $(0,0)$ with a radius of 1.
Since we know that $f(X,Y)$ has the same density over the 2x2 area, $P[(X_i^2+Y_i^2)<1]$ can be computed as:

$$P = \frac{Area(\text{circle}_{x^2+y^2=1})}{Area_{x\in[-1,1],\, Y\in[-1,1]}} = \frac{\pi}{2\times 2} = \boxed{\frac{\pi}{4}}$$

as shown in the following graph:



b. Now we know $D \sim Ber(\frac{\pi}{4})$, which follows a binomial distribution with $n=1$.

$$\sigma^2 = \frac{\pi}{4}(1-\frac{\pi}{4}) = \frac{\pi}{4} - \frac{\pi^2}{16}$$
$$\hookrightarrow \sigma = \sqrt{\frac{\pi}{4} - \frac{\pi^2}{16}}$$

c. $\sigma_{\bar{D}} = \sqrt{V(\sigma_{\bar{D}})} = \sigma/\sqrt{n} = \sqrt{\frac{\pi}{4} - \frac{\pi^2}{16}}/\sqrt{n}$

(since $n > 30$)

d. Using Central Limit Theorem, we know that $\bar{D}$ follows a normal distribution with $\mu_{\bar{D}} = E(D_i)$, $\sigma_{\bar{D}}^2 = \sigma^2/n$

$$P(\bar{D} > 3/4) \approx P\left(Z > \frac{3/4 - \pi/4}{\sqrt{\pi/4 - \pi^2/16} / \sqrt{100}}\right) = P\left(Z > \frac{-0.035}{0.041}\right)$$

$$= 1 - \Phi(-0.862)$$

$$= 0.806$$