Everything You Always Wanted to Know About Multiple Interest Rate Curve Bootstrapping But Were Afraid To Ask

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Abstract

After the credit and liquidity crisis started in summer 2007 the market has recognized that multiple yield curves are required for estimation of both discount and FRA rates with different tenors (e.g. Overnight, Libor 3 months, etc.), consistently with the large basis spreads and the wide diffusion of bilateral collateral agreements and central counterparties for derivatives transactions observed on the market.

This paper recovers and extends our previous work [1] to the modern single-currency multiple-curve bootstrapping of both discounting and FRA yield curves, consistently with the funding of market instruments. The theoretical pricing framework is introduced and modern pricing formulas for plain vanilla interest rate derivatives, such as Deposits, Forward Rate Agreements (FRA), Futures, Swaps, OIS, and Basis Swaps, are derived from scratch.

The concrete EUR market case is worked out, and many details are discussed regarding the selection of market instruments, synthetic market quotes,, smooth interpolation, effect of OIS discounting, possible negative rates, turn of year effect, local vs non local delta sensitivities, performance and yield curve sanity checks.

The implementation of the proposed algorithms is available open source within the QuantLib framework.

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1 Introduction

Pricing complex interest rate derivatives requires modeling the *future* dynamics of the yield curve term structure. Hence, most of the literature assumes the existence of the *current* yield curve as given, and its construction is often neglected, or even obscured, as it is considered more a question of art than a science. Clearly, any yield curve term structure modeling approach will fail to produce good/reasonable prices if the current term structure is not correct.

Financial institutions, software houses and practitioners have developed their own proprietary methodologies in order to extract the current yield curve term structure from quoted prices of a finite number of market instruments (see e.g. [2, 3, 4, 5, 6, 7]). Broadly speaking, there are two classes of yield curve construction algorithms: best-fit and exact-fit.

Best-fit algorithms assume a functional form for the term structure and calibrate its parameters using a selection of calibration instruments quoted on the market, such that to minimize the repricing error. For instance, the European Central Bank publishes yield curves on the basis of the Soderlind and Svensson model [8], which is an extension of the Nelson-Siegel model (see e.g. refs. [9], [10] and [11]). Such approach is popular due to the smoothness of the curve, calibration easiness, intuitive financial interpretation of functional form parameters (level, slope, curvature) and correspondence with principal component analysis. On the other side, the fit quality is typically not good enough for trading purposes in liquid interest rate markets, where 0.25 basis points can make the difference.

In practice "exact-fit" algorithms are often preferred: they fix the yield curve on a time grid of N points (pillars) in order to *exactly* reprice N pre-selected market instruments. The implementation of such algorithms is often incremental, extending the yield curve step-by-step with the increasing maturity of the ordered instruments, in a so called "bootstrap"approach. Intermediate yield curve values are obtained by interpolation on the bootstrapping grid. Different interpolation algorithms are available, but little attention has been devoted in the literature to the fact that interpolation is often used already during bootstrapping, not just after that, and that the interaction between bootstrapping and interpolation can be subtle if not nasty (see e.g. [12], [13], [14]).

Whilst naive algorithms may fail to deal with market subtleties such as date conventions, the intra-day fixing of the first floating payment of a Swap, the turn-of-year effect, the Futures convexity adjustment, etc., even very sophisticated algorithms used in a naive way may fail to estimate correct FRA rates in difficult market conditions, as those observed since the summer of 2007 during the so-called *subprime credit crunch crisis*, or the very low (even negative) rates regimes in late 2012. Namely using just one single yield curve is not enough to account for discount and FRA rates of different tenor, such as 1, 3, 6, 12 months, because of the large Basis Swap spreads presently quoted on the market.

A further critical point triggered by the crisis is the coherent construction and usage of the discounting curve. Since, by no-arbitrage, the present value of any future cash flow must be unique, the *same* discounting curve must be consistently used during the construction of all the FRA curves. The construction of the discounting curve must be consistent with the funding of the bootstrapping instruments quoted on the market. Since Futures are traded on regulated markets, and OTC instruments (Forward Rate

Agreements, Swaps, and Basis Swaps) are traded between counterparties under bilateral collateral agreements, or with central counterparties, with daily margination at the over night collateral rate, the natural bootstrapping instruments selected for the discounting curve construction are the Overnight Indexed Swaps (OIS).

Finally, we stress that bootstrapping interest rate yield curves is nothing else than a recursive application of pricing formulas associated with the bootstrapping financial instruments within some pricing model. Thus a solid theoretical framework for pricing interest rate derivatives under collateral is required.

The plan of the paper is as follows. In section 2 we start by reviewing the market practice for interest rate yield curves construction and pricing and hedging interest rate derivatives, both in the traditional (old style) single-curve version, and in the modern multiple-curve version triggered by the credit crunch crisis. In section 3 we fix the notation and we briefly derive the theoretical framework for pricing derivatives including funding and collateral. In section 4, that constitutes the central contribution of this work, we describe in great detail the modern multiple-curve approach, in the single currency case. In particular in its numerous subsections we discuss the general features of the bootstrapping procedure, we review in detail the (EUR) market instruments available for yield curves construction, and we deal with some issues crucial for bootstrapping, such as interpolation, negative rates, turn of year effect, multiple delta hedging, performance and yield curve checks. Finally, in section 5 we show an example of numerical results for the Euribor1M, 3M, 6M and 12M FRA curves bootstrapping using the open source implementation released within the QuantLib framework. The conclusions are collected in section 6. A few final appendices reports mathematical details used in the main text.

2 Classical and Modern Pricing & Hedging of Interest Rate Derivatives

The evolution of the financial markets after the crisis of 2007 has triggered a general reflection about the methodology used to price and hedge interest rate derivatives, namely those financial instruments whose price depends on the present value of future interest rate-linked cash flows. In the next two sections we briefly review the classical and modern market approaches, from a pure practitioner point of view.

2.1 The Classical Single-Curve Approach

The pre-crisis standard market practice for the construction of the *single* interest rate yield curve can be summarized in the following procedure (see e.g. refs. [15], [12], [16] [13]).

1. Interbank credit/liquidity issues do not matter for pricing, Libors are a good proxy for risk free rates, Basis Swap spreads are negligible (and neglected).

2. The collateral do not matter for pricing, Libor discounting is adopted.

3. Select *one* finite set of the most convenient (i.e. liquid) vanilla interest rate instruments traded in real time on the market, with increasing maturities. For instance,

a very common choice in the EUR market is a combination of short-term EUR Deposit, medium-term Futures on Euribor3M and medium-long-term Swaps on Euribor6M.

4. Build *one* yield curve using the selected instruments plus a set of bootstrapping rules (e.g. pillars, priorities, interpolation, etc.).

5. Compute *on the same curve* FRA rates, cash flows[1], discount factors, and work out the prices by summing up the discounted cash flows.

6. Compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the *same* set of vanillas.

For instance, a 5.5Y maturity EUR floating Swap leg on Euribor1M (not directly quoted on the market) is commonly priced using discount factors and FRA rates calculated on the same Depo-Futures-Swap curve cited above. The corresponding delta sensitivity is calculated by shocking one by one the curve pillars and the resulting delta risk is hedged using the suggested amounts (hedge ratios) of 5Y and 6Y Euribor6M Swaps[2].

We stress that this is a *single-currency-single-curve approach*, in that a *unique* yield curve is built and used to price and hedge any interest rate derivative on a given currency. Thinking in terms of more fundamental variables, e.g. the short rate, this is equivalent to assume that there exist a unique fundamental underlying short rate process able to model and explain the whole term structure of interest rates of any tenor.

It is also a *relative pricing* approach, because both the price and the hedge of a derivative are calculated *relatively* to a set of vanillas quoted on the market. We notice also that the procedure is not strictly guaranteed to be arbitrage-free, because discount factors and FRA rates obtained through interpolation are, in general, not necessarily consistent with the no-arbitrage condition; in practice bid-ask spreads and transaction costs virtually hide any arbitrage possibility.

Finally, we stress that the first key point in the procedure above is much more a matter of art than of science, because there is not an unique financially sound choice of bootstrapping instruments and, in principle, none is better than the others.

The pricing & hedging methodology described above can be extended, in principle, to more complicated cases, in particular when a model of the underlying interest rate evolution is used to calculate the future dynamics of the yield curve and the expected cash flows. The volatility and (eventually) correlation dependence carried by the model implies, in principle, the bootstrapping of a variance/covariance matrix (two or even three dimensional) and hedging the corresponding sensitivities (vega and rho) using volatility and correlation dependent vanilla market instruments. In practice just a small subset of such quotations is available, and thus only some portions of the variance/covariance matrix can be extracted from the market. In this paper we will focus only on the basic matter of yield curves and leave out the volatility/correlation dimensions.

[1]within the present context of interest rate derivatives we focus in particular on cash flows dependent on a single FRA rate.

[2]we refer here to the case of local yield curve bootstrapping methods, for which there are no sensitivity delocalization effect (see sec. 4.5.

2.2 The Modern Multiple-Curve Approach

Unfortunately, the pre-crisis approach outlined above is no longer consistent, at least in this simple formulation, with the present market approach.

First, it does not take into account the market information carried by the Basis Swap spreads, now much larger than in the past and no longer negligible. Second, it does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with different underlying rate tenors, characterized, in principle, by *different* dynamics (e.g. short rate processes). Thus, pricing and hedging an interest rate derivative on a single yield curve mixing different underlying rate tenors can lead to "dirty" results, incorporating the different dynamics, and eventually the inconsistencies, of different market areas, making prices and hedge ratios less stable and more difficult to interpret. On the other side, the more the vanillas and the derivative share the same homogeneous underlying rate, the better should be the relative pricing and the hedging. Third, by no-arbitrage, discounting must be unique: two identical future cash flows of whatever origin must display the *same* present value; hence we need an unique discounting curve. Fourth, there is no mention to the collateral agreements and funding rates associated with the bootstrapping instruments.

The market practice has thus evolved to take into account the new market information cited above, that translate into the additional requirement of *homogeneity* and *funding*. The homogeneity requirement means that interest rate derivatives with a given underlying rate tenor must be priced and hedged using vanilla interest rate market instruments with the *same* underlying. The funding requirement means that the discount rate of any cash flow generated by the derivative must be consistent, by no-arbitrage, with the funding rate associated with that cash flow. In case of OTC derivatives traded between counterparties subject to collateral agreements (CSA, Credit Support Annex to the ISDA Master Agreement), the funding rate is the collateral rate, normally the over night rate in case of daily margination (see sec. 3 for more details).

The post-crisis standard market practice for the construction of *multiple* interest rate yield curves can be summarized in the following procedure.

1. Interbank credit/liquidity issues do matter for pricing, Libors are risky rates, Basis Swap spreads are no longer negligible.

2. The collateral does matter for pricing, OIS discounting is adopted.

3. Decide the appropriate funding rates of the derivatives to be priced, then select the corresponding market instruments and build *one single discounting curve* using the classical single-curve bootstrapping technique.

4. Select *multiple separated* sets of vanilla interest rate instruments traded in real time on the market with increasing maturities, each set *homogeneous* in the underlying rate (typically with 1M, 3M, 6M, 12M tenors).

5. Build *multiple separated FRA curves* using the selected instruments plus their bootstrapping rules and the unique discounting curve.

6. Compute the relevant FRA rates and the corresponding cash flows from the FRA curve with the appropriate tenor.

7. Compute the relevant discount factors from the discounting curve with the appropriate funding characteristics.

8. Work out prices by summing the discounted cash flows;

9. Compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the *corresponding* set of vanillas.

For instance, the 5.5Y floating Swap leg cited in the previous section, in case of standard CSA with daily margination and Eonia rate, must be priced using Euribor1M FRA rates computed on an "pure" 1M FRA curve, bootstrapped on Euribor1M vanillas only, plus discount factors computed on the discounting curve, bootstrapped on Eur OIS. The corresponding delta sensitivity should be calculated by shocking one by one the pillars of both yield curves, and the resulting delta risk hedged using the suggested amounts (hedge ratios) of 5Y and 6Y Euribor1M Swaps plus the suggested amounts of 5Y and 6Y OIS.

The improved approach described above is more consistent with the present market situation, but - there is no free lunch - it does demand much more additional efforts. First, the discounting curve clearly plays a special and fundamental role, and must be built with particular care (see e.g. the discussion in ref. [17]). Second, building multiple curves requires multiple quotations: much more interest rate bootstrapping instruments must be considered (Deposits, Futures, Swaps, Basis Swaps, FRA, etc.), which are available on the market with different degrees of liquidity and can display transitory inconsistencies. Third, non trivial interpolation algorithms are crucial to produce smooth FRA curves (see e.g. refs. [13], [16]). Fourth, multiple bootstrapping instruments implies multiple sensitivities, so hedging becomes more complicated. Last but not least, pricing libraries, platforms, reports, etc. must be extended, configured, tested and released to manage multiple and separated yield curves for forwarding and discounting, not a trivial task for quants, developers and IT people.

3 Notation and Basic Theoretical Framework

In this section we fix the main assumptions and theoretical results underlying the multiple-curve framework. The notation is derived from the classical Brigo and Mercurio set up [18], adapted to the modern market situation.

3.1 Contract Description

In general, we will describe the time structure of financial contracts using time grids, or *schedules*, such as

$$\begin{aligned}
&\boldsymbol{T}_x = \{T_{x,0}, ..., T_{x,m}\}\,, \text{ floating leg schedule,}\\
&\boldsymbol{S} = \{S_0, ..., S_n\}\,, \text{ fixed leg schedule,}\\
&S_0 = T_{x,0}, S_n = T_{x,m},
\end{aligned} \tag{1}$$

roughly corresponding to the schedules of a fixed and floating leg of an Interest Rate Swap (IRS). The schedules will collect all the relevant contract dates (period start/end dates, fixing dates, accrual dates, cash flow dates, etc.) and year fractions known at the beginning of the contract, written in or derived from the termsheet. We will denote the fixed and floating year fractions as

$$\begin{aligned}\tau_{\mathit{fix},j} &:= \tau(S_{j-1}, S_j, dc_{\mathit{fix}}) \\ \tau_{\mathit{float},i} &:= \tau(T_{x,i-1}, T_{x,i}, dc_{\mathit{fix}}),\end{aligned} \tag{2}$$

respectively, where dc is the day count convention, that will be dropped whenever unnecessary. In particular, the floating leg year fractions $\tau_{\mathit{Float},i}$ will be consistent with the corresponding floating Libor rate tenor x, e.g. if the rate is Libor6M, with tenor $x = 6M$ then the floating leg frequency is $2y^{-1}$ and the times $T_{x,i}$ are six-month spaced.

The payoff of any derivative Π at each cash flow date T will be denoted by $\Pi(T;p)$, where p is a generic set of parameters specifying the contract details, such as nominal amount, schedules, etc. The price of the same derivative at any previous date $t < T$ will be denoted consistently by $\Pi(t;p)$. Finally, we will consider only single currency derivatives, dropping any currency index.

3.2 Funding and Collateral

Over The Counter (OTC) derivatives' counterparties may borrow and lend funds on the market through a variety of market operations, such as trading Deposits, repurchase agreements (repo), bonds, etc., and eventually reduce the counterparty risk through the adoption of bilateral collateral agreements.

We may think that the amount of cash borrowed or lent by a counterparty in the market is associated with a generic *funding account* B_α, with value $B_\alpha(t)$ at time t. The index α will denote the specific source of funding. Denoting with $R_\alpha(T_1)$ the funding rate fixed at time T_1 for the finite time interval $[T_1, T_2]$, we have that $B_\alpha(T_1)R_\alpha(T_1)\tau(T_1, T_2, dc_\alpha)$ is the funding interest exchanged at time T_2, related to the initial funding account value $B_\alpha(T_1)$. The value change of the funding account over $[T_1, T_2]$ is thus given by

$$B_\alpha(T_2) - B_\alpha(T_1) = B_\alpha(T_1)R_\alpha(T_1)\tau(T_1, T_2). \tag{3}$$

Hence, for pricing purposes we may assume the following funding account dynamics

$$dB_\alpha(t) = r_\alpha(t)B_\alpha(t)dt, \tag{4}$$

$$B_\alpha(0) = 1, \tag{5}$$

$$B_\alpha(t) = \exp\left[\int_0^t r_\alpha(u)du\right], \tag{6}$$

where $r_\alpha(t)$ is the (short) funding interest rate, related to the cash amount $B_\alpha(t)$, fixed at time t, spanning the time interval $[t, t+dt]$, and exchanged at time $t+dt$.

In particular, we identify two sources of funding associated with interest rate derivatives[3].

- The generic *funding* (or *treasury*) *account*, denoted with B_f, associated with the standard (unsecured) money and bond market funding at rate r_f, typically Libor plus spread, operated by a trading desk through a treasury desk.

[3] we omit here the secured repo funding.

- The *collateral account*, denoted with B_c, associated with a collateral agreement at collateral rate r_c, typically over night, operated by a trading desk through a collateral desk.

Real CSA are regulated mostly under the Credit Support Annex (CSA) of the ISDA Standard Master Agreement. They are characterized by various clauses and parameters, such as margination frequency, margination rate, threshold, minimum transfer amount, eligible collateral (cash, bonds, etc.), collateral currency, collateral asymmetry (e.g. one-way collateral), etc. We refer to ISDA documentation [19] for the precise definition of the terminology used here.

For pricing purposes it is useful to introduce an abstract 'perfect' collateral agreement, characterized as follows.

Definition 3.1 (Perfect CSA). *We define "perfect CSA" an ideal collateral agreement with the following characteristics:*

- *zero initial margin or initial deposit*
- *fully symmetric*
- *cash collateral*
- *zero threshold*
- *zero minimum transfer amount*
- *continuous margination*
- *instantaneous margination rate* $r_c(t)$
- *instantaneous settlement*
- *no collateral re-hypothecation by the collateral holder*

As a consequence we have, in general,

$$B_c(t) = \Pi(t), \quad \forall t \leq T. \tag{7}$$

We stress that, in case of default of one of the counterparties, we assume that the perfect collateral exactly matches the realized loss, and there are neither close-out amount nor legal risk relative to the closing of the deal or availability of the collateral.

The best market proxy available to a perfect CSA is the new ISDA Standard Credit Support Annex (SCSA) as described e.g. in [20], [21]. The main features of the SCSA are daily margination frequency, flat over night margination rate, zero threshold, zero minimum transfer amount, cash collateral in the same currency of the trade, collateral asymmetry.

Collateral Agreements are very diffused on the OTC market, according with the annual margin survey conducted by ISDA [22]. In particular, all the major financial institutions operating in the interest rate money and derivative markets are covered by mutual collateral agreements. As a consequence, we may safely assume that market quotations of derivatives, such as FRA, Swaps, Basis Swaps, OIS, etc. reflect collateralized transactions.

3.3 Pricing Under Collateral

Pricing derivatives under collateral implies that we must take into account, other than the cash flows generated by the derivative, also the cash flows generated by the margination mechanism provided by the CSA. As a consequence, we must extend the basic no-arbitrage framework described in standard textbooks (see e.g. [18, 14, 23, 24]) introducing the collateral account B_c defined in the previous section 3.2, and the corresponding collateral cash flows generated by the margination mechanism. This extension is discussed in a number of papers, see e.g. refs. [25, 26, 27, 28, 29, 30], also in connection with counterparty risk, see e.g. refs. [31, 32, 33, 34, 35]. We report here the basic result, assuming that under perfect CSA the default of both counterparties is irrelevant, and there are no Credit/Debit Valuation Adjustments (CVA/DVA) to the value of the trade.

Proposition 3.2 (Pricing under perfect collateral)**.** *Let* Π *be a derivative with maturity* T *written on a single asset* X *following the Geometric Brownian Motion process*

$$\begin{aligned} &dX(t) = \mu^P(t,X)X(t)dt + \sigma(t,X)X(t)dW^P(t), \\ &X(0) = X_0, \end{aligned} \tag{8}$$

where $t \in \mathbb{R}_+$*,* $X \in \mathbb{R}$*,* $\mu^P : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$*,* $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$*,* $W^P \in \mathbb{R}$ *is a* 1*-dimensional independent standard Brownian motion in the probability space* $(\Omega, \mathcal{F}, P)$*, and* P *is the real, or objective, probability measure. Assuming perfect collateral as in def. 3.1, the derivatives' price* $\Pi(t,X)$ *at time* $t < T$ *obeys the PDE*

$$\begin{aligned} &\hat{\mathcal{D}}_{r_f}\Pi(t,X) = r_c(t)\Pi(t,X), \\ &\hat{\mathcal{D}}_{r_f} = \frac{\partial}{\partial t} + r_f(t)X(t)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t)X^2(t)\frac{\partial^2}{\partial X^2}, \end{aligned} \tag{9}$$

and is given by the expectation

$$\Pi(t,X) = \mathbb{E}_t^{Q_f}\left[D_c(t,T)\Pi(T,X)\right], \tag{10}$$

$$D_c(t,T) = \exp\left[-\int_t^T r_c(u)du\right], \tag{11}$$

where Q_f *is the probability measure associated with the funding account* B_f *such that*

$$dX(t) = r_f(t)X(t)dt + \sigma(t,X)X(t)dW^{Q_f}(t). \tag{12}$$

Proof. See Appendix A. □

The perfect collateral case presented in prop. 3.2 above differs from the standard Black-Scholes-Merton framework without collateral, in how the cash used to replicate the derivative is split among the different sources of funding. In particular, thanks to the perfect collateral hypothesis, the cash in the collateral account B_c exactly funds (secured) the derivative position, $\Pi(t,S) = B_c(t)$, while the hedge $\frac{\partial \Pi}{\partial S}S(t)$ is funded (unsecured) by the funding account B_f, such that $\theta_2(t)B_f(t) = -\frac{\partial \Pi}{\partial S}S(t)$. The term $d\Gamma(t,S)$ in eq. 112 represents the variation of such amount of cash in the funding accounts B_f and B_c (rebalancing excluded), at the funding rates $r_f(t)$ and $r_c(t)$, respectively. The cash is split between the amount $\theta_1(t)S(t)$, borrowed to finance the purchase of $\theta_1(t)$ units of the risky underlying asset $S(t)$, and the amount $\Pi(t,S)$, but the funding rates are different. This term generates the discount and the drift terms in eqs. 10 and 12.

We conclude that the collateral rate is the correct discount rate in case of perfect collateral.

Proposition 3.3 (Forward Measure). *The pricing expression*

$$\Pi(t,X) = P_c(t,T)\mathbb{E}_t^{Q_f^T}\left[\Pi(T,X)\right], \tag{13}$$

$$P_c(t,T) = \mathbb{E}_t^{Q_f}\left[D_c(t,T)\right], \tag{14}$$

holds, where Q_f^T is the probability measure associated with the collateral zero coupon bond $P_c(t,T)$.

Proof. The result is obtained by applying the usual change of numeraire technique (see e.g. [18]), from $B_c(t)$ under Q_f to $P_c(t,T)$ under Q_f^T. □

Eq. 13 above is the general pricing formula for collateralized derivatives.

3.4 Interest Rates

In this section we discuss the particular case of interest rate derivatives in the modern multiple-curve framework.

We denote with $L_x(T_{i-1},T_i) := L_{x,i}$ the spot Libor rate, fixed on the market at time T_{i-1} and covering the time interval $[T_{i-1},T_i]$, where x indexes the different rate tenors. The spot Libor rate is the typical underlying for interest rate derivatives, in particular of plain vanilla derivatives quoted on the market, such as Interest Rate Swaps (IRS), Basis Swaps (BIRS), etc. There are many different flavors of interbank rates, such as Libor (London Interbank Offered Rate), Euribor (Euro Interbank Offered Rate), Eonia (Euro Over Night Index Average), etc., that differs for tenor, fixing mechanics, contribution panel, etc. In general, we will refer to these rates with the generic term “Libor”, discarding further distinctions if not necessary.

These rates are, in general, “risky”, because they are associated with unsecured financial transactions, such as Deposits, between OTC counterparties subject to default risk. The degree of risk carried by each rate depends on the characteristics of the transaction, in particular the tenor and the possible collateral. In general, the smallest degree of risk is carried by rates associated with secured transactions, such as Repo contracts, and to the shortest tenor, which is over night.

Spot Libor rates are normally quoted on the money market in association with (unsecured) Deposit contracts (see sec. 4.3.1). Forward Libor rates are quoted in the OTC derivatives’ market in terms of equilibrium rates of FRA (Forward Rate Agreement) contracts. FRA are OTC contracts in which two counterparties agree to exchange two cash flows, typically tied to a floating Libor rate $L_x(T_{i-1},T_i)$ with tenor x, fixed at time T_{i-1}, and spanning the time interval $[T_{i-1},T_i]$, versus a fixed rate K spanning the same time interval. The Standard FRA payoff at cash flow date T_i is given by

$$\mathbf{FRA}_{\mathrm{Std}}(T_i;\boldsymbol{T},L_{x,i},K,\omega) = \omega N\left[L_x(T_{i-1},T_i) - K\right]\tau_L(T_{i-1},T_i), \tag{15}$$

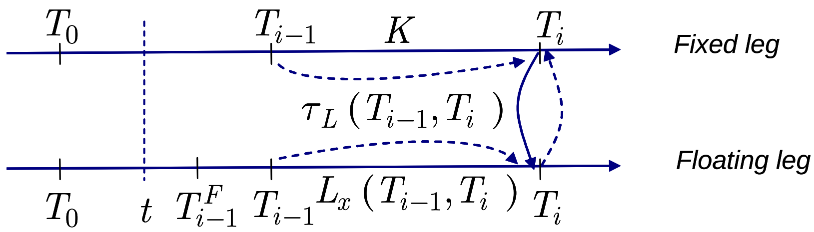


Figure 1: representation of a standard (or textbook) FRA contract. Cash flow date is at T_i, the end of the interest rate period.

where N is the contract's nominal, $\omega = +/-1$ for a payer/receiver FRA (referred to the fixed rate K), respectively, and we have assumed for simplicity that both Libor and fixed rates have the same simply compounded annual convention and share the same year fraction τ_L. The FRA contract described by eq. 15 is called "standard" or "textbook" FRA. The "market" FRA actually quoted on the market is characterized by a different payoff, discussed in sec. 4.3.2. In fig. 1 we show a representation of the standard FRA contract.

Following the discussion in sec. 3.2, we assume that FRA traded on the OTC market involve collateralized counterparties. Hence the price of the standard FRA, using the general pricing expression in eq. 13 for derivatives under collateral, is given by

$$\begin{aligned}
\mathbf{FRA}_{\mathrm{Std}}(t; \boldsymbol{T}, L_{x,i}, K, \omega) &= P_c(t; T_i)\mathbb{E}_t^{Q_f^{T_i}}\left[\mathbf{FRA}_{\mathrm{Std}}(T_i; \boldsymbol{T}, L_{x,i}, K, \omega)\right] \\
&= \omega N P_c(t; T_i)\left\{\mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1}, T_i)\right] - K\right\}\tau_L(T_{i-1}, T_i) \\
&= \omega N P_c(t; T_i)\left[F_{x,i}(t) - K\right]\tau_L(T_{i-1}, T_i), \qquad (16)
\end{aligned}$$

where we have defined

$$F_{x,i}(t) := \mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1}, T_i)\right]. \qquad (17)$$

The FRA rate at time t, denoted by $R_{x,Std}^{\mathrm{FRA}}(t; T_{i-1}, T_i)$, is defined as the FRA contract equilibrium rate, that is the value of the fixed rate K that makes null the FRA price. From eq. 16 we obtain

$$\begin{gathered}
\mathbf{FRA}_{\mathrm{Std}}(t; \boldsymbol{T}, L_{x,i}, K, \omega) = 0, \\
K = R_{x,Std}^{\mathrm{FRA}}(t; T_{i-1}, T_i) = F_{x,i}(t) = \mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1}, T_i)\right]. \qquad (18)
\end{gathered}$$

We conclude that the FRA rate is equal to the expectation $F_{x,i}(t)$ defined in eq. 17 above, that will be called FRA rate as well.

Proposition 3.4 (Properties of FRA rate)**.** *The FRA rate $F_{x,i}(t)$ defined in eq. 17 above has the following properties.*

1. *At fixing date T_{i-1}, $F_{x,i}(t)$ collapses onto the spot Libor rate $L_x(T_{i-1}, T_i)$,*

$$F_{x,i}(T_{i-1}) = \mathbb{E}_{T_{i-1}}^{Q_f^{T_i}}\left[L_x(T_{i-1}, T_i)\right] = L_x(T_{i-1}, T_i). \qquad (19)$$

2. By definition, $F_{x,i}(t)$ is a martingale under $Q_f^{T_i}$ measure

$$F_{x,i}(t) := \mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1},T_i)\right] = \mathbb{E}_t^{Q_f^{T_i}}\left[F_{x,i}(T_{i-1})\right]. \tag{20}$$

3. In the single-curve limit $F_{x,i}(t)$ recovers the classical single-curve value $F_i(t)$,

$$\begin{aligned} F_{x,i}(t) &:= \mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1},T_i)\right] \longrightarrow \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1},T_i)\right] \\ &= \mathbb{E}_t^{Q^{T_i}}\left[F(T_{i-1};T_{i-1},T_i)\right] = F(t;T_{i-1},T_i) := F_i(t). \end{aligned} \tag{21}$$

We observe that FRA contracts are quoted on the market in terms of the FRA equilibrium rates. Thus FRA rates can be directly read "on the screen" (see sec. 4.3.2). FRA rates are also included into Futures, IRS and IRBS rates (see sec. 4.3.3, 4.3.4 and 4.3.6). Hence, a FRA rate term structure can be stripped from market quotations. In conclusion, the FRA rate is the basic building block of the modern theoretical interest rate framework. We do not even need to talk about the classical "forward rates" anymore. The pricing expressions for interest rate derivatives may be derived within this framework, as discussed in sec. 4.2.

Given the Libor rate, we can *define* the corresponding risky zero coupon bond $P_x(t;T)$

$$P_x(T_{i-1};T_i) := \frac{1}{1 + L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}, \tag{22}$$

as a (Libor) discount factor, where $\tau_L(T_{i-1},T_i)$ denotes the year fraction associated with the time interval $[T_{i-1},T_i]$ with some day count convention dc_L. We stress that the zero coupon bond $P_x(t;T_i)$ is risky because it is associated with a risky Libor rate with tenor x. We can think to this bond as issued by an average Libor bank, representing the banks belonging to the Libor panel (see e.g. the discussion in [36]). We stress also that $P_x(t;T)$ differs, in general, from the collateral zero coupon bond $P_c(t;T)$, associated with the collateral account as in eqs. 14.

We may invert the definition of risky zero coupon bond in eq. 22 above,

$$L_x(T_{i-1},T_i) = \frac{1}{\tau_L(T_{i-1},T_i)}\left[\frac{1}{P_x(T_{i-1},T_i)} - 1\right], \tag{23}$$

and generalize it to (risky) forward Libor rates using the classical expression

$$F_{x,i}(t) := F_x(t;T_{i-1},T_i) := \frac{1}{\tau_L(T_{i-1},T_i)}\left[\frac{P_x(t;T_{i-1})}{P_x(t;T_i)} - 1\right]. \tag{24}$$

Clearly eq. 24 reduces to eq. 22 in the limit of spot rates, $t \to T_{i-1}$. We observe that eq. 24 can be rewritten as a no-arbitrage relation between discount factors over the three time intervals $[t,T_{i-1}]$, $[T_{i-1},T_i]$, and $[t,T_i]$,

$$\begin{aligned} P_x(t;T_i) &= P_x(t;T_{i-1})P_x(t;T_{i-1},T_i), \\ P_x(t;T_{i-1},T_i) &= \frac{1}{1 + F_x(t;T_{i-1},T_i)\tau_L(T_{i-1},T_i)}, \end{aligned} \tag{25}$$

as represented in fig. 2, but we stress that the relations 24 and 3.4 above are just a recursive *definition* of risky zero coupon bonds $P_x(t;T)$, not a consequence of no-arbitrage between consecutive spot/forward rates in the classical sense.

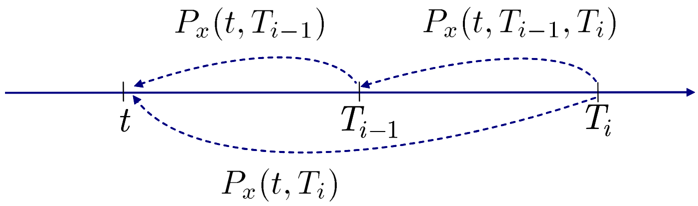


Figure 2: representation of no-arbitrage between discount factors associated with consecutive time intervals.

3.5 Yield Curves

Following the discussion in the previous sections, (see also refs. [37], [38]), we describe a multiplicity of N distinct yield curves $\mathcal{C}_x$, indexed with the subscript x, in the form of a continuous term structure or discount factors (or zero coupon bonds)

$$\mathcal{C}_x^P(t) = \{T \longrightarrow P_x(t;T), T \geq t\}, \tag{26}$$

or FRA rates

$$\mathcal{C}_x^F(t) = \{T \longrightarrow F_x(t;T,T+x), T \geq t\}, \tag{27}$$

where t is the reference date (e.g. today, or spot date). The index x will take values corresponding to the underlying rate tenors, e.g. $x = \{ON, 1M, 3M, 6M, 12M\}$.

We also define continuously compounded *zero coupon rates* $z_x(t,T)$ and simply compounded *instantaneous FRA rates* $f_x(t,T)$ such that

$$P_x(t,T) = \exp\left[-z_x(t,T)\tau_z(t,T)\right] = \exp\left[-\int_t^T f_x(t,u)\, du\right], \tag{28}$$

or, using the equivalent log notation,

$$\log P_x(t,T) = -z_x(t,T)\tau_z(t,T) = -\int_t^T f_x(t,u)du, \tag{29}$$

where

$$\tau_z(T_1,T_2) := \tau(T_1,T_2;dc_z) \tag{30}$$

and dc_z is the day count convention specific for the zero rate. Eq. (28) or (29) allows to define a zero rate curve $\mathcal{C}_x^z$ and an instantaneous FRA rate curve $\mathcal{C}_x^f$, such that

$$\mathcal{C}_x^z(t) = \{T \longrightarrow z_x(t,T), T \geq t\}, \tag{31}$$
$$\mathcal{C}_x^f(t) = \{T \longrightarrow f_x(t,T), T \geq t\}, \tag{32}$$

where

$$z_x(t,T) = -\frac{1}{\tau_z(t,T)} \log P_x(t,T), \tag{33}$$

$$\begin{aligned} f_x(t,T) &= -\frac{\partial}{\partial T} \log P_x(t,T) \\ &= z_x(t,T) + \tau_z(t,T)\frac{\partial}{\partial T} z_x(t,T), \end{aligned} \tag{34}$$

respectively. Swap par rates could also be used to represent yield curves. We do not use them here as they are not frequently used and would not provide additional benefit anyway.

From the relationships above it is immediate to observe that:

- $z_x(t,T)$ is the average of $f_x(t,u)$ over the time interval $[t,T]$,

$$z_x(t,T) = \frac{1}{\tau_z(t,T)} \int_t^T f_x(t,u)\, du; \tag{35}$$

- if rates are non-negative, (log)$P(t,T)$ is a monotone non-increasing function of T such that $0 < P(t,T) \leq 1 \; \forall T > t$. This is generally true, but not always assured, as we will see in the following examples.
- the instantaneous FRA rate curve $\mathcal{C}_x^f$ is the most severe indicator of yield curve smoothness, since anything else is obtained through its integration, therefore being smoother by construction. We will discuss this point in section 4.5.

In the following we will denote with $\mathcal{C}_x$ the generic curve and we will specify the particular typology if necessary. For interest rate conventions see e.g. [39].

4 Bootstrapping Interest Rate Yield Curves

4.1 Classical Bootstrapping of Single Yield Curves

A summary of the classical bootstrapping methodology is given in common textbooks as, for instance, [40] and [41]. The classical single yield curve was usually bootstrapped using a selection from the following market instruments.

1. Deposit contracts, covering the window from today up to 1Y.
2. FRA contracts, covering the window from 1M up to 2Y.
3. Short term interest rate Futures contracts, covering the window from spot/3M (depending on the current calendar date) up to 2Y and more.
4. IRS contracts, covering the window from 2Y-3Y up to 60Y.

The main characteristics of the instruments set above are:

- they are not homogeneous, admitting underlying interest rates with mixed tenors,
- the four blocks overlap by maturity and requires further selection.

The selection was generally done according to the principle of maximum liquidity: Futures with short expiries are the most liquid on the interest rate market, so they was generally preferred with respect to overlapping Deposits, FRA and short term IRS. For longer expiries Futures are not as liquid, so long term IRS were used.

We do not discuss further the traditional single-curve bootstrapping methodology as it is, more or less, history and it can be also viewed as a particular case of the multiple-curve approach described in the next section.

4.2 Modern Bootstrapping of Multiple Yield Curves

An yield curve is a complex object bearing many properties resulting from multiple different choices. We collect here the complete set of features that concur to shape an yield curve. Most of these features will be discussed in great detail in the following sections. We refer in particular to the EUR market case.

Zero coupon rate conventions: since the discount curve is normally observed to be exponentially decreasing, as expected when the interest rate compounding is made so frequent to be practically continuous, the zero rates compounding rule is chosen to be continuous, as in eq. (28). The associated year fraction dc_z in eq. (30) must be monotonically increasing with increasing time intervals (non increasing convention would lead to spurious null FRA rates), and additive, such that

$$\tau_z(T_1, T_2) + \tau_z(T_2, T_3) = \tau_z(T_1, T_3). \tag{36}$$

The day count convention satisfying the above conditions that is normally adopted by the market is the $dc_z = actual/365(fixed)$ [19], such that:

$$\tau_z\left(T_1, T_2\right) := \tau\left[T_1, T_2; actual/365(fixed)\right] = \frac{T_2 - T_1}{365}. \tag{37}$$

FRA rate conventions: chosen to be simply compounded as in eq. 3.4, consistently with spot Libor rates as in eq. 22. The associated year fraction τ_L is computed, for Euribor rates considered in this paper, using the $dc_L = actual/360$ day count convention [19], such that

$$\tau_L\left(T_1, T_2\right) := \tau\left[T_1, T_2; actual/360(fixed)\right] = \frac{T_2 - T_1}{360}. \tag{38}$$

Typology: parameter specifying the type of the yield curve, as defined in section 3.5, i.e.

- discount (or zero coupon bond) curve $\mathcal{C}_x^P$,
- zero rate curve $\mathcal{C}_x^z$,
- FRA rate curve[4] $\mathcal{C}_x^F$,
- instantaneous FRA rate curve $\mathcal{C}_x^f$.

Reference date: parameter t_0 specifying the reference date of the yield curve, such that $P_x\left(t_0, t_0\right) = 1$. It can be, for instance, t_0 = today date, or t_0 = spot date (which in the EUR market is two business days after today according to the chosen calendar) or, in principle, any business day after today (see sec. 4.3).

Time grid: the predetermined vector of dates, also named pillars, or knots, for which the yield curve bootstrapping procedure returns a value. It is defined by the set of maturities associated with the selected bootstrapping instruments.

[4] as discussed in sec. 3.4, we will dismiss the classical expression *forward rate/curve* and we will always talk about *FRA rates/curves*.

Bootstrapping instruments: the set of market instruments selected as input for the bootstrapping procedure (see sec. 4.3).

Currency: parameter specifying the reference currency of the yield curve, corresponding to the currency of the bootstrapping instruments.

Calendar: parameter specifying the calendar used to determine holidays and business days. In the EUR market the standard TARGET[5] calendar is used.

Side: parameter specifying the bid, mid or ask price chosen for the market instruments, if quoted.

Best fit vs exact fit: as discussed in the introduction, best fit and exact fit algorithms can be used to bootstrap an yield curve. We will adopt an exact fit algorithm because it ensures exact repricing of the input bootstrapping instruments.

Interpolation: parameter specifying the particular interpolation algorithm to be used for bootstrapping an the yield curve $\mathcal{C}_x$. The parameter admits two fields: the interpolation scheme (e.g. linear, spline, etc.), and the quantity to be interpolated (discounts, log discounts, zero/FRA/instantaneous FRA rates, see sec. 4.5).

Endogenous vs exogenous bootstrapping: parameter specifying the discounting yield $\mathcal{C}_d$ curve to be used in the bootstrapping procedure of another yield curve $\mathcal{C}_x$ (see sec. 4.7).

4.3 Bootstrapping Instruments Selection

As mentioned in section 2, in the present market situation, distinct interest rate market areas, relative to different underlying rate tenors, are characterized by different internal dynamics, liquidity and credit risk premia, reflecting the different views and interests of the market players. Such more complex market mechanic generates the following features:

- similar market instruments insisting on different underlyings, for instance FRA or Swaps on Euribor3M and Euribor6M, may display very different price levels;
- similar market instruments may display very different relative liquidities;
- even small idiosyncracies, asynchronism and inconsistencies in market quotations may result in erratic FRA rates.

Hence, the first step for multiple yield curve construction is a very careful selection of the corresponding multiple sets of bootstrapping instruments. Different kinds of instruments can be selected for bootstrapping an yield curve term structure, and whilst they roughly cover different maturities, they may overlap in some sections. Therefore it may be impossible to include all the available instruments, and the subset of the mostly non-overlapping contracts is selected, with preference given to the more liquid ones (with tighter bid/ask spreads), used for delta sensitivity representation and hedging purposes. The mispricing

[5] Trans-european Automated Real-time Gross settlement Express Transfer.

level of the market instruments excluded from the bootstrapping must thus be monitored as safety check (see sec. 4.11).

The set of bootstrapping instruments also defines the time grid used for bootstrapping. Market practitioners usually consider bootstrapping time grids from today up to 30Y-60Y. The first point in the time grid is the reference date t_0 of the yield curve $\mathcal{C}_x(t_0)$. While it makes perfectly sense to consider the first point $\{t_0, P_x(t_0, t_0) = 1\}$ for the discount curve $\mathcal{C}_x^P$, the corresponding choices for $\{t_0, z_x(t_0, t_0)\}$, $\{t_0, F_x(t_0, t_0 + x)\}$ and $\{t_0, f_x(t_0, t_0)\}$, for the zero curve $\mathcal{C}_x^z$, FRA curve $\mathcal{C}_x^F$, and the instantaneous FRA curve $\mathcal{C}_x^f$, respectively, are less significant and to some extent arbitrary, being just limits for shrinking $T \to t_0$, and as such must be handled with care. The reference date for all the EUR market bootstrapping instruments (except Over Night and Tomorrow Next Deposit contracts, see section 4.3.1 below) is t_0 = spot date. Once the yield curve at t_0 is available, the corresponding yield curve at t_0 = today date can be obtained using the discount between these two dates implied by ON and TN Deposits.

In the following sections we examine these market instruments in detail. In order to fix the data set once for all, we always refer consistently to the EUR market quotes observed on the Reuters platform as of 11 Dec. 2012, close time (around 16.30 CET[6]. Obviously the discussion holds for other EUR market data sets and can be remapped to other major currencies with small changes.

4.3.1 Deposits (Depos)

Interest Rate Certificates of Deposit (Depo) are standard money market zero coupon contracts where, at start date T_0 (today or spot), counterparty A, called the *Lender*, pays a nominal amount N to counterparty B, called the *Borrower*, and, at maturity date T_i, the Borrower pays back to the Lender the nominal amount N plus the interest accrued over the period $[T_0; T_i]$ at the annual simply compounded *Deposit Rate* $R_x^{\text{Depo}}(T_0^F; \boldsymbol{T}_i)$, fixed at time $T_0^F \leq T_0$, such that $T_0 = \text{spot}(T_0^F)$, with rate tenor x corresponding to the time interval $[T_0; T_i]$. Thus the contract schedule is $\boldsymbol{T}_i = \left\{T_0^F, T_0, T_i\right\}$. The payoff at maturity T_i, from the point of view of the Lender, is given by

$$\mathbf{Depo}(T_i; \boldsymbol{T}_i) = N\left[1 + R_x^{\text{Depo}}(T_0^F; \boldsymbol{T}_i)\tau_L(T_0, T_i)\right], \tag{39}$$

where N is the contract nominal amount.

The price at time t, such that $T_0^F \leq t \leq T_i$, is given by

$$\begin{aligned}\mathbf{Depo}(t; \boldsymbol{T}_i) &= P_x(t; T_i)\mathbb{E}_t^{Q_f^{T_i}}\left[\mathbf{Depo}(T_i; \boldsymbol{T}_i)\right] \\ &= NP_x(t; T_i)\left[1 + R_x^{\text{Depo}}(T_0^F; \boldsymbol{T}_i)\tau_L(T_0, T_i)\right],\end{aligned} \tag{40}$$

where the expectation on the r.h.s. has obviously no effect because the inner value is already fixed at time t. In fig. 3 we show a representation of the Deposit contract.

We stress that the Deposit is not a collateralized contract. Hence, in eq. 40 we have not used the pricing expression for collateralized financial instruments, given in proposition 3.3, but we have used a discount factor $P_x(t; T_i)$ based on a rate tenor x consistent with the Deposit rate tenor, as in eq. 23.

[6]Central European Time, equal to Greenwich Mean Time (GMT) plus 1 hour

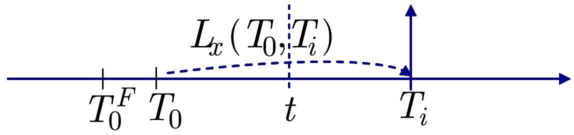


Figure 3: representation of a Deposit contract. The timeline reports the schedule, with fixing, start, maturity and evaluation dates, T_0^F, T_0, T_i, t, respectively.

Deposit contracts are quoted and exchanged on the interbank OTC market for various currencies. The EUR market quotes at time t_0 = today a standard strip of Deposits based on Euribor rates, with fixing date $T_0^F = t_0$, start date T_0 = spot date = t_0 + 2 business days, and maturity dates $T_1, ..., T_n$ from 1 day up to 1 year. Thus the schedule of the Deposit strip is $\boldsymbol{T} = \left\{T_0^F, T_0, T_1, ..., T_n\right\} = \{t_0, T_0, T_1, ..., T_n\}$ and the Deposit rate is, in our notation, $R_x^{\text{Depo}}(T_0^F; \boldsymbol{T}_i) = L_x(T_0, T_i)$, as shown in fig. 3.

In fig. 4 we show an example of the Euro Deposit strip quoted as of t_0 = 11 Dec. 2012. We see that the quotation is expressed directly in terms of the Deposit rate $R^{\text{Depo}}(t_0; \boldsymbol{T}_i)$, instead of the contract value in eq. 40. In particular the first two Deposits in the strip are “irregular”: the first Deposit, denoted with *ON* (*Over-Night Deposit*), starts today ($T_0 = t_0$) and matures tomorrow ($T_1 = t_0 + 1$ business day); the second Deposit, denoted with *TN* (*Tomorrow-Next Deposit*), starts tomorrow ($T_0 = t_0 + 1$ business day) and matures 1 day after ($T_2 = t_0 + 2$ business days). The next Deposit, denoted with *SN* (*Spot-Next Deposit*) is regular. The following Deposits are denoted according with their maturity, e.g. *SW* (*Spot-Week Deposit*), *1M* (*1-Month Deposit*). Thus the Deposit strip covers with neither holes nor overlaps the two business days interval between today and spot dates. The maturity date of Deposits shorter than one month obeys date rolling convention *following*. For longer Deposits the convention is *modified following, end of month*. The TARGET calendar is adopted. Notice that the ON, TN and SN Deposits insist on over night rates, with one day tenor, while the other Deposits in the strip admit underlying rates with increasing tenors, depending on their maturities. For example the 3M Deposit admits Euribor3M, the 6M Deposit admits Euribor6M, etc.

Market Deposits can be selected as bootstrapping instruments for the construction of the short term structure section of the yield curves $\mathcal{C}_x$. Hence each Deposit may be selected, in principle, for the construction of one single different curve with that tenor. The first pillar at time T_i of the FRA curve at spot date T_0, $\mathcal{C}_x^F(T_0)$ (actually a spot rate) is obtained directly using the i-th market Deposit rate $R_x^{Depo}(t_0; \boldsymbol{T}_i) = L_x(T_0, T_i)$ associated with the i-th Deposit with maturity T_i and underlying rate tenor $x = T_i - T_0$ months. Instead, the discount curve $\mathcal{C}_x^P(T_0)$ pillar at T_i is obtained using the following relation

$$P_x(T_0, T_i) = \frac{1}{1 + R_x^{Depo}(t_0; \boldsymbol{T}_i)\, \tau_L(T_0, T_i)}, \quad T_0 < T_i, \tag{41}$$

where the year fraction τ_L is computed using the actual/360 day count convention given in eq. 38.

4.3.2 Forward Rate Agreements (FRA)

Forward Rate Agreement (FRA) contacts are forward starting Deposits. For instance a 3x9 FRA is a six months Deposit starting three months forward. In sec. 3.4 we have

Instrument	Quote (ask, %)	Underlying	Start Date	Maturity Date	Settlement rule	Business Day Convention	End of Month Convention
Depo ON	0.040	Euribor1D	Tue 11 Dec 2012	Wed 12 Dec 2012	Today	Following	False
Depo TN	0.040	Euribor1D	Wed 12 Dec 2012	Thu 13 Dec 2012	Tomorrow	Following	False
Depo SN	0.040	Euribor1D	Thu 13 Dec 2012	Fri 14 Dec 2012	Spot	Following	False
Depo 1W	0.070	Euribor1W	Thu 13 Dec 2012	Thu 20 Dec 2012	Spot	Following	False
Depo 2W	0.080	Euribor2W	Thu 13 Dec 2012	Thu 27 Dec 2012	Spot	Following	False
Depo 3W	0.110	Euribor3W	Thu 13 Dec 2012	Thu 03 Jan 2013	Spot	Following	False
Depo 1M	0.110	Euribor1M	Thu 13 Dec 2012	Mon 14 Jan 2013	Spot	Mod. Following	True
Depo 2M	0.140	Euribor2M	Thu 13 Dec 2012	Wed 13 Feb 2013	Spot	Mod. Following	True
Depo 3M	0.180	Euribor3M	Thu 13 Dec 2012	Wed 13 Mar 2013	Spot	Mod. Following	True
Depo 4M	0.220	Euribor4M	Thu 13 Dec 2012	Mon 15 Apr 2013	Spot	Mod. Following	True
Depo 5M	0.270	Euribor5M	Thu 13 Dec 2012	Mon 13 May 2013	Spot	Mod. Following	True
Depo 6M	0.320	Euribor6M	Thu 13 Dec 2012	Thu 13 Jun 2013	Spot	Mod. Following	True
Depo 7M	0.350	Euribor7M	Thu 13 Dec 2012	Mon 15 Jul 2013	Spot	Mod. Following	True
Depo 8M	0.390	Euribor8M	Thu 13 Dec 2012	Tue 13 Aug 2013	Spot	Mod. Following	True
Depo 9M	0.420	Euribor9M	Thu 13 Dec 2012	Fri 13 Sep 2013	Spot	Mod. Following	True
Depo 10M	0.460	Euribor10M	Thu 13 Dec 2012	Mon 14 Oct 2013	Spot	Mod. Following	True
Depo 11M	0.500	Euribor11M	Thu 13 Dec 2012	Wed 13 Nov 2013	Spot	Mod. Following	True
Depo 12M	0.540	Euribor12M	Thu 13 Dec 2012	Fri 13 Dec 2013	Spot	Mod. Following	True

Figure 4: EUR Deposit strip. Source: Reuters page KLIEM, as of 11 Dec. 2012.

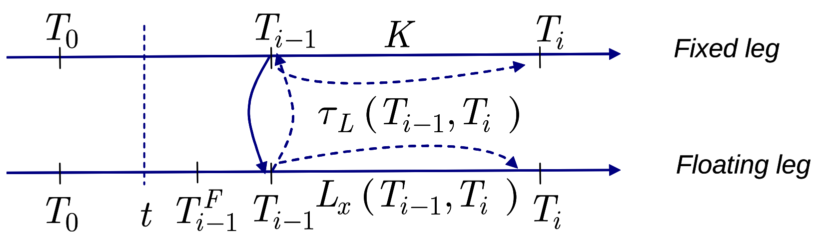


Figure 5: representation of a market FRA contract. Cash flow are exchanged at time T_{i-1}, the beginning of the interest rate period (compare with standard FRA in fig. 1).

discussed the standard (or textbook) FRA, with payoff, price and equilibrium rate given by eq. 15, 16, and 17, respectively. The actual FRA traded on the market, has a different payoff, such that, at payment date T_{i-1} (not T_i), we have

$$\begin{aligned}\mathbf{FRA}_{\text{Mkt}}(T_{i-1};\boldsymbol{T},L_{x,i},K,\omega) &= N\omega\frac{[L_x(T_{i-1},T_i)-K]\,\tau_F(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_F(T_{i-1},T_i)}\\ &= \frac{\mathbf{FRA}_{\text{Std}}(T_i;\boldsymbol{T},L_x,K,\omega)}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}.\end{aligned} \tag{42}$$

In fig. 5 we show a representation of the market FRA contract.

Since FRA are traded OTC between collateralized counterparties, we may apply the pricing under collateral approach discussed in sec. 3.2. The price and the FRA equilibrium rate, derived in appendix C.1, are given by eq. 125 and 126,

$$\mathbf{FRA}_{\text{Mkt}}(t;\boldsymbol{T},K,\omega) = N\omega P_c(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C^{\text{FRA}}_{c,x}(t;T_{i-1})}\right], \tag{43}$$

$$R^{\text{FRA}}_{x,\text{Mkt}}(t;\boldsymbol{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{[1+F_{x,i}(t)\tau_x(T_{i-1},T_i)]\,e^{C^{\text{FRA}}_{c,x}(t;T_{i-1})}-1\right\}, \tag{44}$$

where $C^{\text{FRA}}_{c,x}(t;T_{i-1})$ is a *convexity adjustment*, depending on the particular model adopted for the dynamics of the two rates $F_{d,i}(t)$ and $F_{x,i}(t)$. In ref [42] it is proved that, for typical post credit crunch market situations, the actual size of the convexity adjustment results

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying	Start Date	Maturity Date
FRA Tod3M	0.156	0.206	0.181	Euribor3M	Thu 13 Dec 2012	Wed 13 Mar 2013
FRA Tom3M	0.154	0.204	0.179	Euribor3M	Fri 14 Dec 2012	Thu 14 Mar 2013
FRA 1x4	0.140	0.190	0.165	Euribor3M	Mon 14 Jan 2013	Mon 15 Apr 2013
FRA 2x5	0.116	0.166	0.141	Euribor3M	Wed 13 Feb 2013	Mon 13 May 2013
FRA 3x6	0.104	0.154	0.129	Euribor3M	Wed 13 Mar 2013	Thu 13 Jun 2013
FRA 4x7	0.101	0.151	0.126	Euribor3M	Mon 15 Apr 2013	Mon 15 Jul 2013
FRA 5x8	0.099	0.149	0.124	Euribor3M	Mon 13 May 2013	Tue 13 Aug 2013
FRA 6x9	0.096	0.146	0.121	Euribor3M	Thu 13 Jun 2013	Fri 13 Sep 2013
FRA Tod6M	0.291	0.341	0.316	Euribor6M	Thu 13 Dec 2012	Thu 13 Jun 2013
FRA Tom6M	0.287	0.337	0.312	Euribor6M	Fri 14 Dec 2012	Fri 14 Jun 2013
FRA 1x7	0.268	0.318	0.293	Euribor6M	Mon 14 Jan 2013	Mon 15 Jul 2013
FRA 2x8	0.247	0.297	0.272	Euribor6M	Wed 13 Feb 2013	Tue 13 Aug 2013
FRA 3x9	0.235	0.285	0.260	Euribor6M	Wed 13 Mar 2013	Fri 13 Sep 2013
FRA 4x10	0.231	0.281	0.256	Euribor6M	Mon 15 Apr 2013	Tue 15 Oct 2013
FRA 5x11	0.227	0.277	0.252	Euribor6M	Mon 13 May 2013	Wed 13 Nov 2013
FRA 6x12	0.223	0.273	0.248	Euribor6M	Thu 13 Jun 2013	Fri 13 Dec 2013
FRA 7x13	0.229	0.279	0.254	Euribor6M	Mon 15 Jul 2013	Wed 15 Jan 2014
FRA 8x14	0.236	0.286	0.261	Euribor6M	Tue 13 Aug 2013	Thu 13 Feb 2014
FRA 9x15	0.242	0.292	0.267	Euribor6M	Fri 13 Sep 2013	Thu 13 Mar 2014
FRA 10x16	0.254	0.304	0.279	Euribor6M	Mon 14 Oct 2013	Mon 14 Apr 2014
FRA 11x17	0.266	0.316	0.291	Euribor6M	Wed 13 Nov 2013	Tue 13 May 2014
FRA 12x18	0.278	0.328	0.303	Euribor6M	Fri 13 Dec 2013	Fri 13 Jun 2014
FRA 13x19	0.293	0.343	0.318	Euribor6M	Mon 13 Jan 2014	Mon 14 Jul 2014
FRA 14x20	0.310	0.360	0.335	Euribor6M	Thu 13 Feb 2014	Wed 13 Aug 2014
FRA 15x21	0.327	0.377	0.352	Euribor6M	Thu 13 Mar 2014	Mon 15 Sep 2014
FRA 16x22	0.346	0.396	0.371	Euribor6M	Mon 14 Apr 2014	Tue 14 Oct 2014
FRA 17x23	0.364	0.414	0.389	Euribor6M	Tue 13 May 2014	Thu 13 Nov 2014
FRA 18x24	0.384	0.434	0.409	Euribor6M	Fri 13 Jun 2014	Mon 15 Dec 2014
FRA 12x24	0.482	0.532	0.507	Euribor12M	Fri 13 Dec 2013	Mon 15 Dec 2014
FRA IMMF3	0.264	0.314	99.7110	Euribor6M	Wed 16 Jan 2013	Tue 16 Jul 2013
FRA IMMG3	0.244	0.294	99.7310	Euribor6M	Wed 20 Feb 2013	Tue 20 Aug 2013
FRA IMMH3	0.234	0.284	99.7410	Euribor6M	Wed 20 Mar 2013	Fri 20 Sep 2013
FRA IMMJ3	0.230	0.280	99.7450	Euribor6M	Wed 17 Apr 2013	Thu 17 Oct 2013

Figure 6: EUR FRA strips on Euribor3M, Euribor6M, and Euribor12M. Source: Reuters page ICAPSHORT2, as of 11 Dec. 2012.

to be below 1 bp, even for long maturities. Hence, in any practical situation, we can discard the convexity adjustment and use the classical pricing expressions

$$\begin{aligned}\mathbf{FRA}_{\mathrm{Mkt}}(t;\boldsymbol{T},L_{x,i},K,\omega) &\simeq \mathbf{FRA}_{\mathrm{Std}}(t;\boldsymbol{T},L_{x,i},K,\omega)\\ &= \omega N P_c(t;T_i)\left[F_{x,i}(t)-K\right]\tau_L(T_{i-1},T_i),\end{aligned} \tag{45}$$

$$R^{\mathrm{FRA}}_{x,Mkt}(t;\boldsymbol{T}) \simeq R^{\mathrm{FRA}}_{x,Std}(t;T_{i-1},T_i) = F_{x,i}(t) = \mathbb{E}_t^{Q_f^{T_i}}\left[L_x(T_{i-1},T_i)\right]. \tag{46}$$

Notice that in the single-curve limit we recover the classical single-curve expressions

$$\mathbf{FRA}_{\mathrm{Mkt}}(t;\boldsymbol{T},K,\omega) \longrightarrow N\omega P(t;T_i)\left[F_i(t)-K\right]\tau_L(T_{i-1},T_i), \tag{47}$$

$$R^{\mathrm{FRA}}_{x,\mathrm{Mkt}}(t;\boldsymbol{T}) \longrightarrow F_i(t). \tag{48}$$

FRA contracts are quoted and exchanged on the interbank OTC market for various currencies. For example the EUR market quotes at t_0 = today three standard strips of Euribor FRA, starting at spot date $T_0 = t_0 + 2$ business days with different forward start and end dates, T_{i-1} and T_i, and tenors $x = 3M, 6M, 12M$. The corresponding FRA is denoted shortly with $T_{i-1}\times T_i$ FRA. The year fraction τ_L is computed using the actual/360 day count convention given in eq. 38. The underlying Euribor FRA rate fixes at time

T_{i-1}^F, two business days before the forward start date T_{i-1}, and shares the same tenor of the FRA. The FRA is conventionally denominated payer or receiver with respect to the fixed leg (we enter in a payer FRA if we receive the fixed rate interest). For example, the 3x6 FRA starts at $T = 3$ months and matures at $T = 6$ months, the underlying being Euribor3M. In particular, the FRA end dates are set with the same convention used for the end dates of Deposits, such that FRA dates do concatenate exactly: the 3x6 FRA above matures at $T = 6M$ where the following 6x9 FRA starts. For the single market FRA with maturity T_i we will use the schedule notation $\boldsymbol{T}_i = \left\{t_0, T_{i-1}^F, T_{i-1}, T_i\right\}$, with $t_0 < T_{i-1}^F < T_{i-1} < T_i, i = 1, ..., n$, and $R_{x,\text{Mkt}}^{\text{FRA}}(t_0; \boldsymbol{T}_i)$ for the equilibrium FRA rate.

In fig. 6 we show an example of the four FRA strips on Euribor3M, Euribor6M, and Euribor12M rates, quoted as of 11 Dec. 2012. As for Deposits, the FRA quotation is expressed directly in terms of the equilibrium FRA rate $R_{x,\text{Mkt}}^{\text{FRA}}(t_0, \boldsymbol{T}_i)$, such that the quoted FRA has zero value, $\mathbf{FRA}_{\text{Mkt}}(t_0; \boldsymbol{T_i}) = 0, i = 1, ..., n$. We notice also that the 3M strip includes also four IMM FRA, struck on the fixed IMM dates as Futures (see section 4.3.3 below). Furthermore, there are 2+2 "irregular" FRA at the beginning of the two 3M and 6M strips, respectively, bridging the two business days interval between today and spot dates.

Market FRA quotations in fig. 6 provide direct empirical evidence that a single-curve cannot be used to estimate FRA rates with different tenors. We can observe that, for instance, the level of the market 1x4 FRA3M (spanning from 14th Jan to 15th Apr. 2013, with $\tau_L(\text{1x4}) = 0.25278$) was $R_{3M,\text{Mkt}}^{\text{FRA}}(t_0, \text{1x4}) = 0.165\%$, the level of market 4x7 FRA3M (spanning from 15th Apr. to 15th Jul., $\tau_L(\text{4x7}) = 0.25278$) was $R_{3M,\text{Mkt}}^{\text{FRA}}(t_0, \text{4x7}) = 1.580\%$. If one would compound these two rates to obtain the level of the implied 1x7 FRA6M (spanning from 14th Jan. to 15th Jul., $\tau_L(\text{1x7}) = 0.50556$) would obtain, using eq. 3.4,

$$\begin{aligned} &R_{6M,\text{Implied}}^{\text{FRA}}(t_0, \text{1x7}) \\ &\quad = \frac{\left[1 + R_{3M,\text{Mkt}}^{\text{FRA}}(t_0, \text{1x4})\tau_L(\text{1x4})\right] \times \left[1 + R_{3M,\text{Mkt}}^{\text{FRA}}(t_0, \text{4x7})\tau_L(\text{4x7})\right] - 1}{\tau_L(\text{1x7})} \\ &\quad = 0.146\%, \end{aligned} \tag{49}$$

while the market quote for the $1x7$ FRA6M was $F_{1x7}^{mkt} = 0.293\%$, 14.7 basis point larger. As discussed in section 3.4, this is not surprising, because eq. 3.4 is not a no-arbitrage relation fulfilled by market rates, but just a recursive definition of risky zero coupon bonds. The 14.7 basis points are the price assigned by the market to the different liquidity/default risk implicit in the two investment strategies, using a 1x7 FRA6M or using two 1x4 and 4x7 FRA3M. See [36] for further details.

Market FRA with tenor x can be selected as bootstrapping instruments for the construction of the short term structure section of the yield curve $\mathcal{C}_x$. The FRA curve $\mathcal{C}_x^F(T_0)$ pillar at T_i is obtained directly using the i-th market FRA rate $R_{x,\text{Mkt}}^{\text{FRA}}(t_0, \boldsymbol{T}_i)$. Instead, the discount curve $\mathcal{C}_x^P(T_0)$ pillar at T_i is obtained recursively, given the previous point T_{i-1}, from eq. 46 and 24 as

$$P_x(T_0, T_i) = \frac{P_x(T_0, T_{i-1})}{1 + R_{x,\text{Mkt}}^{\text{FRA}}(t_0, \boldsymbol{T}_i)\tau_L(T_{i-1}, T_i)}, \tag{50}$$

where τ_L is given by eq. 38. Notice that FRA collapse to Depos in the limit $T_{i-1} \to t_0$

$$\lim_{T_{i-1} \to T_0} R_{x,\text{Mkt}}^{\text{FRA}}(t; \boldsymbol{T}_i) = R_x^{Depo}(t_0, T_i), \tag{51}$$

and eq. (50) reduces to eq. (41).

4.3.3 Futures

Interest rate Futures are the exchange-traded and margined contracts equivalent to the over-the-counter (market) FRA. The Futures' payoff at payment date T_{i-1} (as for the market FRA) is given, from the point of view of the counterparty paying the floating rate, by[7]

$$\mathbf{Futures}(T_{i-1};\boldsymbol{T}) = N\left[1 - L_x(T_{i-1}, T_i)\right], \tag{52}$$

where N is the nominal amount. This payoff is a classical example of "mixing apples and oranges" because, clearly, the Libor rate in eq. 52 is not an adimensional number that can be directly summed to 1, and an year fraction $\tau_L(T_{i-1}, T_i)$ is lost. Thus we must consider this definition merely as a rule to compute the margin.

The Futures' price and rate at time $t < T_{i-1}$ are given in app. C.2 as

$$\mathbf{Futures}(t;\boldsymbol{T}) = N\left[1 - R_x^{\mathrm{Fut}}(t;\boldsymbol{T})\right], \tag{53}$$

$$R_x^{\mathrm{Fut}}(t;\boldsymbol{T}) = F_{x,i}(t) + C_x^{\mathrm{Fut}}(t;T_{i-1}). \tag{54}$$

Hence the pricing of Futures requires the computation of the *convexity adjustment* in the Futures' rate. The expression of the convexity adjustment depends on the particular model adopted. Convexity adjustment arises because of the daily marking to market and margination mechanism of Futures. An investor long a Futures contract will have a loss when the Futures price increases (and the Futures rate decreases) but she will finance such loss at lower rate. Viceversa, when the Futures price decreases, the profit will be reinvested at higher rate. This means that the volatility of the FRA rates and their correlation to the spot rates have to be accounted for, and a convexity adjustment is needed to convert the rate $R_x^{\mathrm{Fut}}(t;\boldsymbol{T})$ implied in the Futures price to its corresponding FRA rate $F_{x,i}(t)$ (see e.g. ref. [43]). From a mathematical point of view, the trivial unit discount factor implied by daily margination introduces a pricing measure mismatch that generates a volatility-correlation dependent convexity adjustment (see e.g. ch. 12 in ref. [18]).

Different approaches to Futures' convexity adjustment can be found in the financial literature. Hull-White model approach [44], Libor Market Model [43], [18], two-factors short rate gaussian model [18], stochastic volatility model [45], one-factor HJM model [46]. The most recent and advanced derivation is given by [42] under the multiple-curve Libor Market Model, as discussed in app. C.2, eq. C.2. The most common practitioners' recipe is that of [44], based on a simple short rate 1 factor Hull & White model [47]. This approach has been used in fig. 7 to calculate the adjustments, using the Hull-White parameters values given in table 1.

While FRA are traded OTC and have the advantage of being more customizable, Futures are highly standardized contracts traded on exchanges. In the EUR market the most common contracts (so called *IMM*[8] *Futures*) insist on Euribor3M. The contract's schedule of the *i-th* Futures, $\boldsymbol{T}_i = \left[T_i^F, T_i^e, T_i\right]$, includes, fixing, expiry and maturity dates

[7] See for example the NYSE Euronext definition for Euribor Futures at `https://globalderivatives.nyx.com/en/contract/content/29045/contract-specification`.

[8] International Money Market of the Chicago Mercantile Exchange.

such that $\tau\left(T_i^F, T_i^e\right)$ = settlement lag (two business days), and $\tau_L\left(T_i^e, T_i\right) = 3M$. This schedule is in common with the underlying FRA3M. Standard expiry dates are March, June, September and December, called IMM dates. The IMM Futures fix and stop trading the third Monday of the expiry month, and expiry the following Wednesday. Notice that such date grid is not regular: in general, $T_i \neq T_{i+1}^e$, so Futures' dates do not concatenate exactly. The traded amount is measured in *lots*, with unitary size of 1.000.000€. The market trades also so called *serial Futures*, expiring in the upcoming months not covered by the quarterly Futures. Since these contracts are traded on exchanges, any profit and loss generated by a Futures position and market movements is regulated through daily marking to market. The margination amount is computed, for quarterly Futures, as

$$\Delta(t, t-1; \boldsymbol{T})) = \frac{N'}{n}\left[\mathbf{Futures}(t; \boldsymbol{T}) - \mathbf{Futures}(t-1; \boldsymbol{T})\right], \tag{55}$$

where N' is the number of lots, and n is the Futures' tenor (in months, e.g. $n = 4$ for quarterly futures).

Such standard characteristics reduce the credit risk and the transaction costs of Market Futures, thus enhancing a very high liquidity. The first front contract is one of the most liquid interest rate instruments, with longer expiry contracts having very good liquidity up to the 8th-12th contract. Also the first serial contract is quite liquid, especially when it expires before the front contract.

In fig. 7 we report the quoted Futures strip on Euribor3M rate up to 3 years maturity. As we can see, Futures are quoted in terms of prices instead of rates, the relation being

$$\mathbf{Futures}(t; \boldsymbol{T}) = 100 \times \left[1 - R_x^{\mathrm{Fut}}(t; \boldsymbol{T})\right]. \tag{56}$$

HW parameter	Value
Mean reversion	3%
Volatility	0.3526%

Table 1: Hull-White parameters values for Futures3M convexity adjustment as of 11 Dec. 2012.

Market Futures on x-tenor Euribor can be selected as bootstrapping instruments for the construction of short-medium term structure section of the yield curve $\mathcal{C}_x$. Given the i-th Futures market quote, $\mathbf{Futures}(t_0; \boldsymbol{T}_i)$, the FRA curve $\mathcal{C}_x^F(t_0)$ pillar at T_i is obtained directly from the i-th market Futures rate in eq. 54 as

$$F_{x,i}(t_0) = R_x^{\mathrm{Fut}}(t_0; \boldsymbol{T}_i) - C_x^{\mathrm{Fut}}(t_0; T_i^e). \tag{57}$$

Instead, the discount curve $\mathcal{C}_x^P(t_0)$ pillar at T_i is obtained recursively, given the previous point T_{i-1}, from eq. 46 and 24 as

$$P_x(t_0, T_i) = \frac{P_x\left(t_0, T_{i-1}\right)}{1 + \left[R_x^{\mathrm{Fut}}(t_0; \boldsymbol{T}_i) - C_x^{\mathrm{Fut}}(t_0; T_i^e)\right]\tau_L\left(T_i^e, T_i\right)}. \tag{58}$$

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Convexity adjustment	Underlying	Underlying Start Date	Underlying End Date
FUT 3MZ2	99.8200	99.8250	99.8225	0.0000%	Euribor3M	Wed 19 Dec 2012	Tue 19 Mar 2013
FUT 3MF3	99.8450	99.8500	99.8475	0.0000%	Euribor3M	Wed 16 Jan 2013	Tue 16 Apr 2013
FUT 3MG3	99.8450	99.8650	99.8550	0.0001%	Euribor3M	Wed 20 Feb 2013	Mon 20 May 2013
FUT 3MH3	99.8700	99.8750	99.8725	0.0001%	Euribor3M	Wed 20 Mar 2013	Thu 20 Jun 2013
FUT 3MM3	99.8750	99.8800	99.8775	0.0003%	Euribor3M	Wed 19 Jun 2013	Thu 19 Sep 2013
FUT 3MU3	99.8700	99.8750	99.8725	0.0006%	Euribor3M	Wed 18 Sep 2013	Wed 18 Dec 2013
FUT 3MZ3	99.8400	99.8450	99.8425	0.0009%	Euribor3M	Wed 18 Dec 2013	Tue 18 Mar 2014
FUT 3MH4	99.8000	99.8050	99.8025	0.0013%	Euribor3M	Wed 19 Mar 2014	Thu 19 Jun 2014
FUT 3MM4	99.7400	99.7450	99.7425	0.0018%	Euribor3M	Wed 18 Jun 2014	Thu 18 Sep 2014
FUT 3MU4	99.6850	99.6900	99.6875	0.0024%	Euribor3M	Wed 17 Sep 2014	Wed 17 Dec 2014
FUT 3MZ4	99.6150	99.6200	99.6175	0.0030%	Euribor3M	Wed 17 Dec 2014	Tue 17 Mar 2015
FUT 3MH5	99.5550	99.5600	99.5575	0.0036%	Euribor3M	Wed 18 Mar 2015	Thu 18 Jun 2015
FUT 3MM5	99.4750	99.4800	99.4775	0.0044%	Euribor3M	Wed 17 Jun 2015	Thu 17 Sep 2015
FUT 3MU5	99.3800	99.3850	99.3825	0.0051%	Euribor3M	Wed 16 Sep 2015	Wed 16 Dec 2015
FUT 3MZ5	99.2750	99.2800	99.2775	0.0060%	Euribor3M	Wed 16 Dec 2015	Wed 16 Mar 2016
FUT 3MH6	99.1650	99.1700	99.1675	0.0069%	Euribor3M	Wed 16 Mar 2016	Thu 16 Jun 2016
FUT 3MM6	99.0350	99.0400	99.0375	0.0079%	Euribor3M	Wed 15 Jun 2016	Thu 15 Sep 2016
FUT 3MU6	98.9050	98.9150	98.9100	0.0090%	Euribor3M	Wed 21 Sep 2016	Wed 21 Dec 2016
FUT 3MZ6	98.7700	98.7850	98.7775	0.0100%	Euribor3M	Wed 21 Dec 2016	Tue 21 Mar 2017
FUT 3MH7	98.6500	98.6650	98.6575	0.0111%	Euribor3M	Wed 15 Mar 2017	Thu 15 Jun 2017
FUT 3MM7	98.5200	98.5450	98.5325	0.0124%	Euribor3M	Wed 21 Jun 2017	Thu 21 Sep 2017
FUT 3MU7	98.4000	98.4250	98.4125	0.0136%	Euribor3M	Wed 20 Sep 2017	Wed 20 Dec 2017

Figure 7: EUR Futures on Euribor 3M. Both main cycle Futures (denoted by "H", "M", "U"and "Z", standing for March, June, September and December expiries, respectively) and two serial Futures (denoted as F3, G3 for Jan. and Feb 2013 expiries) are displayed. Source: Reuters page 0#FEI, as of 11 Dec. 2012. In column 5 are reported the corresponding convexity adjustments, calculated as discussed in the text.

The expression above can be used to bootstrap the yield curve $\mathcal{C}_x$ at point T_i once point T_{i-1} is known.

We stress that Futures contracts have fixed, not rolling, expiration dates, gradually shrinking to zero as today's date approaches the expiration date. As such, Futures used as bootstrapping instruments in the yield curve generate *rolling pillars* that periodically jumps and overlap the fixed DepoDeposit and FRA pillars[9]. Hence some *priority* rule must be used in order to decide which instruments must be excluded from the bootstrapping procedure.

4.3.4 Interest Rate Swaps (IRS)

Interest Rate Swaps (IRS) are OTC contracts in which, in general, two counterparties agree to exchange two streams of cash flows in the same currency, typically tied to a floating Libor rate $L_x(T_{i-1}, T_i)$ versus a fixed rate K. These payment streams are called fixed and floating leg of the swap, respectively. Thus an IRS is characterized by a schedule as anticipated in sec. 3.1,

$$\begin{aligned} &\boldsymbol{T} = \{T_0, ..., T_n\}\,, \text{ floating leg schedule,} \\ &\boldsymbol{S} = \{S_0, ..., S_m\}\,, \text{ fixed leg schedule,} \\ &\quad \text{with } T_0 = S_0,\ T_n = S_m, \end{aligned} \tag{59}$$

[9]The contract dates may be *fixed*, as for Futures, or *rolling*, as for Depos, FRA, IRS, etc. Conversely, the corresponding yield curve pillars are rolling or fixed, in the sense that they change or not their tenor day by day.

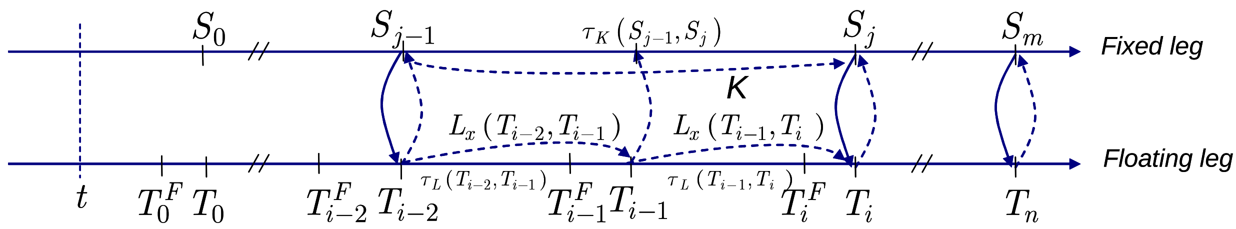


Figure 8: representation of an IRS contract. In this case, the frequency of the floating leg, indexed to Libor L_x with tenor x (bottom), is twice the frequency of the fixed leg (top), indexed to the fixed rate K. The coupon frequency of the floating leg is also consistent with the Libor tenor (e.g. Libor3M, quarterly coupon).

and coupon payoffs

$$\begin{aligned}\mathbf{IRSlet}_{\text{float}}(T_i; T_{i-1}, T_i, L_x) &= N L_x(T_{i-1}, T_i)\tau_L(T_{i-1}, T_i),\ i = 1, ..., n,\\ \mathbf{IRSlet}_{\text{fix}}(S_j; S_{j-1}, S_j, K) &= N K \tau_K(S_{j-1}, S_j),\ j = 1, ..., m,\end{aligned} \tag{60}$$

where N is the contract's nominal and τ_K and τ_L are the year fractions with the associated fixed and floating rate conventions, respectively. In fig. 8 we show a representation of the IRS contract. We may look at an IRS as a portfolio of standard FRA, part of them with a null fixed or floating rate (those with single-direction payment).

Since IRS are traded OTC between collateralized counterparties, we may apply the pricing under collateral approach discussed in sec. 3.2. The IRS price, rate and annuity, derived in appendix C.3, are given by eq. 138, 141 and 142, respectively, as

$$\mathbf{IRS}(t; \boldsymbol{T}, \boldsymbol{S}, L_x, K, \omega) = \omega N \left[R_x^{\text{IRS}}(t; \boldsymbol{T}, \boldsymbol{S}) - K\right] A_c(t; \boldsymbol{S}), \tag{61}$$

$$R_x^{\text{IRS}}(t; \boldsymbol{T}, \boldsymbol{S}) = \frac{\sum_{i=1}^{n} P_c(t; T_i) F_{x,i}(t) \tau_L(T_{i-1}, T_i)}{A_c(t; \boldsymbol{S})}, \tag{62}$$

$$A_c(t; \boldsymbol{S}) = \sum_{j=1}^{m} P_c(t; S_j) \tau_K(S_{j-1}, S_j). \tag{63}$$

We stress that the classical telescopic property of IRS rates, such that

$$\sum_{i=1}^{n} P_c(t; T_i) F_{x,i}(t) \tau_L(T_{i-1}, T_i) \simeq P_c(t; T_0) - P_c(t; T_n), \tag{64}$$

does not hold, even as an approximation, because in the modern multiple-curve world the discount rate and the FRA rate belongs to two different yield curves. As a consequence, each leg of an IRS starting at $T_0 = S_0 =$ today at par (with null total value) does not need to be worth par (when a fictitious exchange of notionals is introduced at maturity), that is

$$\begin{aligned}&\sum_{i=1}^{n} P_c(T_0; T_i) F_{x,i}(t) \tau_L(T_{i-1}, T_i) + P_c(T_0, T_n)\\ &= \sum_{j=1}^{m} P_c(S_0; S_j) \tau_K(S_{j-1}, S_j) + P_c(S_0, S_m) \neq 1.\end{aligned} \tag{65}$$

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying	Start Date	Maturity
AB6E1Y	0.266	0.306	0.286	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2013
AB6E15M	0.225	0.275	0.250	Euribor6M	Thu 13 Dec 2012	Thu 13 Mar 2014
AB6E18M	0.268	0.318	0.293	Euribor6M	Thu 13 Dec 2012	Fri 13 Jun 2014
AB6E21M	0.257	0.307	0.282	Euribor6M	Thu 13 Dec 2012	Mon 15 Sep 2014
AB6E2Y	0.304	0.344	0.324	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2014
AB6E3Y	0.404	0.444	0.424	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2015
AB6E4Y	0.556	0.596	0.576	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2016
AB6E5Y	0.742	0.782	0.762	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2017
AB6E6Y	0.934	0.974	0.954	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2018
AB6E7Y	1.115	1.155	1.135	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2019
AB6E8Y	1.283	1.323	1.303	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2020
AB6E9Y	1.432	1.472	1.452	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2021
AB6E10Y	1.564	1.604	1.584	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2022
AB6E11Y	1.683	1.723	1.703	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2023
AB6E12Y	1.789	1.829	1.809	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2024
AB6E13Y	1.881	1.921	1.901	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2025
AB6E14Y	1.956	1.996	1.976	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2026
AB6E15Y	2.017	2.057	2.037	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2027
AB6E16Y	2.066	2.106	2.086	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2028
AB6E17Y	2.103	2.143	2.123	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2029
AB6E18Y	2.130	2.170	2.150	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2030
AB6E19Y	2.151	2.191	2.171	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2031
AB6E20Y	2.167	2.207	2.187	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2032
AB6E21Y	2.180	2.220	2.200	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2033
AB6E22Y	2.191	2.231	2.211	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2034
AB6E23Y	2.200	2.240	2.220	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2035
AB6E24Y	2.208	2.248	2.228	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2036
AB6E25Y	2.214	2.254	2.234	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2037
AB6E26Y	2.219	2.259	2.239	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2038
AB6E27Y	2.223	2.263	2.243	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2039
AB6E28Y	2.227	2.267	2.247	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2040
AB6E29Y	2.231	2.271	2.251	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2041
AB6E30Y	2.236	2.276	2.256	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2042
AB6E35Y	2.275	2.315	2.295	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2047
AB6E40Y	2.328	2.368	2.348	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2052
AB6E50Y	2.401	2.441	2.421	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2062
AB6E60Y	2.443	2.483	2.463	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2072

Figure 9: EUR IRS on Euribor6M. The codes "AB6En"in col. 1 label IRS receiving yearly a fixed rate and paying semi-annually a floating rate on Euribor6M with maturity in n months/years. Source: Reuters page ICAPEURO, as of 11 Dec. 2012.

However, this is not a problem, since the only requirement for quoted spot-starting IRS is that their total initial value must be equal to zero[10].

The EUR market quotes at t_0 = today standard strips of plain vanilla IRS, with start date T_0 = spot date = $t_0 + 2$ business and maturities ranging from $1Y$ to $60Y$, annual fixed leg versus floating leg indexed to x-months Euribor rate exchanged with x-months frequency. The day count convention for the quoted (fair) IRS rate $R_x^{\text{IRS}}(t; \boldsymbol{T}, \boldsymbol{S})$ is *30/360 (bond basis)* [19]. Also IMM IRS are quoted. In figures 9, 10 and 11 we report the quoted IRS strips on 6M, 3M and 1M Euribor rates, respectively.

Market IRS on x-tenor Euribor can be selected as bootstrapping instruments for the construction of the medium-long term structure section of the yield curve $\mathcal{C}_x(T_0)$. Setting the market IRS schedule as $\boldsymbol{T}_i = \{T_0, \cdots, T_i\}$, $\boldsymbol{S}_j = \{S_0, \cdots, S_j\}$, with $t = T_0 = S_0, T_i =$

[10]both these notions are hard-coded into (classical) financial operators and thus very hard to die.

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying	Start Date	Maturity
AB3E1Y	0.116	0.166	0.141	Euribor3M	Thu 13 Dec 2012	Fri 13 Dec 2013
AB3E15M	0.119	0.169	0.144	Euribor3M	Thu 13 Dec 2012	Thu 13 Mar 2014
AB3E18M	0.128	0.178	0.153	Euribor3M	Thu 13 Dec 2012	Fri 13 Jun 2014
AB3E21M	0.143	0.193	0.168	Euribor3M	Thu 13 Dec 2012	Mon 15 Sep 2014
AB3E2Y	0.161	0.211	0.186	Euribor3M	Thu 13 Dec 2012	Mon 15 Dec 2014
AB3E3Y	0.260	0.310	0.285	Euribor3M	Thu 13 Dec 2012	Mon 14 Dec 2015
AB3E4Y	0.412	0.462	0.437	Euribor3M	Thu 13 Dec 2012	Tue 13 Dec 2016
AB3E5Y	0.598	0.648	0.623	Euribor3M	Thu 13 Dec 2012	Wed 13 Dec 2017
AB3E6Y	0.792	0.842	0.817	Euribor3M	Thu 13 Dec 2012	Thu 13 Dec 2018
AB3E7Y	0.975	1.025	1.000	Euribor3M	Thu 13 Dec 2012	Fri 13 Dec 2019
AB3E8Y	1.146	1.196	1.171	Euribor3M	Thu 13 Dec 2012	Mon 14 Dec 2020
AB3E9Y	1.299	1.349	1.324	Euribor3M	Thu 13 Dec 2012	Mon 13 Dec 2021
AB3E10Y	1.434	1.484	1.459	Euribor3M	Thu 13 Dec 2012	Tue 13 Dec 2022
AB3E11Y	1.557	1.607	1.582	Euribor3M	Thu 13 Dec 2012	Wed 13 Dec 2023
AB3E12Y	1.667	1.717	1.692	Euribor3M	Thu 13 Dec 2012	Fri 13 Dec 2024
AB3E15Y	1.908	1.958	1.933	Euribor3M	Thu 13 Dec 2012	Mon 13 Dec 2027
AB3E20Y	2.074	2.124	2.099	Euribor3M	Thu 13 Dec 2012	Mon 13 Dec 2032
AB3E25Y	2.131	2.181	2.156	Euribor3M	Thu 13 Dec 2012	Mon 14 Dec 2037
AB3E30Y	2.161	2.211	2.186	Euribor3M	Thu 13 Dec 2012	Mon 15 Dec 2042
AB3E40Y	2.263	2.313	2.288	Euribor3M	Thu 13 Dec 2012	Fri 13 Dec 2052
AB3E50Y	2.342	2.392	2.367	Euribor3M	Thu 13 Dec 2012	Wed 13 Dec 2062
AB3EZ2	0.113	0.163	0.138	Euribor3M	Wed 19 Dec 2012	Thu 19 Dec 2013
AB3EH3	0.109	0.159	0.134	Euribor3M	Wed 20 Mar 2013	Thu 20 Mar 2014
AB3EM3	0.126	0.176	0.151	Euribor3M	Wed 19 Jun 2013	Thu 19 Jun 2014
AB3EU3	0.158	0.208	0.183	Euribor3M	Wed 18 Sep 2013	Thu 18 Sep 2014
AB3EZ3	0.158	0.208	0.183	Euribor3M	Wed 18 Dec 2013	Fri 18 Dec 2015
AB3EH4	0.183	0.233	0.208	Euribor3M	Wed 19 Mar 2014	Mon 21 Mar 2016
AB3EZ4	0.258	0.308	0.283	Euribor3M	Wed 17 Dec 2014	Mon 18 Dec 2017

Figure 10: EUR IRS on Euribor3M, receiving yearly a fixed rate and paying quarterly a floating rate on Euribor3M. At the bottom IMM starting IRS are reported, denoted with their corresponding IMM code (see sec. 4.3.3). Source: Reuters page ICAPSHORT2, as of 11 Dec. 2012.

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying	Start Date	Maturity
AB1E2M	0.081	0.131	0.106	Euribor1M	Thu 13 Dec 2012	Wed 13 Feb 2013
AB1E3M	0.071	0.121	0.096	Euribor1M	Thu 13 Dec 2012	Wed 13 Mar 2013
AB1E4M	0.060	0.110	0.085	Euribor1M	Thu 13 Dec 2012	Mon 15 Apr 2013
AB1E5M	0.054	0.104	0.079	Euribor1M	Thu 13 Dec 2012	Mon 13 May 2013
AB1E6M	0.050	0.100	0.075	Euribor1M	Thu 13 Dec 2012	Thu 13 Jun 2013
AB1E7M	0.046	0.096	0.071	Euribor1M	Thu 13 Dec 2012	Mon 15 Jul 2013
AB1E8M	0.044	0.094	0.069	Euribor1M	Thu 13 Dec 2012	Tue 13 Aug 2013
AB1E9M	0.041	0.091	0.066	Euribor1M	Thu 13 Dec 2012	Fri 13 Sep 2013
AB1E10M	0.040	0.090	0.065	Euribor1M	Thu 13 Dec 2012	Mon 14 Oct 2013
AB1E11M	0.039	0.089	0.064	Euribor1M	Thu 13 Dec 2012	Wed 13 Nov 2013
AB1E12M	0.038	0.088	0.063	Euribor1M	Thu 13 Dec 2012	Fri 13 Dec 2013

Figure 11: EUR IRS on Euribor1M, receiving yearly a fixed rate and paying monthly a floating rate on Euribor1M. Source: Reuters page ICAPSHORT2, as of 11 Dec. 2012.

S_j, the FRA curve $\mathcal{C}_x^F(T_0)$ and the discount curve $\mathcal{C}_x^P(T_0)$ at pillar $T_i = S_j$ are obtained, shorting the notation in eqs. 62 and 63 above, as

$$F_{x,i}(T_0) = \frac{R_x^{\mathrm{IRS}}(T_0;T_i)A_c(T_0;T_i) - \sum_{\alpha=1}^{i-1} P_c(t;T_\alpha)F_{x,\alpha}(t)\tau_L(T_{\alpha-1},T_\alpha)}{P_c(T_0;T_i)\tau_L(T_{i-1},T_i)}, \tag{66}$$

$$P_x(T_0;T_i) = \frac{P_c(T_0;T_i)P_x(T_0;T_{i-1})}{R_x^{\mathrm{IRS}}(T_0;T_i)A_c(T_0;T_i) - \sum_{\alpha=1}^{i-1} P_c(t;T_\alpha)F_{x,\alpha}(t)\tau_L(T_{\alpha-1},T_\alpha) + P_c(T_0;T_i)}, \tag{67}$$

$$A_c(T_0;S_j) = \sum_{\beta=1}^{j} P_c(T_0;S_\beta)\tau_K(S_{\beta-1},S_\beta) = A_c(T_0;S_{j-1}) + P_c(T_0;S_j)\tau_K(S_{j-1},S_j). \tag{68}$$

The expressions above can be used, in principle, to bootstrap the yield curve $\mathcal{C}_x$ at pillar T_i recursively, once the curve at previous pillars is known. We stress that both formulas 66, 67 require inputs from the discounting curve $\mathcal{C}_c(T_0)$.

In practice, since in the market IRS the fixed leg frequency is annual and the floating leg frequency is given by the underlying Euribor rate tenor x, we have that $[S_1,...,S_j] \subseteq [T_1,...,T_i]$ for any given fixed leg date S_j. Hence some points $[T_{i-2},T_{i-1},\cdots]$ inside the last fixed leg interval $[S_{j-1},S_j]$ in eq. 67 may be unknown, and one must resort to interpolation and, in general, to numerical solution of eqs. 66, 67. For example the bootstrap of Euribor6M curve $\mathcal{C}_{6M}(T_0)$ from 9Y to 10Y pillars using the quotation $R_{6M}^{\mathrm{IRS}}(T_0;10Y) = 3.488\%$ in fig. 9 is given by

$$F_{6M,10Y}(T_0) = \frac{R_{6M}^{\mathrm{IRS}}(10Y)A_c(10Y) - R_{6M}^{\mathrm{IRS}}(9Y)A_c(9Y) - P_c(9.5Y)\tau_L(9Y,9.5Y)}{P_c(10Y)\tau_L(9.5Y,10Y)}, \tag{69}$$

$$P_{6M}(10Y) = \frac{P_c(10Y)P_{6M}(9.5Y)}{R_{6M}^{\mathrm{IRS}}(10Y)A_c(10Y) - R_{6M}^{\mathrm{IRS}}(9Y)A_c(9Y) - P_c(9.5Y)\tau_L(9Y,9.5Y) + P_c(10Y)}, \tag{70}$$

where $\boldsymbol{T} = [T_0, 0.5Y, ..., 9.5Y, 10Y]$, $\boldsymbol{S} = [S_0, 1Y, ..., 9Y, 10Y]$ are the floating and fixed leg schedules. Since $P_{6M}(T_0, 9.5Y)$ in eq. 70 above is unknown, it must be interpolated between $P_{6M}(T_0, 9Y)$ (known) and $P_{6M}(T_0, 10Y)$ (unknown). We thus see, as anticipated in the introduction, that interpolation is already used during the bootstrapping procedure, not only after that.

4.3.5 Overnight Indexed Swaps (OIS)

Overnight Indexed Swaps (OIS) are a special case of IRS in which the floating leg is tied to over night rates and pays the daily compounded rate over the coupon period. The OIS coupon payoffs are thus given by

$$\begin{aligned}\mathbf{OISlet}_{\mathrm{float}}(T_i;\boldsymbol{T}_i,R_{on}) &= NR_{on}(T_i;\boldsymbol{T}_i)\tau_{on}(T_{i-1},T_i),\ i=1,...,n,\\ \mathbf{OISlet}_{\mathrm{fix}}(S_j;S_{j-1},S_j,K) &= \mathbf{IRSlet}_{\mathrm{fix}}(S_j;S_{j-1},S_j,K),\ j=1,...,m,\end{aligned} \tag{71}$$

where N is the contract's nominal, τ_K, τ_{on} are the year fractions with the fixed and floating rate conventions, and $R_{on}(T_i;\boldsymbol{T}_i)$ is the coupon rate compounded from over night rates

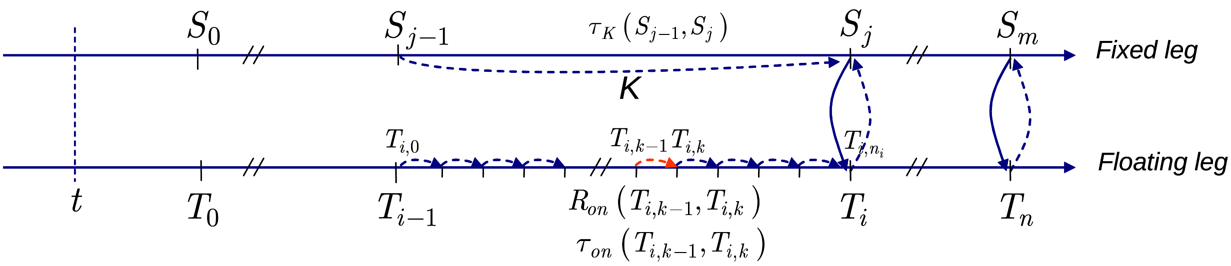


Figure 12: representation of an OIS contract. In this case, the frequency of the two legs is the same. The frequency of the floating leg is not consistent with the frequency of the over night rate. The floating leg (bottom) is indexed to the over night rate $R_{on}(T_{i,k-1}, T_{i,k})$ with daily fixings and complex schedule, as described in eq. 72.

over the $i-$th coupon period $(T_{i-1}; T_i]$, given by

$$R_{on}(T_i; \boldsymbol{T}_i) := \frac{1}{\tau_{on}(T_{i-1}, T_i)} \left\{ \prod_{k=1}^{n_i} \left[1 + R_{on}(T_{i,k-1}, T_{i,k})\tau_{on}(T_{i,k-1}, T_{i,k})\right] - 1 \right\},$$
$$\boldsymbol{T}_i = \{T_{i,0}, ..., T_{i,n_i}\}, \text{ with } T_{i,0} = T_{i-1},\ T_{i,n_i} = T_i,\ i = 1, ..., n,$$
$$(T_{i-1}, T_i] = \bigcup_{k=1}^{n_i} (T_{i,k-1}, T_{i,k}], \tag{72}$$

where $\boldsymbol{T}_i$ denotes the sub-schedule for the coupon rate $R_{on}(T_i; \boldsymbol{T}_i)$, and $R_{on}(T_{i,k-1}, T_{i,k})$ are the single over night rate spanning the over night time intervals $(T_{i,k-1}, T_{i,k}]$. Notice that the $i-$th coupon rate $R_{on}(T_i; \boldsymbol{T}_i)$ is completely fixed only at the fixing date T_{i,n_i-1} of the last rate $R_{on}(T_{i,n_i-1}, T_{i,n_i})$. In fig. 12 we show a representation of the OIS contract.

The price and the equilibrium rate of the OIS are given in app. C.4, eqs. 149 and 152, respectively, as

$$\mathbf{OIS}(t; \boldsymbol{T}, \boldsymbol{S}, R_{on}^c, K, \omega) = N\omega \left[R_{on}^{\mathrm{OIS}}(t; \boldsymbol{T}, \boldsymbol{S}) - K\right] A_c(t; \boldsymbol{S}), \tag{73}$$
$$R_{on}^{\mathrm{OIS}}(t; \boldsymbol{T}, \boldsymbol{S}) = \frac{\sum_{i=1}^{n} P_c(t; T_i) R_{on}(t; \boldsymbol{T}_i)\tau_{on}(T_{i-1}, T_i)}{A_c(t; \boldsymbol{S})} = \frac{P_c(t; T_0) - P_c(t; T_n)}{A_c(t; \boldsymbol{S})}. \tag{74}$$

The last term on the r.h.s of equation 74 above assumes that, under perfect collateral, the over night FRA rates $R_{on}(T_{i,k-1}, T_{i,k})$ may be replicated using collateral zero coupon bonds $P_c(t; T)$. Hence, the OIS floating leg value is drastically simplified to the classical single-curve expression $P_c(t; T_0) - P_c(t; T_n)$ above, to be compared to the multiple-curve expression for the IRS in eq. 62.

The EUR market quotes at t_0 = today standard strips of plain vanilla OIS, with start date T_0 = spot date = $t_0 + 2$ business and maturities ranging from $1W$ to $60Y$, annual fixed leg versus annual floating leg indexed to Eonia rate. The day count convention is the same for IRS. Also Eonia FRA on IMM and ECB dates are quoted. In figure 13 we report the quoted OIS strips on Eonia as of 11 Dec. 2012. We observe very low and even negative quotations for short term OIS, that will be discussed in sec. 4.6.

Market OIS can be selected as bootstrapping instruments for the construction of the yield curve $\mathcal{C}_c(T_0)$ used for discounting collateralised cash flows. In fact, as discussed in

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying	Start Date	Maturity
EON1W	0.020	0.120	0.070	Eonia	Thu 13 Dec 2012	Thu 20 Dec 2012
EON2W	0.019	0.119	0.069	Eonia	Thu 13 Dec 2012	Thu 27 Dec 2012
EON3W	0.028	0.128	0.078	Eonia	Thu 13 Dec 2012	Thu 03 Jan 2013
EON1M	0.049	0.099	0.074	Eonia	Thu 13 Dec 2012	Mon 14 Jan 2013
EON2M	0.036	0.086	0.061	Eonia	Thu 13 Dec 2012	Wed 13 Feb 2013
EON3M	0.022	0.072	0.047	Eonia	Thu 13 Dec 2012	Wed 13 Mar 2013
EON4M	0.008	0.058	0.033	Eonia	Thu 13 Dec 2012	Mon 15 Apr 2013
EON5M	-0.001	0.049	0.024	Eonia	Thu 13 Dec 2012	Mon 13 May 2013
EON6M	-0.007	0.043	0.018	Eonia	Thu 13 Dec 2012	Thu 13 Jun 2013
EON7M	-0.012	0.038	0.013	Eonia	Thu 13 Dec 2012	Mon 15 Jul 2013
EON8M	-0.016	0.034	0.009	Eonia	Thu 13 Dec 2012	Tue 13 Aug 2013
EON9M	-0.020	0.030	0.005	Eonia	Thu 13 Dec 2012	Fri 13 Sep 2013
EON10M	-0.022	0.028	0.003	Eonia	Thu 13 Dec 2012	Mon 14 Oct 2013
EON11M	-0.024	0.026	0.001	Eonia	Thu 13 Dec 2012	Wed 13 Nov 2013
EON1Y	-0.035	0.035	0.000	Eonia	Thu 13 Dec 2012	Fri 13 Dec 2013
EON15M	-0.023	0.027	0.002	Eonia	Thu 13 Dec 2012	Thu 13 Mar 2014
EON18M	-0.017	0.033	0.008	Eonia	Thu 13 Dec 2012	Fri 13 Jun 2014
EON21M	-0.004	0.046	0.021	Eonia	Thu 13 Dec 2012	Mon 15 Sep 2014
EON2Y	0.001	0.071	0.036	Eonia	Thu 13 Dec 2012	Mon 15 Dec 2014
EON3Y	0.092	0.162	0.127	Eonia	Thu 13 Dec 2012	Mon 14 Dec 2015
EON4Y	0.239	0.309	0.274	Eonia	Thu 13 Dec 2012	Tue 13 Dec 2016
EON5Y	0.421	0.491	0.456	Eonia	Thu 13 Dec 2012	Wed 13 Dec 2017
EON6Y	0.612	0.682	0.647	Eonia	Thu 13 Dec 2012	Thu 13 Dec 2018
EON7Y	0.792	0.862	0.827	Eonia	Thu 13 Dec 2012	Fri 13 Dec 2019
EON8Y	0.961	1.031	0.996	Eonia	Thu 13 Dec 2012	Mon 14 Dec 2020
EON9Y	1.112	1.182	1.147	Eonia	Thu 13 Dec 2012	Mon 13 Dec 2021
EON10Y	1.245	1.315	1.280	Eonia	Thu 13 Dec 2012	Tue 13 Dec 2022
EON11Y	1.369	1.439	1.404	Eonia	Thu 13 Dec 2012	Wed 13 Dec 2023
EON12Y	1.481	1.551	1.516	Eonia	Thu 13 Dec 2012	Fri 13 Dec 2024
EON15Y	1.729	1.799	1.764	Eonia	Thu 13 Dec 2012	Mon 13 Dec 2027
EON20Y	1.904	1.974	1.939	Eonia	Thu 13 Dec 2012	Mon 13 Dec 2032
EON25Y	1.968	2.038	2.003	Eonia	Thu 13 Dec 2012	Mon 14 Dec 2037
EON30Y	2.003	2.073	2.038	Eonia	Thu 13 Dec 2012	Mon 15 Dec 2042
EONECBFEB13	0.021	0.071	0.046	Eonia	Wed 16 Jan 2013	Wed 13 Feb 2013
EONECBMAR13	-0.009	0.041	0.016	Eonia	Wed 13 Feb 2013	Wed 13 Mar 2013
EONECBAPR13	-0.032	0.018	-0.007	Eonia	Wed 13 Mar 2013	Wed 10 Apr 2013
EONECBMAY13	-0.038	0.012	-0.013	Eonia	Wed 10 Apr 2013	Wed 08 May 2013
EONECBJUN13	-0.039	0.011	-0.014	Eonia	Wed 08 May 2013	Wed 12 Jun 2013
EONECBJUL13	-0.041	0.009	-0.016	Eonia	Wed 12 Jun 2013	Wed 10 Jul 2013

Figure 13: EUR OIS, receiving yearly a fixed rate and paying yearly a floating rate on Eonia. Bottom section: forward starting OIS at known ECB dates. Negative OIS rates are enlightened in red color. Source: Reuters pages ICAPSHORT1 and ICAPEURO2, as of 11 Dec. 2012.

secs. 3.2 and 3.3, the over night rate underlying the OIS is also the collateral rate for collateralised market instruments. Setting the market OIS schedule as $\boldsymbol{T}_i = \{T_0, \cdots, T_i\} = \boldsymbol{S}_i$, the FRA curve $\mathcal{C}_c^F(T_0)$ and the discount curve $\mathcal{C}_c^P(T_0)$ at pillar T_i are obtained from eq. 74 above as

$$\begin{aligned}
F_c(T_0;T_{i-1},T_i) = & \frac{1}{\tau_{on}(T_{i-1};T_i)} \\
& \times \left\{ \frac{P_c(T_0;T_{i-1})\left[1+R_{on}^{\mathrm{OIS}}(T_0;T_i)\tau_K(T_{i-1};T_i)\right]}{\left[R_{on}^{\mathrm{OIS}}(T_0;T_{i-1}) - R_{on}^{\mathrm{OIS}}(T_0;T_i)\right] A_c(T_0;T_{i-1}) + P_c(T_0;T_{i-1})} - 1 \right\},
\end{aligned} \tag{75}$$

$$P_c(T_0;T_i) = \frac{\left[R_{on}^{\mathrm{OIS}}(T_0;T_{i-1}) - R_{on}^{\mathrm{OIS}}(T_0;T_i)\right] A_c(T_0;T_i) + P_c(T_0;T_{i-1})}{1+R_{on}^{\mathrm{OIS}}(T_0;T_i)\tau_K(T_{i-1};T_i)}. \tag{76}$$

The expressions above may be simplified using the simpler single-curve expression for the OIS rate on the r.h.s. of eq. 74,

$$F_c(T_0;T_{i-1},T_i) = \frac{1}{\tau_{on}(T_{i-1};T_i)} \left\{ \frac{P_c(T_0;T_{i-1})\left[1+R_{on}^{\mathrm{OIS}}(T_0;T_i)\tau_K(T_{i-1};T_i)\right]}{1-R_{on}^{\mathrm{OIS}}(T_0;T_i)A_c(T_0;T_i)} - 1 \right\}, \tag{77}$$

$$P_c(T_0;T_i) = \frac{1-R_{on}^{\mathrm{OIS}}(T_0;T_i)A_c(T_0;T_i)}{1+R_{on}^{\mathrm{OIS}}(T_0;T_i)\tau_K(T_{i-1};T_i)}. \tag{78}$$

4.3.6 Basis Swaps (IRBS)

Interest rate Basis Swaps (IRBS) are OTC contracts in which two counterparties agree to exchange two streams of cash flows in the same currency, tied to two floating Libor rates with different tenors x and y. There are two ways for building IRBS instruments, depending on how the two floating legs are packed together: as a portfolio of two fixed vs floating IRS, or as a single IRS floating vs floating plus spread.

- **IRBS as two IRS**
  In this case the IRBS is a portfolio of two fixed vs floating IRS, payer vs receiver, with identical fixed legs and floating legs indexed to two different Libors. Schedule and coupon payoffs are given, respectively, by

$$\begin{aligned}
\boldsymbol{T}_x &= \left\{T_{x,0}, ..., T_{x,n_x}\right\}, \ x \text{ floating leg schedule,} \\
\boldsymbol{T}_y &= \left\{T_{y,0}, ..., T_{y,n_y}\right\}, \ y \text{ floating leg schedule,} \\
\boldsymbol{S} &= \left\{S_0, ..., S_m\right\}, \ \text{fixed legs schedule,} \\
& \text{with } T_{x,0} = T_{y,0} = S_0, \ T_{x,n_x} = T_{x,n_y} = S_m,
\end{aligned} \tag{79}$$

$$\begin{aligned}
\mathbf{IRBSlet}_{x,\mathrm{float}}(T_{x,i};T_{x,i-1},T_{x,i},L_x) &= N L_x(T_{x,i-1},T_{x,i})\tau_L(T_{x,i-1},T_{x,i}), \ i=1,...,n_x, \\
\mathbf{IRBSlet}_{y,\mathrm{float}}(T_{y,j};T_{y,j-1},T_{y,j},L_y) &= N L_y(T_{y,j-1},T_{y,j})\tau_L(T_{y,j-1},T_{y,j}), \ j=1,...,n_y, \\
\mathbf{IRSlet}_{\mathrm{fix}}(S_k;S_{k-1},S_k,K) &= N K \tau_K(S_{k-1},S_k), \ k=1,...,m,
\end{aligned} \tag{80}$$

  where N is the contract's nominal, $L_x(T_{x,i-1},T_{x,i})$ and $L_y(T_{y,j-1},T_{y,j})$ are the Libors with tenors x and y spanning the coupon periods $(T_{x,i-1},T_{x,i}]$ and $(T_{y,j-1},T_{y,j}]$, respectively, and K is the common fixed rate of the two IRS.

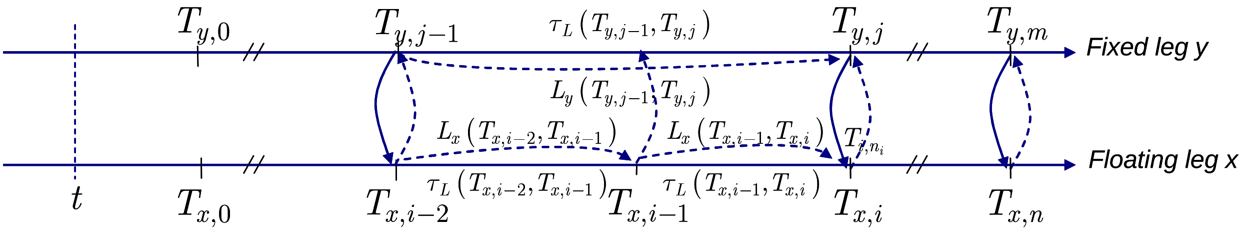


Figure 14: representation of an IRBS contract, in the form of a single IRS with two floating legs indexed to different Libors L_x, L_y with tenors x, y, respectively. In this case, the frequency of the x leg is twice the frequency of the y leg, and they are consistent with their corresponding Libor tenors. The IRBS as two IRS (not represented here) is a portfolio of two IRS as in fig. 8, indexed to Libors L_x and L_y.

The price and the equilibrium basis spread of the IRBS as two IRS are given in app. C.5, eqs. 153, 154, as

$$\mathbf{IRBS}(t; \boldsymbol{T}_x, \boldsymbol{T}_y, \boldsymbol{S}, L_x, L_y, K, \omega) = \mathbf{IRS}_x(t; \boldsymbol{T}_x, \boldsymbol{S}, L_x, K, \omega) - \mathbf{IRS}_y(t; \boldsymbol{T}_y, \boldsymbol{S}, L_y, K, \omega),$$
$$\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y, \boldsymbol{S}) = \frac{\mathbf{IRS}_{x,\text{float}}(t; \boldsymbol{T}_x, L_x) - \mathbf{IRS}_{y,\text{float}}(t; \boldsymbol{T}_y, L_y)}{N A_c(t; \boldsymbol{S})}, \tag{81}$$

where $\mathbf{IRS}_{x,\text{float}}(t; \boldsymbol{T}_x, L_x), \mathbf{IRS}_{y,\text{float}}(t; \boldsymbol{T}_y, L_y)$ are the values of the two IRS floating legs as in eq. 139.

- **IRBS as single IRS**
  In this case the IRBS is a floating vs floating IRS with legs indexed to two different Libors. Schedule and coupon payoffs are given, respectively, by

$$\begin{aligned} \boldsymbol{T}_x &= \{T_{x,0}, ..., T_{x,n_x}\}, \ x \text{ leg schedule}, \\ \boldsymbol{T}_y &= \{T_{y,0}, ..., T_{y,n_y}\}, \ y \text{ leg schedule}, \\ &\text{with } T_{x,0} = T_{y,0}, \ T_{x,n_x} = T_{x,n_y}, \end{aligned} \tag{82}$$

$$\begin{aligned} &\mathbf{IRBSlet}_x(T_{x,i}; T_{x,i-1}, T_{x,i}, L_x) = N L_x(T_{x,i-1}, T_{x,i}) \tau_L(T_{x,i-1}, T_{x,i}), \ i = 1, ..., n_x, \\ &\mathbf{IRBSlet}_y(T_{y,j}; T_{y,j-1}, T_{y,j}, L_y, \Delta_{y,n_y}) \\ &\qquad = N \left[L_y(T_{y,j-1}, T_{y,j}) + \Delta_{y,n_y}\right] \tau_L(T_{y,j-1}, T_{y,j}), \ j = 1, ..., n_y, \end{aligned} \tag{83}$$

where Δ_{y,n_y} in the second leg is a (constant) basis spread on $L_y(T_{y,j-1}, T_{y,j})$ for maturity $T_{x,n_x} = T_{x,n_y}$. The price and the equilibrium basis spread of the IRBS as single IRS are given in app. C.5, eqs. 155, 156,

$$\begin{aligned} \mathbf{IRBS}(t; \boldsymbol{T}_x, \boldsymbol{T}_y, L_x, L_y, \omega, \Delta_{x,y}) = \omega \, [&\mathbf{IRS}_{x,\text{float}}(t; \boldsymbol{T}_x, L_x) - \mathbf{IRS}_{y,\text{float}}(t; \boldsymbol{T}_y, L_y) \\ &- N\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y) A_c(t; \boldsymbol{T}_y)], \end{aligned} \tag{84}$$
$$\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y) := \Delta_{y,n_y} = \frac{\mathbf{IRS}_{x,\text{float}}(t; \boldsymbol{T}_x, L_x) - \mathbf{IRS}_{y,\text{float}}(t; \boldsymbol{T}_y, L_y)}{N A_c(t; \boldsymbol{T}_y)}. \tag{85}$$

In fig. 14 we show a representation of the IRBS contract. A particular case is the OIS-Libor IRBS, where one floating leg is an OIS leg, indexed to compounded over night rate as discussed in sec. 4.3.5.

In both eqs. 81, 85 above, the IRBS spread is positive when the fixings of the first floating rate are higher, on average, than the fixings of the second floating rate on the r.h.s.. The two definitions are financially equivalent, but the two IRBS spreads are slightly different because of the different annuities involved (see eq. 157). The definition of IRBS as two IRS is sometimes preferred because, in this case, the basis rate shares the same rate conventions of the swap rates, while in the definition of IRBS as single IRS the basis rate shares the same conventions of the FRA rate.

The EUR market quotes standard plain vanilla IRBS as two IRS[11] with annual fixed leg vs floating legs indexed to Euribors with different tenors, e.g. 3M vs 6M, 1M vs 6M, 6M vs 12M, etc. Also the 3M vs Eonia IRBS is quoted. In fig. 15 we report three quoted IRBS strips. The quotation convention is to provide the difference (in basis points) between the fixed rate of the higher frequency IRS and the fixed rate of the lower frequency IRS. Normally such difference is positive, reflecting the tenor-dependent counterparty risk included in the underlying rate (e.g. ON instead of 3M, 3M instead of 6M, etc.).

IRBS are a fundamental element for long term multiple-curve bootstrapping, because, starting from the quoted IRS on Libor 6M (e.g. fig. 9), they allow to imply levels for non-quoted IRS on different underlying Libor tenors (e.g. Euribor 1M, 3M, and 12M), to be selected as bootstrapping instruments for the corresponding yield curves construction. If, for example, $\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_{6M}, \boldsymbol{S})$ is the quoted basis spread for an IRBS (as two IRS) receiving Libor xM and paying Libor 6M for maturity T_i, we simply have

$$R_x^{\text{IRS}}(t; \boldsymbol{T}, \boldsymbol{S}) = R_{6M}^{\text{IRS}}(t; \boldsymbol{T}, \boldsymbol{S}) + \Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_{6M}, \boldsymbol{S}), \tag{86}$$

with the obvious caveat that $\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_{6M}, \boldsymbol{S}) = -\Delta(t; \boldsymbol{T}_{6M}, \boldsymbol{T}_x, \boldsymbol{S})$. In fig. 16 we show all the possible basis combinations obtained from figs. 15 and 13. Notice that IRBS in fig. 15 are quoted up to 30 years, while swaps on Euribor6M in fig. 9 are quoted up to 60 years. Thus the bootstrapping of yield curves different from $\mathcal{C}_{6M}$ over 30 years maturity requires extrapolation of basis swap quotations. When the long term shape of the basis curves is smooth and monotonic as in fig. 15, such extrapolation is not particularly critical.

4.4 Synthetic Bootstrapping Instruments

We call "synthetic bootstrapping instruments" any financial instrument that is not directly quoted and traded on the market, but that we are able to build from market instruments with (relatively) simple operations[12]. The reasons for building synthetic bootstrapping instruments are essentially practical: they help to build an yield curve with some desired property. To this purpose, there are at least three types of synthetic financial instruments, described in the following subsections.

[11] the market convention was as single IRS until 2008.

[12] another appropriate name could be "meta-market instruments".

Instrument	Quote (bid, %)	Quote (ask, %)	Quote (mid, %)	Underlying 1st leg	Underlying 2nd leg	Start Date	Maturity Date
1E6E1Y			22.20	Euribor1M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2013
1E6E2Y			22.60	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2014
1E6E3Y			23.80	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2015
1E6E4Y			24.60	Euribor1M	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2016
1E6E5Y			25.00	Euribor1M	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2017
1E6E6Y			25.00	Euribor1M	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2018
1E6E7Y			24.80	Euribor1M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2019
1E6E8Y			24.50	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2020
1E6E9Y			24.10	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2021
1E6E10Y			23.70	Euribor1M	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2022
1E6E11Y			23.30	Euribor1M	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2023
1E6E12Y			22.80	Euribor1M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2024
1E6E15Y			21.10	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2027
1E6E20Y			18.90	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2032
1E6E25Y			17.50	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2037
1E6E30Y			16.30	Euribor1M	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2042
3E6E1Y	14.00	15.00	14.50	Euribor3M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2013
3E6E2Y	13.30	14.30	13.80	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2014
3E6E3Y	13.45	14.45	13.95	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2015
3E6E4Y	13.40	14.40	13.90	Euribor3M	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2016
3E6E5Y	13.45	14.45	13.95	Euribor3M	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2017
3E6E6Y	13.25	14.25	13.75	Euribor3M	Euribor6M	Thu 13 Dec 2012	Thu 13 Dec 2018
3E6E7Y	13.00	14.00	13.50	Euribor3M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2019
3E6E8Y	12.70	13.70	13.20	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2020
3E6E9Y	12.35	13.35	12.85	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2021
3E6E10Y	12.00	13.00	12.50	Euribor3M	Euribor6M	Thu 13 Dec 2012	Tue 13 Dec 2022
3E6E11Y	11.65	12.65	12.15	Euribor3M	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2023
3E6E12Y	11.20	12.20	11.70	Euribor3M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2024
3E6E15Y	9.95	10.95	10.45	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2027
3E6E20Y	8.35	9.35	8.85	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 13 Dec 2032
3E6E25Y	7.30	8.30	7.80	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 14 Dec 2037
3E6E30Y	6.50	7.50	7.00	Euribor3M	Euribor6M	Thu 13 Dec 2012	Mon 15 Dec 2042
3E6E40Y	5.50	6.50	6.00	Euribor3M	Euribor6M	Thu 13 Dec 2012	Fri 13 Dec 2052
3E6E50Y	4.90	5.90	5.40	Euribor3M	Euribor6M	Thu 13 Dec 2012	Wed 13 Dec 2062
6E12E1Y			26.20	Euribor6M	Euribor12M	Thu 13 Dec 2012	Fri 13 Dec 2013
6E12E2Y			20.70	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 15 Dec 2014
6E12E3Y			17.90	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 14 Dec 2015
6E12E4Y			16.40	Euribor6M	Euribor12M	Thu 13 Dec 2012	Tue 13 Dec 2016
6E12E5Y			15.10	Euribor6M	Euribor12M	Thu 13 Dec 2012	Wed 13 Dec 2017
6E12E6Y			13.90	Euribor6M	Euribor12M	Thu 13 Dec 2012	Thu 13 Dec 2018
6E12E7Y			13.00	Euribor6M	Euribor12M	Thu 13 Dec 2012	Fri 13 Dec 2019
6E12E8Y			12.30	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 14 Dec 2020
6E12E9Y			11.80	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 13 Dec 2021
6E12E10Y			11.30	Euribor6M	Euribor12M	Thu 13 Dec 2012	Tue 13 Dec 2022
6E12E11Y			10.90	Euribor6M	Euribor12M	Thu 13 Dec 2012	Wed 13 Dec 2023
6E12E12Y			10.60	Euribor6M	Euribor12M	Thu 13 Dec 2012	Fri 13 Dec 2024
6E12E15Y			9.30	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 13 Dec 2027
6E12E20Y			8.00	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 13 Dec 2032
6E12E25Y			7.20	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 14 Dec 2037
6E12E30Y			6.60	Euribor6M	Euribor12M	Thu 13 Dec 2012	Mon 15 Dec 2042

Figure 15: EUR IRBS. The codes "xEyEnY" in col. 1 label basis swaps receiving Euribor xM and paying Euribor yM plus basis spread with n years maturity. Source: Reuters page ICAPEUROBASIS, as of 11 Dec. 2012.

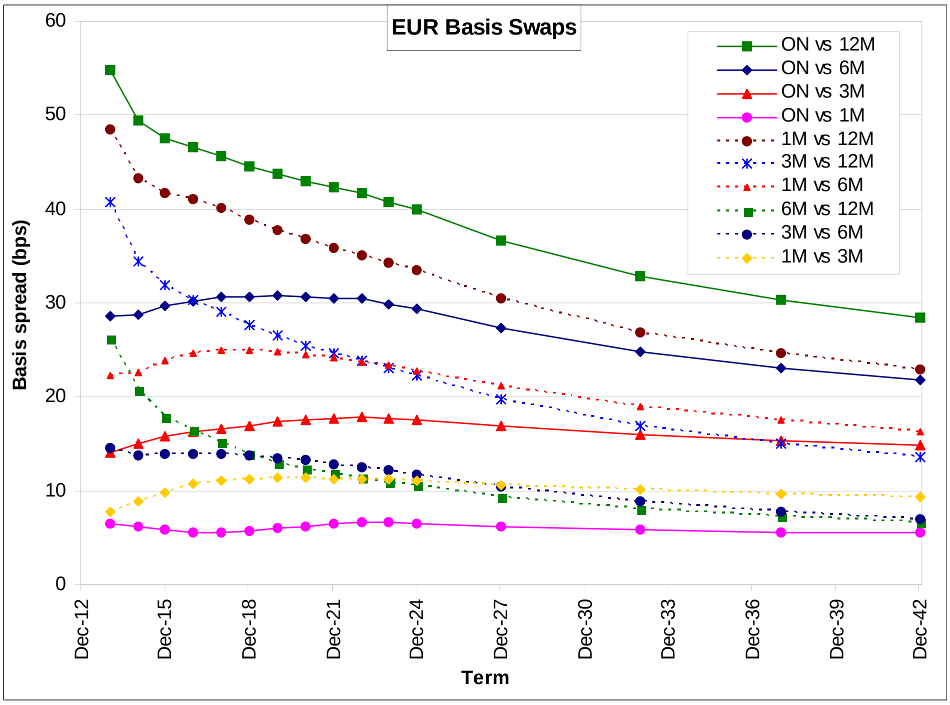


Figure 16: EUR Basis spreads from fig. 15. The spreads not explicitly quoted there have been deduced using eq. (86) and the corresponding IRBS, IRS and OIS quotations. Continuous lines: IRBS Eonia vs Euribor. dashed lines: IRBS Euribor vs Euribor.

4.4.1 Synthetic Interpolated/Extrapolated Instruments

The first and most obvious synthetic financial instruments are obtained by direct interpolation of available market quotes. This is the case, for example, of "in between" Swaps (IRS, OIS, or IRBS), not shown in figs. 9, 10, 13, and 15. These quotations could be inferred from the quoted market pillars using some smooth interpolator, such as monotonic cubic spline (see sec. 4.5). In practice, we build such synthetic IRS, we assign to them the interpolated IRS rates, and use them as input to the bootstrapping procedure. We argue that this is also what some broker does to quote illiquid "in between" IRS below 30Y.

This approach is, obviously, quite rough, but it can be typically useful when we do not want to trust too much our bootstrapping procedure, and we prefer to "fill the gaps" by controlling directly some "in between" bootstrapping pillar. Essentially, we are just shifting the interpolation from the bottom (the bootstrapping procedure, where the interpolator is applied to zero rates, (log)discount factors, or FRA rates), to the top (where the interpolator is applied to market quotes), the only caveat being to keep control of the final result, using the techniques discussed in sec. 4.11. In particular, one must control the smoothness of the yield curve, because non-optimal interpolations of market quotes could generate angles and jumps in FRA rates.

Similarly, we can obtain synthetic bootstrapping instruments by extrapolation of market quotes. A typical case is the extrapolation of IRS quotes after the maximum maturity reported in broker's pages. An example of this extrapolation will be used for yield curves on Euribor1M, 3M and 12M in the 30Y-60Y section, as discussed in the corresponding secs. 5.2, 5.3, and 5.5, respectively. Clearly, extrapolation of market quotes is, in general, much more subjective, and must be handled with care.

4.4.2 Synthetic Deposits

Synthetic Deposits are typically useful for a better coverage of the short end of the yield curve, below the first quoted market pillar (e.g. 6M for yield curve $\mathcal{C}_{6M}$). While the discount curve $\mathcal{C}_{6M}^P$ is naturally hooked to the starting point $P(t_0, t_0) = 1$, the zero and FRA rate curves need a backward extrapolation from the first market pillar to t_0 (e.g. in the window $[t_0, 6M]$ for $\mathcal{C}_{6M}$), which may be inconsistent with other market quotations, typically the OIS-Libor basis. Clearly, we can't use the market Deposits with maturity smaller than the first market pillar for yield curve $\mathcal{C}_x^P$ because they are not based on the same underlying Euribor xM, and we do not want to mix apples and oranges.

The basic idea for building market-consistent synthetic Deposits on Euribor with tenor xM is to use the available market quotes on Euribor xM and the OIS-Euribor xM basis. Starting from the link between zero rates and instantaneous FRA rates in eq. 29 we have

$$\begin{aligned} z_x(T_1, T_2)\tau_z(T_1, T_2) &= \int_{T_1}^{T_2} f_x(T_1, u)\, du \\ &= \int_{T_1}^{T_2} \left[f_{on}(T_1, u) + \Delta_{on,x}(T_1, u)\right] du \\ &= z_{on}(T_1, T_2)\tau_z(T_1, T_2) + \Delta_{on,x}(T_1, T_2). \end{aligned} \tag{87}$$

In case of zero coupon market instruments, such as FRA and OIS, we can introduce the corresponding market quotes

$$R_x^{\text{FRA}}(t_0; T_1, T_2)\tau_x(T_1, T_2) = R_{on}^{\text{OIS}}(t_0; T_1, T_2)\tau_{on}(T_1, T_2) + \Delta_{on,x}(T_1, T_2). \tag{88}$$

The basis in eq. 88 above can be approximated with a polynomial of degree n, such that

$$\Delta_{on,x}(T_1, t) = \alpha_{on,x} + \beta_{on,x}\,(t - T_1) + \gamma_{on,x}\,(t - T_1)^2 + \cdots, \tag{89}$$

such that

$$\begin{aligned} \Delta_{on,x}(T_1, T_2) &= \int_{T_1}^{T_2} \Delta_{on,x}(T_1, u)\, du \\ &= \alpha_{on,x}\,(T_2 - T_1) + \frac{1}{2}\beta_{on,x}\,(T_2 - T_1)^2 + \frac{1}{3}\gamma_{on,x}\,(T_2 - T_1)^3 + \cdots \end{aligned} \tag{90}$$

Eq. 88, with the basis $\Delta_{on,x}(T_1, T_2)$ given by eq. 90, contains n unknowns $\alpha_x, \beta_x, \gamma_x, \cdots$. Using n market quotes on the l.h.s and the over night yield curve $\mathcal{C}_{on}$ to compute the corresponding OIS rates $R_{on}^{\text{OIS}}(t_0; T_1, T_2)$ on the r.h.s. allows to solve for the n unknowns. Once the basis $\Delta_{on,x}(T_1, T_2)$ is known, we can interpolate/extrapolate it to produce any synthetic quote in the interval $[t_0, T_2]$.

A numerical application of the procedure discussed above is reported in fig. 17, where we have computed four strips of synthetic EUR Deposits on Euribor 1M, 3M, 6M, 12M with maturities ranging from 1D to 12M. Since the β coefficients are very small, the constant FRA-OIS basis approximation is sufficient for a good interpolation/extrapolation.

Instrument	Tenor	Start date	Maturity Date	Quote (mid, %)	OIS (%)	FRA-OIS basis (%)	Alpha	Beta
Depo 1M	1M	Thu 13 Dec 2012	Mon 14 Jan 2013	0.1100	0.0740	0.0360		
FRA 1x2	*1M*	*Mon 14 Jan 2013*	*Wed 13 Feb 2013*	*0.1017*	*0.0460*	*0.0557*	*0.0258*	*0.0003*
AB1E2M	1M	Thu 13 Dec 2012	Wed 13 Feb 2013	0.1060	0.0605	0.0455		
FRA Tom3M	3M	Fri 14 Dec 2012	Thu 14 Mar 2013	0.1790	0.0461	0.1329		
Fut 3MZ2	3M	Wed 19 Dec 2012	Tue 19 Mar 2013	0.1775	0.0415	0.1359	0.1148	0.0002
FRA Tom6M	6M	Fri 14 Dec 2012	Fri 14 Jun 2013	0.3120	0.0170	0.2950		
FRA 1x7	6M	Mon 14 Jan 2013	Mon 15 Jul 2013	0.2930	0.0027	0.2903	0.3166	-0.0001
Depo 12M	12M	Thu 13 Dec 2012	Fri 13 Dec 2013	0.5400	0.0042	0.5358		
FRA 12x24	12M	Fri 13 Dec 2013	Mon 15 Dec 2014	0.5070	0.0677	0.4393	0.5839	-0.0001

Instrument	Tenor	Start date	Maturity Date	OIS (%)	1M (%)	3M (%)	6M (%)	12M (%)
Synth. Depo 1D	*1D*	*Thu 13 Dec 2012*	*Fri 14 Dec 2012*	*0.0400*	*0.0261*	*0.1150*	*0.3165*	*0.5838*
Synth. Depo 1W	*1W*	*Thu 13 Dec 2012*	*Thu 20 Dec 2012*	*0.0700*	*0.0280*	*0.1161*	*0.3158*	*0.5830*
Synth. Depo 2W	*2W*	*Thu 13 Dec 2012*	*Thu 27 Dec 2012*	*0.0690*	*0.0303*	*0.1175*	*0.3150*	*0.5821*
Synth. Depo 3W	*3W*	*Thu 13 Dec 2012*	*Thu 03 Jan 2013*	*0.0780*	*0.0325*	*0.1189*	*0.3142*	*0.5811*
Synth. Depo 1M	*1M*	*Thu 13 Dec 2012*	*Mon 14 Jan 2013*	*0.0740*	*0.0360*	*0.1211*	*0.3129*	*0.5797*
Synth. Depo 2M	*2M*	*Thu 13 Dec 2012*	*Wed 13 Feb 2013*	*0.0605*		*0.1270*	*0.3094*	*0.5757*
Synth. Depo 3M	*3M*	*Thu 13 Dec 2012*	*Wed 13 Mar 2013*	*0.0466*		*0.1325*	*0.3061*	*0.5720*
Synth. Depo 4M	*4M*	*Thu 13 Dec 2012*	*Mon 15 Apr 2013*	*0.0320*			*0.3022*	*0.5677*
Synth. Depo 5M	*5M*	*Thu 13 Dec 2012*	*Mon 13 May 2013*	*0.0236*			*0.2989*	*0.5640*
Synth. Depo 6M	*6M*	*Thu 13 Dec 2012*	*Thu 13 Jun 2013*	*0.0172*			*0.2953*	*0.5599*
Synth. Depo 7M	*7M*	*Thu 13 Dec 2012*	*Mon 15 Jul 2013*	*0.0133*				*0.5557*
Synth. Depo 8M	*8M*	*Thu 13 Dec 2012*	*Tue 13 Aug 2013*	*0.0107*				*0.5519*
Synth. Depo 9M	*9M*	*Thu 13 Dec 2012*	*Fri 13 Sep 2013*	*0.0085*				*0.5478*
Synth. Depo 10M	*10M*	*Thu 13 Dec 2012*	*Mon 14 Oct 2013*	*0.0067*				*0.5437*
Synth. Depo 11M	*11M*	*Thu 13 Dec 2012*	*Wed 13 Nov 2013*	*0.0054*				*0.5398*
Synth. Depo 12M	*12M*	*Thu 13 Dec 2012*	*Fri 13 Dec 2013*	*0.0042*				*0.5358*

Figure 17: Construction of synthetic EUR Deposits on Euribor 1M, 3M, 6M, 12M. Top: input instruments and quotes as of 11 Dec. 2012. Coefficients α and β are those of eq. 90. Bottom: output synthetic quotes. Black: market instruments. Black: market instruments. Green italics: synthetic instruments.

Instrument	Value Date	Maturity Date	Quote (mid, %)
Synt Depo 1MD Euribor12M	*Thu 13 Dec 2012*	*Fri 13 Dec 2013*	*0.5400*
FRA 12x24	Fri 13 Dec 2013	Mon 15 Dec 2014	0.5070
IRBS 3Y ON vs 12M	Thu 13 Dec 2012	Mon 14 Dec 2015	0.6030

Instrument	Value Date	Maturity Date	OIS (%)	FRA-OIS basis (%)	FRA (%)
Synt Depo 12M	*Thu 13 Dec 2012*	*Fri 13 Dec 2013*	*0.0011*	*0.5389*	*0.5400*
Synt FRA 1x13	*Mon 14 Jan 2013*	*Tue 14 Jan 2014*	*-0.0052*	*0.5275*	*0.5223*
Synt FRA 2x14	*Wed 13 Feb 2013*	*Thu 13 Feb 2014*	*-0.0088*	*0.5167*	*0.5079*
Synt FRA 3x15	*Wed 13 Mar 2013*	*Thu 13 Mar 2014*	*-0.0091*	*0.5065*	*0.4974*
Synt FRA 4x16	*Mon 15 Apr 2013*	*Tue 15 Apr 2014*	*-0.0064*	*0.4948*	*0.4884*
Synt FRA 5x17	*Mon 13 May 2013*	*Tue 13 May 2014*	*-0.0026*	*0.4851*	*0.4825*
Synt FRA 6x18	*Thu 13 Jun 2013*	*Fri 13 Jun 2014*	*0.0033*	*0.4749*	*0.4783*
Synt FRA 7x19	*Mon 15 Jul 2013*	*Tue 15 Jul 2014*	*0.0116*	*0.4652*	*0.4768*
Synt FRA 8x20	*Tue 13 Aug 2013*	*Wed 13 Aug 2014*	*0.0208*	*0.4572*	*0.4780*
Synt FRA 9x21	*Fri 13 Sep 2013*	*Mon 15 Sep 2014*	*0.0324*	*0.4498*	*0.4822*
Synt FRA 10x22	*Mon 14 Oct 2013*	*Tue 14 Oct 2014*	*0.0440*	*0.4437*	*0.4877*
Synt FRA 11x23	*Wed 13 Nov 2013*	*Thu 13 Nov 2014*	*0.0566*	*0.4392*	*0.4958*
FRA 12x24	Fri 13 Dec 2013	Mon 15 Dec 2014	0.0707	0.4363	0.5070
Synt FRA 13x25	*Mon 13 Jan 2014*	*Tue 13 Jan 2015*	*0.0848*	*0.4343*	*0.5192*
Synt FRA 14x26	*Thu 13 Feb 2014*	*Fri 13 Feb 2015*	*0.1010*	*0.4327*	*0.5337*
Synt FRA 15x27	*Thu 13 Mar 2014*	*Fri 13 Mar 2015*	*0.1165*	*0.4315*	*0.5481*
Synt FRA 16x28	*Mon 14 Apr 2014*	*Tue 14 Apr 2015*	*0.1354*	*0.4304*	*0.5658*
Synt FRA 17x29	*Tue 13 May 2014*	*Wed 13 May 2015*	*0.1532*	*0.4296*	*0.5828*
Synt FRA 18x30	*Fri 13 Jun 2014*	*Mon 15 Jun 2015*	*0.1736*	*0.4290*	*0.6025*
Synt FRA 19x31	*Mon 14 Jul 2014*	*Tue 14 Jul 2015*	*0.1929*	*0.4285*	*0.6214*
Synt FRA 20x32	*Wed 13 Aug 2014*	*Thu 13 Aug 2015*	*0.2131*	*0.4282*	*0.6413*
Synt FRA 21x33	*Mon 15 Sep 2014*	*Tue 15 Sep 2015*	*0.2366*	*0.4280*	*0.6646*
Synt FRA 22x34	*Mon 13 Oct 2014*	*Tue 13 Oct 2015*	*0.2580*	*0.4279*	*0.6859*
Synt FRA 23x35	*Thu 13 Nov 2014*	*Fri 13 Nov 2015*	*0.2833*	*0.4279*	*0.7112*
Synt FRA 24x36	*Mon 15 Dec 2014*	*Tue 15 Dec 2015*	*0.3111*	*0.4279*	*0.7391*

Figure 18: Construction of synthetic EUR FRA 12M. Top: input instruments and quotes as of 11 Dec. 2012. Bottom: output interpolated FRA 12M. Black: market instruments. Green italics: synthetic instruments.

4.4.3 Synthetic FRA

Synthetic FRA on Euribor xM may be constructed using consistently the existing FRA market quotations on Euribor xM, the OIS yield curve $\mathcal{C}_{ON}^{EUR}$, and market quotations for IRBS Eonia vs Euribor xM. In practice, the recipe is simple. We start by focusing the available market pillars for FRA on Euribor xM. Then we compute the corresponding OIS with the same start/maturity dates using the yield curve $\mathcal{C}_{ON}^{EUR}$, and the resulting FRA-OIS basis spreads. Next, we define the desired synthetic pillar dates, and we interpolate the FRA-OIS basis spreads onto such time grid using an appropriate smooth interpolator (see sec. 4.5). Finally, we compute the OIS at the same synthetic pillar dates using the yield curve $\mathcal{C}_{ON}^{EUR}$, and we add the spread above.

An example of this technique will be applied to compute the synthetic FRA 12M used as bootstrapping instruments for yield curve $\mathcal{C}_{12M}^{EUR}$ discussed in sec. 5.5, using the market quotations available for the FRA 12x24, the 3Y IRBS Eonia vs Euribor12M, and the synthetic 1M Deposit (1MD). The results are shown in fig. 18.

4.5 Interpolation

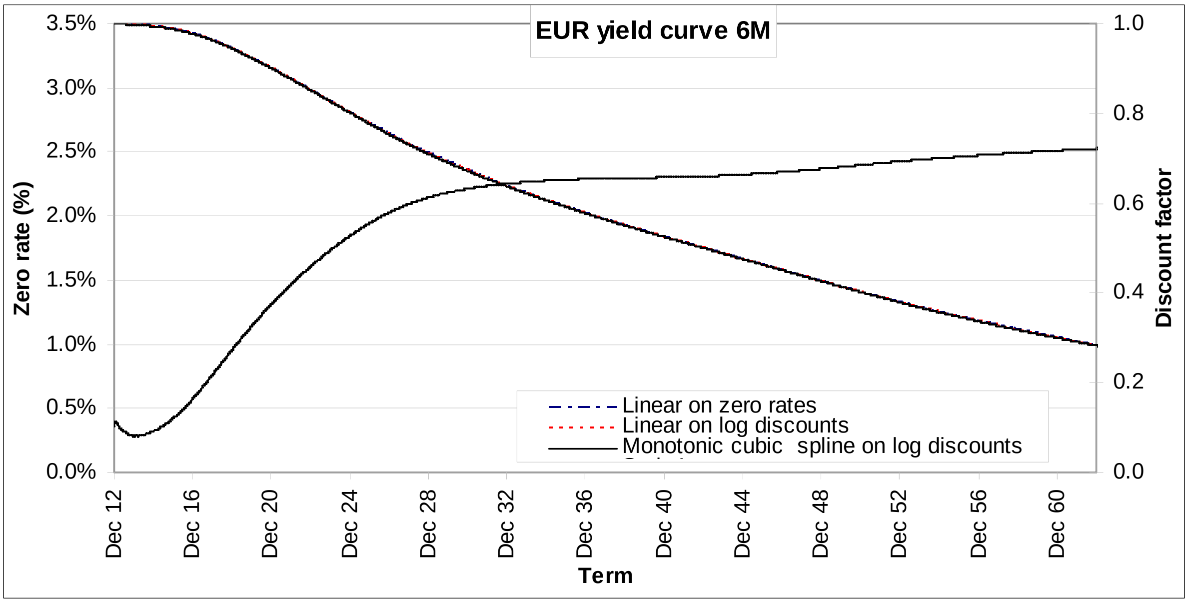
The interpolation scheme adopted for bootstrapping a given parametrization is of paramount importance to obtain reasonable yield curves (see e.g. refs. [12], [13], [16], [14]). For instance, linear interpolation of discount factors is an obvious but extremely poor choice. Linear interpolation of zero rates or log-discounts are popular choices, leading to stable and fast bootstrapping procedures, but unfortunately they produce horrible FRA curves, with a sagsaw or piecewise-constant shape.

We show in fig. 19 some examples of such poor interpolation schemes. Discount zero rate curves (top panel) display similar and smooth behaviors, giving the false impression of a good, interpolation independent, bootstrapping. Simple visual inspection of FRA curves (bottom panel) may reveal non-smooth behaviors, with nasty oscillations up to tens of basis points. Such discontinuities in the FRA curves correspond to angle points in the zero curves, generated by linear interpolation that forces them to suddenly "turn around" a market point. Notice that only the most liquid IRS from fig. 9, with maturities 3-10, 12, 15, 20, 25, 30, 35, 40, 50, 60 years have been included in the bootstrapping of curve $\mathcal{C}_{6M}$. Thus the most irregular sections of the yield curve are those with less dense market data: 13-14, 16-19, 21-24, 26-29, 31-34, 36-39, 41-49, 51-59 years. Often the remaining less liquid quotations for maturity are included in the linear interpolations schemes to reduce the amplitude of the FRA curve oscillations. If the market does not quote an annual time grid, such as for IRS 3M, OIS, and IRBS shown in figs. 10, 13, 15, respectively, the market quotes are often directly interpolated using some smooth algorithm, as discussed in sec. 4.4.

In fig. 19 the monotonic cubic spline interpolation on log-discounts is shown too, clearly ensuring a smooth and financially sound behavior of the FRA curve. This choice is not straightforward, because spline interpolations must be handled with care. Simple splines (see e.g. [49]) suffer of well-documented problems such as spurious inflection points, excessive convexity, and lack of locality after input price perturbations (distributed sensitivities). Recently, Andersen [16] has addressed these issues through the use of shape-preserving splines from the class of generalized tension splines, while Hagan and West [12]-[13] have developed a new scheme based on positive preserving forward interpolation. We found the classic Hyman monotonic cubic filter [48] applied to spline interpolation of log-discounts to be the easiest and best approach: its monotonicity ensures non-negative FRA curves and actually removes most of the unpleasant waviness. Notice that the Hyman filter can be applied to any cubic interpolants: this helps to address the non-locality of spline using alternative more local cubic interpolations. In fig. 20 we show an example of particularly nasty curve taken from Hagan and West [12] (p. 98 and fig. 2, bottom right panel). The FRA curve obtained through Hyman monotonic cubic spline [48] applied on log discounts (bottom panel) is always non negative (there is a unique minimum at 20Y), contrary to the findings in ref. [12].

There are three penalties for using non-local interpolation schemes inside the bootstrapping procedure.

1. During the bootstrapping, the shape of the part of the yield curve already bootstrapped is altered by the addition of further pillars. This is usually remedied by cycling in an iterative fashion: after a first bootstrap, which might even use a local interpolation scheme and build up the pillar grid one point at time, the resulting



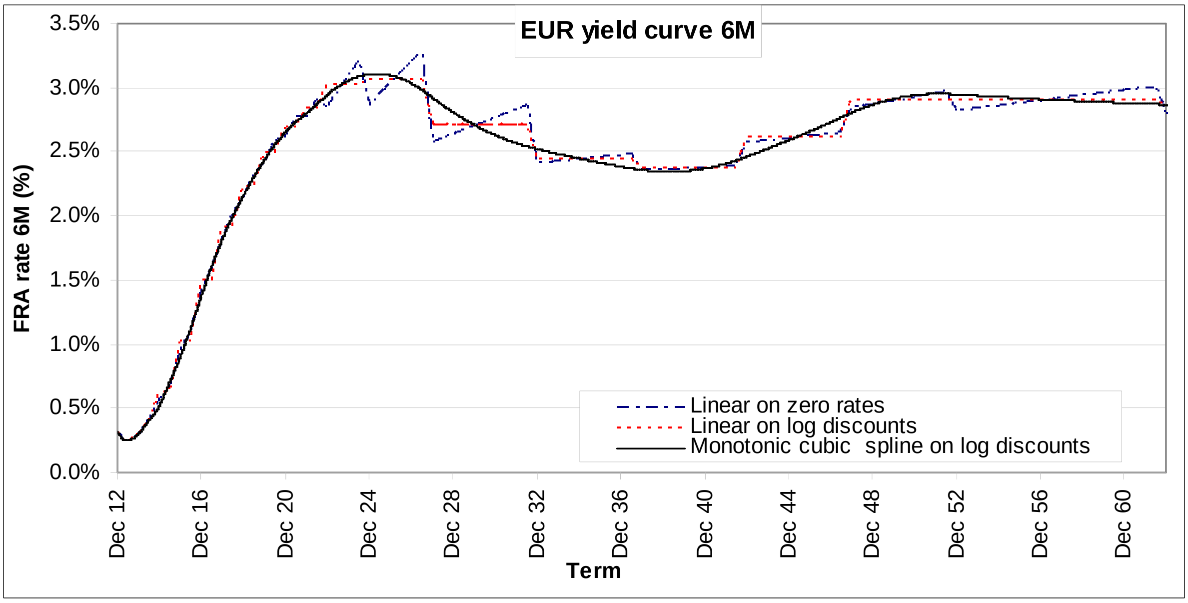


Figure 19: Examples of *bad* (but very popular!) interpolation schemes. Top panel: discount and zero rate curves, $\mathcal{C}^z_{6M}, \mathcal{C}^P_{6M}$, obtained using different interpolation schemes, display very similar and smooth behaviors, giving the false impression of a good, interpolation independent, bootstrapping. Bottom panel: FRA curves $\mathcal{C}^F_{6M}$ may reveal non-smooth behaviors, with nasty oscillations up to tens of basis points. The most affected sections of the yield curve are those with less dense market data. The monotonic cubic spline interpolation on log-discounts (continuous black line) appears to be a better approach.

Term	Zero rate	Capitalization factor	Discount factor	Log Discount factor	Discrete forward	FRA
0.0	0.00%	1.000000	1.000000	0.000000		
0.1	8.10%	1.008133	0.991933	0.008100	8.1000%	8.1329%
1.0	7.00%	1.072508	0.932394	0.070000	6.8778%	7.0951%
4.0	4.40%	1.192438	0.838618	0.176000	3.5333%	3.7274%
9.0	7.00%	1.877611	0.532592	0.630000	9.0800%	11.4920%
20.0	4.00%	2.225541	0.449329	0.800000	1.5455%	1.6846%
30.0	3.00%	2.459603	0.406570	0.900000	1.0000%	1.0517%

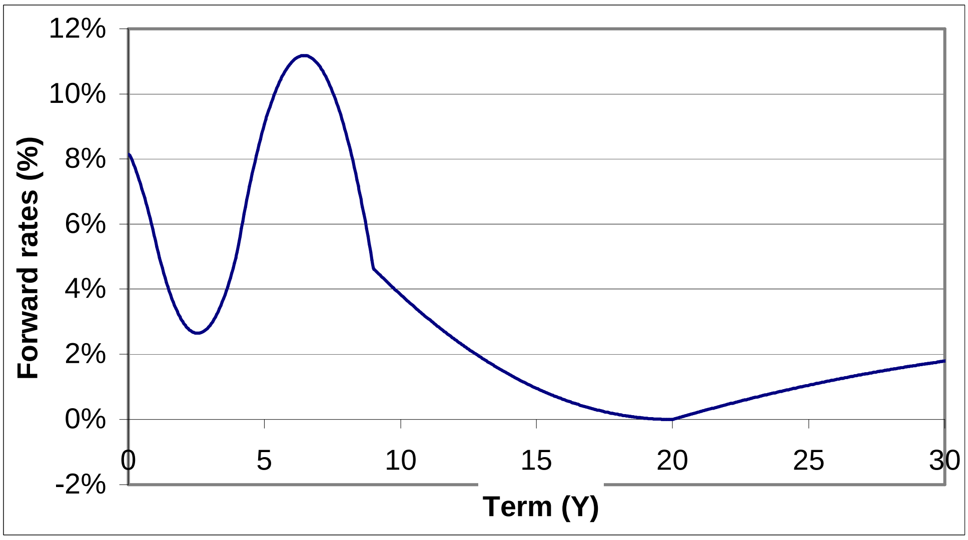


Figure 20: Example of nasty curve taken from ref. [12] (p. 98 and fig. 2, bottom right panel). Top panel: the example curve. Bottom panel: the FRA curve obtained through Hyman monotonic cubic spline [48] applied on log-discounts. The FRA rate at 20Y is null but no negative rates appear, contrary to the findings in ref. [12].

complete grid is altered one pillar at time using again the same bootstrapping algorithm, until convergence is reached within a given precision. The first cycle can be even replaced by a good grid guess, the most natural one being just the grid previous state in a dynamically changing environment.

2. The non-local interpolation implies a non-local delta sensitivity, such that the shock of, e.g., the 10Y market quote produces a delta sensitivity peaked at the 10Y pillar but distributed onto the surrounding pillars (see sec. 4.9).

3. The computational performance of bootstrapping algorithm is lowered, with the most important effects observed during the delta sensitivity calculation.

We stress that the focus on smooth discrete FRA rate is the key point of state-of-the-art bootstrapping. For even the best interpolation schemes to be effective the FRA rate curve must be smooth, i.e. any jump must be removed, and added back only at the end of the smooth curve construction. The most relevant jump in FRA rates is the so-called turn of year effect, discussed in sec 4.8.

4.6 Negative Rates

In July 2012 the European Central Bank (ECB) cut the main policy rate to 0.75% and set to 0% the interest rate charged on the deposit facility[13]. In the following months, growing expectations the ECB will lower interest rates and even set negative rates for its deposit facility started to be reflected in market interest rates. The 1Y Deposit rate in EUR wholesale markets dipped below zero after the ECB downgraded steeply its 2013 eurozone growth forecasts on 6th Dec. 2012 (see e.g. ref. [50]). In particular, on Dec. 11th, 2012, the market quoted very low negative mid OIS rates for forward starting OIS on ECB dates (see fig. 13). A market for interest rate options (Caps, Floors, Swaptions) with null or negative strikes had already developed in the preceding months, for banks willing to mark their short floors position to market, and to hedge, or capitalise on, increasingly low rates [51].

Negative market rates poses new challenges to interest rate derivatives traders, interested into valuing FRA, Swaps and options including negative FRA rates and strikes. Clearly, these pricing challenges are hopeless without an accurate yield curve bootstrapping, able to consistently capture and reflect such extreme market conditions. In fig. 21 we show the short-end section of the OIS curve bootstrapped on Dec. 11th, 2012. Negative FRA rates appear in the 3M-12M window, with a minimum of -1.8 bps in the second week of Aug. 2013. The corresponding discount factor (zero coupon bond) curve (right scale) displays a non-monotonic behavior in the same time window.

We stress that this particular market configuration is rather exceptional, but our bootstrapping framework is robust enough to work as usual.

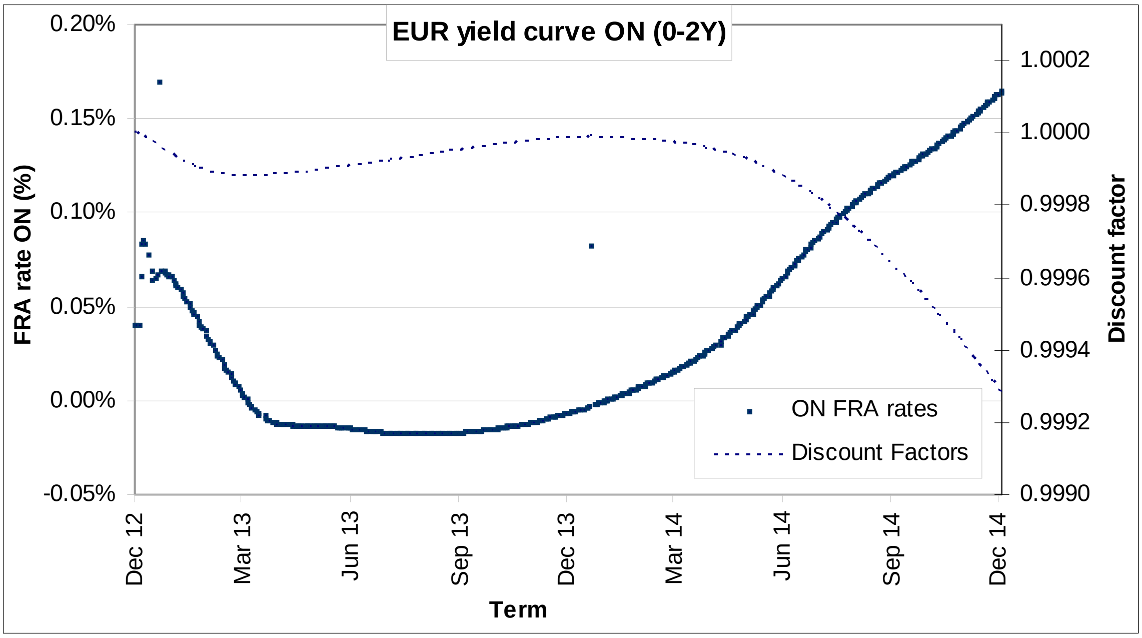


Figure 21: Short term FRA curve $\mathcal{C}_{ON}^{f}$ on Eonia as of 11 Dec. 2012 (from fig. 26, top panel). Left scale: ON-tenor FRA rates $F_{ON}(t_0; t, t+1d, act/360)$, t daily sampled. The two outlier dots are the turn of the year jumps discussed in sec. 4.8. Right scale: discount factors $P_{1D}(t_0; t, act/360)$. Negative FRA rates up to -2 bps and increasing discount factors are visible in the 3M-12M window. See comments in the text.

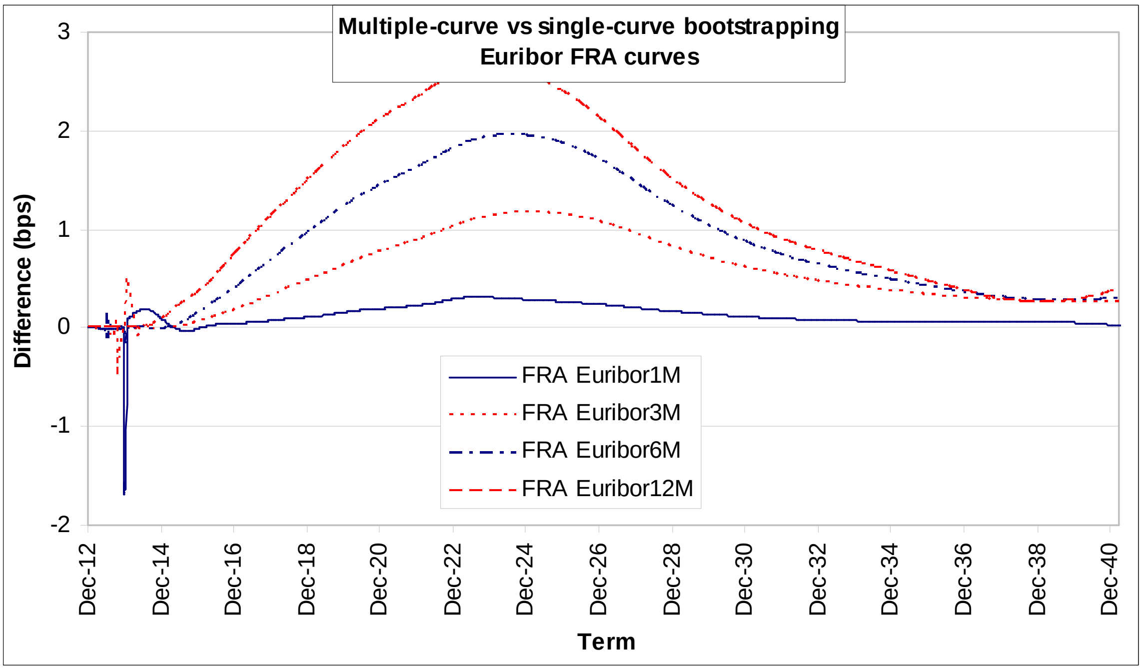
4.7 Endogenous vs Exogenous Yield Curve Bootstrapping

Bootstrapping an yield curve $\mathcal{C}_x$ is said to be *endogenous*, when the discount factors entering into the pricing formulas of bootstrapping instruments are taken from the curve $\mathcal{C}_x^P$ itself, as in the classical single-curve framework. Instead, when the discount factors are taken consistently from a (predetermined) discounting curve $\mathcal{C}_d^P$, the bootstrapping is said to be *exogenous*. For example, IRS single-curve bootstrapping is based on IRS pricing formulas, eqs. 61, 62, 63, with indexes $c = x$. We stress that the exogenous multiple-curve bootstrapping generates a multiple-curve delta sensitivity, as discussed in sec. 4.9.

In fig. 22 we show the effect of exogenous vs endogenous bootstrapping on Euribor FRA curves (top panel) and Swap Euribor6M curves (bottom panel). Overall, the difference in FRA or IRS rates is small but non-negligible, ranging from zero to a couple of basis points.

Clearly, only the exogenous bootstrapping is consistent with the modern multiple-curve pricing framework discussed in the previous sections 4.3.2–4.3.6. On the other side, we stress that the pricing effects of endogenous bootstrapping are rather subtle: using the *wrong* discount factors in the (exact fit) bootstrapping procedure implies the production of *wrong* FRA curves, such that the *right* market quotations (FRA, IRS, IRBS) are reproduced. Only non-bootstrapping instruments are affected, with deviations of the

[13]The ECB's deposit facility is used by European banks for over night funding and provides a floor for market interest rates.



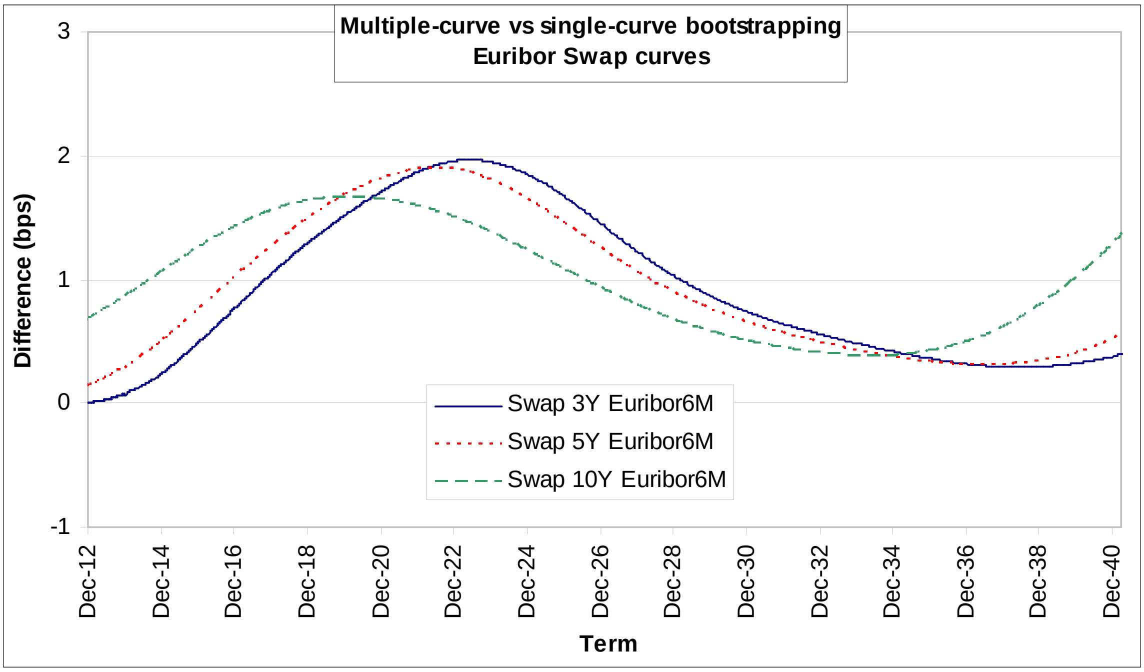


Figure 22: Effect of exogenous bootstrapping. Top panel: difference (in basis points) between exogenous and endogenous bootstrapping for FRA Euribor curves. Each FRA rate has a tenor consistent with the underlying Euribor tenor. The spikes in the 1M and 3M curves are due to turn of the years jumps (see sec. 4.8) included in the exogenous bootstrapping and excluded in the endogenous bootstrapping. Bottom panel: effect on Swap curves.

order of the (non-negligible) differences observed in fig. 22. Nevertheless, such approach is still used in old-style analytics libraries and softwares, sometimes unconsciously.

4.8 Turn of Year Effect

In the interest rate market, the turn of year effect is a jump that may be observed in market quotations and in historical series of rates spanning across technical dates, typically the end of the year. In fig. 23 we display the historical series of Euribor1M fixings in the window Oct. 2007 - Feb 2009. The 2007 turn of year jump (64 bps) is clearly visible on 29th Nov. 2007 (left rectangle), just when the spot starting 1M tenor rate spans the end of 2007, with rates reverting toward the previous levels one month later. The 2008 turn of year jump on 27th Nov. 2008 (22 bps, right oval) is partially hidden by the high market descent realized in that period. Viceversa, during flat market regimes even the much smaller "end of semester effect"may be observable, as seen on 29th of May 2008 (9 bps, middle rhombus). In other periods, not shown in the figure, the jumps have been hardly observed. From a financial point of view, these jumps are caused by a possible excess of liquidity demand from financial institutions in correspondence of technical deadlines for liquidity management with central banks.

Turn of the year jumps, realised or not at the jump date, are preceded by corresponding jumps into the quotations of some market instruments, that reflect the market expectations before the jump date. Looking at EUR market quotations, the larger jump may (or may not) be observed the last working day of the year (e.g. 31th December) for the quote of the Overnight Deposit (OND) maturing the first working day of the next year (e.g. 2nd January). The Tomorrow Next and Spot Next Deposits (TND and SND, respectively) may jump one and two business days before, respectively (e.g. 30th and 29th December). Other instruments with longer underlying rate tenors may display smaller jumps when their maturity crosses the same border: for instance, the 1M Deposit (1MD) quotation jumps 2 business days before the 1st business day of December; the 12M Deposits always include a jump except 2 business days before the end of the year (due to the end of month rule); the December IMM Futures always include a jump, as well as the October and November serial Futures; 2Y Swaps always include two jumps; etc. The effect may be observable at the first two ends of year and becomes negligible at the following crosses. The jump in market quotes may also be observable in advance, e.g. in 1MD quotations at end of November, and not observable later in shorter quotations, e.g. SWD, SND, TND and OND, because the market expectation of the jump dissolves approaching the jump date.

The decreasing jump with increasing underlying rate tenor can be easily understood once we distinguish between jumping rates and non-jumping rates. For instance, we may think to the 1M Deposit as a weighted average of 22 (business days in one month) over night rates (plus a basis). If such Deposit spans an end of year, there must be a single over night rate, weighting 1/22th, that crosses that end of year and displays the jump, while the others do not. Considering rates with longer tenors, there are still single jumping over night rates, but with smaller weights. Hence longer Deposits/FRA display smaller jumps. The same holds for Swaps, as portfolios of Depos/FRA.

Since turn of year jumps are included into market quotations, and may also realise in the historical fixings, a sophisticated trader or treasurer may wish to include them into

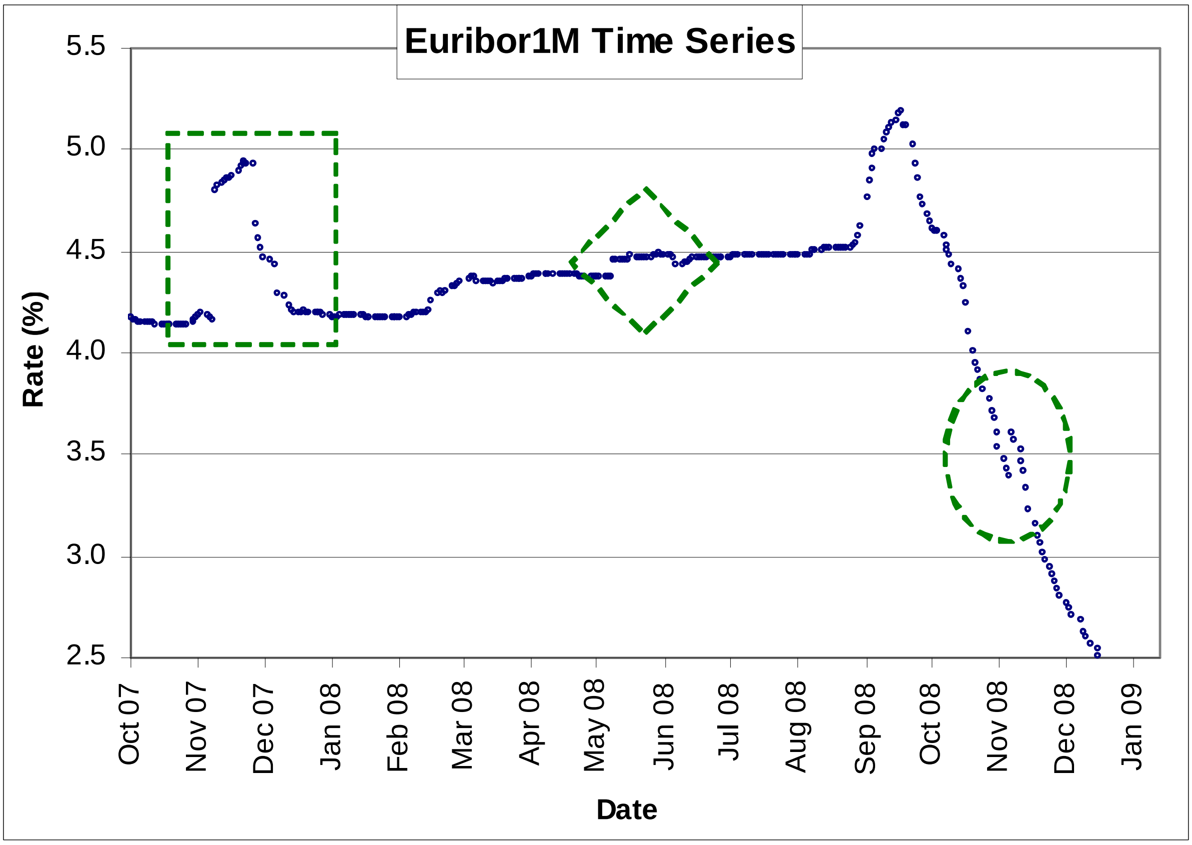


Figure 23: Turn of year effect on Euribor1M. The historical time series in the Oct. 2007 - Feb 2009 window is displayed. Three jumps can be identified: the 2007 turn of year (29th Nov. 2007, 64 bps, left rectangle); the 2008 turn of year (27th Nov. 2008, 22 bps, right oval); a smaller "end of semester effect"(29th May 2008, 9 bps, middle rhombus). Source: Reuters.

the yield curve bootstrapping. An yield curve term structure up to N years including the turn of year effect should contain, in principle, N discontinuities; in practice only the first two jumps expected at the end of the current and of the next year can be taken into account. The jump effect can be modeled simply through a multiplicative coefficient applied to discount factors, or, equivalently, an additive coefficient applied to zero rates, corresponding to all the dates of the yield curve affected by a possible jump date. In this way we are allowed to estimate the coefficient using instruments with a given underlying rate tenor (e.g. those on Euribor3M used for $\mathcal{C}_{3M}$), and to apply it to any other curve $\mathcal{C}_x$ taking into account the proper weights. Notice that, as stressed in the previous section, starting from a smooth and continuous yield curve is crucial for correctly take into account the discontinuity at the turn of year.

The jump coefficient can be estimated from market quotations using different approaches, discussed below.

- Jump in the Futures 3M strip: the (no-jump) end of year crossing FRA rate is obtained through interpolation of non-crossing FRA rates; the rate jump coefficient is given by the difference between the latter and the quoted value. This approach always allows the estimation of the second turn of year. The first turn of year can be obtained only up to the third Wednesday of September, when the corresponding Futures expiries. In the period October-December there are no non-crossing Futures to interpolate and the first turn of year should be extrapolated from the second, making this method not very robust.

- Jump in the FRA 6M strip: this is equivalent to the approach above but it allows the estimation of the first turn of year up to June (included).

- Jump in the IRS 1M strip: this is equivalent to the approaches above and it allows the estimation of the first turn of year up to November (included).

- Jump in the FRA strip quoted by brokers each Monday: this approach is valid all year long, but it allows only a discontinuous weekly update, and is more brokers' dependent.

The empirical approaches above, when available at the same time, normally give estimates in excellent agreement with each other. A numerical example is given in fig. 24, where we collect the short term bootstrapping of FRA curves $\mathcal{C}^f_{ON}, \mathcal{C}^f_{1M}$, and $\mathcal{C}^f_{3M}$ recovered from figs. 26, 28, and 30 in section 5. The $\mathcal{C}^f_{ON}$ yield curve displays both the 2013 (10.2 bps) and the 2014 (8.5 bps) turn of year jumps. The $\mathcal{C}^f_{1M}$ yield curve displays the 2014 turn of year jump between 1st Dec. 2013 (+1.8 bps) and 2nd Jan. 2014 (−1.6 bps) with size roughly equal to 1/20 of the ON jumps. The $\mathcal{C}^f_{3M}$ yield curve displays the 2014 turn of year jump between 1st Oct. 2013 (+0.6 bps) and 2nd Jan. 2014 (−0.5 bps), with size roughly equal to 1/3 of the 1M jump. The jumps are also observable in the zero rate curves $\mathcal{C}^z_{ON}, \mathcal{C}^z_{1M}$, and $\mathcal{C}^z_{3M}$ (not shown in fig. 24). We stress that a single turn of the year induces *one* discontinuity in the zero rate and discount curves, and *two* discontinuities in the FRA rate curve (the FRA rate depending on a ratio of two discounts).

The yield curve discontinuities induced by the turn of year effect may appear, to a non market-driven reader, a fuzzy effect broking the desired yield curve smoothness. On the contrary, we stress that they are neither a strangeness of the market quotations nor an

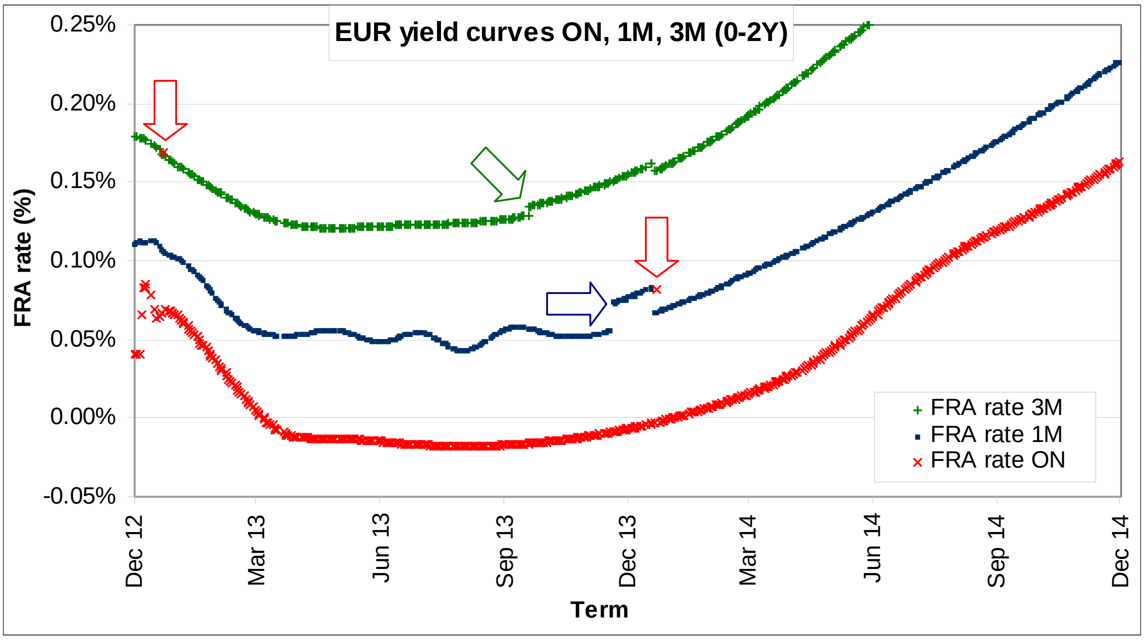


Figure 24: Yield curves $\mathcal{C}_{ON}^F$ (red crosses, bottom), $\mathcal{C}_{1M}^F$ (blue dots, middle) and $\mathcal{C}_{3M}^F$ (green crosses, top), as of 11 Dec. 2012 (details from figs. 26 and 28, and 30, top panels). The the turn of year jumps are indicated by arrows: vertical red for ON, horizontal blue for 1M, and oblique green for 3M.

accident of the bootstrapping, but correspond to true and detectable financial effects that should be included in any yield curve used to mark to market interest rate derivatives.

4.9 Multiple Curves, Multiple Deltas, Multiple Hedging

Hedging in the modern multiple curve world is, not surprisingly, much more complicated. Essentially, we have expanded our set of bootstrapping instruments, with the natural consequence of an expanded set of delta sensitivities and of hedging instruments.

4.9.1 Delta Sensitivity

Let's consider a general portfolio Π of interest rate derivatives depending on multiple yield curves $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$, characterised by time grids $\boldsymbol{T} = \{\boldsymbol{T}_1, \ldots, \boldsymbol{T}_n\}$ and bootstrapping instruments with market quotes $\boldsymbol{R} = \{\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n\}$, where $\boldsymbol{T}_j$ and $\boldsymbol{R}_j$ are the sub-vectors of m_j dates (pillars) and market quotes for yield curve $\mathcal{C}_j$, and $T_{j,k}$, $R_{j,k}$ their elements, respectively. We denote, in this context, the price of the portfolio Π at time t by $\Pi(t; \boldsymbol{R})$.

The delta sensitivity of Π is given by

$$\begin{aligned}
\Delta^{\Pi}(t;\boldsymbol{R}) &= \sum_{j=1}^{n} \Delta_j^{\Pi}(t;\boldsymbol{R}), \quad \text{total delta sensitivity,} \\
\Delta_j^{\Pi}(t;\boldsymbol{R}) &= \sum_{k=1}^{m_j} \Delta_{j,k}^{\Pi}(t;\boldsymbol{R}), \quad \text{partial delta sensitivity for curve } \mathcal{C}_j, \\
\Delta_{j,k}^{\Pi}(t;\boldsymbol{R}) &:= \frac{\partial \Pi}{\partial R_{j,k}}, \quad \text{partial delta sensitivity for pillar } T_{j,k}.
\end{aligned} \tag{91}$$

Actually the value of Π depends indirectly on the bootstrapping market rates $\boldsymbol{R}$ and pillars $\boldsymbol{T}$ through discount factors and FRA rates appearing in the corresponding pricing formulas[14]. Since FRA rates can be written in terms of their associated discount factors in the sense discussed in sec. 3.4, and since discount factors can be written in terms of their corresponding zero coupon rates (see sec. 3.5), we may think, in general, that Π depends directly on a vector of discount factors $\boldsymbol{P} = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n\}$, with $P_{j,k} := P(t;\boldsymbol{R}_j)$, or zero rates $\boldsymbol{z} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$, with $z_{j,k} := z(t;\boldsymbol{R}_j)$. Notice that we take into account possible non-local effects in bootstrapping, such that a single discount factor or zero rate may depend on more than one single component in $\boldsymbol{R}_j$. Since zero rates and market rates are similar quantities, it's common practice to consider zero rates as intermediate variables. In this case the delta sensitivity in eqs. 91 may be written as

$$\Delta_{j,k}^{\Pi}(t;\boldsymbol{R}) = \sum_{\alpha=1}^{m_j} J_{j,k,a} \frac{\partial \Pi}{\partial z_{j,\alpha}}, \tag{92}$$

$$J_{j,k,a} := \frac{\partial z_{j,\alpha}}{\partial R_{j,k}}, \tag{93}$$

or, in compact matrix notation

$$\begin{aligned}
\boldsymbol{\Delta}_j^{\Pi}(t;\boldsymbol{R}) &= \boldsymbol{J}_j \cdot \boldsymbol{\nabla}_j \Pi, \\
\boldsymbol{J}_j := \begin{bmatrix} \frac{\partial z_{j,1}}{\partial R_{j,1}} & \cdots & \frac{\partial z_{j,m_j}}{\partial R_{j,1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_{j,1}}{\partial R_{j,m_j}} & \cdots & \frac{\partial z_{j,m_j}}{\partial R_{j,m_j}} \end{bmatrix}&, \quad \boldsymbol{\nabla}_j := \begin{bmatrix} \frac{\partial}{\partial z_{j,1}} \\ \vdots \\ \frac{\partial}{\partial z_{j,m_j}} \end{bmatrix},
\end{aligned} \tag{94}$$

where $\boldsymbol{J}_j$ and $\boldsymbol{\nabla}_j$ are the jacobian matrix and the gradient operator for yield curve $\mathcal{C}_j$, respectively. Notice that the sum in eq. 92 above may include a few terms surrounding $\frac{\partial z_{j,k}}{\partial R_{j,k}}$ (with index $\alpha \simeq k$), depending on the degree of non-locality of the bootstrapping methodology adopted (see sec. 4.5). In case of local bootstrapping the Jacobian is a diagonal matrix and we have

$$\begin{aligned}
J_{j,k,\alpha} &= \delta_{\alpha,k} \frac{\partial z_{j,k}}{\partial R_{j,k}}, \\
\Delta_{j,k}^{\Pi}(t;\boldsymbol{R}) &= \frac{\partial \Pi}{\partial z_{j,k}} \frac{\partial z_{j,k}}{\partial R_{j,k}},
\end{aligned} \tag{95}$$

[14]this is true also in case of financial instrument more complex than the plain vanillas discussed in secs. 4.3.2-4.3.6.

where $\delta_{\alpha,k}$ is the Kronecker's symbol[15].

Finally, we take into account the multiple-curve (exogenous) bootstrapping (see sec. 4.2), such that the forwarding yield curves $\mathcal{C}_x$ depend on the discounting yield curve $\mathcal{C}_{ON}$. Assuming that $\mathcal{C}_{ON} = \mathcal{C}_1$ in the notation above, we have the Jacobian

$$J_{j,k,\alpha} = \delta_{j,1} \frac{\partial z_{j,\alpha}}{\partial R_{j,k}} + (1 - \delta_{j,1}) \left[\frac{\partial z_{j,\alpha}}{\partial R_{j,k}} + \frac{\partial z_{j,\alpha}}{\partial R_{1,k}} \right]. \tag{96}$$

In case of single-curve (endogenous) bootstrapping we have $\frac{\partial z_{j,\alpha}}{\partial R_{1,k}} = 0$ and eq. 96 reduces to eq. 93.

We observe that the computation of the delta sensitivity in eqs. 92, 94, is factorised into the computation of the Jacobian $\boldsymbol{J}$ and of the gradient $\boldsymbol{\nabla}\Pi$. The computation of the gradient depends on the possible complexity of the portfolio Π. Instead, the computation of the Jacobian depends on yield curve bootstrapping, following the logical chain: shock $R_{j,k}$ $\rightarrow$ bootstrap $\left\{\mathcal{C}_1, \ldots, \mathcal{C}_{m_j}\right\}$ $\rightarrow$ compute $J_{j,k,\alpha}$. Hence, once the yield curves structure has been fixed (pillars, bootstrapping instruments, etc.), the Jacobian $\boldsymbol{J}$ is relatively stable with respect to first order movements of the market rates $\boldsymbol{R}$, and can be recomputed less frequently than the gradient $\boldsymbol{\nabla}\Pi$.

4.9.2 Delta Hedging

Once the delta sensitivity of portfolio Π has been computed, we want to delta hedge Π by trading appropriate amounts $\boldsymbol{H} = \{\boldsymbol{H}_1, \ldots, \boldsymbol{H}_n\}$ of hedging instruments $\boldsymbol{\pi}^H = \left\{\boldsymbol{\pi}_1^H, \ldots, \boldsymbol{\pi}_n^H\right\}$ (with unit nominal amount), where $\boldsymbol{H}_j, \boldsymbol{\pi}_j^H$ are the sub-vectors of amounts and hedging instruments for yield curve $\mathcal{C}_j$, and $H_{j,h}, \pi_{j,h}^H$ their elements, respectively, such that the delta sensitivity of the total portfolio

$$\begin{aligned} \Pi^{\text{Tot}}(t, \boldsymbol{R}^H) &= \Pi(t, \boldsymbol{R}^H) + \Pi^H(t, \boldsymbol{R}^H), \\ \Pi^H(t, \boldsymbol{R}^H) &= \sum_{j=1}^{n} \sum_{h=1}^{h_j} H_{j,h}(t, \boldsymbol{R}^H) \pi_{j,h}^H(t, \boldsymbol{R}^H), \end{aligned} \tag{97}$$

is null. Typically the set of hedging instruments for yield curve $\mathcal{C}_j$ is a subset of the most liquid bootstrapping instruments of $\mathcal{C}_j$, with market quotes denoted by $\boldsymbol{R}^H = \left\{\boldsymbol{R}_1^H, \ldots, \boldsymbol{R}_n^H\right\} \subseteq \boldsymbol{R} = \{\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n\}$, with $\sum_{j=1}^n h_j \leq \sum_{j=1}^n m_j$. The hedging instruments define an hedging time grid $\boldsymbol{T}^H$ such that $\boldsymbol{T}^H = \left\{\boldsymbol{R}_1^H, \ldots, \boldsymbol{T}_n^H\right\} \subseteq \boldsymbol{T} = \{\boldsymbol{T}_1, \ldots, \boldsymbol{T}_n\}$. The selection of the hedging instruments is clearly subjective, more an art of the interest rate trader than a science.

In practice, the delta sensitivity of Π is re-distributed on the hedging time grid $\boldsymbol{T}^H$,

$$\begin{aligned} \Delta^\Pi(t; \boldsymbol{R}) &= \Delta^\Pi(t; \boldsymbol{R}^H) = \sum_{j=1}^{n} \Delta_j^\Pi(t; \boldsymbol{R}^H), \quad \text{total delta sensitivity,} \\ \Delta_j^\Pi(t; \boldsymbol{R}) &= \sum_{h=1}^{h_j} \Delta_{j,h}^\Pi(t; \boldsymbol{R}^H), \quad \text{partial delta sensitivity for curve } \mathcal{C}_j, \\ \Delta_{j,h}^\Pi(t; \boldsymbol{R}^H) &:= \frac{\partial \Pi}{\partial R_{j,h}^H}, \quad \text{partial delta sensitivity for pillar } T_{j,h}. \end{aligned} \tag{98}$$

[15] such that $\delta_{\alpha,k} = 1$ if $\alpha = k$ and 0 otherwise.

Then, the hedge amount (or hedge ratios) are computed as

$$
\begin{aligned}
H_{j,h}(t, \boldsymbol{R}^H) &= -\frac{\Delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H)}{\delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H)}, \\
\delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H) &:= \frac{\partial \pi^H_{j,h}}{\partial R^H_{j,h}},
\end{aligned} \tag{99}
$$

such that the total portfolio satisfies the zero delta condition,

$$
\begin{aligned}
\Delta^{\mathrm{Tot}}(t; \boldsymbol{R}^H) &= \Delta^{\Pi}(t; \boldsymbol{R}^H) + \Delta^{H}(t; \boldsymbol{R}^H) \\
&= \sum_{j=1}^{n} \sum_{h=1}^{h_j} \Delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H) + \sum_{j=1}^{n} \sum_{h=1}^{h_j} H_{j,h}(t, \boldsymbol{R}^H) \frac{\partial \pi^H_{j,h}}{\partial R^H_{j,h}} \\
&= \sum_{j=1}^{n} \sum_{h=1}^{h_j} \left[\Delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H) + H_{j,h}(t, \boldsymbol{R}^H) \delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H) \right] = 0,
\end{aligned} \tag{100}
$$

where in the second line we have computed the delta sensitivity $\Delta^H(t; \boldsymbol{R}^H)$ of the hedge portfolio Π^H assuming static hedge ratios, such that

$$
\frac{\partial H_{j,h}(t, \boldsymbol{R}^H)}{\partial R^H_{j,h}} \simeq 0. \tag{101}
$$

The computation of delta sensitivity components $\Delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H)$ and $\delta^{\Pi}_{j,h}(t; \boldsymbol{R}^H)$ in eq. 100 above can be performed using the decomposition in eq. 92 with the Jacobian as in eq. 96.

4.10 Performance

A good computational performance of yield curve bootstrapping algorithms is a key feature of real-time applications in liquid markets. This is particularly important when delta sensitivities are required. The bootstrapping algorithms discussed in the previous sections may display several bottlenecks, as discussed below.

- Analytical vs numerical interpolation (see sec. 4.5: while simple interpolation schemes may use analytical formulas (e.g. linear interpolation), more general schemes require a solver or zero-finding numerical algorithm. Hence, efficient algorithms (e.g. Newton-Raphson [49]) and careful choice of their parameters (e.g. initial guess and convergence criteria) are required for fast interpolation.

- Non-local interpolation: non-local schemes, such as splines, require an iterative algorithm to converge to the final solution. Again careful control of numerical parameters is required for fast convergence.

- Exogenous bootstrapping (see sec. 4.7: this bootstrapping scheme implies a cross-dependence of the forwarding yield curves $\mathcal{C}_x$ on the discounting yield curve $\mathcal{C}_{on}$. Hence, a market change in a single OIS quotation triggers the recalculation of all the dependent yield curves. Since this dependence may be often very small, the cross recalculation may be disabled.

- Delta sensitivity (see sec. 91): this is clearly the most expensive computation, since each single pillar shock triggers the recalculation of the entire yield curve (or of the whole set of yield curves, in case of OIS shock and exogenous bootstrapping), and the revaluation of the corresponding portfolio of derivatives. This huge task may be made more efficient by pre-computing and storing the Jacobian in eq. 96.

We conclude that a good design and careful test of the basic yield curve bootstrapping component will affect the overall performance of any pricing system.

4.11 Yield Curves Checks

Once the desired set of yield curves has been constructed, one wish to check and monitor their correctness. Obviously the bootstrapping algorithm must be carefully debugged, but even a correct implementation requires a constant monitoring against possible market changes and failures of the basic yield curve construction hypothesis. A typical example of market change is the switch to (exogenous) OIS discounting in 2010 (see e.g. [52], [53]).

There are essentially four methods to check and monitor yield curves.

1. Market knowledge: clearly, any good yield curve assumes there is a good trader behind, with real-time knowledge of the corresponding market bootstrapping instruments. She is the best check that anyone can set up.

2. Visual inspection: the bare shape of the yield curve is a good sentinel of possible bootstrapping problems. All the yield curve typologies (discount, zero/FRA/instantaneous FRA rates) must be monitored all together, since they convey different and complementary information. Typical examples are reported in figs. 19 and 35.

3. Repricing of bootstrapping instruments: by construction, each market instrument selected as input bootstrapping instrument must be repriced exactly (in exact fit approaches) or within a predetermined precision (in best fit approaches).

4. Repricing of market instruments: finally, instruments quoted on the market but not included in the bootstrapping should be repriced within their bid-ask window. A typical example are quotes of long term Futures and forward starting IRS.

We conclude that any good yield curve bootstrapping system should include, for each curve generated, a real-time snapshot of yield curve shapes and a real-time pricer of market instruments.

5 Implementation and Examples of Bootstrapping

In this section we apply the methodology illustrated in the previous sections to the concrete EUR market case, working out and discussing all the details. In particular, we bootstrap five types of yield curves $\mathcal{C}_{ON}$, $\mathcal{C}_{1M}$, $\mathcal{C}_{3M}$, $\mathcal{C}_{6M}$, $\mathcal{C}_{12M}$ on Eonia, Euribor1M, 3M, 6M and 12M, respectively, that we comment in the following subsections 5.1-5.5. A final section 5.6 discusses basis yield curves. Only FRA curves are reported, being the most significative bootstrapping test as discussed in section 4.5. The curves are displayed using the same units and scales in all graphs, allowing a general comparison.

The numerical results have been obtained using the QuantLib framework[16]. The basic classes and methods (iterative bootstrapping, interpolations, market conventions, etc.) are implemented in the object oriented C++ QuantLib library [54]. The QuantLib objects and analytics are exposed to a variety of end-user platforms (including Excel and Calc) through the QuantLibAddin [55] and QuantLibXL [56] libraries. Market data are retrieved from the chosen provider and real time is ensured by the ObjectHandler in-memory repository [57]. The full framework described above is available open source[17].

Clearly, the yield curves reported here are the final result of a complex chain of choices, and many alternatives are possible. We do not claim that our choices are the best solution to any problem, they are just our preferences and experience.

5.1 Eonia Yield Curve

Our Eonia curve $\mathcal{C}_{ON}^{\mathrm{EUR}}$ is bootstrapped using the OIS market instruments shown in fig. 25 and the OIS pricing formulas discussed in sec. 4.3.5. The results are shown in fig. 26.

The first section of the term structure is covered using the first three Deposits from fig. 4 in order to set the yield curve reference date to today's date ($t_0 = 11$ Dec. 2012). This feature is necessary since this curve is used for discounting purposes. Notice that these instruments are based on Euribor1D (one-day tenor) and not properly on Eonia, thus we are introducing a (very small) inconsistency. The other bootstrapping instruments are OIS taken from fig. 13. We include the forward OIS on ECB dates in order to capture the market forecast of ECB monetary policy decisions. The very long-end of the term structure $\mathcal{C}_{ON}^{EUR}$ after 30Y is extrapolated flat.

In the case we show two different interpolations schemes: piecewise constant and monotonic cubic spline, both applied on log discounts. These two yield curves are typically used by market practitioners for different purposes. The smooth and continuous curve is normally used for pricing and hedging any collateralised financial instrument, serving as the discounting curve. For OIS, it also serves as the forwarding curve. The piecewise constant curve is typically used by short term traders (e.g. treasurers) for marking short term OIS consistently with the market expectations on future announcements and monetary policy decisions of the European Central Bank (ECB). These expectations are reflected into the market quotations of forward starting OIS struck on known future ECB announcement dates (see fig. 13, bottom section). Sometimes, market practitioners use a mixed yield curve, combining the discontinuous piecewise constant curve in the short end section and the continuous curve in the medium/long end section.

The two jumps observed in the FRA curves, both piecewise constant and continuous, correspond to the two turn of years for ON tenor FRA rates spot starting at 2nd Jan. 2013 (+10.2 bps) and 2nd Jan. 2014 (+8.5 bps). See the discussion on the turn of year effect in sec. 4.8.

Negative FRA rates observed in the 3M-12M window (top panel) have been commented in sec. 4.6.

[16] precisely, revision 18431 in the QuantLib SVN repository `R01020x-branch`.

[17] Anyone interested in the topic may download and test the implementation, posting questions, comments and suggestions to the QuantLib community forums.

Eonia ON Yield Curve			
Bootstrapping Instrument	**Rate**	**Start Date**	**End Date**
EUR_YC_OND	0.0400%	Tue 11 Dec 2012	Wed 12 Dec 2012
EUR_YC_TND	0.0400%	Wed 12 Dec 2012	Thu 13 Dec 2012
EUR_YC_SND	0.0400%	Thu 13 Dec 2012	Fri 14 Dec 2012
EUR_YC_EONSW	0.0700%	Thu 13 Dec 2012	Thu 20 Dec 2012
EUR_YC_EON2W	0.0690%	Thu 13 Dec 2012	Thu 27 Dec 2012
EUR_YC_EON3W	0.0780%	Thu 13 Dec 2012	Thu 03 Jan 2013
EUR_YC_EON1M	0.0740%	Thu 13 Dec 2012	Mon 14 Jan 2013
EUR_YC_EONECBJAN13	0.0460%	Wed 16 Jan 2013	Wed 13 Feb 2013
EUR_YC_EONECBFEB13	0.0160%	Wed 13 Feb 2013	Wed 13 Mar 2013
EUR_YC_EONECBMAR13	-0.0070%	Wed 13 Mar 2013	Wed 10 Apr 2013
EUR_YC_EONECBAPR13	-0.0130%	Wed 10 Apr 2013	Wed 08 May 2013
EUR_YC_EONECBMAY13	-0.0140%	Wed 08 May 2013	Wed 12 Jun 2013
EUR_YC_EON15M	0.0020%	Thu 13 Dec 2012	Thu 13 Mar 2014
EUR_YC_EON18M	0.0080%	Thu 13 Dec 2012	Fri 13 Jun 2014
EUR_YC_EON21M	0.0210%	Thu 13 Dec 2012	Mon 15 Sep 2014
EUR_YC_EON2Y	0.0360%	Thu 13 Dec 2012	Mon 15 Dec 2014
EUR_YC_EON3Y	0.1270%	Thu 13 Dec 2012	Mon 14 Dec 2015
EUR_YC_EON4Y	0.2740%	Thu 13 Dec 2012	Tue 13 Dec 2016
EUR_YC_EON5Y	0.4560%	Thu 13 Dec 2012	Wed 13 Dec 2017
EUR_YC_EON6Y	0.6470%	Thu 13 Dec 2012	Thu 13 Dec 2018
EUR_YC_EON7Y	0.8270%	Thu 13 Dec 2012	Fri 13 Dec 2019
EUR_YC_EON8Y	0.9960%	Thu 13 Dec 2012	Mon 14 Dec 2020
EUR_YC_EON9Y	1.1470%	Thu 13 Dec 2012	Mon 13 Dec 2021
EUR_YC_EON10Y	1.2800%	Thu 13 Dec 2012	Tue 13 Dec 2022
EUR_YC_EON11Y	1.4040%	Thu 13 Dec 2012	Wed 13 Dec 2023
EUR_YC_EON12Y	1.5160%	Thu 13 Dec 2012	Fri 13 Dec 2024
EUR_YC_EON15Y	1.7640%	Thu 13 Dec 2012	Mon 13 Dec 2027
EUR_YC_EON20Y	1.9390%	Thu 13 Dec 2012	Mon 13 Dec 2032
EUR_YC_EON25Y	2.0030%	Thu 13 Dec 2012	Mon 14 Dec 2037
EUR_YC_EON30Y	2.0380%	Thu 13 Dec 2012	Mon 15 Dec 2042

Figure 25: Bootstrapping instruments selected for yield curve $\mathcal{C}_{ON}^{EUR}$ construction as of 11 Dec. 2012.

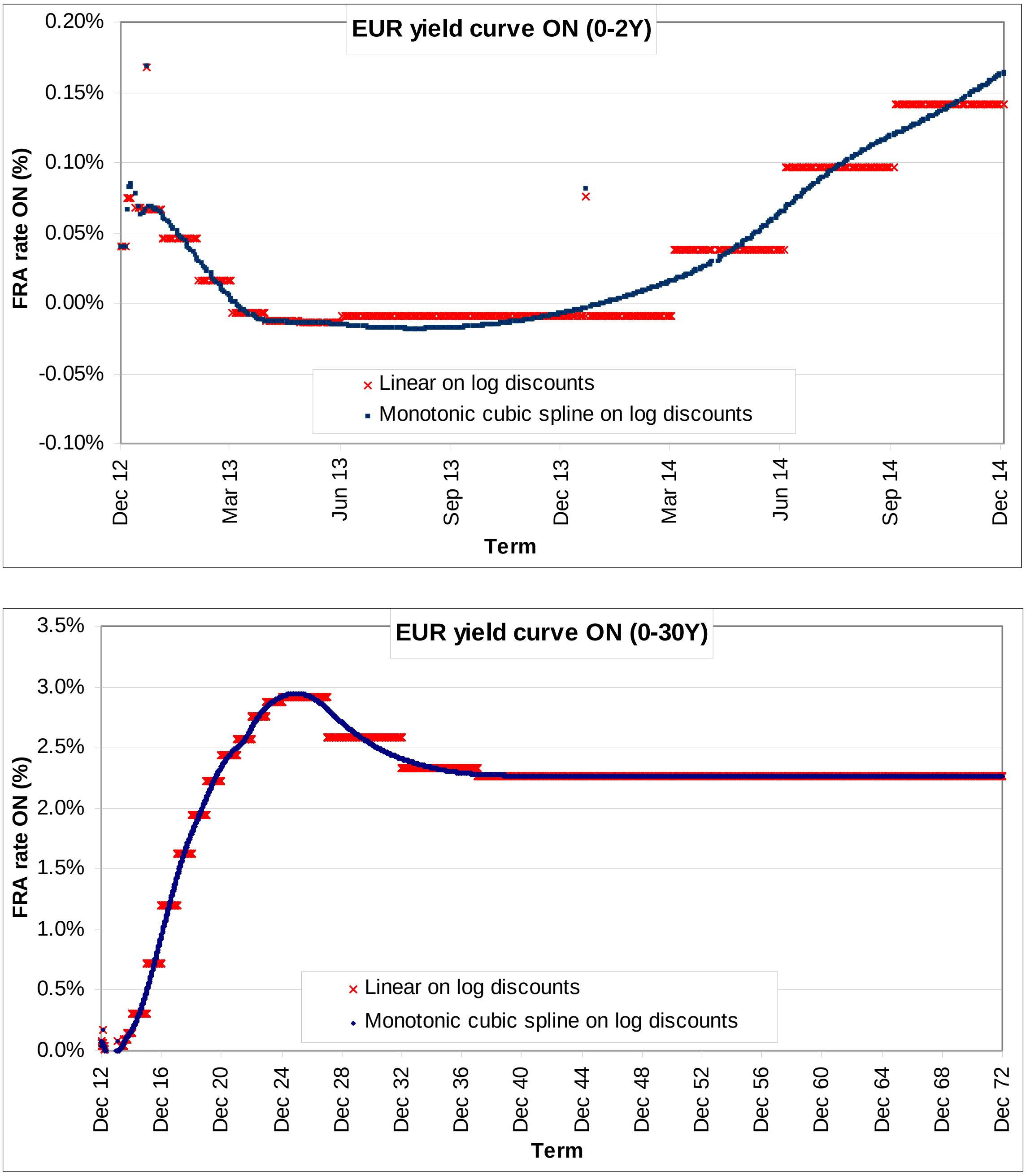


Figure 26: FRA curve $\mathcal{C}_{ON}^{EUR,f}$ on Eonia as of 11 Dec. 2012, plotted with ON-tenor FRA rates $F_{ON}(t_0; t, act/360)$, t daily sampled. Top panel: short term structure up to 2 years. Bottom panel: whole term structure up to 60 years. Blue dots (continuous line): monotonic cubic spline interpolation on log discounts. Red crosses (discontinuous line): piecewise constant interpolation on log discounts. See comments in the text.

5.2 Euribor1M Yield Curve

Our Euribor1M curve $\mathcal{C}_{1M}^{\mathrm{EUR}}$ is bootstrapped using the Euribor1M market instruments selected in fig. 27 and the corresponding pricing formulas discussed in secs. 4.3.1-4.3.6. The results are shown in fig. 28.

The first section of the term structure is covered using four synthetic Deposits, from SND to 3WD, obtained as discussed in sec. 4.4. These instruments are selected to ensure a smooth short curve in the very first section of the term structure, below the first market pillar (1M in case of Euribor 1M underlying). We stress that such particular choice is rather subjective and possibly unconventional, more an art of the interest rate trader than a science. The 1M Deposit, insisting on Euribor 1M, is taken from fig. 4. The term structure up to 1Y is built with short term IRS 1M from fig. 11. The medium/long term structure up to 30Y is built with IRS 1M obtained from IRS 6M in fig. 9 and IRBS 1M vs 6M in fig. 15. The last four pillars between 35Y and 60Y are obtained by flat extrapolation of the 30Y pillar quote of IRBS 1M vs 6M in fig. 15. Notice that the yield curve reference date is spot date (t_0 = 13 Dec. 2012) and not today's date. This is not a problem, since this curve is used for computing FRA rates, not for discounting.

The resulting yield curve term structure displays the mixed upward/downward sloping behavior frequently observed in recent years. The jumps observed in Dec. 13 correspond to the turn of year for 1M tenor FRA rates discussed in sec. 4.8.

5.3 Euribor3M Yield Curve

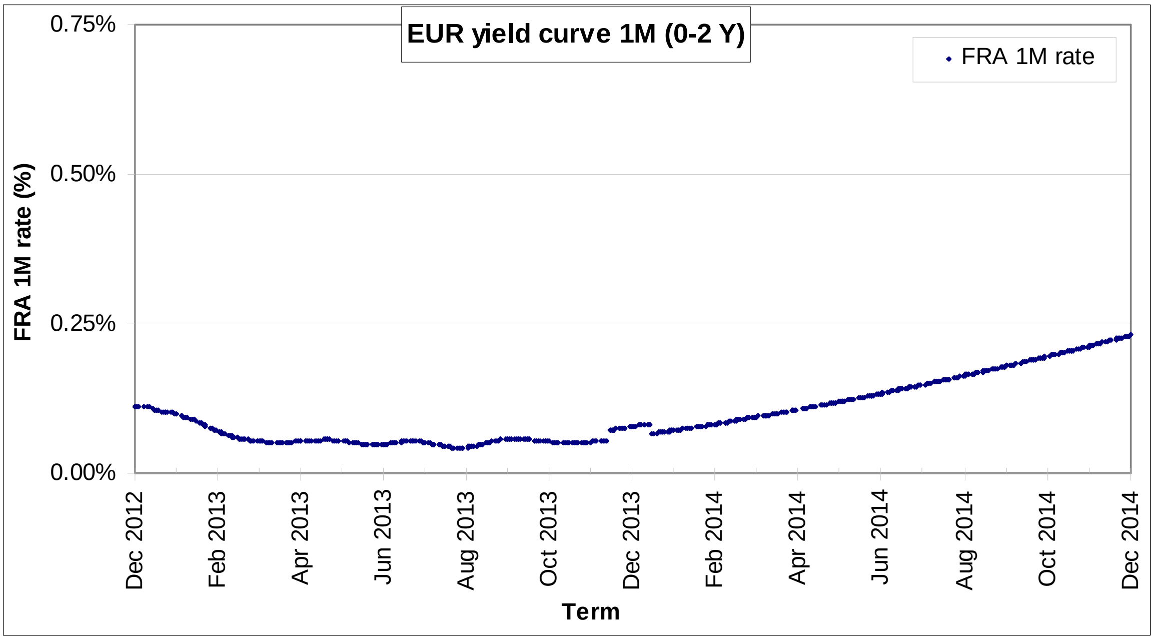
Our Euribor3M curve $\mathcal{C}_{3M}^{\mathrm{EUR}}$ is bootstrapped using the Euribor3M market instruments selected in fig. 29 and the corresponding pricing formulas discussed in secs. 4.3.1-4.3.6. The results are shown in fig. 30.

The first section of the term structure is covered using four synthetic Deposits from 2W to 1M maturity, obtained as discussed in sec. 4.4. Notice that, in this case, we do not include the shortest synthetic Deposits (SND, SWD). As discussed in the previous section, this choice is rather subjective. Next, we include the Tomorrow FRA 3M taken from fig. 6. A common alternative choice would be to select the 3M Deposit (3MD). In general, we prefer FRA with respect to Deposits, because the formers are unfunded[18] and collateralised instruments, more similar to IRS. Next, we cover the term structure up to 2Y maturity including the strip of eight Futures 3M taken from fig. 7. Notice that the Futures dates are fixed by the IMM market, thus the time grid presented in fig. 29 is actually not static, with the Futures strip sliding day by day towards the yield curve origin at t_0. When the the first Futures fixes (in this case, on Monday 17 Dec. 2012), we "roll" the Futures strip, such that the first Futures is eliminated, the second Futures becomes the first Futures, and a new Futures is included at the end of the train. Actually, the Futures rolling can be executed before the first Futures fixing date, in case of lowered market liquidity of the first Futures. Notice that this choice of bootstrapping instruments is free of overlaps between Futures and Deposits, and allows a stable Deposits strip. An

[18]Market practitioners distinguish between "unfunded" instruments, such as FRA and IRS, with vanishing (or low) initial value, and "funded" instruments, such as Deposits, with important directional cash-flows. The price of the latters depends heavily on funding and liquidity issues, since one counterparty is in some way funding the other, while the price of formers is less (or not) dependent, since there is no strong directional funding.

Euribor 1M Yield Curve			
Bootstrapping Instrument	**Rate**	**Start Date**	**End Date**
EUR_YC1M_SND	*0.0661%*	*Thu 13 Dec 2012*	*Fri 14 Dec 2012*
EUR_YC1M_SWD	*0.0980%*	*Thu 13 Dec 2012*	*Thu 20 Dec 2012*
EUR_YC1M_2WD	*0.0993%*	*Thu 13 Dec 2012*	*Thu 27 Dec 2012*
EUR_YC1M_3WD	*0.1105%*	*Thu 13 Dec 2012*	*Thu 03 Jan 2013*
EUR_YC1M_1MD	0.1100%	Thu 13 Dec 2012	Mon 14 Jan 2013
EUR_YC1M_2X1S	0.1060%	Thu 13 Dec 2012	Wed 13 Feb 2013
EUR_YC1M_3X1S	0.0960%	Thu 13 Dec 2012	Wed 13 Mar 2013
EUR_YC1M_4X1S	0.0850%	Thu 13 Dec 2012	Mon 15 Apr 2013
EUR_YC1M_5X1S	0.0790%	Thu 13 Dec 2012	Mon 13 May 2013
EUR_YC1M_6X1S	0.0750%	Thu 13 Dec 2012	Thu 13 Jun 2013
EUR_YC1M_7X1S	0.0710%	Thu 13 Dec 2012	Mon 15 Jul 2013
EUR_YC1M_8X1S	0.0690%	Thu 13 Dec 2012	Tue 13 Aug 2013
EUR_YC1M_9X1S	0.0660%	Thu 13 Dec 2012	Fri 13 Sep 2013
EUR_YC1M_10X1S	0.0650%	Thu 13 Dec 2012	Mon 14 Oct 2013
EUR_YC1M_11X1S	0.0640%	Thu 13 Dec 2012	Wed 13 Nov 2013
EUR_YC1M_12X1S	0.0630%	Thu 13 Dec 2012	Fri 13 Dec 2013
EUR_YC1M_AB1EBASIS2Y	0.0980%	Thu 13 Dec 2012	Mon 15 Dec 2014
EUR_YC1M_AB1EBASIS3Y	0.1860%	Thu 13 Dec 2012	Mon 14 Dec 2015
EUR_YC1M_AB1EBASIS4Y	0.3300%	Thu 13 Dec 2012	Tue 13 Dec 2016
EUR_YC1M_AB1EBASIS5Y	0.5120%	Thu 13 Dec 2012	Wed 13 Dec 2017
EUR_YC1M_AB1EBASIS6Y	0.7040%	Thu 13 Dec 2012	Thu 13 Dec 2018
EUR_YC1M_AB1EBASIS7Y	0.8870%	Thu 13 Dec 2012	Fri 13 Dec 2019
EUR_YC1M_AB1EBASIS8Y	1.0580%	Thu 13 Dec 2012	Mon 14 Dec 2020
EUR_YC1M_AB1EBASIS9Y	1.2110%	Thu 13 Dec 2012	Mon 13 Dec 2021
EUR_YC1M_AB1EBASIS10Y	1.3470%	Thu 13 Dec 2012	Tue 13 Dec 2022
EUR_YC1M_AB1EBASIS11Y	1.4700%	Thu 13 Dec 2012	Wed 13 Dec 2023
EUR_YC1M_AB1EBASIS12Y	1.5810%	Thu 13 Dec 2012	Fri 13 Dec 2024
EUR_YC1M_AB1EBASIS15Y	1.8260%	Thu 13 Dec 2012	Mon 13 Dec 2027
EUR_YC1M_AB1EBASIS20Y	1.9980%	Thu 13 Dec 2012	Mon 13 Dec 2032
EUR_YC1M_AB1EBASIS25Y	2.0590%	Thu 13 Dec 2012	Mon 14 Dec 2037
EUR_YC1M_AB1EBASIS30Y	2.0930%	Thu 13 Dec 2012	Mon 15 Dec 2042
EUR_YC1M_AB1EBASIS35Y	*2.1320%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2047*
EUR_YC1M_AB1EBASIS40Y	*2.1850%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2052*
EUR_YC1M_AB1EBASIS50Y	*2.2580%*	*Thu 13 Dec 2012*	*Wed 13 Dec 2062*
EUR_YC1M_AB1EBASIS60Y	*2.3000%*	*Thu 13 Dec 2012*	*Tue 13 Dec 2072*

Figure 27: Bootstrapping instruments selected for yield curve $\mathcal{C}_{1M}^{EUR}$ construction as of 11 Dec. 2012. Black: market instruments. Green italics: synthetic instruments.



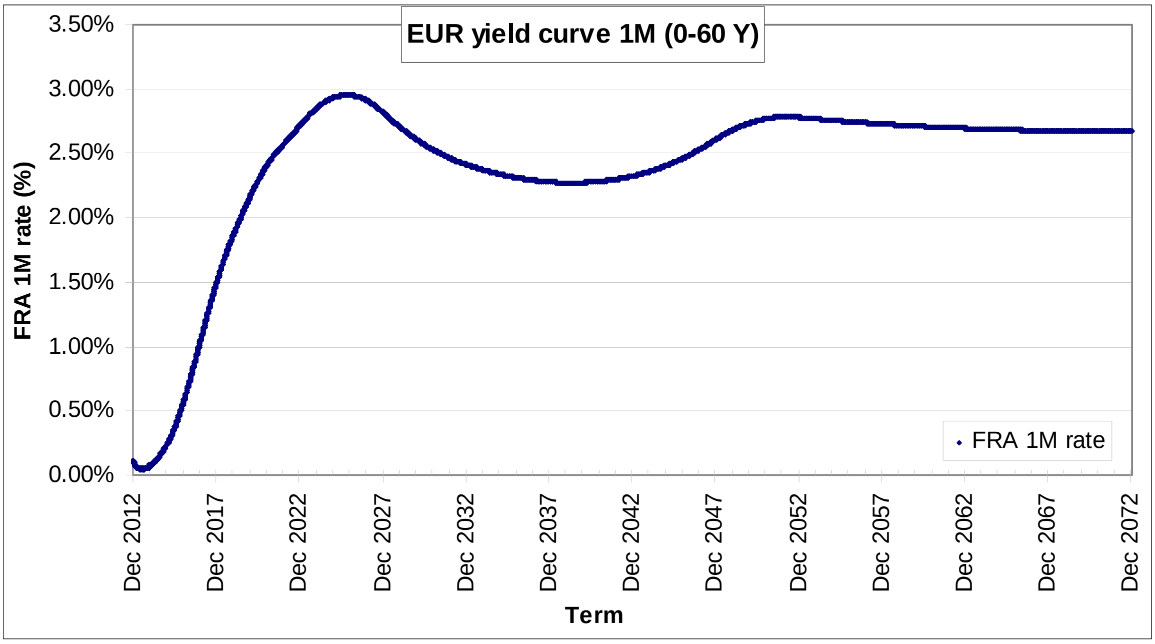


Figure 28: FRA curve $\mathcal{C}_{1M}^{f}$ on Euribor1M as of 11 Dec. 2012, plotted with 1M-tenor FRA rates $F\left(t_0; t, t+1M, act/360\right)$, t daily sampled. Top panel: short term structure up to 2 years. Bottom panel: whole term structure up to 60 years. The jump observed in the curve correspond to the turn of years for 1M tenor FRA rates spot starting at 2nd Dec. 2013.

Euribor 3M Yield Curve			
Bootstrapping Instrument	**Rate**	**Start Date**	**End Date**
EUR_YC3M_2WD	*0.1865%*	*Thu 13 Dec 2012*	*Thu 27 Dec 2012*
EUR_YC3M_3WD	*0.1969%*	*Thu 13 Dec 2012*	*Thu 03 Jan 2013*
EUR_YC3M_1MD	*0.1951%*	*Thu 13 Dec 2012*	*Mon 14 Jan 2013*
EUR_YC3M_2MD	*0.1874%*	*Thu 13 Dec 2012*	*Wed 13 Feb 2013*
EUR_YC3M_TOM3F1	0.1790%	Fri 14 Dec 2012	Thu 14 Mar 2013
EUR_YC3M_FUT3MZ2	0.1775%	Wed 19 Dec 2012	Tue 19 Mar 2013
EUR_YC3M_FUT3MH3	0.1274%	Wed 20 Mar 2013	Thu 20 Jun 2013
EUR_YC3M_FUT3MM3	0.1222%	Wed 19 Jun 2013	Thu 19 Sep 2013
EUR_YC3M_FUT3MU3	0.1269%	Wed 18 Sep 2013	Wed 18 Dec 2013
EUR_YC3M_FUT3MZ3	0.1565%	Wed 18 Dec 2013	Tue 18 Mar 2014
EUR_YC3M_FUT3MH4	0.1961%	Wed 19 Mar 2014	Thu 19 Jun 2014
EUR_YC3M_FUT3MM4	0.2556%	Wed 18 Jun 2014	Thu 18 Sep 2014
EUR_YC3M_FUT3MU4	0.3101%	Wed 17 Sep 2014	Wed 17 Dec 2014
EUR_YC3M_AB3E3Y	0.2850%	Thu 13 Dec 2012	Mon 14 Dec 2015
EUR_YC3M_AB3E4Y	0.4370%	Thu 13 Dec 2012	Tue 13 Dec 2016
EUR_YC3M_AB3E5Y	0.6230%	Thu 13 Dec 2012	Wed 13 Dec 2017
EUR_YC3M_AB3E6Y	0.8170%	Thu 13 Dec 2012	Thu 13 Dec 2018
EUR_YC3M_AB3E7Y	1.0000%	Thu 13 Dec 2012	Fri 13 Dec 2019
EUR_YC3M_AB3E8Y	1.1710%	Thu 13 Dec 2012	Mon 14 Dec 2020
EUR_YC3M_AB3E9Y	1.3240%	Thu 13 Dec 2012	Mon 13 Dec 2021
EUR_YC3M_AB3E10Y	1.4590%	Thu 13 Dec 2012	Tue 13 Dec 2022
EUR_YC3M_AB3E12Y	1.6920%	Thu 13 Dec 2012	Fri 13 Dec 2024
EUR_YC3M_AB3E15Y	1.9330%	Thu 13 Dec 2012	Mon 13 Dec 2027
EUR_YC3M_AB3E20Y	2.0990%	Thu 13 Dec 2012	Mon 13 Dec 2032
EUR_YC3M_AB3E25Y	2.1560%	Thu 13 Dec 2012	Mon 14 Dec 2037
EUR_YC3M_AB3E30Y	2.1860%	Thu 13 Dec 2012	Mon 15 Dec 2042
EUR_YC3M_AB3EBASIS35Y	*2.2308%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2047*
EUR_YC3M_AB3EBASIS40Y	*2.2880%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2052*
EUR_YC3M_AB3EBASIS50Y	*2.3670%*	*Thu 13 Dec 2012*	*Wed 13 Dec 2062*
EUR_YC3M_AB3EBASIS60Y	*2.4126%*	*Thu 13 Dec 2012*	*Tue 13 Dec 2072*

Figure 29: Bootstrapping instruments selected for yield curve $\mathcal{C}_{1M}^{EUR}$ construction as of 11 Dec. 2012. Black: market instruments. Green italics: synthetic instruments.

alternative choice with respect to Futures would be the selection of the market FRA 3M and IMM FRA 3M in fig. 6. In particular, the FRA 3M would allow for a constant time grid. Futures are often preferred due to their high market liquidity. Again, this choice is rather subjective. The other bootstrapping instruments are medium/long term IRS 3M from 3Y to 30Y taken from fig. 10. The very long-end pillars between 35Y and 50Y are obtained from IRS 6M quotes in fig. 9 and IRBS 3M vs 6M quotes in fig. 15. The last pillar at 60Y is obtained through linear extrapolation of previous IRBS quotes. The yield curve reference date is spot date (t_0 = 13 Dec. 2012).

We observe in fig. 30 a behavior similar to Euribor1M yield curve. The jumps observed in the window Oct. 13 - Dec. 13 correspond to the turn of year for 3M tenor FRA rates discussed in sec. 4.8. Finally, we stress that FRA 3M quotes shown in fig. 30 are not included in the bootstrapping, but perfectly repriced. This is the fourth kind of yield curve check discussed in sec. 4.11.

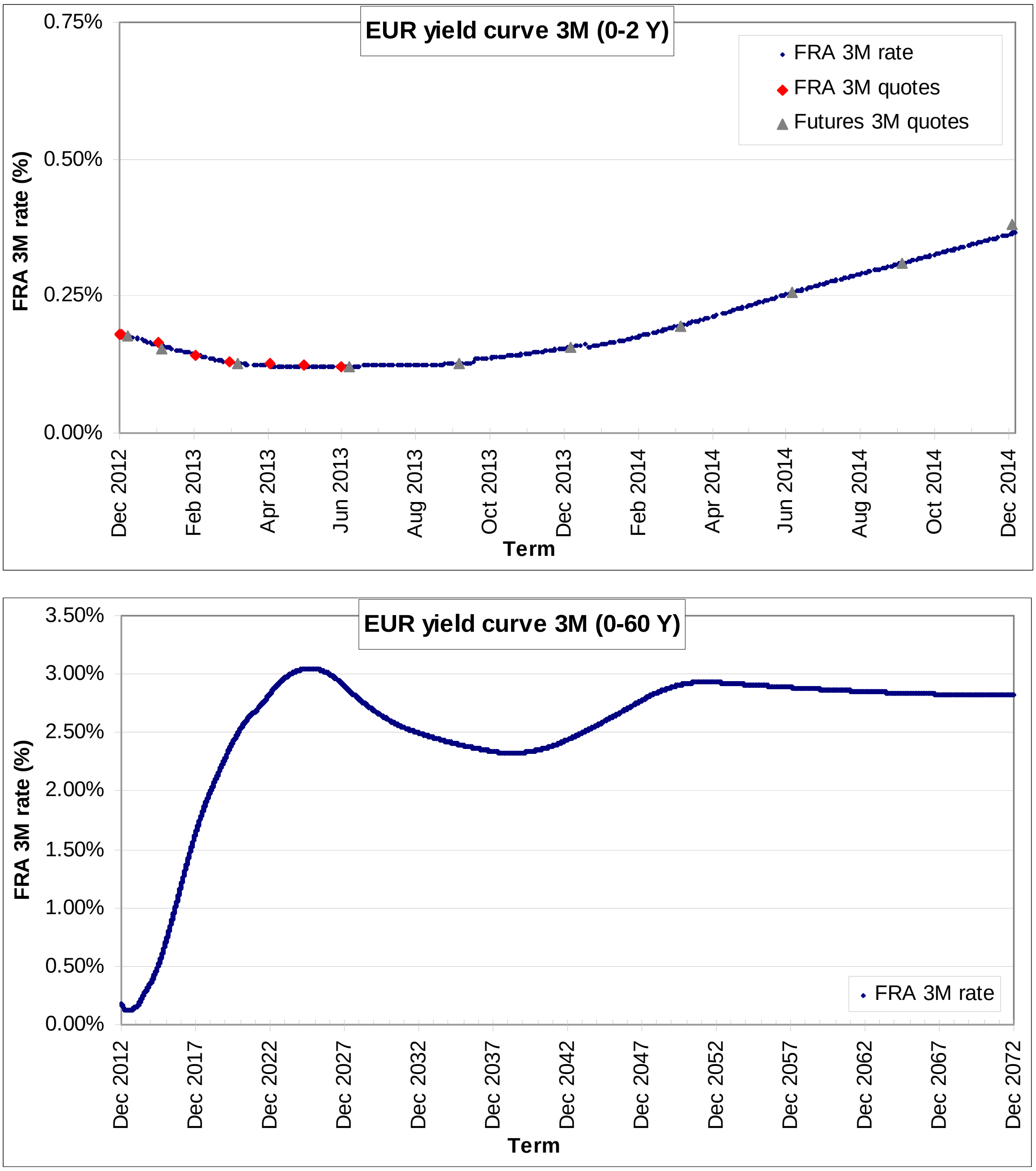


Figure 30: FRA curve $\mathcal{C}_{3M}^{f}$ on Euribor3M as of 11 Dec. 2012, plotted with 3M-tenor FRA rates $F\left(t_0; t, t+3M, act/360\right)$, t daily sampled. Top panel: short term structure up to 2 years. Bottom panel: whole term structure up to 60 years. Quotes for Futures 3M and FRA 3M (not included in bootstrapping) are also reported.

5.4 Euribor6M Yield Curve

Our Euribor6M curve $\mathcal{C}_{6M}^{\text{EUR}}$ is bootstrapped using the Euribor6M market instruments selected in fig. 31 and the corresponding pricing formulas discussed in secs. 4.3.1-4.3.6. The results are shown in fig. 32.

The first section of the term structure is covered using nine synthetic Deposits from SND to 5MD, obtained as discussed in sec. 4.4. In this case, similar to $\mathcal{C}_{!M}^{\text{EUR}}$ bootstrapping, we include also the shortest SND instrument. Next, we include the strip of nineteen market FRA 6M, from FRA Tom6M to FRA 18×24, up to 2Y maturity, taken from fig. 6. We complete the term structure with medium/long term IRS 6M from 3Y to 60Y taken from fig. 9. In this case neither IRBS nor extrapolations are used, since the market quotations cover up to 60Y maturity. The yield curve reference date is spot date (t_0 = 13 Dec. 2012).

We observe in fig. 32 a behavior similar to Euribor1M and 3M yield curves. No turn of the year jumps are included, because their market quotation is negligible. The IMM FRA 6M quotes shown in fig. 32 are not included in the bootstrapping, but perfectly repriced.

5.5 Euribor12M Yield Curve

Our Euribor12M curve $\mathcal{C}_{12M}^{\text{EUR}}$ is bootstrapped using the Euribor12M market instruments selected in fig. 33 and the corresponding pricing formulas discussed in secs. 4.3.1-4.3.6. The results are shown in fig. 34.

The first section of the term structure is covered using four synthetic Deposits from 1M to 9M maturity, obtained as discussed in sec. 4.4. Clearly this is a subjective choice, as discussed in the previous sections. The first market pillar is the 12M Deposit, insisting on Euribor 12M, taken from fig. 4. Next, we include the strip of six FRA 12M, from FRA 3×15 to FRA 18×30, up to 2.5Y maturity. Only the FRA 12x24 is taken from the market, fig. 6, while the other five FRA are synthetic instruments, built as discussed in sec. 4.4. The medium/long term structure up to 30Y is built with IRS 12M obtained from IRS 6M in fig. 9 and IRBS 12M vs 6M in fig. 15. The last four pillars between 35Y and 60Y are obtained by flat extrapolation of the 30Y pillar quote of IRBS 1M vs 6M in fig. 15. The yield curve reference date is spot date (t_0 = 13 Dec. 2012).

We observe in fig. 34 a behavior similar to previous Euribor yield curves. No turn of the year jumps are included, because their market quotation is negligible.

5.6 Basis Yield Curves

We finally present our basis yield curves in fig. 35. These curves are not directly bootstrapped, but they result from differences between FRA rates computed on two different yield curves. In particular, the bottom panel shows, for each future date $t > t_0$, the difference between two FRA rates: the Euribor FRA rate and the OIS FRA rate, both with over night length. This is equivalent to a series of forward starting one-day zero coupon IRBS. This latter representation is the best finite approximation to an instantaneous FRA basis, and is the most sensitive to the details of the bootstrapping. This explains the irregular shapes. The spikes observed in the figure are due to the turn of

Euribor 6M Yield Curve			
Bootstrapping Instrument	**Rate**	**Start Date**	**End Date**
EUR_YC6M_SND	*0.3565%*	*Thu 13 Dec 2012*	*Fri 14 Dec 2012*
EUR_YC6M_SWD	*0.3858%*	*Thu 13 Dec 2012*	*Thu 20 Dec 2012*
EUR_YC6M_2WD	*0.3840%*	*Thu 13 Dec 2012*	*Thu 27 Dec 2012*
EUR_YC6M_3WD	*0.3922%*	*Thu 13 Dec 2012*	*Thu 03 Jan 2013*
EUR_YC6M_1MD	*0.3869%*	*Thu 13 Dec 2012*	*Mon 14 Jan 2013*
EUR_YC6M_2MD	*0.3698%*	*Thu 13 Dec 2012*	*Wed 13 Feb 2013*
EUR_YC6M_3MD	*0.3527%*	*Thu 13 Dec 2012*	*Wed 13 Mar 2013*
EUR_YC6M_4MD	*0.3342%*	*Thu 13 Dec 2012*	*Mon 15 Apr 2013*
EUR_YC6M_5MD	*0.3225%*	*Thu 13 Dec 2012*	*Mon 13 May 2013*
EUR_YC6M_TOM6F1	0.3120%	Fri 14 Dec 2012	Fri 14 Jun 2013
EUR_YC6M_1x7F	0.2930%	Mon 14 Jan 2013	Mon 15 Jul 2013
EUR_YC6M_2x8F	0.2720%	Wed 13 Feb 2013	Tue 13 Aug 2013
EUR_YC6M_3x9F	0.2600%	Wed 13 Mar 2013	Fri 13 Sep 2013
EUR_YC6M_4x10F	0.2560%	Mon 15 Apr 2013	Tue 15 Oct 2013
EUR_YC6M_5x11F	0.2520%	Mon 13 May 2013	Wed 13 Nov 2013
EUR_YC6M_6x12F	0.2480%	Thu 13 Jun 2013	Fri 13 Dec 2013
EUR_YC6M_7x13F	0.2540%	Mon 15 Jul 2013	Wed 15 Jan 2014
EUR_YC6M_8x14F	0.2610%	Tue 13 Aug 2013	Thu 13 Feb 2014
EUR_YC6M_9x15F	0.2670%	Fri 13 Sep 2013	Thu 13 Mar 2014
EUR_YC6M_10x16F	0.2790%	Mon 14 Oct 2013	Mon 14 Apr 2014
EUR_YC6M_11x17F	0.2910%	Wed 13 Nov 2013	Tue 13 May 2014
EUR_YC6M_12x18F	0.3030%	Fri 13 Dec 2013	Fri 13 Jun 2014
EUR_YC6M_13x19F	0.3180%	Mon 13 Jan 2014	Mon 14 Jul 2014
EUR_YC6M_14x20F	0.3350%	Thu 13 Feb 2014	Wed 13 Aug 2014
EUR_YC6M_15x21F	0.3520%	Thu 13 Mar 2014	Mon 15 Sep 2014
EUR_YC6M_16x22F	0.3710%	Mon 14 Apr 2014	Tue 14 Oct 2014
EUR_YC6M_17x23F	0.3890%	Tue 13 May 2014	Thu 13 Nov 2014
EUR_YC6M_18x24F	0.4090%	Fri 13 Jun 2014	Mon 15 Dec 2014
EUR_YC6M_AB6E3Y	0.4240%	Thu 13 Dec 2012	Mon 14 Dec 2015
EUR_YC6M_AB6E4Y	0.5760%	Thu 13 Dec 2012	Tue 13 Dec 2016
EUR_YC6M_AB6E5Y	0.7620%	Thu 13 Dec 2012	Wed 13 Dec 2017
EUR_YC6M_AB6E6Y	0.9540%	Thu 13 Dec 2012	Thu 13 Dec 2018
EUR_YC6M_AB6E7Y	1.1350%	Thu 13 Dec 2012	Fri 13 Dec 2019
EUR_YC6M_AB6E8Y	1.3030%	Thu 13 Dec 2012	Mon 14 Dec 2020
EUR_YC6M_AB6E9Y	1.4520%	Thu 13 Dec 2012	Mon 13 Dec 2021
EUR_YC6M_AB6E10Y	1.5840%	Thu 13 Dec 2012	Tue 13 Dec 2022
EUR_YC6M_AB6E12Y	1.8090%	Thu 13 Dec 2012	Fri 13 Dec 2024
EUR_YC6M_AB6E15Y	2.0370%	Thu 13 Dec 2012	Mon 13 Dec 2027
EUR_YC6M_AB6E20Y	2.1870%	Thu 13 Dec 2012	Mon 13 Dec 2032
EUR_YC6M_AB6E25Y	2.2340%	Thu 13 Dec 2012	Mon 14 Dec 2037
EUR_YC6M_AB6E30Y	2.2560%	Thu 13 Dec 2012	Mon 15 Dec 2042
EUR_YC6M_AB6E35Y	2.2950%	Thu 13 Dec 2012	Fri 13 Dec 2047
EUR_YC6M_AB6E40Y	2.3480%	Thu 13 Dec 2012	Fri 13 Dec 2052
EUR_YC6M_AB6E50Y	2.4210%	Thu 13 Dec 2012	Wed 13 Dec 2062
EUR_YC6M_AB6E60Y	2.4630%	Thu 13 Dec 2012	Tue 13 Dec 2072

Figure 31: Bootstrapping instruments selected for yield curve $\mathcal{C}_{1M}^{EUR}$ construction as of 11 Dec. 2012.

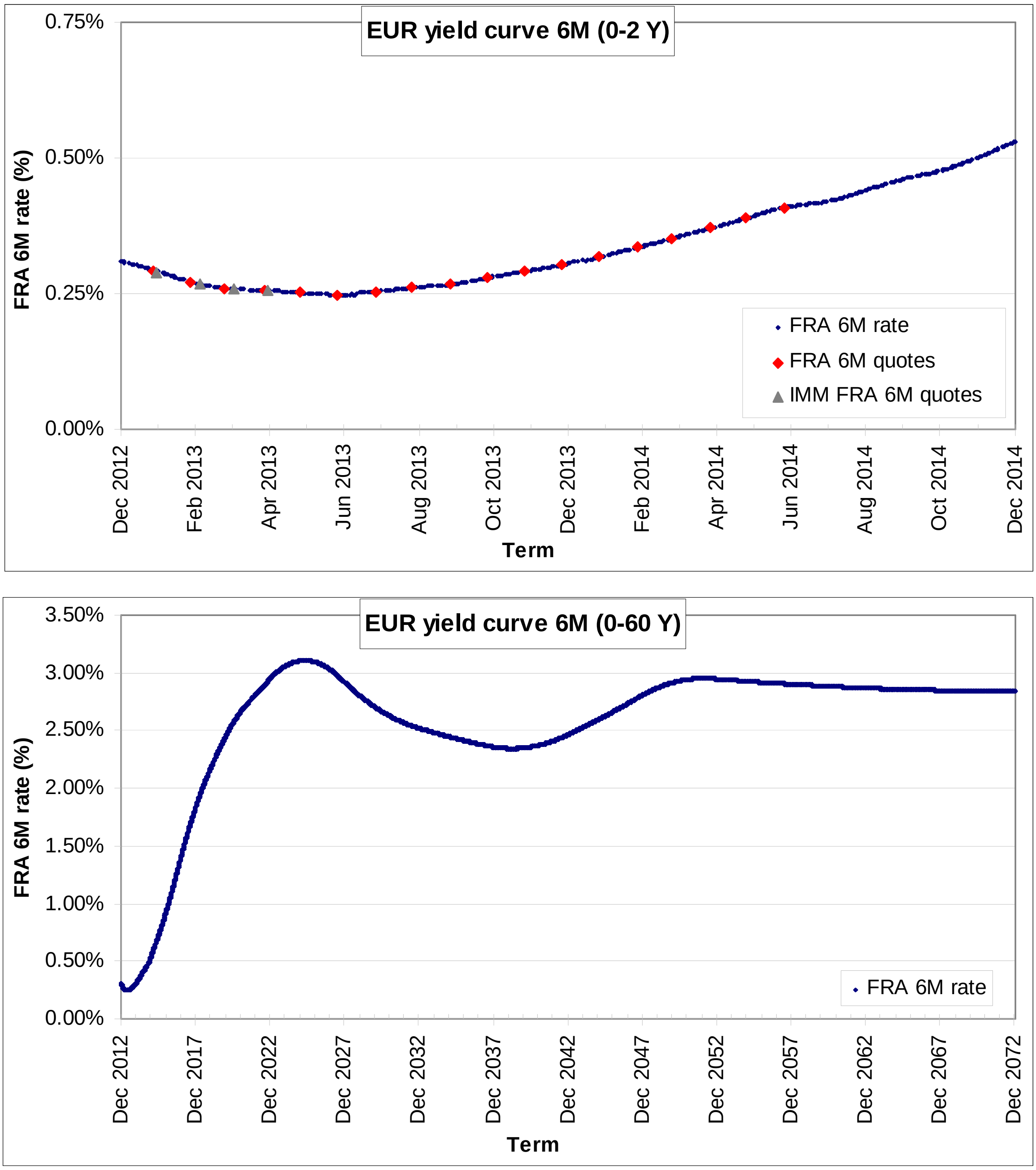


Figure 32: FRA curve $\mathcal{C}^f_{6M}$ on Euribor6M as of 11 Dec. 2012, plotted with 6M-tenor FRA rates $F\,(t_0; t, t+6M, act/360)$, t daily sampled. Top panel: short term structure up to 2 years. Bottom panel: whole term structure up to 60 years. Quoted FRA 6M and IMM FRA 6M (not included in bootstrapping) are also reported.

Euribor 12M Yield Curve			
Bootstrapping Instrument	**Rate**	**Start Date**	**End Date**
EUR_YC12M_1MD	*0.6537%*	*Thu 13 Dec 2012*	*Mon 14 Jan 2013*
EUR_YC12M_3MD	*0.6187%*	*Thu 13 Dec 2012*	*Wed 13 Mar 2013*
EUR_YC12M_6MD	*0.5772%*	*Thu 13 Dec 2012*	*Thu 13 Jun 2013*
EUR_YC12M_9MD	*0.5563%*	*Thu 13 Dec 2012*	*Fri 13 Sep 2013*
EUR_YC12M_12MD	0.5400%	Thu 13 Dec 2012	Fri 13 Dec 2013
EUR_YC12M_3x15F	*0.4974%*	*Wed 13 Mar 2013*	*Thu 13 Mar 2014*
EUR_YC12M_6x18F	*0.4783%*	*Thu 13 Jun 2013*	*Fri 13 Jun 2014*
EUR_YC12M_9x21F	*0.4822%*	*Fri 13 Sep 2013*	*Mon 15 Sep 2014*
EUR_YC12M_12x24F	0.5070%	Fri 13 Dec 2013	Mon 15 Dec 2014
EUR_YC12M_15x27F	*0.5481%*	*Thu 13 Mar 2014*	*Fri 13 Mar 2015*
EUR_YC12M_18x30F	*0.6025%*	*Fri 13 Jun 2014*	*Mon 15 Jun 2015*
EUR_YC12M_AB12EBASIS3Y	0.6030%	Thu 13 Dec 2012	Mon 14 Dec 2015
EUR_YC12M_AB12EBASIS4Y	0.7400%	Thu 13 Dec 2012	Tue 13 Dec 2016
EUR_YC12M_AB12EBASIS5Y	0.9130%	Thu 13 Dec 2012	Wed 13 Dec 2017
EUR_YC12M_AB12EBASIS6Y	1.0930%	Thu 13 Dec 2012	Thu 13 Dec 2018
EUR_YC12M_AB12EBASIS7Y	1.2650%	Thu 13 Dec 2012	Fri 13 Dec 2019
EUR_YC12M_AB12EBASIS8Y	1.4260%	Thu 13 Dec 2012	Mon 14 Dec 2020
EUR_YC12M_AB12EBASIS9Y	1.5700%	Thu 13 Dec 2012	Mon 13 Dec 2021
EUR_YC12M_AB12EBASIS10Y	1.6970%	Thu 13 Dec 2012	Tue 13 Dec 2022
EUR_YC12M_AB12EBASIS12Y	1.9150%	Thu 13 Dec 2012	Fri 13 Dec 2024
EUR_YC12M_AB12EBASIS15Y	2.1300%	Thu 13 Dec 2012	Mon 13 Dec 2027
EUR_YC12M_AB12EBASIS20Y	2.2670%	Thu 13 Dec 2012	Mon 13 Dec 2032
EUR_YC12M_AB12EBASIS25Y	2.3060%	Thu 13 Dec 2012	Mon 14 Dec 2037
EUR_YC12M_AB12EBASIS30Y	2.3220%	Thu 13 Dec 2012	Mon 15 Dec 2042
EUR_YC12M_AB12EBASIS35Y	*2.3610%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2047*
EUR_YC12M_AB12EBASIS40Y	*2.4140%*	*Thu 13 Dec 2012*	*Fri 13 Dec 2052*
EUR_YC12M_AB12EBASIS50Y	*2.4870%*	*Thu 13 Dec 2012*	*Wed 13 Dec 2062*
EUR_YC12M_AB12EBASIS60Y	*2.5290%*	*Thu 13 Dec 2012*	*Tue 13 Dec 2072*

Figure 33: Bootstrapping instruments selected for yield curve $\mathcal{C}_{1M}^{EUR}$ construction as of 11 Dec. 2012.

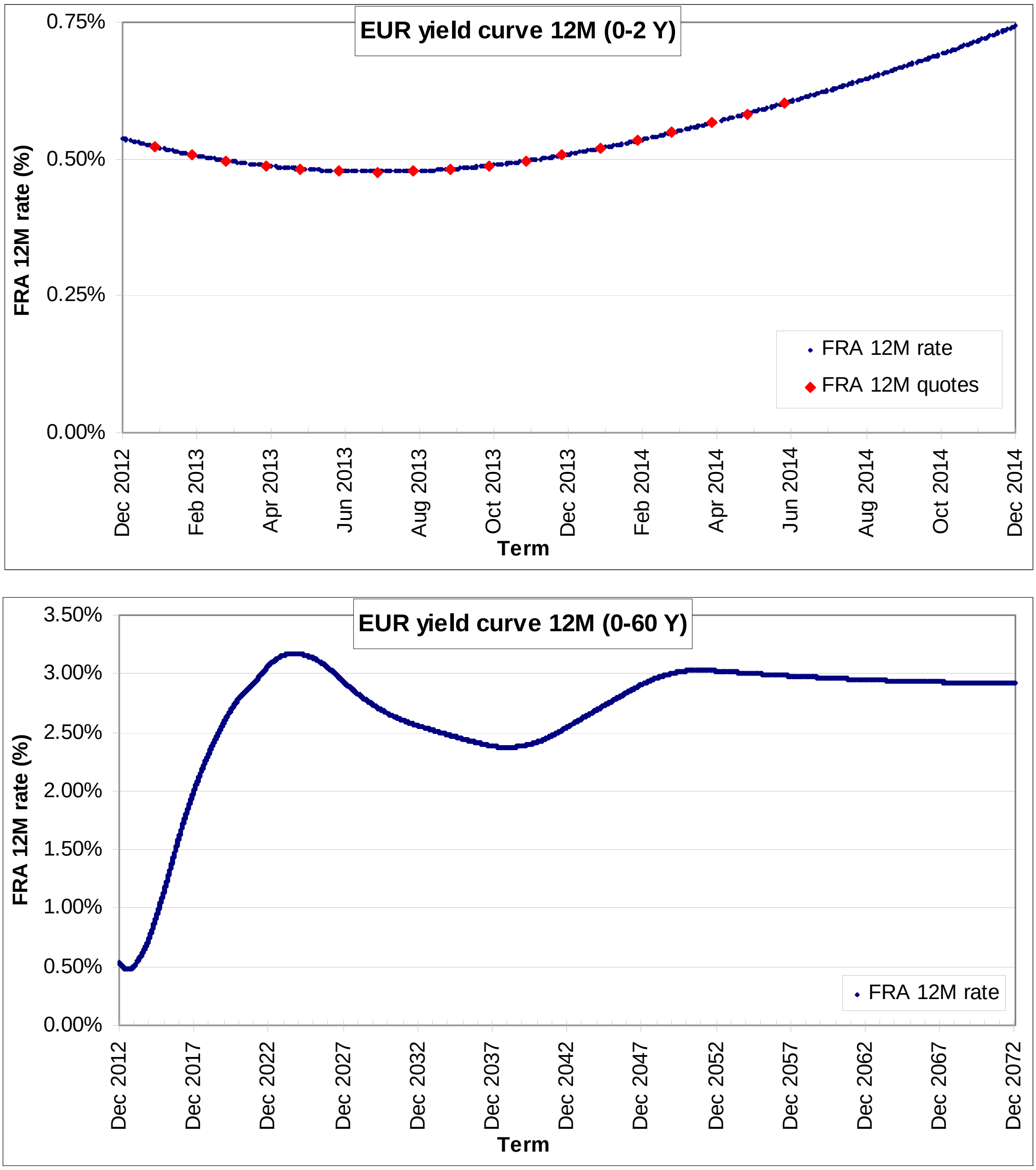
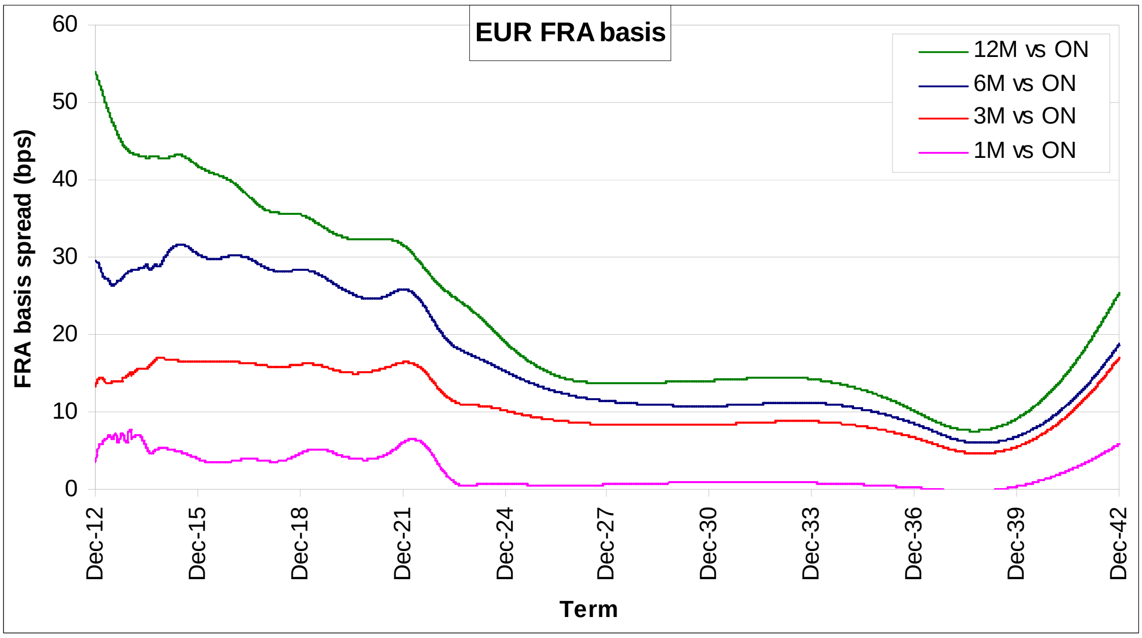


Figure 34: FRA curve $\mathcal{C}^f_{12M}$ on Euribor12M as of 11 Dec. 2012, plotted with 12M-tenor FRA rates $F(t_0; t, t+12M, act/360)$, t daily sampled. Top panel: short term structure up to 2 years. Bottom panel: whole term structure up to 30 years. FRA 12M are also reported.



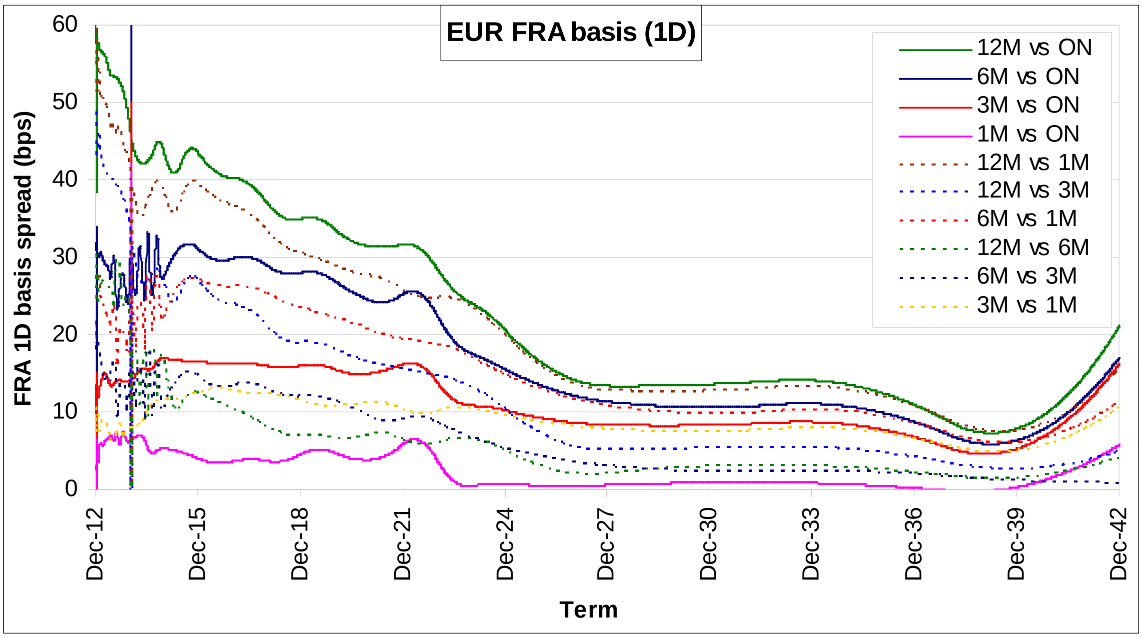


Figure 35: Forward IRBS curves as of 11 Dec. 2012. Top panel: difference between OIS vs Euribor FRA rates, with FRA tenor equal to Euribor tenor. Bottom panel: difference between FRA rates both with over night tenor.

year jumps included in some yield curves, and to the over night FRA rate tenor. On the other side, this representation is less significative from a financial point of view, because we compute a FRA rate with tenor x using an Euribor curve $\mathcal{C}_y^F$ with tenor $y \neq x$.

The top panel shows the same difference between two FRA rates, but in this case the length of both FRA is equal to the tenor x of the Euribor rate. This is equivalent to a series of forward starting zero coupon IRBS with length x. This representation is financially sound and much more smooth, with a residual irregularity due the different local shapes of the pair of yield curves involved.

6 Conclusions and Directions of Future Work

We have reviewed the fundamental pricing formulas for plain vanilla interest rate derivatives, extending the classical framework to the modern market situation with the funding and collateral We have illustrated a methodology for bootstrapping both discounting and FRA yield curves, consistently with the funding of market instruments and homogeneous in the underlying rate tenor.

Results for the concrete EUR market case have been analyzed in detail, showing how real quotations for interest rate instruments admitting Over Night, Euribor1M, 3M, 6M and 12M rates can be used in practice to construct market coherent, stable, robust and smooth yield curves for pricing and hedging interest rate derivatives. The full implementation of the present work, comprehensive of C++ code and Excel workbooks, is available open source in QuantLib.

The work presented here can be extended in many directions. An alternative approach to multiple-curve construction is the direct bootstrapping of multiple basis curves, on the top of the basic OIS yield curve. This technique promises smooth basis curves as discussed e.g. in [58]. An important extension is the multiple-currency-multiple-curves case, in which the currency of the collateral may differ from the currency of the deal, or we consider cross currency yield curves, as discussed e.g. in [59, 60, 61, 27, 62, 26, 63]. Finally, when forward bootstrapping is not possible, due to non-local interpolation OIS-discounting, and multiple-currencies, a more general calibration approach is needed, as suggested e.g. in [6, 64].

A Appendix A: Pricing Under Collateral

In this section we proof proposition 3.2. We use the replication approach, as discussed in standard textbooks, such as [23] and [24] (see also the original references [65, 66, 67]), extended to multiple sources of funding.

The assets available in the market for replicating the derivative Π are three: the risky underlying asset X, the treasury (funding) account B_f and the collateral account B_c. Having no dividends, we obtain the assets price, dividend and gain processes

$$\boldsymbol{\mathcal{X}}(t) = \begin{bmatrix} X(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}, \quad \boldsymbol{D}(t) = \boldsymbol{0}, \quad \boldsymbol{G}(t) = \boldsymbol{\mathcal{X}}(t). \tag{102}$$

Using eqs. 8 and 4, we obtain the corresponding price, dividend and gain dynamics

$$\boldsymbol{\mathcal{X}}(t) = \begin{bmatrix} \mu^P(t)S(t)dt + \sigma(t)S(t)dW^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix}, \quad d\boldsymbol{D}(t) = \boldsymbol{0}, \quad d\boldsymbol{G}(t) = d\boldsymbol{\mathcal{X}}(t). \tag{103}$$

The derivative's price dynamics $d\Pi(t,S)$ is obtained by Ito's Lemma as

$$\begin{aligned} d\Pi(t,X) &= \hat{\mathcal{D}}_\mu(t)\Pi(t,X)dt + \sigma(t)X(t)\frac{\partial \Pi}{\partial X}(t,X)dW^P(t), \\ \hat{\mathcal{D}}_\mu(t,X) :&= \frac{\partial}{\partial t} + \mu^P(t)X(t)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t)X^2(t)\frac{\partial^2}{\partial X^2}. \end{aligned} \tag{104}$$

We replicate the derivatives' price $\Pi(t,X)$, $\forall t \leq T$ with the trading strategy

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix}, \tag{105}$$

such that the corresponding strategy's price, dividend and gain dynamics are given, respectively, by

$$\begin{aligned} d\mathcal{X}(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}) &= d\left[\boldsymbol{\theta}(t)\cdot\boldsymbol{\mathcal{X}}(t)\right] = \boldsymbol{\theta}(t)\cdot d\boldsymbol{\mathcal{X}}(t) \\ &= \theta_1(t)dX(t) + \theta_2(t)dB_f(t) + \theta_3(t)dB_c(t) \\ &= \left[\theta_1(t)\mu^P(t)X(t) + \theta_2(t)r_f(t)B_f(t) + \theta_3(t)r_c(t)B_c(t)\right]dt \\ &+ \theta_1(t)\sigma(t)X(t)dW^P(t), \\ dD(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}) &= 0, \\ dG(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}) &= d\mathcal{X}(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}). \end{aligned} \tag{106}$$

We apply now the relevant conditions: self-financing, replication, perfect collateral, and risk free. The self-financing condition, $D(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}) = 0$, is automatically satisfied by the strategy by construction, thanks to the absence of dividends. The replication conditions are

$$\Pi(t,\boldsymbol{X}) = X(t,\boldsymbol{\theta},\boldsymbol{X}), \quad \forall\, t \in [0,T], \tag{107}$$

$$G_\Pi(t,\boldsymbol{X}) = G(t,\boldsymbol{\theta},\boldsymbol{X}), \quad \forall\, t \in [0,T], \tag{108}$$

where $G_\Pi(t,\boldsymbol{X})$ is the cumulative gain process associated with derivative Π. Using the first replication condition in eq. 107 we obtain

$$\begin{aligned} \Pi(t,S) = \mathcal{X}(t,\boldsymbol{\mathcal{X}},\boldsymbol{\theta}) &= \theta_1(t)X(t) + \theta_2(t)B_f(t) + \theta_3(t)B_c(t) \\ &= \theta_1(t)X(t) + \theta_2(t)B_f(t) + \theta_3(t)\Pi(t,X), \end{aligned} \tag{109}$$

where we have used the perfect collateral condition, $\Pi(t,X) = B_c(t)$, given in Def. 3.1. Setting, without loss of generality, $\theta_3(t) = 1$, we obtain the equation

$$\theta_2(t)B_f(t) = -\theta_1(t)X(t). \tag{110}$$

Using the second replication condition in eq. 108, the derivative's price dynamics in eq. 104, the strategy's gain dynamics in eq. 106, and eliminating $\theta_2(t)$ using eq. 109 above, we obtain

$$\begin{aligned}
dG_\Pi(t,X) = d\Pi(t,X) &= \hat{\mathcal{D}}_{\mu^P}\Pi(t,X)dt + \sigma(t)X(t)\frac{\partial\Pi}{\partial X}(t,X)dW^P(t) \\
&= dG(t,\boldsymbol{\theta},\boldsymbol{X}) = dX(t,\boldsymbol{X},\boldsymbol{\theta}) \\
&= \theta_1(t)\mu^P(t)X(t)dt + \theta_1(t)\sigma(t)X(t)dW^P(t) + d\Gamma(t,X), \qquad (111)
\end{aligned}$$

where we have defined the cash dynamics as

$$\frac{d\Gamma(t,X)}{dt} := -r_f(t)\theta_1(t)X(t) + r_c(t)\Pi(t,X). \qquad (112)$$

Rearranging the terms we obtain

$$\begin{aligned}
&\left\{\frac{\partial\Pi}{\partial t} + \mu^P(t)X(t)\left(\frac{\partial\Pi}{\partial X} - \theta_1(t)\right) + \frac{1}{2}\sigma^2(t)X^2(t)\frac{\partial^2\Pi}{\partial X^2}\right\}dt \\
&+ \sigma(t)X(t)\left[\frac{\partial\Pi}{\partial X} - \theta_1(t)\right]dW^P(t) = d\Gamma(t,X). \qquad (113)
\end{aligned}$$

Applying the market risk neutral condition $\theta_1(t) = \frac{\partial\Pi}{\partial X}$ we obtain the PDE

$$\begin{aligned}
&\hat{\mathcal{D}}_{r_f}\Pi(t,X) = r_c(t)\Pi(t,X), \\
&\hat{\mathcal{D}}_{r_f} = \frac{\partial}{\partial t} + r_f(t)X(t)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t)X^2(t)\frac{\partial^2}{\partial X^2}. \qquad (114)
\end{aligned}$$

The PDE above has the form of a backward Kolmogorov equation, given in eq. 117, with $A(t,S) = r_c(t)$ and $B(t,S) = 0$. By applying the Feynman-Kac theorem as in prop. B.1, we obtain the thesis in sec. 3.3.

B Appendix B: Feynman-Kac Theorem

The Feynman-Kac theorem establishes a relationship between (parabolic) partial differential equations (PDE) and stochastic differential equations (SDE). In this section we report a version of the theorem from [23], slightly more general than what can be found in other references, such as [24, 68]. See also [14]. We also adapt the notation to our context, for an easier usage in sec. 2.

Proposition B.1 (Feynman-Kac theorem)**.** *Given the boundary value problem*

$$\begin{aligned}
&\hat{\mathcal{D}}_\mu(t,X)f(t,X) = A(t,X)f(t,X) + B(t,X), \qquad (115)\\
&\hat{\mathcal{D}}_\mu(t,X) := \frac{\partial}{\partial t} + \mu(t,X)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t,X)\frac{\partial^2}{\partial X^2}, \qquad (116)\\
&f(T,X) = \psi(X),
\end{aligned}$$

also known as backward Kolmogorov equation*, where* $t \in [0,T]$*,* $X \in \mathbb{R}$*,* $\psi : \mathbb{R} \to \mathbb{R}$*,* $f \in \mathscr{C}^{1,2}\{[0,T]\times\mathbb{R}\to\mathbb{R}\}$*,* $A, B : [0,T]\times\mathbb{R}\to\mathbb{R}$*,* $\mu : [0,T]\times\mathbb{R}\to\mathbb{R}$*,* $\sigma : [0,T]\times\mathbb{R}\to\mathbb{R}_+$*, we have, under appropriate technical conditions on functions* $f, \psi, A, B, \mu, \sigma$*,*

$$f(t,X) = \mathbb{E}_t^Q\left[D(t,T,X)\psi(X) - \int_t^T D(u,T,X)B(u,X)du\right], \tag{117}$$

$$D(t,T,X) := \exp\left[-\int_t^T A(u,X)du\right], \tag{118}$$

under the probability measure Q *such that the asset* X *follows the Geometric Brownian Motion process*

$$\begin{aligned} &dX(t) = \mu(t,X)X(t)dt + \sigma(t,X)X(t)dW^Q(t),\\ &X(0) = X_0, \end{aligned} \tag{119}$$

where $W^Q \in \mathbb{R}$ *is a* 1*-dimensional independent standard Brownian motion in the probability space* $(\Omega, \mathcal{F}, Q)$*.*

Remark B.2 (Probability measure)**.** *We stress that the dynamics in eq. 119 differs from those in eq. 8 for the drift,* μ *instead of* μ^P*, and the associated probability measure,* Q *(risk neutral) instead of* P *(real). In particular, the risk neutral drift* μ *in eq. 119 is dictated by the coefficient* μ *in the differential operator* $\hat{\mathcal{D}}_\mu$ *in eq. 116, while the real drift* μ^P *disappears in the final eqs. 117 and 119. Since the stochastic terms in eq. 119 and 8 are, instead, the same, by Girsanov theorem (see e.g. [23], [24]) the Feynman-Kac theorem implies a change of probability measure.*

C Appendix C: Pricing Formulas for Plain Vanilla Interest Rate Derivatives

In this section we derive the pricing formulas for FRA, Futures, IRS, OIS and BIRS.

C.1 FRA

The payoff of the market FRA is given in eq. 42. Using the pricing under collateral approach discussed in sec. 3.2, proposition 3.3, we have that the price of the market FRA at time $t < T_{i-1}$ is given, under the payment forward measure $Q_f^{T_{i-1}}$, by

$$\begin{aligned} \mathbf{FRA}_{\text{Mkt}}(t;\boldsymbol{T},K,\omega) &= P_c(t;T_{i-1})\mathbb{E}_t^{Q_f^{T_{i-1}}}\left[\mathbf{FRA}_{\text{Mkt}}(T_{i-1};\boldsymbol{T},K,\omega)\right]\\ &= N\omega P_c(t;T_{i-1})\mathbb{E}_t^{Q_f^{T_{i-1}}}\left\{\frac{\left[L_x(T_{i-1},T_i)-K\right]\tau_L(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right\}\\ = N\omega P_c(t;T_{i-1})&\left\{1-\left[1+K\tau_L(T_{i-1},T_i)\right]\mathbb{E}_t^{Q_f^{T_{i-1}}}\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right]\right\}. \end{aligned} \tag{120}$$

Switching from $Q_f^{T_{i-1}}$ to $Q_f^{T_i}$, the expectation above becomes

$$
\begin{aligned}
\mathbb{E}_t^{Q_f^{T_{i-1}}}&\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right] \\
&= \frac{P_c(t,T_{i-1})}{P_c(t,T_i)}\mathbb{E}_t^{Q_f^{T_i}}\left[\frac{1}{P_c(T_{i-1},T_i)}\frac{1}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right], \\
&= \frac{1}{1+F_{c,i}(t)\tau_c(T_{i-1},T_i)}\mathbb{E}_t^{Q_f^{T_i}}\left[\frac{1+L_c(T_{i-1},T_i)\tau_c(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right]. \quad (121)
\end{aligned}
$$

Thus we have the pricing expression

$$
\begin{aligned}
\mathbf{FRA}_{\mathrm{Mkt}}(t;\boldsymbol{T},K,\omega) &= N\omega P_c(t;T_{i-1}) \\
&\times\left\{1-\frac{1+K\tau_L(T_{i-1},T_i)}{1+F_{c,i}(t)\tau_c(T_{i-1},T_i)}\mathbb{E}_t^{Q_f^{T_i}}\left[\frac{1+L_c(T_{i-1},T_i)\tau_c(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}\right]\right\}, \quad (122)
\end{aligned}
$$

and the market FRA equilibrium rate

$$
R_{x,Mkt}^{\mathrm{FRA}}(t;T_{i-1},T_i) = \frac{1}{\tau_L(T_{i-1},T_i)}\left\{\frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{\mathbb{E}_t^{Q_f^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]}-1\right\}. \quad (123)
$$

We observe that both the price and the equilibrium rate of the market FRA depend on the expectation of the ratio between the two rates $L_c(T_{i-1},T_i)$ and $L_x(T_{i-1},T_i)$ under the forward measure $Q_f^{T_i}$. This quantity depends on the particular model chosen for the joint distribution of these two rates. In general, for any model, we have

$$
\mathbb{E}_t^{Q_f^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] = \frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_{c,x}^{\mathrm{FRA}}(t;T_{i-1})}, \quad (124)
$$

such that

$$
\mathbf{FRA}_{\mathrm{Mkt}}(t;\boldsymbol{T},K,\omega) = N\omega P_c(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_{c,x}^{\mathrm{FRA}}(t;T_{i-1})}\right], \quad (125)
$$

$$
R_{x,\mathrm{Mkt}}^{\mathrm{FRA}}(t;\boldsymbol{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\left[1+F_{x,i}(t)\tau_x(T_{i-1},T_i)\right]e^{C_{c,x}^{\mathrm{FRA}}(t;T_{i-1})}-1\right\}, \quad (126)
$$

where $C_x^{\mathrm{FRA}}(t;T_{i-1})$ is a *convexity adjustment*, whose detailed expression depends on the chosen model for the dynamics of $F_{d,i}(t)$ and $F_{x,i}(t)$. A possible modeling choice is that of [42], in which the two FRA rates are modeled as shifted lognormal martingales under the collateral forward measure $Q_c^{T_i}$,

$$
\frac{dF_{c,i}(t)}{F_{c,i}(t)+\frac{1}{\tau_c(T_{i-1},T_i)}} = \sigma_{c,i}dW_c^{Q_f^{T_i}}(t), \quad (127)
$$

$$
\frac{dF_{x,i}(t)}{F_{x,i}(t)+\frac{1}{\tau_L(T_{i-1},T_i)}} = \sigma_{x,i}dW_x^{Q_f^{T_i}}(t), \quad (128)
$$

$$
dW_c^{Q_f^{T_i}}(t)dW_x^{Q_f^{T_i}}(t) = \rho_{c,x,i}dt. \quad (129)
$$

In this case the convexity adjustment assumes the form

$$C^{\mathrm{FRA}}_{c,x}(t;T_{i-1}) = \sigma^2_{x,i} - \sigma_{x,i}\sigma_{c,i}\rho_{c,x,i}\tau(t,T_{i-1}). \tag{130}$$

For typical post credit crunch market situations, the actual size of the convexity adjustment results to be below 1 bp, even for very long maturities (see [42]).

C.2 Futures

The payoff of Futures is given in eq. 52. Since Futures are collateralized financial instruments, we may use proposition 3.2 to obtain the price of the Futures at time $t < T_{i-1}$ as

$$\begin{aligned}
\mathbf{Futures}(t;\boldsymbol{T}) &= \mathbb{E}^{Q_f}_t\left[D_c(t;t)\mathbf{Futures}(T_{i-1};\boldsymbol{T})\right] && (131)\\
&= N\left\{1 - \mathbb{E}^{Q_f}_t\left[L_x(T_{i-1},T_i)\right]\right\} = N\left[1 - R^{\mathrm{Fut}}_x(t;\boldsymbol{T})\right], && (132)
\end{aligned}$$

where $R^{\mathrm{Fut}}_x(t;\boldsymbol{T})$ is the Futures' rate and Q_f is the risk neutral funding measure. Notice that the Futures daily margination mechanism implies that the payoff is regulated everyday, thus generating the unitary stochastic discount factor $D_c(t;t) = 1$ appearing in the first line above. Hence the pricing of Futures requires the computation of the Futures' rate

$$R^{\mathrm{Fut}}_x(t;\boldsymbol{T}) := \mathbb{E}^{Q_f}_t\left[L_x(T_{i-1},T_i)\right], = \mathbb{E}^{Q_f}_t\left[F_{x,i}(T_{i-1})\right]. \tag{133}$$

Since the FRA rate $F_{x,i}(t)$ is not a martingale under the risk neutral measure Q_f, such computation requires the adoption of a model for the dynamics of $F_{x,i}(t)$, similarly to the case of the market FRA discussed in app. C.1. In general, we obtain that the Futures' rate is given by the corresponding FRA rate corrected with a convexity adjustment

$$\begin{aligned}
R^{\mathrm{Fut}}_x(t;\boldsymbol{T}) := \mathbb{E}^{Q_f}_t\left[L_x(T_{i-1},T_i)\right] &= \mathbb{E}^{Q^{T_i}_f}_t\left[L_x(T_{i-1},T_i)\right] + C^{\mathrm{Fut}}_x(t;T_{i-1})\\
&= F_{x,i}(t) + C^{\mathrm{Fut}}_x(t;T_{i-1}). && (134)
\end{aligned}$$

The expression of the convexity adjustment will depend on the particular model adopted, and will contain, in general, the model's volatilities and correlations. We report here the result of the most recent derivation given by [42] under the multiple-curve

Libor Market Model, where the convexity adjustment takes the form

$$
\begin{aligned}
C_x^{\mathrm{Fut}}(t; T_{i-1}) &\simeq F_{x,i}(t) \left[\exp \int_t^{T_{i-1}} \mu_{x,i}(u)\, du - 1 \right], \\
\frac{dF_{c,i}(t)}{F_{c,i}(t)} &= \mu_{c,i}(t) dt + \sigma_{c,i} dW_c^{Q_f^{T_i}}(t), \\
\frac{dF_{x,i}(t)}{F_{x,i}(t)} &= \mu_{x,i}(t) dt + \sigma_{x,i} dW_x^{Q_f^{T_i}}(t), \\
\mu_{c,i}(t) &:= \sigma_{c,i} \sum_{j=1}^{i} \frac{\sigma_{c,j} \rho_{c,c,j,i} \tau_{c,j} F_{c,j}(t)}{1 + \tau_{c,j} F_{c,j}(t)}, \\
\mu_{x,i}(t) &:= \sigma_{x,i} \sum_{j=1}^{i} \frac{\sigma_{x,j} \rho_{x,c,j,i} \tau_{x,j} F_{x,j}(t)}{1 + \tau_{x,j} F_{x,j}(t)}, \\
\int_t^{T_{i-1}} \mu_{x,i}(u)\, du &\simeq \sigma_{x,i} \sum_{j=1}^{i} \frac{\sigma_{c,j} \rho_{x,c,j,i} \tau_{c,j} F_{c,j}(t)}{1 + \tau_{c,j} F_{c,j}(t)} \tau(t, T_{j-1}), \\
F_{c,j}(t) &:= \mathbb{E}_t^{Q_f^{T_j}} \left[L_c(T_{j-1}, T_j) \right] = \frac{1}{\tau_{c,j}} \left[\frac{P_c(t; T_{j-1})}{P_c(t; T_j)} - 1 \right],
\end{aligned} \tag{135}
$$

where $\sigma_{x,i},\ \sigma_{c,j},\ \rho_{x,c,i,j}\ \rho_{c,c,i,j}$ are the instantaneous (deterministic) volatilities and correlations of $F_{x,i}(t), F_{c,j}(t)$, respectively.

C.3 IRS

The payoff of IRSlets are given in eq. 60. Using the pricing under collateral approach discussed in sec. 3.2, proposition 3.3, we obtain that the price of the IRSlets at time $t < T_{i-1}$ is given, under the payment forward measure $Q_f^{T_i}$, by

$$
\begin{aligned}
\mathbf{IRSlet}_{\mathrm{float}}(t; T_{i-1}, T_i, L_x) &= P_c(t; T_i) \mathbb{E}_t^{Q_f^{T_i}} \left[\mathbf{IRSlet}_{\mathrm{float}}(T_{i-1}; T_{i-1}, T_i, L_x) \right] \\
&= N P_c(t; T_i) F_{x,i}(t) \tau_L(T_{i-1}, T_i),
\end{aligned} \tag{136}
$$

$$
\begin{aligned}
\mathbf{IRSlet}_{\mathrm{fix}}(t; S_{j-1}, S_j, K) &= P_c(t; S_j) \mathbb{E}_t^{Q_f^{S_j}} \left[\mathbf{IRSlet}_{\mathrm{fix}}(S_{j-1}; S_{j-1}, S_j, K) \right] \\
&= N P_c(t; S_j) K \tau_K(S_{j-1}, S_j).
\end{aligned} \tag{137}
$$

Thus the price of the complete IRS is given by

$$
\begin{aligned}
\mathbf{IRS}(t; \boldsymbol{T}, \boldsymbol{S}, L_x, K, \omega) :&= \omega \left[\mathbf{IRS}_{\mathrm{float}}(t; \boldsymbol{T}, L_x) - \mathbf{IRS}_{\mathrm{fix}}(t; \boldsymbol{S}, K) \right] \\
&= \omega N \left[R_x^{\mathrm{IRS}}(t; \boldsymbol{T}, \boldsymbol{S}) - K \right] A_c(t; \boldsymbol{S}),
\end{aligned} \tag{138}
$$

where $\omega = +/-1$ for a payer/receiver IRS (referred to the fixed rate K), and the legs values, the equilibrium IRS rate and the annuity have been defined as

$$\begin{aligned}
\mathbf{IRS}_{\text{float}}(t;\boldsymbol{T},L_x) &:= \sum_{i=1}^{n}\mathbf{IRSlet}_{\text{float}}(t;T_{i-1},T_i,L_x) \\
&= N\sum_{i=1}^{n}P_c(t;T_i)F_{x,i}(t)\tau_L(T_{i-1},T_i), && (139)\\
\mathbf{IRS}_{\text{fix}}(t;\boldsymbol{S},K) &:= \sum_{j=1}^{m}\mathbf{IRSlet}_{\text{fix}}(t;S_{j-1},S_j,K) \\
&= NKA_c(t;\boldsymbol{S}), && (140)\\
R_x^{\text{IRS}}(t;\boldsymbol{T},\boldsymbol{S}) &:= \frac{\sum_{i=1}^{n}P_c(t;T_i)F_{x,i}(t)\tau_L(T_{i-1},T_i)}{A_c(t;\boldsymbol{S})}, && (141)\\
A_c(t;\boldsymbol{S}) &:= \sum_{j=1}^{m}P_c(t;S_j)\tau_K(S_{j-1},S_j), && (142)
\end{aligned}$$

respectively.

C.4 OIS

The payoff of the OIS is given in eqs. 71, 72. Using the pricing under collateral approach discussed in sec. 3.2, we obtain that the prices of OISlets at time $t < T_{i-1}$ are given by

$$\begin{aligned}
\mathbf{OISlet}_{\text{float}}(t;T_{i-1},T_i,R_{on}) &= P_c(t;T_i)\mathbb{E}_t^{Q_f^{T_i}}\left[\mathbf{OISlet}_{\text{float}}(T_{i-1};T_{i-1},T_i,R_{on})\right] \\
&= NP_c(t;T_i)R_{on}(t;\boldsymbol{T}_i)\tau_{on}(T_{i-1},T_i), && (143)\\
\mathbf{OISlet}_{\text{fix}}(t;S_{j-1},S_j,K) &= \mathbf{IRSlet}_{\text{fix}}(t;S_{j-1},S_j,K), && (144)
\end{aligned}$$

where the floating coupon rate is given by

$$\begin{aligned}
R_{on}(t;\boldsymbol{T}_i) &:= \mathbb{E}_t^{Q_f^{T_i}}\left[R_{on}(T_i;\boldsymbol{T}_i)\right] \\
&= \frac{1}{\tau_{on}(T_{i-1},T_i)}\mathbb{E}_t^{Q_f^{T_i}}\left\{\prod_{k=1}^{n_i}\left[1+R_{on}(T_{i,k-1},T_{i,k})\tau_{on}(T_{i,k-1},T_{i,k})\right]-1\right\} \\
&= \frac{1}{\tau_{on}(T_{i-1},T_i)}\left\{\prod_{k=1}^{n_i}\left[1+\mathbb{E}_t^{Q_f^{T_{i,k}}}\left[R_{on}(T_{i,k-1},T_{i,k})\right]\tau_{on}(T_{i,k-1},T_{i,k})\right]-1\right\} \\
&= \frac{1}{\tau_{on}(T_{i-1},T_i)}\left\{\prod_{k=1}^{n_i}\left[1+R_{on}(t;T_{i,k-1},T_{i,k})\tau_{on}(T_{i,k-1},T_{i,k})\right]-1\right\}, && (145)
\end{aligned}$$

where we have used the tower rule for nested conditioned expectations, and we have defined the over night FRA rate, insisting onto the over night time interval $[T_{i,k-1},T_{i,k}]$, as

$$R_{on}(t;T_{i,k-1},T_{i,k}) := \mathbb{E}_t^{Q_f^{T_i}}\left[R_{on}(T_{i,k-1},T_{i,k})\right]. \tag{146}$$

Under the perfect collateral assumption the over night FRA rates $R_{on}(T_{i,k-1}, T_{i,k})$ may be considered martingales under the funding measure $Q_f^{T_i}$, such that the classical single-curve expression,

$$R_{on}(t; T_{i,k-1}, T_{i,k}) = \frac{1}{\tau_{on}(T_{i,k-1}, T_{i,k})} \left[\frac{P_c(t; T_{i,k-1})}{P_c(t; T_{i,k})} - 1 \right], \tag{147}$$

holds. In other words, eq. 147 assumes that the over night FRA rates $R_{on}(T_{i,k-1}, T_{i,k})$ may be replicated using collateral zero coupon bonds $P_c(t;T)$. In this case the over night FRA rates assume a simpler expression

$$\begin{aligned}
R_{on}(t; \boldsymbol{T}_i) &= \frac{1}{\tau_{on}(T_{i-1}, T_i)} \left\{ \prod_{k=1}^{n_i} \left[\frac{P_c(t; T_{i,k-1})}{P_c(t; T_{i,k})} \right] - 1 \right\} \\
&= \frac{1}{\tau_R(T_{i-1}, T_i)} \left[\frac{P_c(t; T_{i,0})}{P_c(t; T_{i,1})} \frac{P_c(t; T_{i,1})}{P_c(t; T_{i,2})} \cdots \frac{P_c(t; T_{i,n_i-1})}{P_c(t; T_{i,n_i})} - 1 \right] \\
&= \frac{1}{\tau_{on}(T_{i-1}, T_i)} \left[\frac{P_c(t; T_{i-1})}{P_c(t; T_i)} - 1 \right] \\
&= F_c(t; T_{i-1}, T_i),
\end{aligned} \tag{148}$$

where we have used the telescopic rule for the zero coupon bonds inside the product. The price of the complete OIS is thus given by

$$\begin{aligned}
\mathbf{OIS}(t; \boldsymbol{T}, \boldsymbol{S}, R_{on}^c, K, \omega) &= \omega \left[\mathbf{OIS}_{\text{float}}(t; \boldsymbol{T}, R_{on}) - \mathbf{OIS}_{\text{fix}}(t; \boldsymbol{S}, K) \right] \\
&= N\omega \left[R_{on}^{\text{OIS}}(t; \boldsymbol{T}, \boldsymbol{S}) - K \right] A_c(t; \boldsymbol{S}),
\end{aligned} \tag{149}$$

where the OIS legs values and the equilibrium OIS rate are

$$\begin{aligned}
\mathbf{OIS}_{\text{float}}(t; \boldsymbol{T}, R_{on}) :&= \sum_{i=1}^{n} \mathbf{OISlet}_{\text{float}}(t; T_{i-1}, T_i, R_{on}) \\
&= N \sum_{i=1}^{n} P_c(t; T_i) R_{on}(t; \boldsymbol{T}_i) \tau_{on}(T_{i-1}, T_i) \\
&= N \sum_{i=1}^{n} \left[P_c(t; T_{i-1}) - P_c(t; T_i) \right] \\
&= N \left[P_c(t; T_0) - P_c(t; T_n) \right],
\end{aligned} \tag{150}$$

$$\begin{aligned}
\mathbf{OIS}_{\text{fix}}(t; \boldsymbol{S}, K) :&= \sum_{i=1}^{n} \mathbf{OISlet}_{\text{fix}}(t; S_{j-1}, S_j, K) \\
&= \mathbf{IRS}_{\text{fix}}(t; \boldsymbol{S}, K) \\
&= NKA_c(t; \boldsymbol{S}),
\end{aligned} \tag{151}$$

$$\begin{aligned}
R_{on}^{\text{OIS}}(t; \boldsymbol{T}, \boldsymbol{S}) &= \frac{\sum_{i=1}^{n} P_c(t; T_i) R_{on}(t; \boldsymbol{T}_i) \tau_{on}(T_{i-1}, T_i)}{A_c(t; \boldsymbol{S})} \\
:&= \frac{P_c(t; T_0) - P_c(t; T_n)}{A_c(t; \boldsymbol{S})}.
\end{aligned} \tag{152}$$

C.5 IRBS

IRBS as Two IRS

The schedule and payoff of IRBS as two IRS are given in eqs. 79, 80, respectively. The IRBS price is given, using eq. 138, by

$$
\begin{aligned}
\mathbf{IRBS}(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S},L_x,L_y,K,\omega) &= \mathbf{IRS}_x(t;\boldsymbol{T}_x,\boldsymbol{S},L_x,K,\omega) - \mathbf{IRS}_y(t;\boldsymbol{T}_y,\boldsymbol{S},L_y,K,\omega),\\
\mathbf{IRS}_x(t;\boldsymbol{T}_x,\boldsymbol{S},L_x,K,\omega) &= N\omega\left[R_x^{\mathrm{IRS}}(t;\boldsymbol{T}_x,\boldsymbol{S}) - K\right]A_c(t,\boldsymbol{S}),\\
\mathbf{IRS}_y(t;\boldsymbol{T}_y,\boldsymbol{S},L_y,K,\omega) &= N\omega\left[R_y^{\mathrm{IRS}}(t;\boldsymbol{T}_y,\boldsymbol{S}) - K\right]A_c(t,\boldsymbol{S}),
\end{aligned}
\tag{153}
$$

and the corresponding equilibrium IRBS spread is defined as the difference between the two IRS rates

$$
\begin{aligned}
\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S}) : &= R_x^{\mathrm{IRS}}(t;\boldsymbol{T}_x,\boldsymbol{S}) - R_y^{\mathrm{IRS}}(t;\boldsymbol{T}_y,\boldsymbol{S})\\
&= \frac{\mathbf{IRS}_{x,\mathrm{float}}(t;\boldsymbol{T}_x,L_x) - \mathbf{IRS}_{y,\mathrm{float}}(t;\boldsymbol{T}_y,L_y)}{NA_c(t;\boldsymbol{S})}.
\end{aligned}
\tag{154}
$$

IRBS as Single IRS

The schedule and payoff of IRBS as single IRS are given in eqs. 82, 83, respectively. The IRBS price is given, using eq. 138 and factorising out the spread in the second floating leg, by

$$
\begin{aligned}
\mathbf{IRBS}(t;\boldsymbol{T}_x,\boldsymbol{T}_y,L_x,L_y,\omega,\Delta_{x,y}) : &= \omega\left[\mathbf{IRS}_{x,\mathrm{float}}(t;\boldsymbol{T}_x,L_x) - \mathbf{IRS}_{y,\mathrm{float}}(t;\boldsymbol{T}_y,L_y,\Delta_{x,y})\right]\\
&= \omega\left[\mathbf{IRS}_{x,\mathrm{float}}(t;\boldsymbol{T}_x,L_x) - \mathbf{IRS}_{y,\mathrm{float}}(t;\boldsymbol{T}_y,L_y)\right.\\
&\quad\left. -\ N\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)A_c(t;\boldsymbol{T}_y)\right],
\end{aligned}
\tag{155}
$$

and the corresponding equilibrium IRBS spread may be obtained as

$$
\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y) = \frac{\mathbf{IRS}_{x,\mathrm{float}}(t;\boldsymbol{T}_x,L_x) - \mathbf{IRS}_{y,\mathrm{float}}(t;\boldsymbol{T}_y,L_y)}{NA_c(t;\boldsymbol{T}_y)}.
\tag{156}
$$

We observe that the two definitions of IRBS are equivalent, but the two basis spreads in eqs. 154 and 156 are not equal, because of the two different annuities involved

$$
\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S}) = \frac{A_c(t;\boldsymbol{T}_y)}{A_c(t;\boldsymbol{S})}\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y).
\tag{157}
$$

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