# 筆記

陳定善

## 1 logic 邏輯

這一章只會簡單介紹會用到的邏輯符號及基本的一些公理 (axiom) 與定理 (theorem)。
通常上，「對」會表示爲 $T$ 或是 $\top$，而「錯」會表達爲 $F$ 或是 $\bot$。在這篇文章當中，會以 $\top$ 及 $\bot$ 表達。
首先，「公理」是對該話題的預先假設，而「定理」是從假設中推論出來的，而定理會附帶證明。通常，推論會寫成 $A, B \vdash C$，意味著以 $A, B$ 爲前提推論出 $C$。
若是以 $\vdash A$ 表達，則代表除了此定理或公理外，不需要其他前提，就可以推導出 $A$。
公理：

$$\vdash A \to (B \to A) \tag{1}$$

公理：

$$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{2}$$

公理：

$$(A \to B), A \vdash B \tag{MP}$$

以上三個公理中，$A, B, C$ 是任意敘述。以上公理，可以理解成是在對 $\to$ 做定義，只要符合以上公理形式的概念，都是可以使用的。
在定理的證明當中，我會以以下格式書寫：

| | | |
|---|---|---|
| 條目 | 內容 | (前提) |
| 條目 | 內容 | (前提) |
| 條目 | 內容 | (前提) |
| ⋯ | ⋯ | (⋯) |

定理：

$$A \vdash B \to A \tag{3}$$

證明：

| | | |
|---|---|---|
| i | $A$ | (前提) |
| ii | $A \to (B \to A)$ | (1) |
| iii | $B \to A$ | (i, ii, MP) |

定理：

$$A \to (B \to C) \vdash (A \to B) \to (A \to C) \tag{4}$$

證明：

| | | |
|---|---|---|
| i | $A \to (B \to C)$ | (前提) |
| ii | $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ | (2) |
| iii | $(A \to B) \to (A \to C)$ | (i, ii, MP) |

定理：

$$\vdash A \to A \tag{5}$$

證明：

| | | |
|---|---|---|
| i | $A \to ((A \to A) \to A)$ | (1) |
| ii | $(A \to (A \to A)) \to (A \to A)$ | (i, 4) |
| iii | $A \to (A \to A)$ | (1) |
| iv | $A \to A$ | (ii, iii, MP) |

定理：

$$(A \to B), (B \to C) \vdash (A \to C) \tag{6}$$

證明：

| | | |
|---|---|---|
| i | $(A \to B)$ | (前提) |
| ii | $(B \to C)$ | (前提) |
| iii | $A \to (B \to C)$ | (ii, 3) |
| iv | $(A \to B) \to (A \to C)$ | (iii, 4) |
| v | $A \to C$ | (i, iv, MP) |

定理：

$$A \to (B \to C), B \vdash (A \to C) \tag{7}$$

證明：

| | | |
|---|---|---|
| i | $A \to (B \to C)$ | (前提) |
| ii | $B$ | (前提) |
| iii | $(A \to B) \to (A \to C)$ | (i, 4) |
| iv | $A \to B$ | (ii, 3) |
| v | $A \to C$ | (iii, iv, MP) |

公理：

$$\vdash ((\neg A) \to (\neg B)) \to (B \to A) \tag{8}$$

這個公理是對 $\neg$ 的定義。

定理：

$$(\neg A) \to (\neg B) \vdash B \to A \tag{9}$$

證明：

| | | |
|---|---|---|
| i | $(\neg A) \to (\neg B)$ | (前提) |
| ii | $((\neg A) \to (\neg B)) \to (B \to A)$ | (8) |
| iii | $B \to A$ | (i, ii, MP) |

定理：

$$\vdash (\neg\neg A) \to A \tag{10}$$

證明：

| | | |
|---|---|---|
| i | $(\neg\neg A) \to ((\neg\neg\neg\neg A) \to (\neg\neg A))$ | (1) |
| ii | $((\neg\neg\neg\neg A) \to (\neg\neg A)) \to ((\neg A) \to (\neg\neg\neg A))$ | (8) |
| iii | $(\neg\neg A) \to ((\neg A) \to (\neg\neg\neg A))$ | (i, ii, 6) |
| iv | $((\neg A) \to (\neg\neg\neg A)) \to ((\neg\neg A) \to A)$ | (8) |
| v | $(\neg\neg A) \to ((\neg\neg A) \to A)$ | (iii, iv, 6) |
| vi | $((\neg\neg A) \to (\neg\neg A)) \to ((\neg\neg A) \to A)$ | (v, 4) |
| viii | $(\neg\neg A) \to (\neg\neg A)$ | (5) |
| ix | $(\neg\neg A) \to A$ | (vii, viii, MP) |

定理：

$$\vdash A \to (\neg\neg A) \tag{11}$$

證明：

| | | |
|---|---|---|
| i | $(\neg\neg\neg A) \to (\neg A)$ | (10) |
| iii | $A \to (\neg\neg A)$ | (i, 9) |

定理：

$$\vdash (A \to B) \to ((\neg B) \to (\neg A)) \tag{12}$$

證明：

| | | |
|---|---|---|
| i | $(A \to B) \to ((\neg\neg A) \to (A \to B))$ | (1) |
| ii | $((\neg\neg A) \to (A \to B)) \to (((\neg\neg A) \to A) \to ((\neg\neg A) \to B))$ | (2) |
| iii | $(A \to B) \to (((\neg\neg A) \to A) \to ((\neg\neg A) \to B))$ | (i, ii, 6) |
| iv | $(\neg\neg A) \to A$ | (10) |
| v | $(A \to B) \to ((\neg\neg A) \to B)$ | (iii, iv, 7) |
| vi | $B \to (\neg\neg B)$ | (11) |
| vii | $(\neg\neg A) \to (B \to (\neg\neg B))$ | (vi, 3) |
| viii | $((\neg\neg A) \to B) \to ((\neg\neg A) \to (\neg\neg B))$ | (vii, 4) |
| ix | $(A \to B) \to ((\neg\neg A) \to (\neg\neg B))$ | (v, viii, 6) |
| x | $((\neg\neg A) \to (\neg\neg B)) \to ((\neg B) \to (\neg A))$ | (8) |
| xi | $(A \to B) \to ((\neg B) \to (\neg A))$ | (ix, x, 6) |

定理：

$$A \to B \vdash (\neg B) \to (\neg A) \tag{13}$$

證明：

| | | |
|---|---|---|
| i | $A \to B$ | (前提) |
| ii | $(A \to B) \to ((\neg B) \to (\neg A))$ | (12) |
| iii | $(\neg B) \to (\neg A)$ | (i, ii, MP) |

定理：

$$\vdash ((\neg A) \to A) \to A \tag{14}$$

證明：

| | | |
|---|---|---|
| i | $((\neg A) \to A) \to ((\neg A) \to A)$ | (5) |
| ii | $(((\neg A) \to A) \to (\neg A)) \to (((\neg A) \to A) \to A)$ | (i, 4) |
| iii | $(\neg A) \to (((\neg A) \to A) \to (\neg A))$ | (1) |
| iv | $(\neg A) \to (((\neg A) \to A) \to A)$ | (ii, iii, 6) |
| v | $(((\neg A) \to A) \to A) \to ((\neg A) \to (\neg((\neg A) \to A)))$ | (12) |
| vi | $(\neg A) \to ((\neg A) \to (\neg((\neg A) \to A)))$ | (v, vi, 6) |
| vii | $((\neg A) \to (\neg A)) \to ((\neg A) \to (\neg((\neg A) \to A)))$ | (vi, 4) |
| viii | $(\neg A) \to (\neg A)$ | (5) |
| ix | $(\neg A) \to (\neg((\neg A) \to A))$ | (vii, viii, MP) |
| x | $((\neg A) \to A) \to A$ | (ix, 9) |

定理：

$$\vdash ((\neg A) \to (\neg B)) \to (((\neg A) \to B) \to A) \tag{15}$$

證明：

| | | |
|---|---|---|
| i | $((\neg A) \to A) \to A$ | (14) |
| ii | $(B \to A) \to ((\neg A) \to (B \to A))$ | (1) |
| iii | $((\neg A) \to (B \to A)) \to (((\neg A) \to B) \to ((\neg A) \to A))$ | (2) |
| iv | $(B \to A) \to (((\neg A) \to B) \to ((\neg A) \to A))$ | (ii, iii, 6) |
| v | $((\neg A) \to B) \to (((\neg A) \to A) \to A)$ | (i, 3) |
| vi | $(B \to A) \to (((\neg A) \to B) \to (((\neg A) \to A) \to A))$ | (v, 3) |
| vii | $(((\neg A) \to B) \to (((\neg A) \to A) \to A)) \to$ $((((\neg A) \to B) \to ((\neg A) \to A)) \to (((\neg A) \to B) \to A))$ | (2) |
| viii | $(B \to A) \to ((((\neg A) \to B) \to ((\neg A) \to A)) \to (((\neg A) \to B) \to A))$ | (vi, vii, 6) |
| ix | $((B \to A) \to (((\neg A) \to B) \to ((\neg A) \to A))) \to$ $((B \to A) \to (((\neg A) \to B) \to A))$ | (viii, 4) |
| x | $(B \to A) \to (((\neg A) \to B) \to A)$ | (iv, ix, MP) |

定理：

$$A \to B, (\neg A) \to B \vdash B \tag{16}$$

證明：

| | | |
|---|---|---|
| i | $A \to B$ | (前提) |
| ii | $(\neg A) \to B$ | (前提) |
| iii | $(\neg B) \to (\neg A)$ | (i, 13) |
| iv | $(\neg B) \to B$ | (i, ii, 6) |
| v | $((\neg B) \to B) \to B$ | (14) |
| vi | $B$ | (iv, v, MP) |

定理：

$$A, \neg A \vdash B \tag{17}$$

證明：

| | | |
|---|---|---|
| i | $A$ | (前提) |
| ii | $\neg A$ | (前提) |
| iii | $(\neg B) \to A$ | (i, 3) |
| iv | $(\neg B) \to (\neg A)$ | (ii, 3) |
| v | $(\neg A) \to (\neg\neg B)$ | (iii, 13) |
| vi | $(\neg\neg B) \to B$ | (10) |
| vii | $(\neg A) \to B$ | (v, vi, 6) |
| viii | $A \to B$ | (iv, 9) |
| ix | $B$ | (vii, viii, 16) |

定理：

$$A, \neg B \vdash \neg(A \to B) \tag{18}$$

證明：

i $A$ (前提)

ii $\neg B$ (前提)

iii $(A \to B) \to (A \to B)$ (5)

iv $((A \to B) \to A) \to ((A \to B) \to B)$ (iii, 4)

v $(A \to B) \to A$ (i, 3)

vi $(A \to B) \to B$ (iv, v, MP)

vii $(\neg B) \to (\neg(A \to B))$ (vi, 13)

viii $\neg(A \to B)$ (ii, vii, MP)

定理：

$$\neg(A \to B) \vdash \neg B \tag{19}$$

證明：

i $\neg(A \to B)$ (前提)

ii $B \to (A \to B)$ (1)

iii $(\neg(A \to B)) \to (\neg B)$ (ii, 13)

iv $\neg B$ (i, iii, MP)

定理：

$$\neg(A \to B) \vdash A \tag{20}$$

證明：

i $\neg(A \to B)$ (前提)

ii $(\neg A) \to ((\neg B) \to (\neg A))$ (1)

iii $((\neg B) \to (\neg A)) \to (A \to B)$ (8)

iv $(\neg A) \to (A \to B)$ (ii, iii, 6)

v $(\neg(A \to B)) \to (\neg\neg A)$ (iv, 13)

vi $(\neg\neg A)$ (i, v, MP)

vii $(\neg\neg A) \to A$ (10)

viii $A$ (vi, vii, MP)

定理：

$$A \to (\neg B), B \vdash \neg A \tag{21}$$

證明：

i $A \to (\neg B)$ (前提)

ii $B$ (前提)

iii $B \to (\neg\neg B)$ (11)

iv $(\neg\neg B) \to (\neg A)$ (i, 13)

v $B \to (\neg A)$ (iii, iv, 6)

vi $\neg A$ (ii, v, MP)

定理：

$$(\neg A) \to B, \neg B \vdash A \tag{22}$$

證明：

i $(\neg A) \to B$ (前提)

ii $\neg B$ (前提)

iii $(\neg B) \to (\neg\neg A)$ (i, 13)

iv $(\neg\neg A) \to A$ (10)

v $\neg\neg A$ (ii, iii, MP)

vi $A$ (iv, v, MP)

定義：

$$A \wedge B := \neg(A \to (\neg B)) \tag{23}$$

這種 $:=$ 符號稱爲定義，例如 $A := B$ 就是將 $A$ 定義爲 $B$。
定義也可以看成是一種縮寫，將一個概念包裝成新的符號。
定理：

$$\vdash (A \wedge B) \to (B \wedge A) \tag{24}$$

證明：

i $$((\neg\neg A) \to (\neg B)) \to (A \to ((\neg\neg A) \to (\neg B))) \quad (1)$$
ii $$(A \to ((\neg\neg A) \to (\neg B))) \to ((A \to (\neg\neg A)) \to (A \to (\neg B))) \quad (2)$$
iii $$((\neg\neg A) \to (\neg B)) \to ((A \to (\neg\neg A)) \to (A \to (\neg B))) \quad \text{(i, ii, 6)}$$
iv $$A \to (\neg\neg A) \quad (11)$$
v $$((\neg\neg A) \to (\neg B)) \to (A \to (\neg B)) \quad \text{(iii, iv, 7)}$$
vi $$(B \to (\neg A)) \to ((\neg\neg A) \to (\neg B)) \quad (12)$$
vii $$(B \to (\neg A)) \to (A \to (\neg B)) \quad \text{(v, vi, 6)}$$
viii $$(\neg(A \to (\neg B))) \to (\neg(B \to (\neg A))) \quad \text{(vii, 13)}$$
ix $$(A \wedge B) \to (B \wedge A) \quad \text{(viii, 23)}$$

定理：

$$A \wedge B \vdash B \wedge A \quad (25)$$

證明：

i $$A \wedge B \quad \text{(前提)}$$
ii $$(A \wedge B) \to (B \wedge A) \quad (24)$$
iii $$B \wedge A \quad \text{(i, ii, MP)}$$

定理：

$$\vdash (A \to (B \to C)) \to ((A \wedge B) \to C) \quad (26)$$

證明：

i $$(B \to C) \to ((\neg C) \to (\neg B)) \quad (12)$$
ii $$A \to ((B \to C) \to ((\neg C) \to (\neg B))) \quad \text{(i, 3)}$$
iii $$(A \to (B \to C)) \to (A \to ((\neg C) \to (\neg B))) \quad \text{(ii, 4)}$$
iv $$(A \to ((\neg C) \to (\neg B))) \to ((A \to (\neg C)) \to (A \to (\neg B))) \quad (2)$$
v $$(A \to (B \to C)) \to ((A \to (\neg C)) \to (A \to (\neg B))) \quad \text{(iii, iv, 6)}$$
vi $$((A \to (\neg C)) \to (A \to (\neg B))) \to ((\neg C) \to ((A \to (\neg C)) \to (A \to (\neg B)))) \quad (1)$$
vii $$(A \to (B \to C)) \to ((\neg C) \to ((A \to (\neg C)) \to (A \to (\neg B)))) \quad \text{(v, vi, 6)}$$
viii $$((\neg C) \to ((A \to (\neg C)) \to (A \to (\neg B)))) \to (((\neg C) \to (A \to (\neg C))) \to ((\neg C) \to (A \to (\neg B)))) \quad (2)$$
ix $$(A \to (B \to C)) \to (((\neg C) \to (A \to (\neg C))) \to ((\neg C) \to (A \to (\neg B)))) \quad \text{(vii, viii, 6)}$$
x $$(\neg C) \to (A \to (\neg C)) \quad (1)$$
xi $$(A \to (B \to C)) \to ((\neg C) \to (A \to (\neg B))) \quad \text{(ix, x, 7)}$$
xii $$(A \to (\neg B)) \to (\neg\neg(A \to (\neg B))) \quad (11)$$
xiii $$(\neg C) \to ((A \to (\neg B)) \to (\neg\neg(A \to (\neg B)))) \quad \text{(xii, 3)}$$
xiv $$((\neg C) \to (A \to (\neg B))) \to ((\neg C) \to (\neg\neg(A \to (\neg B)))) \quad \text{(xiii, 4)}$$
xv $$(A \to (B \to C)) \to ((\neg C) \to (\neg\neg(A \to (\neg B)))) \quad \text{(xi, xiv, 6)}$$
xvi $$((\neg C) \to (\neg\neg(A \to (\neg B)))) \to ((\neg(A \to (\neg B))) \to C) \quad (8)$$
xvii $$(A \to (B \to C)) \to ((\neg(A \to (\neg B))) \to C) \quad \text{(xv, xvi, 6)}$$
xviii $$(A \to (B \to C)) \to ((A \wedge B) \to C) \quad \text{(xvii, 23)}$$

定理：

$$\vdash ((A \wedge B) \to C) \to (A \to (B \to C)) \quad (27)$$

證明：

i $C \to (\neg\neg C)$ (11)

ii $(\neg(A \to (\neg B))) \to (C \to (\neg\neg C))$ (i, 3)

iii $((\neg(A \to (\neg B))) \to C) \to ((\neg(A \to (\neg B))) \to (\neg\neg C))$ (ii, 4)

iv $((A \wedge B) \to C) \to ((\neg(A \to (\neg B))) \to (\neg\neg C))$ (iii, 23)

v $((\neg(A \to (\neg B))) \to (\neg\neg C)) \to ((\neg C) \to (A \to (\neg B)))$ (8)

vi $((A \wedge B) \to C) \to ((\neg C) \to (A \to (\neg B)))$ (iv, v, 6)

vii $((\neg C) \to (A \to (\neg B))) \to (((\neg C) \to A) \to ((\neg C) \to (\neg B)))$ (2)

viii $((A \wedge B) \to C) \to (((\neg C) \to A) \to ((\neg C) \to (\neg B)))$ (vi, vii, 6)

ix $(((\neg C) \to A) \to ((\neg C) \to (\neg B))) \to (A \to (((\neg C) \to A) \to ((\neg C) \to (\neg B))))$ (1)

x $((A \wedge B) \to C) \to (A \to (((\neg C) \to A) \to ((\neg C) \to (\neg B))))$ (viii, ix, 6)

xi $(A \to (((\neg C) \to A) \to ((\neg C) \to (\neg B)))) \to$ (2)
$(((A \to (\neg C) \to A)) \to (A \to ((\neg C) \to (\neg B))))$

xii $A \to ((\neg C) \to A)$ (1)

xiii $(A \to (((\neg C) \to A) \to ((\neg C) \to (\neg B)))) \to (A \to ((\neg C) \to (\neg B)))$ (xi, xii, 7)

xiv $((A \wedge B) \to C) \to (A \to ((\neg C) \to (\neg B)))$ (x, xiii, 6)

xv $((\neg C) \to (\neg B)) \to (B \to C)$ (8)

xvi $A \to ((\neg C) \to (\neg B)) \to (B \to C)$ (xv, 3)

xvii $(A \to ((\neg C) \to (\neg B))) \to (A \to (B \to C))$ (xvi, 4)

xviii $((A \wedge B) \to C) \to (A \to (B \to C))$ (xiv, xvii, 6)

定理：

$$\vdash (A \wedge B) \to A \tag{28}$$

證明：

i $(A \to (B \to A)) \to ((A \wedge B) \to A)$ (27)

ii $A \to (B \to A)$ (1)

iii $(A \wedge B) \to A$ (i, ii, MP)

定理：

$$\vdash A \to (B \to (A \wedge B)) \tag{29}$$

證明：

i $((A \wedge B) \to (A \wedge B)) \to (A \to (B \to (A \wedge B)))$ (26)

ii $((A \wedge B) \to (A \wedge B))$ (5)

iii $A \to (B \to (A \wedge B))$ (i, ii, MP)

定理：

$$\vdash (A \wedge (B \wedge C)) \to ((A \wedge B) \wedge C) \tag{30}$$

證明：

i $(A \wedge B) \to (C \to ((A \wedge B) \wedge C))$ (29)

ii $((A \wedge B) \to (C \to ((A \wedge B) \wedge C))) \to (A \to (B \to (C \to ((A \wedge B) \wedge C))))$ (27)

iii $A \to (B \to (C \to ((A \wedge B) \wedge C)))$ (i, ii, MP)

iv $(B \to (C \to ((A \wedge B) \wedge C))) \to ((B \wedge C) \to ((A \wedge B) \wedge C))$ (26)

v $A \to ((B \wedge C) \to ((A \wedge B) \wedge C))$ (iii, iv, 6)

vi $(A \to ((B \wedge C) \to ((A \wedge B) \wedge C))) \to ((A \wedge (B \wedge C)) \to ((A \wedge B) \wedge C))$ (26)

vii $(A \wedge (B \wedge C)) \to ((A \wedge B) \wedge C)$ (v, vi, MP)

定理：

$$\vdash ((A \wedge B) \wedge C) \to (A \wedge (B \wedge C)) \tag{31}$$

證明：

i $A \to ((B \wedge C) \to (A \wedge (B \wedge C)))$ (29)

ii $((B \wedge C) \to (A \wedge (B \wedge C))) \to (B \to (C \to (A \wedge (B \wedge C))))$ (27)

iii $A \to (B \to (C \to (A \wedge (B \wedge C))))$ (i, ii, 6)

iv $(A \to (B \to (C \to (A \wedge (B \wedge C))))) \to ((A \wedge B) \to (C \to (A \wedge (B \wedge C))))$ (26)

v $(A \wedge B) \to (C \to (A \wedge (B \wedge C)))$ (iii, iv, MP)

vi $((A \wedge B) \to (C \to (A \wedge (B \wedge C)))) \to (((A \wedge B) \wedge C) \to (A \wedge (B \wedge C)))$ (26)

vii $((A \wedge B) \wedge C) \to (A \wedge (B \wedge C))$ (v, vi, MP)

定義：

$$A \vee B := (\neg A) \to B \tag{32}$$

定理：

$$\vdash (A \vee B) \to (B \vee A) \tag{33}$$

證明：

| | | |
|---|---|---|
| i | $B \to (\neg\neg B)$ | (11) |
| ii | $(\neg A) \to (B \to (\neg\neg B))$ | (i, 3) |
| iii | $((\neg A) \to B) \to ((\neg A) \to (\neg\neg B))$ | (ii, 4) |
| iv | $((\neg A) \to (\neg\neg B)) \to ((\neg B) \to A)$ | (8) |
| v | $((\neg A) \to B) \to ((\neg B) \to A)$ | (iii, iv, 6) |
| vi | $(A \vee B) \to (B \vee A)$ | (v, 32) |

定理：

$$\vdash A \to (A \vee B) \tag{34}$$

證明：

| | | |
|---|---|---|
| i | $A \to ((\neg B) \to A)$ | (1) |
| ii | $A \to (B \vee A)$ | (i, 32) |
| iii | $(B \vee A) \to (A \vee B)$ | (33) |
| iv | $A \to (A \vee B)$ | (ii, iii, 6) |

定理：

$$\vdash (A \vee (B \vee C)) \to ((A \vee B) \vee C) \tag{35}$$

證明：

| | | |
|---|---|---|
| i | $B \to (\neg\neg B)$ | (11) |

## 2 set 集合

在集合當中，引入了兩個新的符號 $\in, =$，及一個概念 $S(\text{set})$。
以下是關於這些的公理：
公理：

$$\vdash (A = B) \leftrightarrow (\forall x)[(A \in x) \leftrightarrow (B \in x)] \tag{36}$$

公理：

$$\vdash (A = B) \leftrightarrow (\forall x)[(x \in A) \leftrightarrow (x \in B)] \tag{37}$$